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APPLICATIONS OF GENERALIZED RUSCHEWEYH DERIVATIVE TO UNIVALENT FUNCTIONS WITH FINITELY MANY COEFFICIENTS

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Abstract. By making use of the generalized Ruscheweyh derivative, the authors investigate several interesting properties of certain subclasses of univalent functions having the form

$$f(z) = z - \sum_{n=2}^{m} \frac{(1-\beta)e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} z^n - \sum_{k=m+1}^{\infty} a_k z^k.$$

1 Introduction

Let $A(n,p)$ denote the class of functions f normalized by $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$, $(p, n \in \mathbb{N} = \{1,2,3,4,....\})$, which are analytic and multivalent in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. The function $f(z)$ is said to be starlike of order $\delta(0 \le \delta < p)$ if and only if $Re\left(\frac{zf'(z)}{f(z)}\right) > \delta, (z \in \mathcal{U})$. On the other hand $f(z)$ is said to be convex of order $\delta(0 \le \delta < p)$ if and only if $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta, (z \in \mathcal{U})$.

Now for $\alpha \ge 0, 0 \le \beta < 1$ and $\lambda > -1, \mu, \nu \in \mathbb{R}$, the following classes

$$\mathcal{M}_p^{\lambda,\mu}(\alpha,\beta) = \left\{ f(z) \in A(n,p) : Re\left\{ \frac{\mathcal{J}_p^{\lambda,\mu} f(z)}{z(\mathcal{J}_p^{\lambda,\mu} f(z))'} \right\} > \alpha \left| \frac{\mathcal{J}_p^{\lambda,\mu} f(z)}{z(\mathcal{J}_p^{\lambda,\mu} f(z))'} - p \right| + \beta \right\} \tag{1.1}$$

$$\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta) = \left\{ f(z) \in A(1,1) : Re\left\{ \frac{\mathcal{J}_1^{\lambda,\mu} f(z)}{z(\mathcal{J}_1^{\lambda,\mu} f(z))'} \right\} > \alpha \left| \frac{\mathcal{J}_1^{\lambda,\mu} f(z)}{z(\mathcal{J}_1^{\lambda,\mu} f(z))'} - 1 \right| + \beta \right\} \tag{1.2}$$

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where $\mathcal{J}_1^{\lambda,\mu} f(z)$ is the generalized Ruscheweyh derivative of f defined by

$$\mathcal{J}_1^{\lambda,\mu} f(z) = \frac{\Gamma(\mu-\lambda+\nu+2)}{\Gamma(\nu+2)\Gamma(\mu+1)} z J_{0,z}^{\lambda,\mu,\nu}(z^{\mu-1}f(z)) \tag{1.3}$$

where $J_{0,z}^{\lambda,\mu,\nu} f(z)$ generalized fractional derivative operator of order λ (see eg.[1], [5], [7], [8]).

We can write (1.3) as

$$\mathcal{J}_1^{\lambda,\mu} f(z) = z - \sum_{k=n+1}^{\infty} a_k B_1^{\lambda,\mu}(k) z^k \tag{1.4}$$

where

$$B_1^{\lambda,\mu}(k) = \frac{\Gamma(k+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu+1)}{\Gamma(k)\Gamma(k+\nu+1+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}. \tag{1.5}$$

Consider the class $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$ is subclass of $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta)$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{m} \frac{(1-\beta)e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} z^n - \sum_{k=m+1}^{\infty} a_k z^k, \tag{1.6}$$

where

$$e_n = \frac{[1-n\beta+\alpha(1-n)]B_1^{\lambda,\mu}(n)}{1-\beta} a_n.$$

Many authors have studied the different cases (e.g. [2], [4], [6]), and for $\lambda = \mu, \nu = 1$, we get the case was studied by [3].

2 Coefficient Bounds

In order to prove our results, we need the following Lemma due to A.R.S.Juma and S. R. Kulkarni [2].

Lemma 1. *Let $f(z) = z^p - \sum\limits_{k=m+p}^{\infty} a_k z^k$. Then $f(z) \in \mathcal{M}_p^{\lambda,\mu}(\alpha,\beta)$ if and only if*

$$\sum_{k=m+p}^{\infty} \frac{[(1+\alpha)-k(\alpha p+\beta)]}{[(1+\alpha)-p(\alpha p+\beta)]} B_p^{\lambda,\mu}(k) a_k < 1. \tag{2.1}$$

First we prove

Theorem 2. *Let the function* $f(z)$ *defined by* (1.6). *Then* $f \in \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$ *if and only if*

$$\sum_{k=m+1}^{\infty} \frac{(1-k\beta)+\alpha(1-k)}{1-\beta} B_1^{\lambda,\mu}(k) a_k < 1 - \sum_{n=2}^{m} e_n. \tag{2.2}$$

Proof. Let

$$a_n = \frac{(1-\beta)e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)}. \tag{2.3}$$

We can say that $f \in \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m) \subset \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{m} \frac{[(1-n\beta)+\alpha(1-n)]B_1^{\lambda,\mu}(n)}{1-\beta} a_n + \sum_{k=m+1}^{\infty} \frac{[1-k\beta+\alpha(1-k)]B_1^{\lambda,\mu}(k)}{1-\beta} a_k < 1$$

or

$$\sum_{k=m+1}^{\infty} \frac{[1-k\beta+\alpha(1-k)]B_1^{\lambda,\mu}(k)}{1-\beta} a_k < 1 - \sum_{n=2}^{m} e_n.$$

□

Corollary 3. *Let* $f(z)$ *defined by* (1.6) *be in* $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$. *Then for* $k \geq m+1$ *we have*

$$a_k \leq \frac{(1-\beta)(1-\sum_{n=2}^{\infty} e_n)}{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}, \tag{2.4}$$

this result is sharp for

$$h(z) = z - \sum_{n=2}^{m} \frac{(1-\beta)e_n}{[(1-n\beta)+\alpha(1-n)]B_1^{\lambda,\mu}(n)} z^n - \frac{(1-\beta)(1-\sum_{n=2}^{\infty} e_n)}{[(1-k\beta)+\alpha(1-k)]B_1^{\lambda,\mu}(k)} z^k. \tag{2.5}$$

Corollary 4. *Let* f *defined by* (1.6). *Then* $f \in \mathcal{M}_{1,1}^{0,0}(0,\beta,e_m)$, *that is,*

$$f(z) = z - \sum_{n=2}^{m} \frac{e_n(1-\beta)}{1-n\beta} z^n - \sum_{k=m+1}^{\infty} a_k z^k,$$

if and only if

$$\sum_{k=m+1}^{\infty} \frac{1-k\beta}{1-\beta} a_k < 1 - \sum_{n=2}^{m} e_n.$$

Corollary 5. *Let f defined by (1.6). Then $f \in \mathcal{M}_{1,1}^{1,0}(0,\beta,e_m)$, that is,*

$$f(z) = z - \sum_{n=2}^{m} \frac{(1-\beta)(\nu+1)e_n}{(1-n\beta)\Gamma(n+\nu)} = z^n - \sum_{k=m+1}^{\infty} a_k z^k$$

if and only if

$$\sum_{k=m+1}^{\infty} \frac{(1-k\beta)\Gamma(k+\nu)}{(1-\beta)(\nu+1)} a_k < 1 - \sum_{n=2}^{m} e_n.$$

Corollary 6. *Let f defined by (1.6). Then $f \in \mathcal{M}_{1,1}^{0,0}(\alpha,0,e_m)$, that is,*

$$f(z) = z - \sum_{n=2}^{m} \frac{e_n}{1+\alpha(1-n)} z^n - \sum_{k=m+1}^{\infty} a_k z^k$$

if and only if

$$\sum_{k=m+1}^{\infty} (1+\alpha(1-k))a_k < 1 - \sum_{n=2}^{m} e_n.$$

Corollary 7. *Let f defined by (1.6). Then $f \in \mathcal{M}_{1,1}^{1,0}(\alpha,0,e_m)$, that is,*

$$f(z) = z - \sum_{n=2}^{m} \frac{e_n(\nu+1)}{(1+\alpha(1-n))\Gamma(n+\nu)} z^n - \sum_{k=m+1}^{\infty} a_k z^k$$

if and only if

$$\sum_{k=m+1}^{\infty} \frac{[1+\alpha(1-k)]\Gamma(k+\nu)}{(\nu+1)} a_k < 1 - \sum_{n=2}^{m} e_n.$$

We claim that all these results are entirely new.

3 Extreme points and other results

Theorem 8. *Let $f_1(z), f_2(z), ..., f_\ell(z)$ defined by*

$$f_i(z) = z - \sum_{n=2}^{m} \frac{(1-\beta)e_n}{[1-n\beta+\alpha(1-n)]B_1^{\lambda,\mu}(n)} z^n - \sum_{k=m+1}^{\infty} a_{k,i} z^k, \tag{3.1}$$

$(i = 1, \cdots, \ell)$ be in the class $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$. Then $G(z)$ defined by

$$G(z) = \sum_{i=1}^{\ell} \lambda_i f_i \quad and \quad \sum_{i=1}^{\ell} \lambda_i = 1, 0 \le \sum_{n=2}^{m} e_n \le 1, 0 \le e_n \le 1$$

is also in this class.

Proof. In view of Theorem 2, we have

$$\sum_{k=m+1}^{\infty} \frac{[1-k\beta-\alpha(1-k)]B_1^{\lambda,\mu}(k)}{1-\beta} a_{k,i} < 1-\sum_{n=2}^{m} e_n$$

for every $i=1,2,\cdots,\ell$. Here

$$G(z)=\sum_{i=1}^{\ell}\lambda_i f_i = z-\sum_{n=2}^{m}\frac{(1-\beta)e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} - \sum_{k=m+1}^{\infty}(\sum_{i=1}^{\ell}\lambda_i a_{k,i})z^k.$$

Thus,

$$\begin{aligned}
&\sum_{k=m+1}^{\infty}\frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{1-\beta}(\sum_{i=1}^{\ell}\lambda_i a_{k,i})\\
&= \sum_{i=1}^{\ell}\sum_{k=m+1}^{\infty}\left(\frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{1-\beta}a_{k,i}\right)\lambda_i\\
&< \sum_{i=1}^{\ell}(1-\sum_{n=2}^{m}e_n)\lambda_i = 1-\sum_{n=2}^{m}e_n.
\end{aligned}$$

□

Remark 9. *The function* $G(z)=\frac{1}{2}[f_1(z)+f_2(z)]$ *belongs to* $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$ *if* $f_1(z), f_2(z)$ *are in the class* $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$*.*

Remark 10. *The class* $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$ *is convex set.*

Theorem 11. *Let* $f_i(z), (i=1,\cdots,\ell)$ *defined by* (3.1) *be in* $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_n)$*. Then the function*

$$F(z)=z-\sum_{n=2}^{m}\frac{e_n(1-\beta)}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)}z^n - \sum_{k=m+1}^{\infty} b_k z^k \quad (b_k\geq 0)$$

is also in the class $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$*, where* $b_k=\frac{1}{\ell}\sum_{i=1}^{\ell}a_{k,i}$*.*

Proof. It is clear that

$$\begin{aligned}
&\sum_{k=m+1}^{\infty}\frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{1-\beta}b_k = \sum_{k=m+1}^{\infty}\frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{\ell(1-\beta)}(\sum_{i=1}^{\ell}a_{k,i})\\
&= \frac{1}{\ell}\sum_{i=1}^{\ell}\left(\sum_{k=m+1}^{\infty}\frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{1-\beta}a_{k,i}\right),
\end{aligned}$$

by Theorem 2, we have the last expression is less than $\frac{1}{\ell}\sum\limits_{i=1}^{\ell}(1-\sum\limits_{n=2}^{m}e_n) = 1 - \sum\limits_{n=2}^{m}e_n$. □

The next Theorem is very useful to obtain the extreme points of the class $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$.

Theorem 12. *Let*

$$f_m(z) = z - \sum_{n=2}^{m}\frac{e_n(1-\beta)}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)}z^n \tag{3.2}$$

and for $k \geq m+1$

$$f_k(z) = z - \sum_{n=2}^{m}\frac{e_n(1-\beta)}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)}z^n - \frac{(1-\beta)(1-\sum\limits_{n=2}^{m}e_n)}{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}z^k. \tag{3.3}$$

Then the function $f(z) \in \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$ *if and only if it can be expressed in the form* $f(z) = \sum\limits_{k=m}^{\infty}\delta_k f_k(z)$, *where* $\delta_k \geq 0, (k \geq m)$ *and* $\sum\limits_{k=m}^{\infty}\delta_k = 1$.

Proof. Let

$$\begin{aligned}
f(z) &= \sum_{k=m+1}^{\infty}\delta_k f_k(z) + \delta_m f_m(z)\\
&= \delta_m z - \delta_m\sum_{n=2}^{m}\frac{e_n(1-\beta)}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)}z^n\\
&\quad + \sum_{k=m+1}^{\infty}\delta_k z - \sum_{k=m+1}^{\infty}\delta_k\left(\sum_{n=2}^{m}\frac{e_n(1-\beta)}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)}z^n\right)\\
&\quad - \sum_{k=m+1}^{\infty}\delta_k\left(\frac{(1-\sum\limits_{n=2}^{m}e_n)(1-\beta)}{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}z^k\right)\\
&= (\delta_m + \sum_{k=m+1}^{\infty}\delta_k)z - (\delta_m + \sum_{k=m+1}^{\infty}\delta_k)\sum_{n=2}^{m}\frac{e_n(1-\beta)}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)}z^n\\
&\quad - \sum_{k=m+1}^{\infty}\frac{(1-\sum\limits_{n=2}^{m}e_n)(1-\beta)}{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}\delta_k z^k
\end{aligned}$$

$$= \quad z - \sum_{n=2}^{m} \frac{e_n(1-\beta)}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} z^n - \sum_{k=m+1}^{\infty} \frac{(1-\sum\limits_{n=2}^{m} e_n)(1-\beta)}{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)} \delta_k z^k,$$

therefore at the end we can write $\sum\limits_{k=m+1}^{\infty} \frac{(1-k\beta+\alpha(1-k))(1-\sum\limits_{n=2}^{m} e_n)(1-\beta)}{(1-\beta)(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)} \delta_k B_1^{\lambda,\mu}(k)$

$$= (1 - \sum_{n=2}^{m} e_n) \sum_{k=m+1}^{\infty} \delta_k = (1 - \sum_{n=2}^{m} e_n)(1-\delta_m) < 1 - \sum_{n=2}^{m} e_n.$$

Thus $f \in \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$. □

Conversely, let $f \in \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$, that is,

$$f(z) = z - \sum_{n=2}^{m} \frac{e_n(1-\beta)}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} z^n - \sum_{k=m+1}^{\infty} a_k z^k,$$

put

$$\delta_k = \frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{(1-\beta)(1-\sum\limits_{n=2}^{m} e_n)} a_k \quad (k \geq m+1)$$

we have $\delta_k \geq 0$ and if we set $\delta_m = 1 - \sum\limits_{k=m+1}^{\infty} \delta_k$, then we have

$$\begin{aligned} f(z) &= z - \sum_{n=2}^{m} \frac{e_n(1-\beta)}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} z^n - \sum_{k=m+1}^{\infty} \frac{(1-\sum\limits_{n=2}^{m} e_n)(1-\beta)}{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)} \delta_k z^k \\ &= f_m(z) - \sum_{k=m+1}^{\infty} \left(z - \sum_{n=2}^{m} \frac{e_n(1-\beta)}{(1-\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} z^n - f_k(z) \right) \delta_k \\ &= f_m(z) - \sum_{k=m+1}^{\infty} (f_m(z) - f_k(z))\delta_k \\ &= (1 - \sum_{k=m+1}^{\infty} \delta_k) f_m(z) + \sum_{k=m+1}^{\infty} \delta_k f_k(z) = \sum_{k=m+1}^{\infty} \delta_k f_k(z). \end{aligned}$$

Corollary 13. *The extreme points of the class $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$ are the functions $f_k(z), (k \geq m)$, defined by (3.2), (3.3).*

Theorem 14. *Let $f(z)$ defined by (1.6) be in the class $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$. Then $f(z)$ is starlike of order $\delta (0 \le \delta < 1)$ in $|z| < r$, where r is the largest value such that*

$$\sum_{n=2}^{m} \frac{e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} r^{n-1} + \sum_{k=m+1}^{\infty} \frac{(1-\sum_{n=2}^{m} e_n)}{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)} r^{k-1}$$
$$< \frac{1}{1-\beta}, (k \ge m+1). \tag{3.4}$$

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta. \tag{3.5}$$

Therefore

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{z - \sum_{n=2}^{m} n \frac{(1-\beta)e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} z^n - \sum_{k=m+1}^{\infty} k a_k z^k}{z - \sum_{n=2}^{m} \frac{(1-\beta)e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} z^n - \sum_{k=m+1}^{\infty} a_k z^k} - 1 \right|$$

$$\le \frac{\sum_{n=2}^{m} \frac{(n-1)(1-\beta)e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} |z|^{n-1} + \sum_{k=m+1}^{\infty} (k-1) a_k |z|^{k-1}}{1 - \sum_{n=2}^{m} \frac{(1-\beta)e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} |z|^{n-1} - \sum_{k=m+1}^{\infty} a_k |z|^{k-1}}$$

$$< \frac{\sum_{n=2}^{m} \frac{(n-1)(1-\beta)e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} r^{n-1} + \sum_{k=m+1}^{\infty} \frac{(k-1)(1-\beta)(1-\sum_{n=2}^{m} e_n)}{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)} r^{k-1}}{1 - \sum_{n=2}^{m} \frac{(1-\beta)e_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} r^{n-1} - \sum_{k=m+1}^{\infty} \frac{(1-\beta)(1-\sum_{n=2}^{m} e_n)}{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)} r^{k-1}}.$$

Thus (3.5) holds true if the last expression is less than $1-\delta$ or,

$$\sum_{n=2}^{m} \frac{(n-\delta)(1-\beta)e_n}{(1-\delta)(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} r^{n-1}$$
$$+ \sum_{k=m+1}^{\infty} \frac{(k-\delta)(1-\beta)(1-\sum_{n=2}^{m} e_n)}{(1-\delta)(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)} r^{k-1} < 1$$

at the end we find (3.4). □

Making use the following Theorem we obtain the next corollary.

Theorem 15. *[**Alexander's Theorem**] Let f be analytic in $\mathcal{U}$ with $f(0) = f'(0) - 1 = 0$. Then f is convex function if and only if zf' is starlike function.*

Corollary 16. *Let $f \in \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha, \beta, e_m)$. Then $f(z)$ is convex of order $\delta (0 \leq \delta < 1)$ in $|z| < r$ where r is the largest value such that*

$$\sum_{n=2}^{m} \frac{n e_n}{(1 - n\beta + \alpha(1-n))B_1^{\lambda,\mu}(n)} r^{n-1} + \frac{k(1 - \sum_{n=2}^{m} e_n)}{(1 - k\beta + \alpha(1-k))B_1^{\lambda,\mu}(k)} r^{k-1} < \frac{1}{1-\beta}.$$

Theorem 17. *Let $f \in \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha, \beta, e_m)$ and $\lambda > 0$, if*

$$d_n = \frac{(1-\beta)e_n^2}{(1 - n\beta + \alpha(1-n))B_1^{\lambda,\mu}(n)} \qquad (2 \leq n \leq m), \tag{3.6}$$

then the function

$$H(z) = z - \sum_{n=2}^{m} \frac{(1-\beta)d_n}{((1 - n\beta) + \alpha(1-n))B_1^{\lambda,\mu}(n)} z^n - \sum_{k=m+1}^{\infty} a_k z^k$$

is also in the class $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha, \beta, e_m)$.

Proof. By assumption we have $(1 - n\beta + \alpha(1-n))B_1^{\lambda,\mu}(n) > 1$ therefore,

$$d_n = \frac{(1-\beta)e_n^2}{(1 - n\beta + \alpha(1-n))B_1^{\lambda,\mu}(n)} < e_n \leq 1.$$

So, $0 \leq \sum_{n=2}^{m} d_n < \sum_{n=2}^{m} e_n \leq 1$, thus

$$\sum_{k=m+1}^{\infty} \frac{(1 - k\beta + \alpha(1-k))B_1^{\lambda,\mu}(k)}{(1-\beta)(1 - \sum_{n=2}^{m} d_n)} a_k < \sum_{k=m+1}^{\infty} \frac{(1 - k\beta + \alpha(1-k))B_1^{\lambda,\mu}(k)}{(1-\beta)(1 - \sum_{n=2}^{m} e_n)} < 1.$$

□

Theorem 18. *Let $f, g \in \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha, \beta, e_m)$ and $\lambda > 0$. Then*

$$(f * g)(z) = z - \sum_{n=2}^{m} \frac{(1-\beta)^2 e_n^2}{(1 - n\beta + \alpha(1-n))^2 (B_1^{\lambda,\mu}(n))^2} z^n - \sum_{k=m+1}^{\infty} a_k b_k z^k$$

is also in the class $\mathcal{M}_{1,1}^{\lambda,\mu}(\alpha, \beta, d_m)$ if $\lambda_1 \leq \inf_k \left[\frac{(B_1^{\lambda,\mu}(k))^2}{(1 - \sum_{n=2}^{m} d_n)} - 1 \right]$, where $d_n (2 \leq n \leq m)$ are defined by (3.6).

Proof. By making use of (3.6), we have

$$(f*g)(z) = z - \sum_{n=2}^{m} \frac{(1-\beta)d_n}{(1-n\beta+\alpha(1-n))B_1^{\lambda,\mu}(n)} z^n - \sum_{k=m+1}^{\infty} a_k b_k z^k.$$

By Theorem 17, we have

$$\sum_{k=m+1}^{\infty} \frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{(1-\beta)(1-\sum\limits_{n=2}^{m} d_n)} a_k < 1$$

and

$$\sum_{k=m+1}^{\infty} \frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{(1-\beta)(1-\sum\limits_{n=2}^{\infty} d_n)} b_k < 1.$$

Therefore, by Cauchy-Schwarz inequality we obtain

$$\sum_{k=m+1}^{\infty} \frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{(1-\beta)(1-\sum\limits_{n=2}^{m} d_n)} \sqrt{a_k b_k} < 1. \tag{3.7}$$

Now, we want to prove that

$$\sum_{k=m+1}^{\infty} \frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{(1-\beta)(1-\sum\limits_{n=2}^{m} d_n)} a_k b_k < 1. \tag{3.8}$$

In view of (3.7) the inequality (3.8) holds true if

$$\sqrt{a_k b_k} \frac{B_1^{\lambda_1,\mu}(k)}{B_1^{\lambda,\mu}(k)} < 1. \tag{3.9}$$

But we have

$$\frac{B_1^{\lambda,\mu}(k)}{1-\sum\limits_{n=2}^{m} d_n} \sqrt{a_k b_k} < \frac{(1-k\beta+\alpha(1-k))B_1^{\lambda,\mu}(k)}{(1-\beta)(1-\sum\limits_{n=2}^{m} d_n)} \sqrt{a_k b_k} < 1.$$

Therefore (3.9) holds true if

$$\frac{1-\sum\limits_{n=2}^{m} d_n}{B_1^{\lambda,\mu}(k)} < \frac{B_1^{\lambda,\mu}(k)}{B_1^{\lambda_1,\mu}(k)}, \quad \text{or} \quad B_1^{\lambda_1,\mu}(k) < \frac{(B_1^{\lambda,\mu}(k))^2}{1-\sum\limits_{n=2}^{m} d_n}.$$

Then $\lambda_1 < \frac{(B_1^{\lambda,\mu}(k))^2}{1-\sum\limits_{n=2}^{m} d_n} - 1.$ □

Theorem 19. *Let* $f(z) \in \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$. *Then*

$$\mathcal{J}_1^{\lambda,\mu} f(z) = \exp\left(\int_0^z \frac{E(t)-\alpha}{(\beta E(t)-\alpha)t}dt\right), |E(z)| < 1, z \in \mathcal{U}.$$

Proof. The case $\alpha = 0$ is obvious. Therefore, suppose that $\alpha \neq 0$. Then for $f \in \mathcal{M}_{1,1}^{\lambda,\mu}(\alpha,\beta,e_m)$ and let $w = \frac{\mathcal{J}_1^{\lambda,\mu} f(z)}{z(\mathcal{J}_1^{\lambda,\mu} f(z))'}$ we have $Re(w) > \alpha|w-1| + \beta$, therefore, $\left|\frac{w-1}{w-\beta}\right| < \frac{1}{\alpha}$ or $\frac{w-1}{w-\beta} = \frac{E(z)}{\alpha}$, where $|E(z)| < 1, z \in \mathcal{U}$. This yields

$$\frac{\frac{\mathcal{J}_1^{\lambda,\mu} f(z)}{z(\mathcal{J}_1^{\lambda,\mu} f(z))'} - 1}{\frac{\mathcal{J}_1^{\lambda,\mu} f(z)}{z(\mathcal{J}_1^{\lambda,\mu} f(z))'} - \beta} = \frac{E(z)}{\alpha} \quad \text{or} \quad \frac{\mathcal{J}_1^{\lambda,\mu} f(z) - z(\mathcal{J}_1^{\lambda,\mu} f(z))'}{\mathcal{J}_1^{\lambda,\mu} f(z) - \beta z(\mathcal{J}_1^{\lambda,\mu} f(z))'} = \frac{E(z)}{\alpha}.$$

Thus $\dfrac{(\mathcal{J}_1^{\lambda,\mu} f(z))'}{(\mathcal{J}_1^{\lambda,\mu} f(z))} = \dfrac{E(z)-\alpha}{(\beta E(z)-\alpha)z}$. □

By the integration we get the result.

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