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POSITIVE DEFINITE SOLUTION OF TWO KINDS OF NONLINEAR MATRIX EQUATIONS

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Abstract. Based on the elegant properties of the Thompson metric, we prove that the following two kinds of nonlinear matrix equations $X = \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i$ and $X = \sum_{i=1}^{m} (A_i^* X A_i)^{\delta_i}, \quad (0 < |\delta_i| < 1)$ always have a unique positive definite solution. Iterative methods are proposed to compute the unique positive definite solution. We show that the iterative methods are more effective as $\delta = max\{|\delta_i|, \quad i = 1, 2, \cdots, m\}$ decreases. Perturbation bounds for the unique positive definite solution are derived in the end.

Full text

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