Twist Automorphisms and Poisson Structures

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Abstract. We introduce (quantum) twist automorphisms for upper cluster algebras and cluster Poisson algebras with coefficients. Our constructions generalize the twist automorphisms for quantum unipotent cells. We study their existence and their compatibility with Poisson structures and quantization. The twist automorphisms always permute well-behaved bases for cluster algebras. We explicitly construct (quantum) twist automorphisms of Donaldson–Thomas type and for principal coefficients.

Key words: cluster algebras; twist automorphisms; Poisson algebras

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1 Introduction

1.1 Background

Cluster algebras

The theory of cluster algebras were introduced by Fomin and Zelevinsky [13] as a combinatorial framework to study the dual canonical bases of quantum groups [28, 34, 35]. In this theory, one has the cluster A-variety $\mathbb{A}$ (also called the cluster K_2-variety). It is a scheme equipped with a cluster structure: $\mathbb{A}$ is the union of many tori which are glued by birational maps called mutations [24]. Let $\mathcal{U}^A$ denote the "function ring" of $\mathbb{A}$ (by which we mean the ring of global sections of its structure sheaf; it is the coordinate ring when $\mathbb{A}$ is affine). Then $\mathcal{U}^A$ is called the upper cluster algebra (or upper cluster A-algebra). Under a mild assumption (*full rank assumption*), one can endow $\mathbb{A}$ with a Poisson structure [19, 20]. Correspondingly, the upper cluster algebra $\mathcal{U}^A$ becomes a Poisson algebra, which can be naturally quantized [5].

One also has the cluster X-variety $\mathbb{X}$ (also called the cluster Poisson variety). It is a scheme equipped with the same cluster structure: $\mathbb{X}$ is the union of many dual tori which are glued by mutations. It has a canonical Poisson structure. Let $\mathcal{U}^X$ denote its function ring, which is called the cluster Poisson algebra (or upper cluster X-algebra). Then $\mathcal{U}^X$ is a Poisson algebra, and one can quantize it naturally.

Fock and Goncharov [9, 10] found that the cluster varieties $\mathbb{A}$ and $\mathbb{X}$ naturally arise in the study of the (higher) Teichmüller theory of a surface. They further conjectured that the upper cluster algebra $\mathcal{U}^A$ should possess a basis naturally parametrized by the tropical points of the cluster X-variety associated to the *Langlands dual* cluster structure and, conversely, the cluster Poisson algebra $\mathcal{U}^X$ should possess a basis naturally parametrized by the tropical points of the

cluster A-variety associated to the Langlands dual cluster structure. (By [25], Fock–Goncharov conjecture is true for many cases, but might not be true in general.)

In view of Fock–Goncharov conjecture, it is important to understand the upper cluster A-algebras, upper cluster X-algebras and their bases.

On the one hand, many well-known cluster A-varieties (strictly speaking, the sets of their rational points) are smooth manifolds. Examples include the unipotent cells N_-^w [15], double Bruhat cells [3], and top dimensional cells of the Grassmannians $\mathsf{Gr}(k, n)$, $k \leq n \in \mathbb{N}$ [50]. On the other hand, there exists few literature on cluster X-varieties. [26, 49, 51] embedded the Drinfeld double quantum groups of Dynkin types to quantized $\mathcal{U}^X$. [48] embedded a subalgebra of a K-theoretic Coulomb branch to quantized $\mathcal{U}^X$.

The bases for (quantized) $\mathcal{U}^A$ have been extensively studied and they have been related to representation theory and geometry (see the survey [46]). Not much is known for the bases of (quantized) $\mathcal{U}^X$ (see [1, 9, 25] for some results).

Twist automorphisms

[2, 4] introduced an automorphism $\widetilde{\eta}_w$ on the unipotent cell N_-^w. Its gives rise to an automorphism η_w on the coordinate ring $\mathbb{C}[N_-^w]$. η_w was called a twist automorphism and has been studied via cluster algebras [16]. A quantum unipotent cell $\mathsf{A}_{\mathsf{q}}[N_-^w]$ is a quantum analog of the coordinate ring $\mathbb{C}[N_-^w]$, which is defined for all Kac–Moody types. In a joint work [33], the first author introduced the quantum twist automorphism η_w on $\mathsf{A}_{\mathsf{q}}[N_-^w]$ and further showed that the dual canonical basis B^* of $\mathsf{A}_{\mathsf{q}}[N_-^w]$ is permuted by η_w.

By [15, 17, 21, 23], $\mathsf{A}_{\mathsf{q}}[N_-^w]$ is a (quantum) upper cluster algebra $\mathcal{U}^A$. By a motivational conjecture of Fomin and Zelevinsky [13] and its natural generalization [32], the dual canonical basis of $\mathsf{A}_{\mathsf{q}}[N_-^w]$ should contain all cluster monomials, i.e., the monomials of coordinate functions in some toric chart which are globally regular on $\mathbb{A}$. With the help of the existence of the twist automorphism η_w and the fact that it permutes the dual canonical basis B^*, the second author gave a proof of the generalized conjecture for all cases [47]. (See [27, 37, 44] for other approaches and the corresponding results.)

In view of the successful application of the twist automorphism η_w to the (quantum) upper cluster algebra $\mathsf{A}_{\mathsf{q}}[N_-^w]$, it is natural to ask for a twist automorphism in general cases.

1.2   Main results

Preliminaries

Let I denote a finite set of vertices and $I = I_{\mathsf{uf}} \sqcup I_{\mathsf{f}}$ a partition of I. Choose skew-symmetrizers $d_i \in \mathbb{Z}_{>0}$, $i \in I$. The vertices in I_{uf} and I_{f} are called unfrozen and frozen, respectively. A seed t is a collection $((A_i)_{i \in I}, (X_i)_{i \in I}, B)$ whose matrix $B = (b_{ij})_{i,j \in I}$ is $\mathbb{Z}$-valued such that $\frac{1}{d_i} b_{ij} = -\frac{1}{d_j} b_{ji}$, and A_i, X_i are indeterminates called cluster A-variables and cluster X-variables, respectively.

Choose a base ring $\Bbbk = \mathbb{Z}$ or $\mathbb{Z}\big[v^{\pm \frac{1}{\mathsf{d}'}}\big]$ for the classical cases or the quantum cases, respectively, where d' is a sufficiently divisible positive integer. We associate to t two Laurent polynomial rings $\mathcal{LP}^A = \Bbbk\big[A_i^{\pm}\big]_{i \in I}$, $\mathcal{LP}^X = \Bbbk\big[X_i^{\pm}\big]_{i \in I}$. Then, if we take $\Bbbk = \mathbb{C}$, $\mathcal{LP}^{\bullet}$ is the coordinate ring of the corresponding torus $\mathcal{T}^{\bullet} := \mathsf{Spec}\, \mathcal{LP}^{\bullet}$, where $\bullet$ stands for A or X. We denote the Laurent monomials by A^m for any $m = (m_i)_{i \in I} \in \mathbb{Z}^I$ and, similarly, X^n for any $n = (n_i)_{i \in I} \in \mathbb{Z}^I$.

Unless otherwise specified, we will use a v-twisted product on $\mathcal{LP}^{\bullet}$ (see Section 2.3). We will set $v = 1$ if we work at the classical level. Let $\mathcal{F}^{\bullet}$ denote the skew-field of fractions of $\mathcal{LP}^{\bullet}$. A monomial map between $\mathcal{LP}^{\bullet}$ is an algebra homomorphism sending Laurent monomials to Laurent monomials. We have a canonical a monomial map $p^* \colon \mathcal{LP}^X \to \mathcal{LP}^A$. Following [43, 44], we consider the pointed elements, i.e., the elements of the form $S_m^A = A^m \cdot F_m|_{Z_k \mapsto p^* X_k}$ in $\mathcal{LP}^A$ and $S_n^X = X^n \cdot F_n|_{Z_k \mapsto X_k}$ in $\mathcal{LP}^X$, where F_m and F_n are polynomials in indeterminates Z_k,

$k \in I_{\mathsf{uf}}$, with constant term 1 (called F-polynomials). Here, $Z_k \mapsto p^* X_k$ and $Z_k \mapsto X_k$ denote the evaluation of Z_k, respectively, and $\cdot$ denotes the commutative product.

For any unfrozen vertex $k \in I_{\mathsf{uf}}$, we have an algorithm called mutation which generates a new seed

$$\mu_k t = ((A_i(\mu_k t))_{i \in I}, (X_i(\mu_k t))_{i \in I}, B(\mu_k t)).$$

In particular, we have an isomorphism $\mu_k^\bullet \colon \mathcal{F}^\bullet(\mu_k t) \xrightarrow{\sim} \mathcal{F}^\bullet(t)$ called a mutation birational map.

Let Δ_t^+ denote the set of seeds $t' = \boldsymbol{\mu} t$ obtained from t by any finite sequence of mutations $\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ takes the form $\mu_{k_s} \cdots \mu_{k_2} \mu_{k_1}$ and starts from the seed t. Note that the mutation sequence $\boldsymbol{\mu}$ depends on t, but we omit the symbol t as in standard literature. Recall that the cluster $\bullet$-variety is the union $\cup_{t' \in \Delta_t^+} \mathcal{T}^\bullet(t')$ such that the tori (local charts) are glued by mutation birational maps $(\boldsymbol{\mu}^\bullet)^*$ (coordinate change). The upper cluster $\bullet$-algebra $\mathcal{U}^\bullet$ is defined to be its function ring, which turns out to be $\mathcal{U}^\bullet(t) = \cap_{t' \in \Delta_t^+} \boldsymbol{\mu}^* \mathcal{LP}^\bullet(t')$ by choosing any initial seed t (local chart). It is known that, up to identification by mutation maps, these objects are independent of the choice of the initial seed t. So we can simply write $\mathcal{U}^\bullet$.

Constructions and results

Let $t = ((A_i), (X_i), B)$ and $t' = \boldsymbol{\mu} t = ((A_i'), (X_i'), B')$ denote two similar seeds, i.e., there exists a permutation σ on I_{uf} such that $b_{i,j} = b'_{\sigma i, \sigma j}$, $d_i = d_{\sigma i}$, $\forall i, j \in I_{\mathsf{uf}}$. We can relabel the vertices I_{uf} when working with t', so that we can assume $\sigma = \mathsf{Id}$ from now on.

Let $\mathsf{pr}_{I_{\mathsf{uf}}}$ denote the natural projection from $\mathbb{Z}^I$ to $\mathbb{Z}^{I_{\mathsf{uf}}}$. Following [43, 44], two pointed elements $S_m^A \in \mathcal{LP}^A(t)$ and $S_{m'}^A \in \mathcal{LP}^A(t')$ are said to be similar if $\mathsf{pr}_{I_{\mathsf{uf}}} m = \mathsf{pr}_{I_{\mathsf{uf}}} m'$ and they have the same F-polynomial. Similarly, we define two pointed elements $S_n^X \in \mathcal{LP}^X(t)$ and $S_{n'}^X \in \mathcal{LP}^X(t')$ to be similar if $\mathsf{pr}_{I_{\mathsf{uf}}} B(t) n = \mathsf{pr}_{I_{\mathsf{uf}}} B(t') n'$ and they have the same F-polynomial.

Following [43, 44], a variation map (or a correction map, a coefficient twist map) $\mathsf{var}_t^\bullet$ is a map sending pointed elements in $\mathcal{LP}^\bullet(t)$ to similar pointed elements in $\mathcal{LP}^\bullet(t')$, where $\bullet$ stands for A or X, see Remarks 3.5 and 4.5. For the purpose of this paper, we require that $\mathsf{var}_t^\bullet$ is an algebra homomorphism from $\mathcal{LP}^\bullet(t)$ to $\mathcal{LP}^\bullet(t')$.

We define a twist endomorphism $\mathsf{tw}_t^\bullet$ on $\mathcal{U}^\bullet(t)$ to be the composition $\boldsymbol{\mu}^* \mathsf{var}_t^\bullet$ of the mutation map $\boldsymbol{\mu}^*$ with a variation map $\mathsf{var}_t^\bullet$. In the classical case, it is called Poisson if it preserves the Poisson structure. We can show that the construction is independent of the choice of the initial seed, so we can simply say that $\mathsf{tw}^\bullet$ acts on $\mathcal{U}^\bullet$ (Propositions 3.14 and 4.12). Note that different choices of variation maps still give different twist endomorphisms.

Let $\mathcal{Z}^\mathbb{Q}$ denote the multiplicative group generated by the roots of coefficients (frozen cluster A-variables A_j, $j \in I_{\mathsf{f}}$). Let $\mathcal{U}_{\mathcal{Z}^\mathbb{Q}}^A$ denote the ring $\mathcal{U}^A \otimes \Bbbk[\mathcal{Z}^\mathbb{Q}]$. The notion of the twist endomorphism tw^A can be naturally generalized for $\mathcal{U}_{\mathcal{Z}^\mathbb{Q}}^A$.

In the classical case, the existence of the twist endomorphisms tw^A is given by Theorem 3.19 (see Remark 4.21 for the existence of tw^X):

- Assume that the full rank assumption holds. Then the twist endomorphisms tw^A on $\mathcal{U}_{\mathcal{Z}^\mathbb{Q}}^A$ exist, and they are in bijection with the solutions of an inhomogeneous linear equation system.

Let $\mathsf{var}_{t'}^X$ denote a monomial map from $\mathcal{LP}^X(t')$ to $\mathcal{LP}^X(t)$ and var_t^A the corresponding monomial map from $\mathcal{LP}^A(t)_{\mathcal{Z}^\mathbb{Q}}$ to $\mathcal{LP}^A(t')_{\mathcal{Z}^\mathbb{Q}}$ constructed using the pullback (see Theorem 4.19). Let $\mathsf{tw}^\bullet$ denote their compositions with the mutation maps, respectively. Then $\mathsf{tw}^\bullet$ are related by Theorem 4.19:

- tw^X is a twist endomorphism (resp. twist automorphism) on $\mathcal{U}^X$ if and only if tw^A is a twist endomorphism (resp. twist automorphism[1]) on $\mathcal{U}^A_{\mathcal{Z}^\mathbb{Q}}$.

- Assume that tw^X is a twist automorphism on $\mathcal{U}^X$. Then it is Poisson if and only if tw^A and $\left(\mathsf{tw}^X\right)^{-1}$ commute with the natural homomorphism $p^* \colon \mathcal{U}^X \to \mathcal{U}^A$, i.e., $p^*\left(\mathsf{tw}^X\right)^{-1} = \mathsf{tw}^A\, p^*$.

Note that a variation map $\mathsf{var}^X_{t'}$ is Poisson if and only if the corresponding linear map satisfies the quadratic equation in Lemma 4.10.

We also explicitly construct Poisson (or quantum) twist automorphisms $\mathsf{tw}^\bullet$ on $\mathcal{U}^\bullet$ in the following cases:

- The case $t' = t[1]$ (see Definition 2.19 and Theorem 5.2): the corresponding twist automorphisms are said to be of Donaldson–Thomas type (DT-type for short). The original twist automorphism η_w is of this type [47].

- The case when there is a seed of principal coefficients (see Section 2.10 and Theorem 5.4).

We prove general results that well-behaved bases for $\mathcal{U}^\bullet$ satisfying Assumptions 6.1 or 6.3 are permuted by twist automorphisms $\mathsf{tw}^\bullet$ (see Theorems 6.2 and 6.4). In addition, we proposed a method for constructing bases of $\mathcal{U}^X$ in Theorem 6.6.

Recall that we can naturally quantize $\mathcal{U}^\bullet$ if a compatible Poisson structure is given. Then we can lift a Poisson twist endomorphism to a quantum twist endomorphism. Conversely, the classical limit of a quantum twist endomorphism is a Poisson twist endomorphism. See Remarks 3.7 and 4.7.

Remark 1.1 (a comparison with the previous literature)**.** The second author introduced the correction technique to compare similar pointed elements for similar seeds t and t' in [43]. In order to facilitate the comparison, in [44], he introduced a variation map var^A_t sending pointed elements in $\mathcal{LP}^A(t)$ to similar pointed elements in $\mathcal{LP}^A(t')$. The variation map $\mathsf{var}^A \colon \mathcal{LP}^A(t)_{\mathcal{Z}^\mathbb{Q}} \to \mathcal{LP}^A(t')_{\mathcal{Z}^\mathbb{Q}}$ considered in this paper is defined in the same spirit, but chosen slightly differently, such that it becomes an algebra homomorphism. The variation map $\mathsf{var}^X \colon \mathcal{LP}^X(t') \to \mathcal{LP}^X(t)$ has not been considered before.

In the classical case $\Bbbk = \mathbb{Z}$, it is easy to see that a variation map $\mathsf{var}^A \colon \mathcal{LP}^A(t) \to \mathcal{LP}^A(t')$ in this paper is the same as a quasi-homomorphism introduced by Fraser [14]. When one identifies $\mathcal{F}^A(t)$ and $\mathcal{F}^A(t')$ using the mutation map $\boldsymbol{\mu}^*$, a twist endomorphism tw^A on $\mathcal{U}^A$ becomes the same as a quasi-homomorphism for a normalized seed pattern in the sense of [14]. [6] studied the group of the twist automorphisms on $\mathcal{U}^A$ for principal coefficient cases and several other special cases.

To the best of the authors' knowledge, the Poisson or quantum twist endomorphisms as well as the twist endomorphisms tw^X have not been introduced in the previous literature (although specific examples have risen from Lie theory and from higher Teichmüller theory, see Section 7).

Remark 1.2 (morphisms for schemes)**.** Consider the classical cases.

On the one hand, our twist endomorphism $\mathsf{tw}^\bullet$ provides an endomorphism for the affine scheme $\mathsf{Spec}\,\mathcal{U}^\bullet$.

On the other hand, a cluster variety might not be affine, and we do not know if $\mathsf{tw}^\bullet$ provide an endomorphism for it. While the variation map is always a scheme morphism, the mutations might be not.

See [24] for the comparison between the schemes.

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[1]While we only consider upper cluster algebras in this paper, Proposition 3.14 implies that our twist automorphism tw^A restricts to an automorphism on the $\Bbbk[\mathcal{Z}^\mathbb{Q}]$-subalgebra (cluster algebra) generated by the cluster variables.

1.3 Contents

We provide preliminaries for the theory of cluster algebras in Section 2.

In Sections 3 and 4, we introduce (Poisson, quantum) twist endomorphisms for upper cluster A-algebras and upper cluster X-algebras, respectively. We discuss their existence.

In Section 5, we explicitly construct (quantum) twist automorphisms for two special cases: the Donaldson–Thomas type and the principal coefficients.

In Section 6, we prove that a basis with nice properties is permuted by a twist automorphism. We also construct bases for cluster Poisson algebras.

In Section 7, we give some explicit examples for Poisson twist automorphisms.

1.4 Convention

We fix a finite set of vertices I together with a partition $I = I_{\mathsf{uf}} \sqcup I_{\mathsf{f}}$. The elements in I_{uf} and I_{f} are said to be unfrozen and frozen, respectively. We also fix skew-symmetrizers $d_i \in \mathbb{Z}_{>0}$ for $i \in I$. Let D denote the diagonal matrix whose diagonal entries are $\frac{1}{d_i}$, $i \in I$.

We choose a base ring $\Bbbk$ of characteristic 0 and a unit $v \in \Bbbk$. For classical cluster algebras, we choose $\Bbbk = \mathbb{Z}$ and $v = 1$. For quantum cluster algebras, we choose $\Bbbk = \mathbb{Z}\big[v^{\pm\frac{1}{\mathsf{d}'}}\big]$ where v is an indeterminate and $\mathsf{d}' \in \mathbb{N}_{>0}$ is sufficiently divisible. The ring multiplication for cluster algebras will be the v-twisted product $*$. We will also use the commutative product $\cdot$.

An $I \times I$ matrix H is said to be skew-symmetrizable by D if DH is skew-symmetric.

For any $I \times I$-matrix H and any $J_1, J_2 \subset I$, let H_{J_1,J_2} denote the $J_1 \times J_2$-submatrix of H. Then we can denote H as a block matrix:

$$H = \begin{pmatrix} H_{I_{\mathsf{uf}},I_{\mathsf{uf}}} & H_{I_{\mathsf{uf}},I_{\mathsf{f}}} \\ H_{I_{\mathsf{f}},I_{\mathsf{uf}}} & H_{I_{\mathsf{f}},I_{\mathsf{f}}} \end{pmatrix} =: \begin{pmatrix} H_{\mathsf{uf}} & H_{\mathsf{uf},\mathsf{f}} \\ H_{\mathsf{f},\mathsf{uf}} & H_{\mathsf{f}} \end{pmatrix}.$$

Let $\mathsf{col}_i H$ and $\mathsf{row}_i H$ denote the i-th column and the i-th row of H, respectively. For any permutation σ of I (resp. of I_{uf}), let P_σ denote the $I \times I$-matrix (resp. the $I_{\mathsf{uf}} \times I_{\mathsf{uf}}$-matrix) such that $\mathsf{col}_i \mathsf{P}_\sigma$ is the σi-th unit vector. Then we have $\mathsf{col}_k(HP_\sigma) = \mathsf{col}_{\sigma k} H$ as column vectors and $\mathsf{row}_i(P_\sigma H) = \mathsf{row}_{\sigma^{-1} i} H$ as row vectors. We work with column vectors unless otherwise specified.

Assume that σ is a permutation on I_{uf}. If $w = (w_k)_{k \in I_{\mathsf{uf}}}$ is a vector in $\mathbb{R}^{I_{\mathsf{uf}}}$, we define $\sigma w \in \mathbb{R}^{I_{\mathsf{uf}}}$ such that $(\sigma w)_{\sigma k} = w_k$. Similarly, if $\boldsymbol{\mu} = \mu_{k_r} \cdots \mu_{k_2} \mu_{k_1}$ is a sequence of mutations with $k_i \in I_{\mathsf{uf}}$, then $\sigma\boldsymbol{\mu}$ denotes $\mu_{\sigma k_r} \cdots \mu_{\sigma k_2} \mu_{\sigma k_1}$.

For a $\mathbb{Z}$-lattice L, we denote $L_\mathbb{Q} = L \otimes_\mathbb{Z} \mathbb{Q}$. If L has an I-labeled basis $\{u_i \mid i \in I\}$, let L_{uf} and L_{f} denote its sublattices spanned by $\{u_i \mid i \in I_{\mathsf{uf}}\}$ and $\{u_i \mid i \in I_{\mathsf{f}}\}$, respectively.

Let $\mathsf{pr}_{I_{\mathsf{uf}}}$ denote the natural projection from $\mathbb{Z}^I$ to $\mathbb{Z}^{I_{\mathsf{uf}}}$.

2 Preliminaries

2.1 Seeds and tori

Definition 2.1 (seeds)**.** A seed t is a collection $((A_i)_{i \in I}, (X_i)_{i \in I}, B)$, where A_i and X_i are indeterminates called cluster A-variables and cluster X-variables, respectively, and $B = (b_{ij})_{i,j \in I}$ is a $\mathbb{Z}$-valued matrix such that DB is skew-symmetric.

Define Laurent polynomial rings $\mathcal{LP}^A = \Bbbk\big[A_i^\pm\big]_{i \in I}$ and $\mathcal{LP}^X = \Bbbk\big[X_i^\pm\big]_{i \in I}$. For any vector $m, n \in \mathbb{Z}^I$, denote the Laurent monomials $A^m = \prod A_i^{m_i}$ and $X^n = \prod X_i^{n_i}$. We use $\cdot$ to denote their commutative multiplication, which is often omitted for simplicity.

We associate to t a lattice $N = \mathbb{Z}^I$ with the natural basis $\mathbf{e}_i$, $i \in I$, whose elements $n = \sum n_i \mathbf{e}_i = (n_i) \in \mathbb{Z}^I$ are viewed as the Laurent degrees for $X^n \in \mathcal{LP}^X$. Let M denote its

dual lattice with the dual basis $\mathbf{e}_i^*$. Let $\langle\ ,\ \rangle$ denote the natural pairing between $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$. Define $\mathbf{f}_i = \frac{1}{d_i}\mathbf{e}_i^*$ and the sublattice $M^\circ = \oplus_{i\in I}\mathbb{Z}\mathbf{f}_i \subset M_{\mathbb{Q}}$. We identify M° with $\mathbb{Z}^I$ such that $\mathbf{f}_i$ become the unit vectors. View its elements $m = \sum m_i\mathbf{f}_i = (m_i) \in \mathbb{Z}^I$ as the Laurent degrees for $A^m \in \mathcal{LP}^A$.

Let $\underline{\mathbf{e}}$ and $\underline{\mathbf{f}}$ denote the I-labeled bases $(\mathbf{e}_i)_{i\in I}$ and $(\mathbf{f}_i)_{i\in I}$, respectively. We can view them as matrices whose columns are the basis elements. Define the linear map $p^*\colon N \to M^\circ$ such that $p^*\mathbf{e}_j = \sum_i b_{ij}\mathbf{f}_i, \forall j$. It has the following matrix presentation:

$$p^*(\underline{\mathbf{e}}) = \underline{\mathbf{f}}B.$$

In the classical case, extend $\Bbbk$ to a field containing $\mathbb{Z}$. Denote the affine schemes $\mathcal{T}^A = \mathsf{Spec}\,\mathcal{LP}^A$ and $\mathcal{T}^X = \mathsf{Spec}\,\mathcal{LP}^X$. We will call $\mathcal{T}^A$ and $\mathcal{T}^X$ the tori associated to t, since $\mathcal{T}^A(\Bbbk)$ and $\mathcal{T}^X(\Bbbk)$ coincide with the split algebraic torus $(\Bbbk^\times)^I$.

2.2   Poisson structures

Recall that d is the least common multiplier of all d_i. We can associate to N a $\frac{1}{d}\mathbb{Z}$-valued canonical skew-symmetric bilinear form $-\omega$ such that $\omega(\mathbf{e}_i, \mathbf{e}_j) = \frac{1}{d_j}b_{ji}, \forall i, j$. The corresponding canonical Poisson structure on $\mathcal{T}^X$ is given by

$$\left\{X^n, X^{n'}\right\} = -\omega(n, n')X^{n+n'}. \tag{2.1}$$

Let W denote the $\mathbb{Q}$-valued matrix $(\omega(\mathbf{e}_i, \mathbf{e}_j))_{i,j}$. Recall that the diagonal entries of D are $\frac{1}{d_i}$. Then we have $WD^{-1} = B^{\mathrm{T}}$.

Remark 2.2. For any $i \in I$, we have $p^*\mathbf{e}_i(\ ) = \omega(\mathbf{e}_i,\ )$.

Remark 2.3. Following [19, 20], the Poisson structure of the form (2.1) is usually called log-canonical in the sense that the following holds (for $\Bbbk = \mathbb{R}$):

$$\left\{\ln X^n, \ln X^{n'}\right\} = -\omega(n, n').$$

We often impose the following assumption. Denote the matrix $\widetilde{B} := (b_{ik})_{i\in I, k\in I_{\mathsf{uf}}}$.

Assumption 2.4 (full rank assumption)**.** We assume that the linear map p^* restricts to an injective map on N_{uf}. Equivalently, the matrix $\widetilde{B}$ is of full rank.

Definition 2.5 ([5])**.** By a (compatible) Poisson structure on $\mathcal{T}^A$, we mean a collection of strictly positive numbers $\mathsf{d}_k' \in \mathbb{Q}_{>0}$, $k \in I_{\mathsf{uf}}$, and a $\mathbb{Q}$-valued skew-symmetric bilinear form λ on M° such that

$$\lambda(\mathbf{f}_i, p^*\mathbf{e}_k) = -\delta_{ik}\,\mathsf{d}_k'$$

for any $i \in I$, $k \in I_{\mathsf{uf}}$.

The corresponding Poisson bracket on $\mathcal{T}^A$ is defined as

$$\left\{A^m, A^{m'}\right\} = \lambda(m, m')A^{m+m'}.$$

The bilinear form λ is represented by the Λ-matrix $\Lambda = (\Lambda_{ij})_{i,j\in I} := (\lambda(\mathbf{f}_i, \mathbf{f}_j))_{i,j\in I}$.

Note that the existence of a compatible Poisson structure implies the full rank assumption (see Assumption 2.4). Conversely, by [19, 20], when the full rank assumption is satisfied, we can always choose a (not necessarily unique) compatible Poisson structure.

From now on, we will make the choice such that $d'_k = \frac{1}{d_k}$, $\forall k \in I_{uf}$ (see [5, Proposition 3.3]). Correspondingly, λ is $\frac{1}{d'}\mathbb{Z}$-valued for some sufficiently divisible $d' \in \mathbb{N}_{>0}$. We choose d' such that $d | d'$.

Note that the bilinear form λ on $M^\circ(t)$ naturally induces a bilinear form λ on N_{uf} such that

$$\lambda(n, n') := \lambda(p^*n, p^*n').$$

It is easy to check that λ on N_{uf} coincides with $-\omega$ (see [47, Lemma 2.1.11 (2)]).

Definition 2.6 (connected matrix). An $I \times I$ matrix B is said to be connected, if for any $i, j \in I$, there exists finitely many vertices $i_s \in I$, $0 \le s \le l$, such that $i_0 = i$, $i_l = j$, and $b_{i_s i_{s+1}} \ne 0$ for any $0 \le s \le l - 1$.

It is straightforward to check the following result.

Lemma 2.7. *Assume that B is a connected $I \times I$ matrix. If DB and $D'B$ are both skew-symmetric for some invertible diagonal matrices D, D', then there exists some $0 \ne \alpha \in \mathbb{Q}$ such that $D = \alpha D'$.*

2.3 Quantum torus algebras

Recall that we have chosen $d | d'$ and ω on N is $\frac{1}{d}\mathbb{Z}$-valued. Using the canonical Poisson structure on $\mathcal{T}^X$, we equip $\mathcal{LP}^X$ with an extra multiplication $*$, called the v-twisted product, such that

$$X^n * X^{n'} = v^{-\omega(n,n')} X^{n+n'}.$$

Then $\mathcal{LP}^X$ is called a quantum torus algebra for the quantum case $\Bbbk = \mathbb{Z}\big[v^{\pm \frac{1}{d'}}\big]$. The canonical Poisson bracket (2.1) can be recovered from the twisted product by

$$\big\{X^n, X^{n'}\big\} = \lim_{v \to 1} \frac{1}{2(v-1)} (X^n * X^{n'} - X^{n'} * X^n).$$

If there exists a $\frac{1}{d'}\mathbb{Z}$-valued compatible Poisson structure λ on $\mathcal{T}^A$ (see Definition 2.5), we can equip $\mathcal{LP}^A$ with an extra multiplication $*$, called the v-twisted product, such that

$$A^m * A^{m'} = v^{\lambda(m,m')} A^{m+m'}.$$

Then $\mathcal{LP}^A$ is called a quantum torus algebra for the quantum case $\Bbbk = \mathbb{Z}\big[v^{\pm \frac{1}{d'}}\big]$. As before, the compatible Poisson structure can be recovered from the twisted product by

$$\big\{A^m, A^{m'}\big\} = \lim_{v \to 1} \frac{1}{2(v-1)} (A^m * A^{m'} - A^{m'} * A^m).$$

For the classical case $\Bbbk = \mathbb{Z}$, we have $v = 1$ and we simply define the v-twisted product $*$ of $\mathcal{LP}^\bullet$ to be the commutative products $\cdot$. From now on, we always view $\mathcal{LP}^\bullet$ as a $\Bbbk$-algebra whose multiplication is the v-twisted product.

Let $\mathcal{LP} = \Bbbk[L] = \oplus \Bbbk_{u \in L} \chi^u$ and $\mathcal{LP}' = \Bbbk[L'] = \oplus \Bbbk_{u' \in L'} \chi^{u'}$ be two quantum torus algebras as above, viewed as $\Bbbk$-modules. Here L and L' denote the lattices of Laurent degrees, respectively, and χ can denote X or A. By a monomial map Φ from $\mathcal{LP}$ to $\mathcal{LP}'$ we mean a $\Bbbk$-linear map such that, there exists some linear map $\Psi\colon L \to L'$ satisfying $\Phi(\chi^u) = \chi^{\Psi(u)}$ $\forall u \in L$. In this case, Φ is called the monomial map associated to Ψ and Ψ the linear map associated to Φ.

In particular, the linear map $p^*\colon N \to M^\circ$ determines a monomial map $\mathcal{LP}^X \to \mathcal{LP}^A$ sending X^n to A^{p^*n}, which is still denoted by p^* for simplicity. Then p^* is an algebra homomorphism preserving the v-twisted products.

Let T denote the subalgebra $\Bbbk[X_k]_{k \in I_{uf}}$ of the quantum torus algebra $\mathcal{LP}^X$. We introduce the $\Bbbk$-algebra $\widehat{\mathcal{LP}}^X = \mathcal{LP}^X \otimes_T \widehat{T}$, where $\widehat{T}$ is the completion of T with respect to its maximal ideal generated by X_k, $k \in I_{uf}$. Similarly, define $\widehat{\mathcal{LP}}^A := \mathcal{LP}^A \otimes_{p^*T} p^*\widehat{T}$. The elements in $\widehat{\mathcal{LP}}^\bullet$ will be called formal Laurent series. Note that p^* extends to a homomorphism from $\widehat{\mathcal{LP}}^X$ to $\widehat{\mathcal{LP}}^A$.

2.4 Mutation maps

Denote $[\ ]_+ = \max(\ , 0)$. For any vector (g_i), denote $[(g_i)]_+ = ([g_i]_+)$.

Let t denote a given seed and k any chosen unfrozen vertex. Choose any sign $\varepsilon \in \{1, -1\}$. We define a new seed $t' = \mu_k t = ((A'_i), (X'_i), B')$, such that the matrix $B' = (b'_{ij})$ is given by

$$
b'_{ij} = \begin{cases} -b_{ij}, & k \in \{i,j\}, \\ b_{ij} + b_{ik}[\varepsilon b_{kj}]_+ + [-\varepsilon b_{ik}]_+ b_{kj}, & k \notin \{i,j\}. \end{cases}
$$

Let $\mathcal{F}^\bullet$ denote the fraction fields of the Laurent polynomial rings $\mathcal{LP}^\bullet$ for classical cases and the skew fields of fractions of the quantum torus algebras $\mathcal{LP}^\bullet$ for quantum cases. We always call $\mathcal{F}^\bullet$ fraction fields for simplicity.

We further relate the cluster variables for t' with those for t, by introducing isomorphisms for the fraction fields $(\mu_k^X)^* \colon (\mathcal{F}^X)' \simeq \mathcal{F}^X$ and $(\mu_k^A)^* \colon (\mathcal{F}^A)' \simeq \mathcal{F}^A$. For classical cases, we define

$$
(\mu_k^X)^* X'_i = \begin{cases} X_i X_k^{[\varepsilon b_{ki}]_+}(1 + X_k^\varepsilon)^{-b_{ki}}, & i \neq k, \\ X_k^{-1}, & i = k \end{cases}
$$

$$
= \begin{cases} X_i(1 + X_k)^{-b_{ki}}, & b_{ki} \leq 0, \\ X_i X_k^{b_{ki}} \dfrac{1}{(1 + X_k)^{b_{ki}}}, & b_{ki} > 0, \\ X_k^{-1}, & i = k, \end{cases}
$$

$$
(\mu_k^A)^* A'_i = \begin{cases} A_i, & i \neq k, \\ A_k^{-1} \prod_j A_j^{[-\varepsilon b_{jk}]_+}(1 + p^* X_k^\varepsilon), & i = k \end{cases}
$$

$$
= \begin{cases} A_i, & i \neq k, \\ A_k^{-1} \prod_j A_j^{[-b_{jk}]_+}(1 + p^* X_k), & i = k. \end{cases}
$$

Following [45, equation (2.6)] and [47, equations (2.2) and (2.4)], for quantum cases, we define $(\mu_k^A)^*$ such that

$$
(\mu_k^A)^* A'_i = \begin{cases} A_i, & i \neq k, \\ A^{-\mathbf{f}_k + \sum_j [-b_{jk}]_+ \mathbf{f}_j} + A^{-\mathbf{f}_k + \sum_i [b_{ik}]_+ \mathbf{f}_i}, & i = k. \end{cases}
$$

We define $(\mu_k^X)^*$ such that

$$
(\mu_k^X)^* X'_i = \begin{cases} X_i \cdot \displaystyle\sum_{s=0}^{-b_{ki}} \binom{-b_{ki}}{s}_{v_k} X_k^s, & b_{ki} \leq 0, \\ \left(\displaystyle\sum_{s=0}^{b_{ki}} \binom{b_{ki}}{s}_{v_k} X_i^{-1} \cdot X_k^{-b_{ki}} \cdot X_k^s \right)^{-1}, & b_{ki} > 0, \\ X_k^{-1}, & i = k, \end{cases}
$$

where $\cdot$ denotes the commutative product. Here, we denote $v_k := v^{\mathsf{d}'_k} = v^{\frac{1}{d_k}}$. The quantum numbers for $b \leq a \in \mathbb{N}$ are defined as

$$
[a]_v := \frac{v^a - v^{-a}}{v - v^{-1}}, \qquad [a]_v! := [a]_v [a-1]_v \cdots [1]_v, \qquad \binom{a}{b}_{v_k} := \frac{[a]_v!}{[b]_v![a-b]_v!}.
$$

The maps $(\mu_k^\bullet)^*$ are called the *mutation maps*, where $\bullet$ stands for A or X. For classical cases, they induce birational maps $\mu_k^\bullet \colon \mathcal{T}^\bullet(t) \dashrightarrow \mathcal{T}^\bullet(t')$.[2]

It is straightforward to check the well-known fact that B' does not depend on the choice of the sign ε. Moreover, mutation is an involution on the seeds, and the compositions

$$((\mu_k^\bullet)^*)^2 \colon \ (\mathcal{F}^\bullet)' \simeq \mathcal{F}^\bullet \simeq (\mathcal{F}^\bullet)'$$

are the identity. One can also check that mutations commute with the monomial map p^*:

$$\left(\mu_k^A\right)^* \circ p^*(t')(X_i(t')) = p^*(t) \circ \left(\mu_k^X\right)^*(X_i(t')), \qquad \forall i \in I. \tag{2.2}$$

For simplicity, we might omit the symbol $\bullet$ when there is no confusion.

Hamiltonian formalism

For classical cases, let us recall the Hamiltonian formalism for the mutation maps following [18] (see also [11, 25]). Recall that the Euler dilogarithm function is given by

$$\mathsf{Li}_2(z) = \sum_{n \geq 1} \frac{z^n}{n^2}.$$

Let $\left(\rho_{k,\varepsilon}^X\right)^*$ denote the automorphism $\exp\{\varepsilon d_k \, \mathsf{Li}_2(-X_k^\varepsilon), \ \}$ on $\mathcal{F}^X$. When $\mathcal{F}^A$ possesses a compatible Poisson structure, let $\left(\rho_{k,\varepsilon}^A\right)^*$ denote its automorphism $\exp\left\{\varepsilon \, \mathsf{Li}_2\left(-A^{\varepsilon p^* \mathbf{e}_k}\right), \ \right\}$. One can check that

$$\left(\rho_{k,\varepsilon}^X\right)^*(X^n) = X^n\left(1 + X_k^\varepsilon\right)^{\omega(n, -d_k \mathbf{e}_k)}, \qquad \left(\rho_{k,\varepsilon}^A\right)^*(A^m) = A^m\left(1 + A^{\varepsilon p^* \mathbf{e}_k}\right)^{-m_k}.$$

In particular, we have

$$\left(\rho_{k,\varepsilon}^X\right)^*(X_i) = \begin{cases} X_i(1 + X_k^\varepsilon)^{-b_{ki}}, & i \neq k, \\ X_k, & i = k, \end{cases}$$

$$\left(\rho_{k,\varepsilon}^A\right)^*(A_i) = \begin{cases} A_i, & i \neq k, \\ A_k(1 + (p^* X_k)^\varepsilon)^{-1}, & i = k. \end{cases}$$

We define monomial maps $\psi_{k,\varepsilon}^X \colon \left(\mathcal{LP}^X\right)' \simeq \mathcal{LP}^X$ and $\psi_{k,\varepsilon}^A \colon \left(\mathcal{LP}^A\right)' \simeq \mathcal{LP}^A$ such that

$$\psi_{k,\varepsilon}^X X_i' = \begin{cases} X_i \cdot X_k^{[\varepsilon b_{ki}]_+}, & i \neq k, \\ X_k^{-1}, & i = k, \end{cases} \qquad \psi_{k,\varepsilon}^A A_i' = \begin{cases} A_i, & i \neq k, \\ A_k^{-1} \cdot \prod_j A_j^{[-\varepsilon b_{jk}]_+}, & i = k, \end{cases}$$

where the commutative products are used. Then the mutation map $(\mu_k^\bullet)^*$ decomposes as the composition $(\rho_{k,\varepsilon}^\bullet)^* \circ \psi_{k,\varepsilon}^\bullet$. The factor $\psi_{k,\varepsilon}^\bullet$ is called its monomial part and $(\rho_{k,\varepsilon}^\bullet)^*$ its Hamiltonian part. Note that we have $(\rho_{k,\varepsilon}^\bullet)^* \circ \psi_{k,\varepsilon}^\bullet = \psi_{k,\varepsilon}^\bullet \circ (\rho_{k,-\varepsilon}^\bullet)^*$.

By [5], if t is endowed with a compatible Poisson structure, then we have a unique compatible Poisson structure for t' such that the homomorphism $\psi_{k,\varepsilon}^A$ is a Poisson homomorphism. In addition, it is independent of ε. We still use λ to denote the corresponding skew-symmetric bilinear form on $(M^\circ)'$, and denote the corresponding matrix by $\Lambda' = (\lambda(f_i', f_j'))_{i,j \in I}$.

---

[2] We use dashed arrows to denote rational maps.

Mutation sequences

Let $\underline{k} = (k_1, \ldots, k_l)$ denote a finite sequence of unfrozen vertices. A sequence of mutations $\boldsymbol{\mu} = \boldsymbol{\mu}_{\underline{k}}$ is the composition of mutations $\mu_{k_l} \cdots \mu_{k_2} \mu_{k_1}$ (read from right to left). The corresponding mutation map is $(\boldsymbol{\mu}^\bullet)^* = (\mu_{k_1}^\bullet)^* (\mu_{k_2}^\bullet)^* \cdots (\mu_{k_l}^\bullet)^*$, which we often write $\boldsymbol{\mu}^*$ for simplicity. Note that $\boldsymbol{\mu}^{-1} = \mu_{k_1} \mu_{k_2} \cdots \mu_{k_l}$ and $(\boldsymbol{\mu}^*)^{-1} = (\boldsymbol{\mu}^{-1})^*$.

We deduce the following equality in $\mathcal{F}^A(t)$ from (2.2):

$$(\boldsymbol{\mu}^A)^* \circ p^*(t')(X(t')^n) = p^*(t) \circ (\boldsymbol{\mu}^X)^*(X(t')^n), \qquad \forall n \in \mathbb{Z}^I. \tag{2.3}$$

Formal Laurent series expansions

We recall the maps $\iota^\bullet$ taking formal Laurent series expansions following [45, Section 3.3]. They will only be used in Lemma 6.5.

Take $t' = \boldsymbol{\mu} t$ for any mutation sequence $\boldsymbol{\mu}$. The mutation map $(\mu^\bullet)^* \colon \mathcal{F}^\bullet(t') \to \mathcal{F}^\bullet(t)$ induces an algebra homomorphism $\iota^\bullet(t')$ from $\mathcal{LP}^\bullet(t')$ to $\widehat{\mathcal{LP}}^\bullet(t)$, sending the Laurent monomials $(X')^n$ to $(\boldsymbol{\mu}^X)^*(X')^n$ $\big($or $(A')^m$ to $(\boldsymbol{\mu}^A)^*(A')^m\big)$. It is injective, see [45, Lemma 3.3.7 (1)]. In addition, the image of the Laurent monomials are pointed elements in the sense of Section 2.11, see [45, Lemma 3.3.6].

Lemma 2.8. *For any $Z \in \mathcal{LP}^\bullet(t')$, the following statements are true.*

(1) *If $(\mu^\bullet)^* Z \in \mathcal{LP}^\bullet(t)$, then $\iota(Z) = (\mu^\bullet)^* Z$.*

(2) *If $\iota Z \in \mathcal{LP}^\bullet(t)$, then $\iota(Z) = (\mu^\bullet)^* Z$.*

Proof. (1) See [45, Lemma 3.3.7 (2)].

(2) We prove it for the X-side, and the proof for the A-side is the same. Our proof is similar to that of [45, Lemma 3.3.7 (2)].

We can write $(X')^n * Z = F$ for some $F \in \Bbbk[X'_i]_{i \in I}$, $n \in \mathbb{N}^I$. On the one hand, we have $\iota(X')^n * \iota(Z) = \iota(F)$ in $\mathcal{LP}^X(s)$. On the other hand, we have $\boldsymbol{\mu}^*(X')^n * \boldsymbol{\mu}^*(Z) = \boldsymbol{\mu}^*(F)$ in $\mathcal{F}^X(s)$. By the definition of ι, we have $\iota(X')^n = \boldsymbol{\mu}^*(X')^n$ and $\iota(F) = \boldsymbol{\mu}^*(F)$. It follows that $\iota(Z) = \boldsymbol{\mu}^*(Z)$. ∎

2.5 Cluster algebras

Let there be any given initial seed t_0. We use $\Delta^+ = \Delta^+_{t_0}$ to denote the set of seeds $t = ((A_i(t))_{i \in I}, (X_i(t))_{i \in I}, B(t))$ obtained from t_0 by sequences of mutations. If we work with the quantum cases, we also choose a compatible Poisson structure and consider the associated quantization as in Section 2.3. Recall that the cluster variables $A_j(t_0)$, $j \in I_f$, are unchanged by mutations, which are denoted by A_j and are called the frozen variables. We use $\mathcal{Z}$ to denote the multiplicative group generated by $A_j^\pm$, $j \in I_f$.

Using the v-twisted product $*$, the partially compactified (quantum) cluster algebra $\overline{\mathcal{A}}(t_0)$ with the initial seed t_0 is defined to be the $\Bbbk$-subalgebra of $\mathcal{F}^A(t_0)$ generated by all the cluster variables $\boldsymbol{\mu}^* A_i(t)$, $i \in I$, $t = \boldsymbol{\mu} t_0 \in \Delta^+_{t_0}$. The (localized) cluster algebra $\mathcal{A}(t_0)$ is defined to be its localization $\overline{\mathcal{A}}(t_0)[A_j^{-1}]_{j \in I_f}$.

The upper cluster algebra (or upper cluster A-algebra) $\mathcal{U}^A(t_0)$ with the initial seed t_0 is defined as the intersection $\cap_{t = \boldsymbol{\mu} t_0 \in \Delta^+} \boldsymbol{\mu}^* \mathcal{LP}^A(t)$ inside $\mathcal{F}^A(t_0)$. By the Laurent phenomenon [5, 13], it contains the cluster algebra $\mathcal{A}(t_0)$. For classical cases, if a compatible Poisson structure on $\mathcal{LP}^A(t_0)$ is given, then $\mathcal{U}^A(t_0)$ inherits the Poisson structure.

The cluster Poisson algebra (or upper cluster X-algebra) $\mathcal{U}^X(t_0)$ with the initial seed t_0 is defined as the intersection $\cap_{t = \boldsymbol{\mu} t_0 \in \Delta^+} \boldsymbol{\mu}^* \mathcal{LP}^X(t)$ inside $\mathcal{F}^X(t_0)$. For classical cases, it inherits the canonical Poisson structure from that of $\mathcal{LP}^X(t_0)$.

We often identify fraction fields $\mathcal{F}^\bullet(t)$ and $\mathcal{F}^\bullet(t_0)$ via the mutation map $\boldsymbol{\mu}^*$ for simplicity. Correspondingly, we omit the symbol t_0 and $\boldsymbol{\mu}^*$ in the above notations, and we can write $\mathcal{U}^A = \cap_t \mathcal{LP}^A(t)$ and $\mathcal{U}^X = \cap_t \mathcal{LP}^X(t)$.

2.6 Cluster varieties

Let us work at the classical cases. Given two seeds $t' = \boldsymbol{\mu} t$. Using the mutation birational maps $\boldsymbol{\mu} \colon \mathcal{T}^A(t) \dashrightarrow \mathcal{T}^A(t')$, we can glue the tori $\mathcal{T}^A(t)$, $t \in \Delta^+$, into a scheme $\mathbb{A}$ (see [24, Proposition 2.4]), which is called the cluster A-variety or cluster K2 variety. Note that the upper cluster algebra $\mathcal{U}^A$ is the ring of global sections of its structure sheaf. A choice of compatible Poisson structure on $\mathcal{LP}^A(t_0)$ gives rise to a Poisson structure on $\mathbb{A}$. It often happens that $\mathbb{A}$ is a smooth manifold, for example, for many well-known the cluster algebra arising from Lie theory (unipotent cells [15], double Bruhat cells [3, 22]).

Similarly, using the mutation birational maps $\boldsymbol{\mu} \colon \mathcal{T}^X(t) \dashrightarrow \mathcal{T}^X(t')$, we can glue the tori $\mathcal{T}^X(t)$, $t \in \Delta^+$, into a scheme $\mathbb{X}$ called the cluster X-variety or the cluster Poisson variety. The cluster Poisson algebra $\mathcal{U}^X$ is the ring of global sections of its structure sheaf. Note that $\mathbb{X}$ has the canonical Poisson structure.

2.7 Transition matrices

We use $(\ )^{\mathrm{T}}$ to denote matrix transpose.

Given seeds $t' = \mu_k t$ and a mutation sign $\varepsilon \in \{+, -\}$. Let us describe the monomial part of mutation using transition matrices.[3]

Define the following $I \times I$-matrix $\mathsf{P}^N_{k,\varepsilon}(t)$:

$$\left(\mathsf{P}^N_{k,\varepsilon}(t)\right)_{ij} = \begin{cases} -1, & i = j = k, \\ [\varepsilon b_{kj}]_+, & i = k, \ j \neq k, \\ \delta_{ij}, & \text{else.} \end{cases}$$

Then it represents the linear map

$$\psi^N_{k,\varepsilon} \colon \ N(t') \to N(t), \qquad \psi^N_{k,\varepsilon}(\underline{\mathbf{e}}') = \underline{\mathbf{e}}\,\mathsf{P}^N_{k,\varepsilon}(t).$$

We see that $\psi^N_{k,\varepsilon}$ induces the monomial part $\psi^X_{k,\varepsilon}$ of the mutation from $\mathcal{F}^X(t')$ to $\mathcal{F}^X(t)$. Similarly, define the following $I \times I$-matrix $\mathsf{P}^M_{k,\varepsilon}(t)$:

$$\left(\mathsf{P}^M_{k,\varepsilon}(t)\right)_{ij} = \begin{cases} -1, & i = j = k, \\ [-\varepsilon b_{ik}]_+, & i \neq k, \ j = k, \\ \delta_{ij}, & \text{else.} \end{cases}$$

Then it represents the linear map

$$\psi^M_{k,\varepsilon} \colon \ M^\circ(t') \simeq M^\circ(t), \qquad \psi^M_{k,\varepsilon}(\underline{\mathbf{f}}') = \underline{\mathbf{f}}\,\mathsf{P}^M_{k,\varepsilon}(t).$$

We see that $\psi^M_{k,\varepsilon}$ induces the monomial part $\psi^A_{k,\varepsilon}$ of the mutation from $\mathcal{F}^A(t')$ to $\mathcal{F}^A(t)$.

The following results were known by [5, 41], see [31, Section 5.6] for a summary.

Proposition 2.9. *We have the following equalities:*

(1) $\mathsf{P}^N_{k,\varepsilon}(t)^2 = \mathsf{Id}$, $\mathsf{P}^N_{k,-\varepsilon}(t') = \mathsf{P}^N_{k,\varepsilon}(t)$, $\mathsf{P}^N_{k,-\varepsilon}(t')\,\mathsf{P}^N_{k,\varepsilon}(t) = \mathsf{Id}$.

---

[3]In [5], our matrices P^N_ε and P^M_ε are denoted by F_ε and E_ε, respectively.

(2) $\mathsf{P}_{k,\varepsilon}^M(t)^2 = \mathsf{Id}$, $\mathsf{P}_{k,-\varepsilon}^M(t') = \mathsf{P}_{k,\varepsilon}^M(t)$, $\mathsf{P}_{k,-\varepsilon}^M(t')\,\mathsf{P}_{k,\varepsilon}^M(t) = \mathsf{Id}$.

(3) $D\,\mathsf{P}_{k,-\varepsilon}^M(t')D^{-1} = D\,\mathsf{P}_{k,\varepsilon}^M(t)D^{-1} = \big(\mathsf{P}_{k,\varepsilon}^N(t)\big)^{-\mathrm{T}}$.

(4) $B' = \mathsf{P}_{k,\varepsilon}^M(t)\cdot B\cdot \mathsf{P}_{k,\varepsilon}^N(t)$.

(5) $\Lambda' = \mathsf{P}_{k,\varepsilon}^M(t)^{\mathrm{T}}\cdot\Lambda\cdot\mathsf{P}_{k,\varepsilon}^M(t)$, where Λ is the matrix of the Poisson structure.

Remark 2.10. It is straightforward to check that we have

$$\big(\mathsf{P}_{k,\varepsilon}^M(t')\,\mathsf{P}_{k,\varepsilon}^M(t)\big)_{ij} = \begin{cases} -\varepsilon b_{ik}, & i\neq k,\ j=k, \\ \delta_{ij}, & \text{else.} \end{cases}$$

and similarly

$$\big(\mathsf{P}_{k,\varepsilon}^N(t')\,\mathsf{P}_{k,\varepsilon}^N(t)\big)_{ij} = \begin{cases} -\varepsilon b_{kj}, & i=k,\ j\neq k, \\ \delta_{ij}, & \text{else.} \end{cases}$$

2.8   Cluster expansions, *g*-vectors, *c*-vectors

Let there be given an initial seed $t_0 = ((A_i),(X_i),B)$. Take any sequence of unfrozen vertices $\underline{k} = (k_0,k_1,\ldots,k_r)$. Denote the corresponding mutation sequence by $\boldsymbol{\mu} = \boldsymbol{\mu}_{\underline{k}} = \mu_{k_r}\cdots\mu_{k_0}$ and the resulting seed by $t = \boldsymbol{\mu}t_0$.

Recall that a vector is said to be sign-coherent if its coordinates are all non-negative or all non-positive.

Theorem 2.11 ([7, 25, 52])**.** *There exist $I\times I$ invertible $\mathbb{Z}$-matrices*

$$E(t) = \begin{pmatrix} E(t)_{\mathsf{uf}} & E(t)_{\mathsf{uf,f}} \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \qquad and \qquad F(t) = \begin{pmatrix} F(t)_{\mathsf{uf}} & 0 \\ F(t)_{\mathsf{f,uf}} & \mathsf{Id}_{\mathsf{f}} \end{pmatrix},$$

such that the following statements hold.

(1) *Any cluster A-variable $A_i(t)$, $i\in I$, has the following Laurent expansion in $\mathcal{LP}^A(t_0)$:*

$$\boldsymbol{\mu}^* A_i(t) = A^{\mathsf{col}_i F(t)} * \sum_{n\in\mathbb{N}^{I_{\mathsf{uf}}}} c_n A^{p^* n},$$

such that $c_0 = 1$.

(2) *Any cluster X-variable $X_i(t)$, $i\in I$, has the following expression in $\mathcal{F}^X(t_0)$:*

$$\boldsymbol{\mu}^* X_i(t) = X^{\mathsf{col}_i E(t)} * P * Q^{-1},$$

where P, Q are polynomials in $\Bbbk[X_k]_{k\in I_{\mathsf{uf}}}$ with constant term 1.

(3) *The row vectors of $F(t)$ are sign-coherent.*

(4) *The column vectors of $E(t)$ are sign-coherent.*

The $I_{\mathsf{uf}}\times I_{\mathsf{uf}}$-submatrices $C(t) := E(t)_{\mathsf{uf}}$ and $G(t) := F(t)_{\mathsf{uf}}$ are usually called the C-matrix and the G-matrix of t (with respect to the initial seed t_0), respectively.

Definition 2.12 ([45])**.** For any pair of seeds $t,t'\in\Delta^+$, let t be the initial seed and $E(t')$ (resp. $F(t')$) denote the corresponding E-matrix (resp. F-matrix) of t'. We define the linear map:

$$\psi_{t,t'}^N\colon\ N(t')\to N(t), \qquad \psi_{t,t'}^N(\underline{\mathbf{e}}(t')) = \underline{\mathbf{e}}(t)E(t');$$
$$\psi_{t,t'}^M\colon\ M^\circ(t')\to M^\circ(t), \qquad \psi_{t,t'}^M(\underline{\mathbf{f}}(t')) = \underline{\mathbf{f}}(t)F(t').$$

Example 2.13. Take the index set $I = \{1, 2\}$ such that 1 is the only unfrozen vertex. Choose $d_1 = 2$, $d_2 = 1$. The initial seed $t = ((X_1, X_2), B)$ is given such that $B = \left(\begin{smallmatrix} 0 & 2 \\ -1 & 0 \end{smallmatrix} \right)$.

Then $W = B^T \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$. We choose $\Lambda = -\frac{1}{2} W$, so that $(\Lambda B)_{i1} = -\delta_{i,1} \frac{1}{d_1}$ for $i = 1, 2$.

The seed $t' = \mu_1 t$ is the only non-initial seed. Using the commutative product $\cdot$, we can write $A'_1 = A^{-\mathbf{f}_1 + \mathbf{f}_2} + A^{-\mathbf{f}_1} = A_1^{-1} \cdot A_2 \cdot (1 + A_2^{-1}) = A_1^{-1} \cdot A_2 \cdot (1 + p^* X_1)$ and $A'_2 = A_2$. It follows that $\psi_{t,t'}^M : M^\circ(t') \to M^\circ(t)$ is represented by $F(t') = \left(\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix} \right)$. We also have

$$X'_1 = X_1^{-1}, \qquad (X'_2)^{-1} = X_2^{-1} \cdot X_1^{-2} \cdot \left(1 + \left(v^{\frac{1}{2}} + v^{-\frac{1}{2}} \right) X_1 + X_1^2 \right)$$

or, equivalently,

$$X'_2 = \left(X^{-2\mathbf{e}_1 - \mathbf{e}_2} + \left(v^{\frac{1}{2}} + v^{-\frac{1}{2}} \right) X^{-\mathbf{e}_1 - \mathbf{e}_2} + X^{-\mathbf{e}_2} \right)^{-1}.$$

So $\psi_{t,t'}^N : N(t') \to N(t)$ is represented by $E(t') = \left(\begin{smallmatrix} -1 & 2 \\ 0 & 1 \end{smallmatrix} \right)$.

Note that, for any $k \in I_{\mathsf{uf}}$, $\psi_{t,\mu_k t}^\bullet$ is represented by the matrix $\mathsf{P}_{k,+}^\bullet(t)$, where $\bullet$ stands for M or N. In particular, $\psi_{t,\mu_k t}^\bullet \psi_{\mu_k t,t}^\bullet$ might not be the identity by Remark 2.10.

Identify $N(t)$ with $N(t_0)$ by using the linear map $\psi_{t_0,t}^N$. Then the basis vector $\mathbf{e}_i(t)$ has the coordinate vector $\mathsf{col}_i E(t)$ in $N(t_0) \simeq \mathbb{Z}^I$. Similarly, identify $M(t)$ with $M(t_0)$ by using the linear map $\psi_{t_0,t}^M$. Then the basis vector $\mathbf{f}_i(t)$ has the coordinate vector $\mathsf{col}_i F(t)$ in $M^\circ(t_0) \simeq \mathbb{Z}^I$.

We will often work with vectors, linear maps and bilinear forms in the fixed lattices $N(t_0)$ and $M(t_0)$. We refer the reader to [24] for more details on the fixed data.

2.9   Canonical mutation signs

For any $k \in I_{\mathsf{uf}}$, recall that the k-th c-vector $\mathsf{col}_k C(t)$ is sign coherent. Then we define the canonical mutation sign ε for the mutation of t at the vertex k to be the sign of $\mathsf{col}_k C(t)$. From now on, we always choose the canonical mutation sign unless otherwise specified.

For $\underline{k} = (k_0, k_1, \ldots, k_r)$, denote $\underline{k}_{\leq s} = (k_0, \ldots, k_s)$ for any $0 \leq s \leq r - 1$ and $t_s = \boldsymbol{\mu}_{\underline{k}_{\leq s-1}} t_0$. Let ε_s denote the canonical sign for the mutation of t_s at the direction k_s.

Proposition 2.14 ([25, 41]). *We have the following*

$$E(t) = \mathsf{P}_{k_0,\varepsilon_0}^N(t_0) \, \mathsf{P}_{k_1,\varepsilon_1}^N(t_1) \cdots \mathsf{P}_{k_{r-1},\varepsilon_{r-1}}^N(t_{r-1}),$$
$$F(t) = \mathsf{P}_{k_0,\varepsilon_0}^M(t_0) \, \mathsf{P}_{k_1,\varepsilon_1}^M(t_1) \cdots \mathsf{P}_{k_{r-1},\varepsilon_{r-1}}^M(t_{r-1}).$$

By Proposition 2.14, the matrices $C(t)$ and $G(t)$ only depend on B_{uf} and $\underline{k}$.
Using Proposition 2.9, we obtain

$$E(t)^{\mathrm{T}} = DF(t)^{-1} D^{-1}. \tag{2.4}$$

Its restriction gives $C(t)^{\mathrm{T}} = D_{\mathsf{uf}} G(t)^{-1} D_{\mathsf{uf}}^{-1}$.

Lemma 2.15. *The following statements are true.*

(1) $F(t) B(t) (E(t))^{-1} = B(t_0)$ *or, equivalently,* $\psi_{t_0,t}^M(p^* \underline{\mathbf{e}}(t)) = p^* \big(\psi_{t_0,t}^N \underline{\mathbf{e}}(t) \big)$.

(2) $F(t)^{\mathrm{T}} \Lambda(t_0) F(t) = \Lambda(t)$ *or, equivalently,* $\lambda \big(\psi_{t_0,t}^M(\underline{\mathbf{f}}(t)), \psi_{t_0,t}^M(\underline{\mathbf{f}}(t)) \big) = \lambda(\underline{\mathbf{f}}(t), \underline{\mathbf{f}}(t))$.

(3) $E(t)^{\mathrm{T}} W(t_0) E(t) = W(t)$ *or, equivalently,* $\omega \big(\psi_{t_0,t}^N(\underline{\mathbf{e}}(t)), \psi_{t_0,t}^N(\underline{\mathbf{e}}(t)) \big) = \omega(\underline{\mathbf{e}}(t), \underline{\mathbf{e}}(t))$.

Proof. (1)–(2) follow from Propositions 2.9 and 2.14.
(3) follow from (1), $W(t) D^{-1} = B(t)^{\mathrm{T}}$ and $E(t)^{\mathrm{T}} = DF(t)^{-1} D^{-1}$.   ∎

2.10   Principal coefficients

Let us recall the seeds with principal coefficients and their relation with the C-matrix and the G-matrix. This part will only be used in Sections 5.2 and 6.3.

Denote a copy of I_{uf} by $I'_{\mathsf{uf}} = \{k' | k' \in I_{\mathsf{uf}}\}$. We extend the corresponding principal B-matrix B_{uf} to the $(I_{\mathsf{uf}} \sqcup I'_{\mathsf{uf}}) \times (I_{\mathsf{uf}} \sqcup I'_{\mathsf{uf}})$ matrix $B^{\mathsf{prin}} = \begin{pmatrix} B_{\mathsf{uf}} & -\mathsf{Id}_{I_{\mathsf{uf}}} \\ \mathsf{Id}_{I_{\mathsf{uf}}} & 0 \end{pmatrix}$, which is called the principal co-efficient B-matrix, where $\mathsf{Id}_{I_{\mathsf{uf}}}$ represents the natural isomorphism $I_{\mathsf{uf}} \simeq I'_{\mathsf{uf}}$. The corresponding diagonal matrices are

$$D^{\mathsf{prin}} = \begin{pmatrix} D_{\mathsf{uf}} & \\ & D_{\mathsf{uf}} \end{pmatrix} \qquad \text{and} \qquad W^{\mathsf{prin}} = \begin{pmatrix} W_{\mathsf{uf}} & D_{\mathsf{uf}} \\ -D_{\mathsf{uf}} & 0 \end{pmatrix}.$$

We denote by t_0^{prin} the seed obtained from t_0 by changing the fixed data as above.

Let $\underline{k}$ denote a sequence of vertices and $t = \boldsymbol{\mu}_{\underline{k}} t_0$. Then it is known that the C-matrix and the G-matrix can be computed using principle coefficients:

$$F\big(\boldsymbol{\mu}_{\underline{k}} t_0^{\mathsf{prin}}\big) = \begin{pmatrix} G(t) & 0 \\ 0 & \mathsf{Id}_{I_{\mathsf{uf}}} \end{pmatrix},$$

$$E\big(\boldsymbol{\mu}_{\underline{k}} t_0^{\mathsf{prin}}\big) = \big(D^{\mathsf{prin}}\big)^{-1} F\big(\boldsymbol{\mu}_{\underline{k}} t_0\big)^{-\mathrm{T}} D^{\mathsf{prin}} = \begin{pmatrix} C(t) & 0 \\ 0 & \mathsf{Id}_{I_{\mathsf{uf}}} \end{pmatrix},$$

where $C(t) = D_{\mathsf{uf}}^{-1} G(t)^{-\mathrm{T}} D_{\mathsf{uf}}$. Then, using Lemma 2.15, we obtain

$$B\big(\boldsymbol{\mu}_{\underline{k}} t_0^{\mathsf{prin}}\big) = F\big(\boldsymbol{\mu}_{\underline{k}} t_0^{\mathsf{prin}}\big)^{-1} B\big(t_0^{\mathsf{prin}}\big) E\big(\boldsymbol{\mu}_{\underline{k}} t_0\big) = \begin{pmatrix} G(t)^{-1} B_{\mathsf{uf}} C(t) & -G(t)^{-1} \\ C(t) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} B(t)_{\mathsf{uf}} & -G(t)^{-1} \\ C(t) & 0 \end{pmatrix} = \begin{pmatrix} B(t)_{\mathsf{uf}} & -D_{\mathsf{uf}}^{-1} C(t)^{\mathrm{T}} D_{\mathsf{uf}} \\ C(t) & 0 \end{pmatrix}.$$

2.11   Degrees and pointedness

We endow $N_{\mathsf{uf}}(t) = \oplus \mathbb{Z}\mathbf{e}_k \simeq \mathbb{Z}^{I_{\mathsf{uf}}}$ with the natural partial order such that $n \geq n'$ if $n - n' \geq 0$. Denote $N_{\mathsf{uf}}^{\geq 0}(t) = \oplus \mathbb{N}\mathbf{e}_k \simeq \mathbb{N}^{I_{\mathsf{uf}}}$. Recall that we have the linear map $p^* \colon N_{\mathsf{uf}}(t) \to M^\circ(t)$ represented by the matrix $\widetilde{B} = \begin{pmatrix} B_{\mathsf{uf}} \\ B_{\mathsf{f,uf}} \end{pmatrix}$.

Definition 2.16 (pointedness). A formal Laurent series $Z \in \widehat{\mathcal{LP}}^{X}(t)$ is said to have degree n, denoted by $\mathsf{deg}_t Z = n$, if it takes the form $Z = X(t)^n \cdot \big(\sum_{n' \in N_{\mathsf{uf}}^{\geq 0}(t)} c_{n'} X(t)^{n'}\big)$ for some $c_0 \neq 0$, $c_{n'} \in \Bbbk$. Its F-function is defined as $\sum_{n' \in N_{\mathsf{uf}}^{\geq 0}(t)} c_{n'} X(t)^{n'}$. It is further said to be pointed at n or n-pointed if $c_0 = 1$.

Now assume that t is a seed such that $\ker p^* \cap N_{\mathsf{uf}}^{\geq 0}(t) = 0$. This condition is satisfied when the seed t satisfies the full rank assumption (see Assumption 2.4). We recall the degrees and pointedness introduced in [44, 45].

Definition 2.17 (dominance order [44]). For any $m, m' \in M^\circ(t)$, we say m' is dominated by m, denoted by $m' \preceq_t m$, if $m' = m + p^* n$ for some $n \in N_{\mathsf{uf}}^{\geq 0}(t)$.

Definition 2.18 (pointedness [44]). A formal Laurent series $Z \in \widehat{\mathcal{LP}}^{A}(t)$ is said to have de-gree m, denoted by $\mathsf{deg}_t Z = m$, if it takes the form $Z = A(t)^m \cdot \big(\sum_{n'' \in N_{\mathsf{uf}}^{\geq 0}(t)} c_n A(t)^{p^* n''}\big)$ for some $c_0 \neq 0$, $c_{n''} \in \Bbbk$, i.e., its $\prec_t$-maximal Laurent degree is m. Its F-function is defined as $\sum_{n'' \in N_{\mathsf{uf}}^{\geq 0}(t)} c_{n''} X(t)^{n''}$. It is further said to be pointed at m or m-pointed if $c_0 = 1$.

Theorem 2.11 implies that $\mathsf{deg}_{t_0} X_i(t) = \mathsf{col}_i E(t)$ and $\mathsf{deg}_{t_0} A_i(t) = \mathsf{col}_i F(t)$. By definition, $\psi_{t_0,t}^N$ is the linear map sending $\mathbf{e}_i(t) = \mathsf{deg}_t X_i(t)$ to $\mathsf{deg}_{t_0} \boldsymbol{\mu}^* X_i(t)$. Similarly, $\psi_{t_0,t}^M$ is the linear map sending $\mathbf{f}_i(t) = \mathsf{deg}_t A_i(t)$ to $\mathsf{deg}_{t_0} \boldsymbol{\mu}^* A_i(t)$.

2.12 Injective-reachable

Definition 2.19 (injective-reachable [44, Section 2.3]). A seed t_0 is said to be injective-reachable, if there exists another seed $t_0[1] = \boldsymbol{\mu} t_0 \in \Delta_{t_0}^+$ and a permutation σ of I_{uf}, such that for any $k \in I_{\mathsf{uf}}$, we have

$$\psi_{t_0,t_0[1]}^N \mathbf{e}_{\sigma k}(t_0[1]) = -\mathbf{e}_k(t_0). \tag{2.5}$$

The sequence $\boldsymbol{\mu}$ is also called a green to red sequence from t_0 to $t_0[1]$ in the sense of Keller [30].

Assume that t_0 is injective-reachable. Then all $t \in \Delta_{t_0}^+$ are injective-reachable, i.e., we can always find a seed $t[1]$, see [44, Proposition 5.1.4] or [38].

By [44, Proposition 2.3.3], for any $k \in I_{\mathsf{uf}}$, we have

$$\psi_{t_0,t_0[1]}^N \mathbf{f}_{\sigma k}(t_0[1]) \in -\mathbf{f}_{k_0}(t_0) + \oplus_{j \in I_{\mathsf{f}}} \mathbb{Z} \mathbf{f}_j, \qquad d_k = d_{\sigma k}.$$

We will see that $t_0[1] \in \Delta^+$ is similar to t_0 up to σ in the sense of Definition 3.1.

3 Twist endomorphisms for upper cluster A-algebras

In this section, we introduce the notion of a twist endomorphisms for a pair of similar seeds t, t', which is defined as the composition of a mutation map with a monomial map called a variation map.

3.1 Similar seeds and variation maps

We first recall the definition of similar seeds.

Definition 3.1 (similar seeds [43, 44]). Two seeds t, t' are called similar, if there is a permutation σ of I_{uf} such that for any $i, j \in I_{\mathsf{uf}}$, we have $b_{ij}(t) = b_{\sigma i, \sigma j}(t')$ and skew-symmetrizers $d_i = d_{\sigma i}$, $\forall i, j \in I_{\mathsf{uf}}$.

If t and t' are similar, in our choices of compatible Poisson structures, we automatically have $\mathsf{d}_k'(t) = \mathsf{d}_{\sigma k}'(t')$ for $k \in I_{\mathsf{uf}}$ (see Definition 2.5).

Note that we can always trivially extends σ to a permutation on I.

Proposition 3.2. *Assume that B_{uf} is connected Definition 2.6. Let $t = \boldsymbol{\mu} t_0$ be a given seed. If there is a permutation σ of I_{uf} such that $b_{ij}(t) = b_{\sigma i, \sigma j}(t_0)$ for any $i, j \in I_{\mathsf{uf}}$, then $d_k = d_{\sigma k}$ for any $k \in I_{\mathsf{uf}}$.*

Proof. Let P_σ be the permutation matrix associated to σ of rank I_{uf}, i.e., the i-th column of P_σ is the σi-th unit vector. Notice that $P_\sigma^{\mathrm{T}} = P_\sigma^{-1} = P_{\sigma^{-1}}$.

Rewrite the equality $b_{\sigma i, \sigma j}(t) = b_{ij}(t_0)$ as $P_\sigma^{-1} B(t)_{\mathsf{uf}} P_\sigma = B(t_0)_{\mathsf{uf}}$. Since $D_{\mathsf{uf}} B(t_0)_{\mathsf{uf}}$ is skew-symmetric,

$$\left(P_\sigma D_{\mathsf{uf}} P_\sigma^{-1} \right) B(t)_{\mathsf{uf}} = P_\sigma D_{\mathsf{uf}} B(t_0)_{\mathsf{uf}} P_\sigma^{-1}$$

is skew-symmetric.

On the other hand, we have $B(t) = F(t)^{-1} B(t_0) E(t)$ by Lemma 2.15. Using $DF(t)^{-1} = E(t)^{\mathrm{T}} D$ by (2.4), we obtain

$$DB(t) = E(t)^{\mathrm{T}} DB(t_0) E(t).$$

Since $DB(t_0)$ is skew-symmetric, so is $DB(t)$. It follows that $D_{\mathsf{uf}} B(t)_{\mathsf{uf}}$ is also skew-symmetric.

Then Lemma 2.7 implies that $P_\sigma^{-1} D_{\mathsf{uf}} P_\sigma = \alpha D_{\mathsf{uf}}$ for some $\alpha \neq 0$. Since D_{uf} is of full rank, by taking the determinant, we see that $\alpha = 1$. The claim follows. ∎

Assume t, t' are similar up to a permutation σ. In view of the correction technique [43], it is often useful to compare pointed elements in $\mathcal{LP}^A(t)$ with those in $\mathcal{LP}^A(t')$. As in [44], we define a variation map sending pointed elements in $\mathcal{LP}^A(t)$ to those in $\mathcal{LP}^A(t')$, which differ only by the frozen variables.

In this paper, we will allow roots of frozen variables. For any given integer number $r > 0$, we define $\mathcal{Z}^{\frac{1}{r}}$ to be the multiplicative group generated by the r-th roots $A_j^{\pm\frac{1}{r}}$, $j \in I_{\mathsf{f}}$. Define $\mathcal{Z}^{\mathbb{Q}}$ to be the multiplicative group generated by any $A_j^{\pm\frac{1}{r}}$, $j \in I_{\mathsf{f}}$, $r > 0$.

For any $\Bbbk[\mathcal{Z}]$-algebra R, we use $R_{\mathcal{Z}^{\mathbb{Q}}}$ to denote $R \otimes_{\Bbbk[\mathcal{Z}]} \Bbbk[\mathcal{Z}^{\mathbb{Q}}]$. Define $R_{\mathcal{Z}^{\frac{1}{r}}}$ similarly.

Definition 3.3. Let t, t' be two seeds similar up to a permutation σ. A $\Bbbk$-algebra homomorphism $\mathsf{var}_t^A \colon \mathcal{LP}(t)_{\mathcal{Z}^{\mathbb{Q}}} \to \mathcal{LP}(t')_{\mathcal{Z}^{\mathbb{Q}}}$ is called a (monomial) variation map from t to t' if, $\forall k \in I_{\mathsf{uf}}$, $j \in I_{\mathsf{f}}$, it takes the form

$$\mathsf{var}_t^A(A_k) = p_{\sigma k} \cdot A'_{\sigma k}, \qquad \mathsf{var}_t^A\left(A^{\mathsf{col}_k B}\right) = (A')^{\mathsf{col}_{\sigma k} B'}, \qquad \mathsf{var}_t^A(A_j) = p_j,$$

for some $p_{\sigma k}, p_j \in \mathcal{Z}^{\mathbb{Q}}$.

For classical cases, if t and t' are similar seeds with compatible Poisson structures, then var_t^A is called a Poisson variation map if it further preserves the Poisson structures:

$$\mathsf{var}_t^A\{A_i, A_j\} = \left\{\mathsf{var}_t^A A_i, \mathsf{var}_t^A A_j\right\}, \qquad \forall i, j \in I.$$

Definition 3.4. In $\widehat{\mathcal{LP}}^A(t)$, take any pointed formal Laurent series $Z = A^m \cdot F|_{X^n \mapsto A^{p^*n}}$, where $m \in \mathbb{Z}^I$, $F = \sum_{n \in \mathbb{N}_{\mathsf{uf}}^{\geq 0}} c_n X^n$, $c_n \in \Bbbk$, $c_0 = 1$. We say Z and $Z' \in \widehat{\mathcal{LP}}^A(t')_{\mathcal{Z}^{\mathbb{Q}}}$ are similar if $Z' = (A')^{m'} \cdot F|_{X^n \mapsto (A')^{p^*(t')(\sigma n)}}$ with $\mathsf{pr}_{I_{\mathsf{uf}}} m' = \sigma \, \mathsf{pr}_{I_{\mathsf{uf}}} m$.

Remark 3.5. The variation map var_t^A is a $\Bbbk$-algebra homomorphism such that it sends a pointed element $Z \in \mathcal{LP}^A(t)$ to a similar element in $\mathcal{LP}^A(t')_{\mathcal{Z}^{\mathbb{Q}}}$.

The variation map in Definition 3.3 is the monomial map associated to the following linear variation map.

Denote $M_{\mathsf{uf}}^\circ(t) = \oplus_{k \in I_{\mathsf{uf}}} \mathbb{Z}\mathbf{f}_k$, $M_{\mathsf{f}}^\circ(t) = \oplus_{j \in I_{\mathsf{f}}} \mathbb{Z}\mathbf{f}_j$.

Definition 3.6. Let t, t' be two seeds similar up to a permutation σ. A linear map $\mathsf{var}_t^M \colon M^\circ(t)_{\mathbb{Q}} \to M^\circ(t')_{\mathbb{Q}}$ is called a (linear) variation map from t to t', for any $k \in I_{\mathsf{uf}}$, $j \in I_{\mathsf{f}}$, if it takes the following form

$$\mathsf{var}_t^M(\mathbf{f}_k) = \mathbf{f}'_{\sigma k} + \mathbf{u}_{\sigma k}, \qquad \mathsf{var}_t^M(\mathbf{f}_j) = \mathbf{u}_j, \qquad \mathsf{var}_t^M\left(\sum b_{ik}\mathbf{f}_i\right) = \sum b'_{i,\sigma k}\mathbf{f}'_i, \qquad (3.1)$$

for some $\mathbf{u}_k, \mathbf{u}_j \in (M_{\mathsf{f}}^\circ(t'))_{\mathbb{Q}}$.

If t and t' are similar seeds with compatible Poisson structures λ, then it is called a Poisson variation map if we have

$$\lambda(\mathbf{f}_i, \mathbf{f}_j) = \lambda\left(\mathsf{var}_t^M \mathbf{f}_i, \mathsf{var}_t^M \mathbf{f}_j\right), \qquad \forall i, j \in I.$$

Note that (3.1) can be written as

$$\mathsf{var}_t^M(p^*\mathbf{e}_k) = p^*\mathbf{e}'_{\sigma k}. \qquad (3.2)$$

Remark 3.7. Assume the existence of compatible Poisson structures, i.e, Assumption 2.4 holds. Then the following statements are equivalent:

- A linear map $\Psi \colon M^\circ(t) \to M^\circ(t')$ is a Poisson variation map.
- For the quantum case $\Bbbk = \mathbb{Z}\left[v^{\pm\frac{1}{d'}}\right]$, the monomial map Φ_v associated to Ψ is a variation map. In particular, it preserves the v-twisted products.

- For the classical case $\Bbbk = \mathbb{Z}$, the monomial map Φ_1 associated to Ψ is a Poisson variation map.

Therefore, a Poisson variation map is the classical limit of a quantum variation map at $v = 1$, see Section 2.3. Conversely, a Poisson variation map gives rise to a quantum variation map by the above equivalent statements.

Let P_σ denote the permutation matrix associated to σ of rank I_{uf}.

Lemma 3.8.

(1) *The variation map* var_t^M *has the following matrix representation in the bases* $\underline{\mathbf{f}}$, $\underline{\mathbf{f}}'$:

$$\mathsf{var}_t^M(\underline{\mathbf{f}}) = \underline{\mathbf{f}}' \cdot \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ U_{\mathsf{f,uf}} & U_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{P}_\sigma & \\ & \mathsf{Id}_{\mathsf{f}} \end{pmatrix},$$

where $\mathsf{col}_k U_{\mathsf{f,uf}}$ *(resp.* $\mathsf{col}_j U_{\mathsf{f}}$*) are the coordinates of* $\mathbf{u}_k$ *(resp.* $\mathbf{u}_j$*) in the basis* $\{\mathbf{f}_{j'}, j' \in I_{\mathsf{f}}\}$, *for* $k \in I_{\mathsf{uf}}$ *(resp.* $j \in I_{\mathsf{f}}$*).*

(2) *Moreover, equation* (3.1) *is equivalent to the following*

$$\begin{pmatrix} U_{\mathsf{f,uf}} \, \mathsf{P}_\sigma & U_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} B_{\mathsf{uf}} \\ B_{\mathsf{f,uf}} \end{pmatrix} = B'_{\mathsf{f,uf}} \, \mathsf{P}_\sigma . \tag{3.3}$$

Proof. (1) Recall that $\mathsf{col}_k(H\mathsf{P}_\sigma) = \mathsf{col}_{\sigma k} H$ for any matrix H. The first statement follows.
(2) For any $k \in I_{\mathsf{uf}}$, we have

$$\mathsf{var}_t^M(p^*\mathbf{e}_k) = \mathsf{var}_t^M(\underline{\mathbf{f}} \, \mathsf{col}_k B) = \underline{\mathbf{f}}' \cdot \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ U_{\mathsf{f,uf}} & U_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{P}_\sigma & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \mathsf{col}_k B$$

$$= \underline{\mathbf{f}}' \cdot \begin{pmatrix} \mathsf{P}_\sigma & 0 \\ U_{\mathsf{f,uf}} \, \mathsf{P}_\sigma & U_{\mathsf{f}} \end{pmatrix} \mathsf{col}_k \begin{pmatrix} B_{\mathsf{uf}} \\ B_{\mathsf{f,uf}} \end{pmatrix} = \underline{\mathbf{f}}' \cdot \mathsf{col}_k \left(\begin{pmatrix} \mathsf{P}_\sigma & 0 \\ U_{\mathsf{f,uf}} \, \mathsf{P}_\sigma & U_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} B_{\mathsf{uf}} \\ B_{\mathsf{f,uf}} \end{pmatrix} \right).$$

We have

$$p^*\mathbf{e}'_{\sigma k} = \underline{\mathbf{f}}' \, \mathsf{col}_{\sigma k} B' = \underline{\mathbf{f}}' \, \mathsf{col}_k \left(B' \begin{pmatrix} \mathsf{P}_\sigma & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \right) = \underline{\mathbf{f}}' \, \mathsf{col}_k \begin{pmatrix} B'_{\mathsf{uf}} \, \mathsf{P}_\sigma \\ B'_{\mathsf{f,uf}} \, \mathsf{P}_\sigma \end{pmatrix}.$$

Note that $\mathsf{P}_\sigma^{-1} \, B'_{\mathsf{uf}} \, \mathsf{P}_\sigma = B_{\mathsf{uf}}$ since $b'_{\sigma i, \sigma j} = b_{i,j}$ for $i, j \in I_{\mathsf{uf}}$. So $\mathsf{P}_\sigma \, B_{\mathsf{uf}} = B'_{\mathsf{uf}} \, \mathsf{P}_\sigma$. The claim follows. ∎

Lemma 3.9. *If a (Poisson) variation map* var_t^M *is invertible, its inverse is still a (Poisson) variation map.*

Proof. Since var_t^M is represented by an invertible matrix

$$\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ U_{\mathsf{f,uf}} & U_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{P}_\sigma & \\ & \mathsf{Id}_{\mathsf{f}} \end{pmatrix},$$

its inverse $\left(\mathsf{var}_t^M \right)^{-1}$ is represented by

$$\begin{pmatrix} \mathsf{P}_{\sigma^{-1}} & \\ & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ -U_{\mathsf{f}}^{-1} U_{\mathsf{f,uf}} & U_{\mathsf{f}}^{-1} \end{pmatrix} = \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ -U_{\mathsf{f}}^{-1} U_{\mathsf{f,uf}} \, \mathsf{P}_\sigma & U_{\mathsf{f}}^{-1} \end{pmatrix} \begin{pmatrix} \mathsf{P}_{\sigma^{-1}} & \\ & \mathsf{Id}_{\mathsf{f}} \end{pmatrix}.$$

Thus $\left(\mathsf{var}_t^M \right)^{-1}$ is a variation map with permutation σ^{-1}.

Since var_t^M is a variation map, we have $\mathsf{var}_t^M \, p^*\mathbf{e}_k = p^*\mathbf{e}'_{\sigma k}$ for any $k \in I_{\mathsf{uf}}$, it follows that $\left(\mathsf{var}_t^M \right)^{-1} p^*\mathbf{e}'_k = p^*\mathbf{e}_{\sigma^{-1}k}$. So (3.2) holds. Finally, if var_t^M preserves the compatible Poisson structures, so does its inverse. ∎

Lemma 3.10. *A variation map* $\mathsf{var}_t^A \colon \mathcal{LP}^A(t) \to \mathcal{LP}^A(t')$ *restricts to an algebra homomorphism from* $\mathcal{U}^A(t)$ *to* $\mathcal{U}^A(t')$.

Proof. Take any mutation sequence $\boldsymbol{\mu} = \mu_{k_r} \cdots \mu_{k_1}$, which is the identify if the sequence $(k_r, \ldots, k_1)$ is empty. Denote $\sigma\boldsymbol{\mu} = \mu_{\sigma k_r} \cdots \mu_{\sigma k_1}$. Then $s = \boldsymbol{\mu}t$ and $s' = (\sigma\boldsymbol{\mu})t'$ are similar up to σ. To prove the claim, it suffices to show that, for any $z \in \mathcal{LP}^A(t) \cap \boldsymbol{\mu}^* \mathcal{LP}^A(s)$, we have $\mathsf{var}_t^A(z) \in (\sigma\boldsymbol{\mu})^* \mathcal{LP}^A(s')$. If so, we obtain that, for any

$$z \in \mathcal{U}^A(t) = \mathcal{LP}^A(t) \bigcap (\cap_{s = \boldsymbol{\mu}t} \boldsymbol{\mu}^* \mathcal{LP}^A(s)),$$

$$\mathsf{var}_t^A(z) \in \cap_{s' = (\sigma\boldsymbol{\mu})t'} (\sigma\boldsymbol{\mu})^* \mathcal{LP}^A(s') = \mathcal{U}^A(t').$$

By the Laurent expansion of z in the seed s, we have $z * (\boldsymbol{\mu}^* A(s)^d) = \sum c_m (\boldsymbol{\mu}^* A(s)^m)$ for some $c_m \in \mathbb{k}$, $d, m \in \mathbb{N}^I$. By [44, Lemma 4.2.2 (ii)], the variation map var_t^A sends cluster monomials of s to the cluster monomials of s' up to frozen factors: we have $\mathsf{var}_t^A (\boldsymbol{\mu}^* A(s)^d) = (\sigma\boldsymbol{\mu})^* A(s')^{d'}$ and $\mathsf{var}_t^A(\boldsymbol{\mu}^* A(s)^m) = (\sigma\boldsymbol{\mu})^* A(s')^{m'}$, $\mathsf{pr}_{I_{\mathsf{uf}}} d'$, $\mathsf{pr}_{I_{\mathsf{uf}}} m' \geq 0$. Therefore, $\mathsf{var}_t^A(z)$ is contained in $(\sigma\boldsymbol{\mu})^* \mathcal{LP}^A(s')$. $\blacksquare$

3.2 Existence of variation maps

Assume that there is a linear variation map $\mathsf{var}_t^M \colon M^\circ(t)_{\mathbb{Q}} \to M^\circ(t')_{\mathbb{Q}}$. Then it restricts to a linear map from $M^\circ(t)$ to $M_{\mathsf{uf}}^\circ(t') \oplus \frac{1}{r} M_{\mathsf{f}}^\circ(t')$ for some $r > 0$. Similarly, the corresponding variation map $\mathsf{var}_t^A \colon \mathcal{LP}^A(t)_{\mathcal{Z}\mathbb{Q}} \to \mathcal{LP}^A(t')_{\mathcal{Z}\mathbb{Q}}$ restricts to an algebra homomorphism $\mathsf{var}_t^A \colon \mathcal{LP}^A(t) \to \mathcal{LP}^A(t')_{\mathcal{Z}\frac{1}{r}}$. The restriction of a linear variation map to sublattices (resp. the restriction of a monomial variation maps to subalgebras) will be still called a variation map.

Proposition 3.11. *Let there be given two seeds* t, t' *similar up to a permutation* σ. *Assume that the full rank assumption holds.*

(1) *There exists a variation map* $\mathsf{var}_t^M \colon M^\circ(t)_{\mathbb{Q}} \to M^\circ(t')_{\mathbb{Q}}$.

(2) *The set of variation maps from* $M^\circ(t)_{\mathbb{Q}}$ *to* $M^\circ(t')_{\mathbb{Q}}$ *takes the form* $\{\mathsf{var}_t^M + z \mid z \in Z_{\mathbb{Q}}\}$, *where* $Z_{\mathbb{Q}}$ *is a* $\mathbb{Q}$-vector space of dimension $|I_{\mathsf{f}}| \cdot |I_{\mathsf{uf}}|$.

(3) *If an order* $|I_{\mathsf{uf}}|$ *minor of* $\widetilde{B}$ *is* ± 1, *then there exists a variation map* $\mathsf{var}_t^M \colon M^\circ(t) \to M^\circ(t')$. *Moreover, the set of variation maps from* $M^\circ(t)$ *to* $M^\circ(t')$ *takes the form* $\{\mathsf{var}_t^M + z \mid z \in Z_{\mathbb{Z}}\}$, *where* $Z_{\mathbb{Z}}$ *is a lattice of rank* $|I_{\mathsf{f}}| \cdot |I_{\mathsf{uf}}|$.

Proof. Relabeling the vertices σk by k for t' if necessary, we assume $\sigma = \mathsf{Id}$ in this proof.

(1) We need to construct a linear map $\mathsf{var}_t^M \colon M^\circ(t)_{\mathbb{Q}} \to M^\circ(t')_{\mathbb{Q}}$ represented by a matrix $\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ U_{\mathsf{f},\mathsf{uf}} & U_{\mathsf{f}} \end{pmatrix}$ in the bases $\underline{\mathbf{f}}$, $\underline{\mathbf{f}}'$ such that (3.3) holds. The unknown matrix $U := \begin{pmatrix} U_{\mathsf{f},\mathsf{uf}} & U_{\mathsf{f}} \end{pmatrix}$ needs to satisfy the following inhomogeneous equation:

$$U\widetilde{B} = B'_{\mathsf{f},\mathsf{uf}}. \tag{3.4}$$

Notice that $\widetilde{B} = \begin{pmatrix} B_{\mathsf{uf}} \\ B_{\mathsf{f},\mathsf{uf}} \end{pmatrix}$ is of full rank. Then we can always choose a (not necessarily unique) size-$|I_{\mathsf{uf}}|$ subset $J \subset I$ such that the submatrix $B_{J,I_{\mathsf{uf}}}$ is full rank. Its inverse matrix $(B_{J,I_{\mathsf{uf}}})^{-1}$ has entries in $\frac{1}{\det B_{J,I_{\mathsf{uf}}}} \mathbb{Z}$. Let u_i, $i \in I$, denote the (unknown) columns of U and b_i, $i \in I$, denote the rows of $\widetilde{B}$, then

$$U\widetilde{B} = \sum_{i \in I} u_i b_i.$$

Taking $u_i = 0$ for $i \in I \backslash J$ and letting u_i, $i \in J$ equal the columns of the $I_{\mathsf{f}} \times J$-matrix $B'_{\mathsf{f},\mathsf{uf}} (B_{J,I_{\mathsf{uf}}})^{-1}$, we obtain a special solution U_0 for (3.4), whose entries lie in $\frac{1}{\det B_{J,I_{\mathsf{uf}}}} \mathbb{Z}$.

(2) Let U_0 denote the special solution in (1). Let $Z_{\mathbb{Q}}$ denote the set of solutions of the homogeneous linear system

$$U\widetilde{B} = 0, \qquad \text{or equivalently} \quad \sum_{i \in J} u_i b_i = -\sum_{i \in I \setminus J} u_i b_i. \tag{3.5}$$

Equation (3.5) has a unique solution u_i, $i \in I$, for any given u_i, $i \in I \setminus J$. Hence the set of $\mathbb{Q}$-solutions is a vector space of dimension $|I_{\mathsf{f}}| \cdot |I_{\mathsf{uf}}|$. It follows that the solutions for (3.4) take the form $U + U_0$, $U \in Z_{\mathbb{Q}}$.

(3) It is a direct consequence of the above argument since $\det B_{J,I_{\mathsf{uf}}}$ is ± 1.   ∎

3.3   Change of seeds

We treat lattices $M^\circ(t)$ and the corresponding quantum torus algebras $\mathcal{LP}^A(t)$ in this subsection. Our arguments and results remain valid after the extension to $M^\circ(t)_{\mathbb{Q}}$ and $\mathcal{LP}^A(t)_{\mathcal{Z}^{\mathbb{Q}}}$.

Let $t', t \in \Delta^+$ denote seeds similar up to a permutation σ. Let there be given a linear variation map $\mathsf{var}_t^M \colon M^\circ(t) \to M^\circ(t')$. Choose any $k \in I_{\mathsf{uf}}$ and denote $s = \mu_k t$, $s' = \mu_{\sigma k} t'$. Recall that we have linear isomorphisms $\psi_{t,s}^M \colon M^\circ(s) \simeq M^\circ(t)$ and $\psi_{t',s'}^M \colon M^\circ(s') \simeq M^\circ(t')$, which are represented by the matrices $\mathsf{P}_{k,+}^M(t)$ and $\mathsf{P}_{\sigma k,+}^M(t')$, respectively. Define

$$\mathsf{var}_s^M = \left(\psi_{t',s'}^M\right)^{-1} \mathsf{var}_t^M \, \psi_{t,s}^M \colon \ M^\circ(s) \simeq M^\circ(s')$$

so that the following diagram commutes:

$$
\begin{array}{ccc}
M^\circ(t) & \xrightarrow{\ \mathsf{var}_t^M\ } & M^\circ(t') \\
\uparrow \psi_{t,s}^M & & \uparrow \psi_{t',s'}^M \\
M^\circ(s) & \xrightarrow{\ \mathsf{var}_s^M\ } & M^\circ(s').
\end{array}
\tag{3.6}
$$

The following properties for var_s^M are analogous to those in [47, Proposition 5.1.6].

Proposition 3.12.

(1) *The linear map var_s^M is a variation map.*

(2) *If var_t^M is invertible, so is var_s^M.*

(3) *If var_t^M is a Poisson variation map, so is var_s^M.*

Proof. As before, we can assume $\sigma = \mathsf{Id}$ by relabeling the vertices for t' and s'. Then $\mathsf{P}_{k,+}^M(t)_{\mathsf{uf}}$ coincides with $\mathsf{P}_{k,+}^M(t')_{\mathsf{uf}}$ by the similarity between t and t'. It is of the block matrix form

$$
\begin{pmatrix}
P_{\mathsf{uf}} & 0 \\
P_{\mathsf{f},\mathsf{uf}} & \mathsf{Id}_{\mathsf{f}}
\end{pmatrix}.
$$

(1) The statement can be translated from [47, Proposition 5.1.6] by using Lemma 3.13. Let us give a direct proof.

The linear map var_s^M has the following matrix representation:

$$
\begin{aligned}
\mathsf{var}_s^M(\underline{\mathbf{f}}(s)) &= \left(\psi_{t',s'}^M\right)^{-1} \mathsf{var}_t^M \, \psi_{t,s}^M(\underline{\mathbf{f}}(s)) = \left(\psi_{t',s'}^M\right)^{-1} \mathsf{var}_t^M(\underline{\mathbf{f}}(t)\, \mathsf{P}_{k,+}^M(t)) \\
&= \left(\psi_{t',s'}^M\right)^{-1}\left(\underline{\mathbf{f}}(t')\begin{pmatrix}\mathsf{Id}_{\mathsf{uf}} & 0 \\ U_{\mathsf{f},\mathsf{uf}} & U_{\mathsf{f}}\end{pmatrix}\mathsf{P}_{k,+}^M(t)\right) = \underline{\mathbf{f}}(s')\,\mathsf{P}_{k,+}^M(t')^{-1}\begin{pmatrix}\mathsf{Id}_{\mathsf{uf}} & 0 \\ U_{\mathsf{f},\mathsf{uf}} & U_{\mathsf{f}}\end{pmatrix}\mathsf{P}_{k,+}^M(t).
\end{aligned}
$$

We see that the matrix $\mathsf{P}_{k,+}^M(t')^{-1}\begin{pmatrix}\mathsf{Id}_{\mathsf{uf}} & 0 \\ U_{\mathsf{f},\mathsf{uf}} & U_{\mathsf{f}}\end{pmatrix}\mathsf{P}_{k,+}^M(t)$ is block diagonal, such that its $I_{\mathsf{uf}} \times I_{\mathsf{uf}}$ submatrix is $\mathsf{Id}_{\mathsf{uf}}$ (by the similarity between t, t') and its $I_{\mathsf{uf}} \times I_{\mathsf{f}}$ submatrix is 0.

We need to check (3.2). Let $\underline{\mathbf{e}}_{\mathsf{uf}}$ denote the matrix $(\mathbf{e}_i)_{i \in I_{\mathsf{uf}}}$ where $\mathbf{e}_i$ are column vectors. Then $\mathsf{P}^N_{k,+}(t)_{\mathsf{uf}}$ coincides with $\mathsf{P}^N_{k,+}(t')_{\mathsf{uf}}$ by the similarity between t and t'. Recall that, for $j \in I_{\mathsf{uf}}$, we have

$$\mathsf{col}_j\left(\underline{\mathbf{e}}(t)\,\mathsf{P}^N_{k,+}(t)\right) = \begin{cases} \mathbf{e}_j(t) + [b_{kj}]_+\mathbf{e}_k(t), & j \neq k, \\ -\mathbf{e}_k(t), & j = k. \end{cases}$$

We deduce that $\mathsf{col}_j\left(\underline{\mathbf{e}}(t)\,\mathsf{P}^N_{k,+}(t)\right) = \mathsf{col}_j\left(\underline{\mathbf{e}}(t)_{\mathsf{uf}}\,\mathsf{P}^N_{k,+}(t)_{\mathsf{uf}}\right)$. Using Lemma 2.15, we have

$$\begin{aligned}
\psi^M_{t',s'}\,\mathsf{var}^M_s(p^*\mathbf{e}_j(s)) &= \mathsf{var}^M_t\,\psi^M_{t,s}(p^*\mathbf{e}_j(s)) = \mathsf{var}^M_t\,p^*\left(\psi^N_{t,s}\mathbf{e}_j(s)\right) \\
&= \mathsf{var}^M_t\,p^*\left(\mathsf{col}_j(\underline{\mathbf{e}}(t)\,\mathsf{P}^N_{k,+}(t))\right) = \mathsf{var}^M_t\,p^*\left(\mathsf{col}_j(\underline{\mathbf{e}}(t)_{\mathsf{uf}}\,\mathsf{P}^N_{k,+}(t)_{\mathsf{uf}})\right) \\
&= \mathsf{col}_j\,\mathsf{var}^M_t\,p^*\left(\underline{\mathbf{e}}(t)_{\mathsf{uf}} \cdot \mathsf{P}^N_{k,+}(t)_{\mathsf{uf}}\right).
\end{aligned}$$

We also have

$$\begin{aligned}
\psi^M_{t',s'}(p^*\mathbf{e}_j(s')) &= p^*\left(\psi^N_{t',s'}\mathbf{e}_j(s')\right) = p^*\left(\mathsf{col}_j\left(\underline{\mathbf{e}}(t')\,\mathsf{P}^N_{k,+}(t')\right)\right) \\
&= p^*\left(\mathsf{col}_j\left(\underline{\mathbf{e}}(t')_{\mathsf{uf}}\,\mathsf{P}^N_{k,+}(t')_{\mathsf{uf}}\right)\right) = \mathsf{col}_j\,p^*\left(\underline{\mathbf{e}}(t')_{\mathsf{uf}} \cdot \mathsf{P}^N_{k,+}(t')_{\mathsf{uf}}\right).
\end{aligned}$$

Recall that $\mathsf{var}^M_t(p^*\mathbf{e}_j(t)) = p^*\mathbf{e}_j(t')$ for any $j \in I_{\mathsf{uf}}$. Then

$$\mathsf{var}^M_t\left(p^*(\underline{\mathbf{e}}(t)_{\mathsf{uf}} \cdot \mathsf{P}^N_{k,+}(t)_{\mathsf{uf}})\right) = p^*\left(\underline{\mathbf{e}}(t')_{\mathsf{uf}} \cdot \mathsf{P}^N_{k,+}(t')_{\mathsf{uf}}\right).$$

Therefore $\psi^M_{t',s'}\,\mathsf{var}^M_s(p^*\mathbf{e}_j(s)) = \psi^M_{t',s'}(p^*\mathbf{e}_j(s'))$, $\forall j \in I_{\mathsf{uf}}$. It follow that var^M_s is a variation map.

(2) The statement is obvious.

(3) By Lemma 2.15, $\psi^M_{t,s}$ and $\psi^M_{t',s'}$ preserve the bilinear form λ. The claim follows. ∎

Let $\mathsf{var}^A_t \colon \mathcal{LP}^A(t) \to \mathcal{LP}^A(t')$ denote the monomial map associated to var^M_t. We further assume that it is a variation map (equivalently, var^M_t needs to be Poisson if we work at the quantum level, see Remark 3.7).

Lemma 3.13. *We have* $\mathsf{var}^M_s\,\mathbf{f}_i(s) = \deg_{s'}\left(\left(\mu^A_{\sigma k}\right)^*\right)^{-1}\mathsf{var}^A_t\left(\mu^A_k\right)^* A_i(s)$.

Proof. On the one hand, the i-th cluster variable $\left(\mu^A_k\right)^* A_i(s)$ of s, $i \in I$, is pointed at $\deg_t\left(\mu^A_k\right)^* A_i(s) = \psi^M_{t,s}\mathbf{f}_i(s)$ in $\mathcal{LP}^A(t)$. Since var^A_t is a variation map, $\mathsf{var}^A_t\left(\mu^A_k\right)^* A_i(s)$ is pointed at $\mathsf{var}^M_t\,\psi^M_{t,s}\mathbf{f}_i(s)$ in $\mathcal{LP}(t')$. In particular, we have

$$\deg_{t'}\,\mathsf{var}^A_t\left(\mu^A_k\right)^* A_i(s) = \mathsf{var}^M_t\,\psi^M_{t,s}\mathbf{f}_i(s).$$

On the other hand, by the similarity between t and t', the image $\mathsf{var}^A_t\left(\mu^A_k\right)^* A_i(s)$ of the i-th cluster variable $\left(\mu^A_k\right)^* A_i(s)$ agrees with the σi-th cluster variable $\left(\mu^A_{\sigma k}\right)^* A_{\sigma i}(s')$ up to a frozen factor, see [44, Lemma 4.2.2]. It follows that, similar to the cluster variable $\left(\mu^A_{\sigma k}\right)^* A_{\sigma i}(s')$, the degree of $\mathsf{var}^A_t\left(\mu^A_k\right)^* A_i(s)$ in t' can be computed from the degree of its Laurent expansion $\left(\left(\mu^A_{\sigma k}\right)^*\right)^{-1}\mathsf{var}^A_t\left(\mu^A_k\right)^* A_i(s)$ in s' by using the linear map $\psi^M_{t',s'}$:

$$\deg_{t'}\,\mathsf{var}^A_t\left(\mu^A_k\right)^* A_i(s) = \psi^M_{t',s'}\,\deg_{s'}\left(\left(\mu^A_{\sigma k}\right)^*\right)^{-1}\mathsf{var}^A_t\left(\mu^A_k\right)^* A_i(s).$$

Therefore, we obtain $\mathsf{var}^M_t\,\psi^M_{t,s}\mathbf{f}_i(s) = \psi^M_{t',s'}\,\deg_{s'}\left(\left(\mu^A_{\sigma k}\right)^*\right)^{-1}\mathsf{var}^A_t(\mu^A_k)^* A_i(s)$. The desired claim follows from the definition $\mathsf{var}^M_s = \left(\psi^M_{t',s'}\right)^{-1}\mathsf{var}^M_t\,\psi^M_{t,s}$. ∎

We define the monomial map $\mathsf{var}^A_s \colon \mathcal{LP}^A(s) \to \mathcal{LP}^A(s')$ to be the monomial map associated to $\mathsf{var}^M_s = \left(\psi^M_{t',s'}\right)^{-1}\mathsf{var}^M_t\,\psi^M_{t,s}$. Since var^A_t is a variation map, it is also a variation map, see Remark 3.7.

Proposition 3.14. *The map* $\mathsf{var}_t^A \left(\mu_k^A\right)^*$ *agrees with* $\left(\mu_{\sigma k}^A\right)^* \mathsf{var}_s^A$, *i.e., the following diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{U}^A(t) & \xrightarrow{\ \mathsf{var}_t^A\ } & \mathcal{U}^A(t') \\
\uparrow \left(\mu_k^A\right)^* & & \uparrow \left(\mu_{\sigma k}^A\right)^* \\
\mathcal{U}^A(s) & \xrightarrow{\ \mathsf{var}_s^A\ } & \mathcal{U}^A(s').
\end{array}
$$

Proof. It suffices to show $\left(\left(\mu_{\sigma k}^A\right)^*\right)^{-1} \mathsf{var}_t^A \left(\mu_k^A\right)^* A_i(s) = \mathsf{var}_s^A A_i(s)$, $\forall i$.

Both $\left(\left(\mu_{\sigma k}^A\right)^*\right)^{-1} \mathsf{var}_t^A \left(\mu_k^A\right)^* A_i(s)$ and $\mathsf{var}_s^A A_i(s)$ are cluster variables in s' up to a frozen factor by [44, Lemma 4.2.2]. By Lemma 3.13, they have the same degree. The claim follows. ∎

We can given a different proof for Proposition 3.14 by straightforward computation (see Proposition 4.12).

Take any mutation sequence $\boldsymbol{\mu}$ and denote $s = \boldsymbol{\mu}t$, $s' = (\sigma\boldsymbol{\mu})t'$. By using Proposition 3.14 recursively, var_t^A and $\boldsymbol{\mu}$ uniquely determine a variation map var_s^A such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{U}^A(t) & \xrightarrow{\ \mathsf{var}_t^A\ } & \mathcal{U}^A(t') \\
\uparrow (\boldsymbol{\mu})^* & & \uparrow (\sigma\boldsymbol{\mu})^* \\
\mathcal{U}^A(s) & \xrightarrow{\ \mathsf{var}_s^A\ } & \mathcal{U}^A(s').
\end{array}
$$

By tracking the degrees of cluster variables, we see that the variation map var_s^M associated to var_s^A is determined by var_t^M and $\boldsymbol{\mu}$. It still makes (3.6) commutative.

3.4   Twist endomorphisms

We are now ready to define twist endomorphisms. Notice that the mutation maps always preserves the compatible Poisson structures and the v-twisted products. Let t and $t' = \boldsymbol{\mu}t$ be two seeds similar up to σ as before. Let there be given a variation map var_t^A from $\mathcal{LP}^A(t)_{\mathcal{Z}^\mathbb{Q}}$ to $\mathcal{LP}^A(t')_{\mathcal{Z}^\mathbb{Q}}$. Note that it induces a homomorphism $\mathsf{var}_t^A \colon \mathcal{U}^A(t)_{\mathcal{Z}^\mathbb{Q}} \to \mathcal{U}^A(t')_{\mathcal{Z}^\mathbb{Q}}$.

Definition 3.15. The composition $\boldsymbol{\mu}^* \mathsf{var}_t^A \colon \mathcal{U}^A(t)_{\mathcal{Z}^\mathbb{Q}} \to \mathcal{U}^A(t)_{\mathcal{Z}^\mathbb{Q}}$ is called a twist endomorphism passing through the seed $t' = \boldsymbol{\mu}t$, which is denoted by tw_t^A.

For classical cases, if t and t' are equipped with compatible Poisson structures which are preserved by tw_t^A, then tw_t^A is called a Poisson twist endomorphism.

Note that var_t^A restricts to a variation map $\mathsf{var}_t^A \colon \mathcal{LP}(t) \to \mathcal{LP}(t')_{\mathcal{Z}^{\frac{1}{r}}}$ for some integer $r > 0$. So the twist endomorphism tw_t^A restricts to an algebra homomorphism from $\mathcal{U}^A(t)$ to $\mathcal{U}^A(t)_{\mathcal{Z}^{\frac{1}{r}}}$.

Propositions 3.12 and 3.14 have the following consequence.

Corollary 3.16. *A twist endomorphism* tw_t^A *on* $\mathcal{U}^A(t)_{\mathcal{Z}^\mathbb{Q}}$ *(resp. on* $\mathcal{U}^A(t)$*) gives rise to twist endomorphisms* tw_s^A *on* $\mathcal{U}^A(s)_{\mathcal{Z}^\mathbb{Q}}$ *(resp. on* $\mathcal{U}^A(s)$*) for all seeds* $s \in \Delta^+$ *via the mutation maps. For classical cases, if* tw_t^A *is Poisson, so are* tw_s^A.

Correspondingly, we can simply denote the twist endomorphisms tw_t^A by tw^A when we do not want to choose a specific seed t.

Proposition 3.17. *The twist endomorphism* tw^A *restricts to an endomorphism on* $\mathcal{A}_{\mathcal{Z}^\mathbb{Q}}$.

Proof. Recall that var_t^A sends cluster variables to cluster variables up to a frozen factor, see [44, Lemma 4.2.2]. The claim follows. ∎

If var_t^A is an invertible variation map, then we have a twist automorphism $\mathsf{tw}_t^A := \boldsymbol{\mu}^* \, \mathsf{var}_{t_1}^A$ on $\mathcal{U}^A(t)_{\mathcal{Z}\mathbb{Q}}$. Lemma 3.9 implies that $(\mathsf{var}_t^A)^{-1}$ is still a variation map. So $\eta := (\boldsymbol{\mu}^*)^{-1}(\mathsf{var}_t^A)^{-1}$ is a twist automorphism on $\mathcal{U}^A(t')_{\mathcal{Z}\mathbb{Q}}$.

Lemma 3.18. *We have $\boldsymbol{\mu}^* \eta (\boldsymbol{\mu}^*)^{-1} = (\mathsf{tw}_t^A)^{-1}$. Namely, by identifying the fraction fields by mutations, the twist automorphisms associated to invertible variation maps $\mathsf{var}_t^A \colon \mathcal{LP}^A(t)_{\mathcal{Z}\mathbb{Q}} \to \mathcal{LP}^A(t')_{\mathcal{Z}\mathbb{Q}}$ and $(\mathsf{var}_t^A)^{-1} \colon \mathcal{LP}^A(t')_{\mathcal{Z}\mathbb{Q}} \to \mathcal{LP}^A(t)_{\mathcal{Z}\mathbb{Q}}$ are inverse to each other.*

Proof. We have

$$\boldsymbol{\mu}^* \eta (\boldsymbol{\mu}^*)^{-1} = \boldsymbol{\mu}^* \circ (\boldsymbol{\mu}^*)^{-1} (\mathsf{var}_t^A)^{-1} \circ (\boldsymbol{\mu}^*)^{-1} = (\mathsf{var}_t^A)^{-1} \circ (\boldsymbol{\mu}^*)^{-1} = (\mathsf{tw}_t^A)^{-1}. \qquad \blacksquare$$

Proposition 3.11 implies the following result.

Theorem 3.19. *Consider the classical case $\Bbbk = \mathbb{Z}$. Assume that $t, t' \in \Delta^+$ are similar and the full rank assumption holds, then the following statements are true.*

(1) *There exists a twist endomorphism tw_t^A on $\mathcal{U}_{\mathcal{Z}\mathbb{Q}}^A$ passing through the seed t'. The set of such twist endomorphisms is in bijection with a $\mathbb{Q}$-vector space of dimension $|I_{\mathsf{uf}}| \cdot |I_{\mathsf{f}}|$.*

(2) *If an order $|I_{\mathsf{uf}}|$ minor of $\widetilde{B}$ is ± 1, then there exists a twist endomorphism tw_t^A on $\mathcal{U}^A$ passing through the seed t'. The set of such twist endomorphisms is in bijection with a lattice of rank $|I_{\mathsf{uf}}| \cdot |I_{\mathsf{f}}|$.*

4 Twist endomorphisms for upper cluster X-algebras

4.1 Variation maps

Let t, t' be similar seeds as before. In the following calculation, we choose t as the initial seed. We want to construct twist endomorphism on $\mathcal{U}^X(t')$ preserving the canonical Poisson structure (2.1).

Let us first investigate linear maps between lattices $N(t)$ which arise as the pullback of those between $M^\circ(t)$. Let var_t^M denote a $\mathbb{Z}$-linear map $M^\circ(t) \to M^\circ(t')_{\mathbb{Q}}$, its pullback between the dual lattices gives rise to a $\mathbb{Z}$-linear map $(\mathsf{var}_t^M)^*$ from $N(t')$ to $N(t)_{\mathbb{Q}}$. Recall that the diagonal entries of D are $\frac{1}{d_i}$, $i \in I$.

Lemma 4.1. *The pullback $(\mathsf{var}_t^M)^* \colon N(t') \to N(t)_{\mathbb{Q}}$ is represented by the matrix*

$$D^{-1} \begin{pmatrix} \mathsf{P}_{\sigma^{-1}} & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & U_{\mathsf{f},\mathsf{uf}}^{\mathrm{T}} \\ 0 & U_{\mathsf{f}}^{\mathrm{T}} \end{pmatrix} D$$

with respect to the bases $\underline{\mathbf{e}}$ and $\underline{\mathbf{e}}'$.

Proof. By assumption, we have $\mathsf{var}_t^M(\underline{\mathbf{f}}) = \underline{\mathbf{f}}' \cdot \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ U_{\mathsf{f},\mathsf{uf}} & U_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{P}_\sigma & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix}$. Then its pullback with respect to the dual bases is represented by the transpose:

$$(\mathsf{var}_t^M)^*((\underline{\mathbf{f}}')^*) = (\underline{\mathbf{f}}^*) \cdot \begin{pmatrix} \mathsf{P}_{\sigma^{-1}} & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & U_{\mathsf{f},\mathsf{uf}}^{\mathrm{T}} \\ 0 & U_{\mathsf{f}}^{\mathrm{T}} \end{pmatrix}.$$

Since $\mathbf{f}_i^* = d_i \mathbf{e}_i$ and the diagonal entries of D are $\frac{1}{d_i}\cdot$, we have $\underline{\mathbf{f}}^* = \underline{\mathbf{e}}D^{-1}$ and, similarly, $(\underline{\mathbf{f}}')^* = (\underline{\mathbf{e}}')D^{-1}$. So we obtain

$$(\mathsf{var}_t^M)^*(\underline{\mathbf{e}}'D^{-1}) = \underline{\mathbf{e}}D^{-1} \cdot \begin{pmatrix} \mathsf{P}_{\sigma^{-1}} & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & U_{\mathsf{f},\mathsf{uf}}^{\mathrm{T}} \\ 0 & U_{\mathsf{f}}^{\mathrm{T}} \end{pmatrix},$$

and, equivalently, $(\mathsf{var}_t^M)^*(\underline{\mathbf{e}}') = \underline{\mathbf{e}}D^{-1} \begin{pmatrix} \mathsf{P}_{\sigma^{-1}} & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & U_{\mathsf{f},\mathsf{uf}}^{\mathrm{T}} \\ 0 & U_{\mathsf{f}}^{\mathrm{T}} \end{pmatrix} D.$ $\qquad \blacksquare$

Since t and t' are similar, we have $d_k = d_{\sigma k}$ by definition. Then it follows that

$$D^{-1} \begin{pmatrix} \mathsf{P}_{\sigma^{-1}} & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} = \begin{pmatrix} \mathsf{P}_{\sigma^{-1}} & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} D^{-1}.$$

In view of Lemma 4.1, we propose the following definition.

Definition 4.2. A linear map $\mathsf{var}^N_{t'} \colon N(t')_{\mathbb{Q}} \to N(t)_{\mathbb{Q}}$ is said to be a variation map, if it has the following $\mathbb{Q}$-valued matrix representation in the basis $\underline{\mathbf{e}}' = \{\mathbf{e}'_i \mid i \in I\}$, $\underline{\mathbf{e}} = \{\mathbf{e}_i \mid i \in I\}$:

$$\mathsf{var}^N_{t'}(\underline{\mathbf{e}}') = \underline{\mathbf{e}} \cdot \begin{pmatrix} \mathsf{P}_{\sigma^{-1}} & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & V_{\mathsf{uf},\mathsf{f}} \\ 0 & V_{\mathsf{f}} \end{pmatrix}, \tag{4.1}$$

and it satisfies

$$\omega\big(\mathsf{var}^N_{t'} \mathbf{e}'_i, \mathsf{var}^N_{t'} \mathbf{e}'_k\big) = \omega(\mathbf{e}'_i, \mathbf{e}'_k), \qquad \forall i \in I, \quad k \in I_{\mathsf{uf}}. \tag{4.2}$$

It is said to be Poisson if it further preserves the Poisson structures:

$$\omega\big(\mathsf{var}^N_{t'} \mathbf{e}'_i, \mathsf{var}^N_{t'} \mathbf{e}'_j\big) = \omega(\mathbf{e}'_i, \mathbf{e}'_j), \qquad \forall i, j \in I.$$

We will see in Proposition 4.8 that (4.2) is natural.

Definition 4.2 will be naturally deduced from the definition of the following monomial variation map, see Remark 4.6. Note that we always have $\mathsf{var}^N_{t'}(\mathbf{e}'_{\sigma k}) = \mathbf{e}_k$ for $k \in I_{\mathsf{uf}}$. In particular, (4.2) is an inhomogeneous linear system of equations on $\mathsf{var}^N_{t'}$.

Definition 4.3. A $\Bbbk$-algebra homomorphism $\mathsf{var}^X_{t'} \colon \mathcal{LP}^X(t') \to \mathcal{LP}^X(t)$ is called a (monomial) variation map if, for any $k \in I_{\mathsf{uf}}$, $j \in I_{\mathsf{f}}$, we have

$$\mathsf{var}^X_{t'}((X')_k) = X_{\sigma^{-1}k}, \qquad \mathsf{var}^X_{t'}((X')_j) = X^{n^{(j)}},$$

for some $n^{(j)} \in \mathbb{Z}^I$ satisfying, $\forall n' \in \mathbb{Z}^I$,

$$\sigma \, \mathsf{pr}_{I_{\mathsf{uf}}} B(t)n = \mathsf{pr}_{I_{\mathsf{uf}}} B(t')(n'), \tag{4.3}$$

where $X^n := \mathsf{var}^X_{t'}(X')^{n'}$.

For classical cases, $\mathsf{var}^X_{t'}$ is called a Poisson variation map if it further preserves the Poisson structures:

$$\mathsf{var}^X_{t'}\{(X')_i, (X')_j\} = \big\{\mathsf{var}^X_{t'}(X')_i, \mathsf{var}^X_{t'}(X')_j\big\}, \qquad \forall i, j \in I.$$

Definition 4.4. In $\widehat{\mathcal{LP}}^X(t')$, take any pointed formal Laurent series $Z' = (X')^{n'} \cdot F$, where

$$n' \in \mathbb{Z}^I, \qquad F = \sum_{n'' \in N^{\geq 0}_{\mathsf{uf}}} c_{n''}(X')^{n''}, \qquad c_{n''} \in \Bbbk, \qquad c_0 = 1.$$

We say Z' and $Z \in \widehat{\mathcal{LP}}^X(t)$ are similar if Z takes the form

$$X^n \cdot F\big|_{(X')^{n''} \mapsto (X)^{\sigma^{-1}n''}} \qquad \text{with} \quad \sigma \, \mathsf{pr}_{I_{\mathsf{uf}}} B(t)n = \mathsf{pr}_{I_{\mathsf{uf}}} B(t')n'.$$

Remark 4.5. The variation map $\mathsf{var}^X_{t'}$ is a $\Bbbk$-algebra homomorphism such that it sends a pointed element $Z' \in \mathcal{LP}^X(t')$ to a similar element in $\mathcal{LP}^X(t)$.

Remark 4.6. Let $\mathsf{var}_{t'}^X$ be a monomial variation map and $\mathsf{var}_{t'}^N$ the associated linear map. Then $\mathsf{var}_{t'}^N$ must satisfies (4.1). We can study condition (4.3) by taking n' to be $\mathbf{e}_i'$ and consider the k-th rows of both sides for all $i \in I$, $k \in I_{\mathsf{uf}}$. Recall that $\omega(n, \mathbf{e}_{\sigma^{-1}k}) = \frac{1}{d_{\sigma^{-1}k}}(Bn)_{\sigma^{-1}k}$, $\omega(\mathbf{e}_i', \mathbf{e}_k') = \frac{1}{d_k}b_{ki}'$, and $d_k = d_{\sigma^{-1}k}$. Then condition (4.3) is equivalent to

$$\omega\big(\mathsf{var}_{t'}^N \mathbf{e}_i', \mathbf{e}_{\sigma^{-1}k}\big) = \omega(\mathbf{e}_i', \mathbf{e}_k').$$

We see that this equation is equivalent to (4.2) by using $\mathbf{e}_{\sigma^{-1}k} = \mathsf{var}_{t'}^N \mathbf{e}_k'$.

Remark 4.7. By using Remark 4.6, we obtain the following equivalent statements:

- A linear map $\Psi \colon N(t') \to N(t)$ is a Poisson variation map.

- For the quantum case $\Bbbk = \mathbb{Z}\big[v^{\pm\frac{1}{d'}}\big]$, the monomial map Φ_v associated to Ψ is a variation map.

- For the classical case $\Bbbk = \mathbb{Z}$, the monomial map Φ_1 associated to Ψ is a Poisson variation map.

Therefore, a Poisson variation map is the classical limit of a quantum variation map at $v = 1$, see Section 2.3. Conversely, a Poisson variation map gives rise to a quantum variation map by the above equivalent statements.

Proposition 4.8. *A linear map $\mathsf{var}_{t'}^N \colon N(t')_\mathbb{Q} \to N(t)_\mathbb{Q}$ is a variation map if and only if its pullback $\big(\mathsf{var}_{t'}^N\big)^* \colon M^\circ(t)_\mathbb{Q} \to M^\circ(t')_\mathbb{Q}$ is a variation map.*

Proof. Relabeling the vertices σk by k for t' if necessary, we assume that σ is the identity. Denote the pullback of $\mathsf{var}_{t'}^N$ by var_t^M.

By Lemma 4.1, $\mathsf{var}_{t'}^N$ takes the form of (4.1) if and only if var_t^M takes the form of (3.2). It remains to show that $\mathsf{var}_t^M p^*(\mathbf{e}_k) = p^* \mathbf{e}_k'$ is equivalent to $\omega\big(\mathsf{var}_{t'}^N \mathbf{e}_i', \mathsf{var}_{t'}^N \mathbf{e}_k'\big) = \omega(\mathbf{e}_i', \mathbf{e}_k')$, $\forall i \in I$, $k \in I_{\mathsf{uf}}$.

Recall that $p^* \mathbf{e}_i(\ ) = \omega(\mathbf{e}_i,\ )$, we have

$$\omega\big(\mathsf{var}_{t'}^N \mathbf{e}_i', \mathsf{var}_{t'}^N \mathbf{e}_k'\big) = \omega\big(\mathsf{var}_{t'}^N \mathbf{e}_i', \mathbf{e}_k\big) = -p^*(\mathbf{e}_k)\big(\mathsf{var}_{t'}^N \mathbf{e}_i'\big) = -\mathsf{var}_t^M p^*(\mathbf{e}_k)(\mathbf{e}_i').$$

On the other hand, $\omega(\mathbf{e}_i', \mathbf{e}_k') = -p^*(\mathbf{e}_k')(\mathbf{e}_i')$.

Therefore, the condition

$$\omega\big(\mathsf{var}_{t'}^N \mathbf{e}_i', \mathsf{var}_{t'}^N \mathbf{e}_k'\big) = \omega(\mathbf{e}_i', \mathbf{e}_k'), \qquad \forall i \in I,$$

is equivalent to $\mathsf{var}_t^M p^*(\mathbf{e}_k) = p^*(\mathbf{e}_k')$. ∎

The following result is analogous to Lemma 3.9.

Lemma 4.9. *An invertible linear map $\mathsf{var}_{t'}^N \colon N(t) \to N(t')$ is a (Poisson) variation map if and only if its inverse is.*

Proof. Assume that $\mathsf{var}_{t'}^N \colon N(t') \to N(t)$ is represented by an invertible matrix

$$\begin{pmatrix} \mathsf{P}_{\sigma^{-1}} & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & V_{\mathsf{uf},\mathsf{f}} \\ 0 & V_{\mathsf{f}} \end{pmatrix}$$

as in 4.1. Then its inverse $\big(\mathsf{var}_{t'}^N\big)^{-1}$ is represented by

$$\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & -V_{\mathsf{uf},\mathsf{f}} V_{\mathsf{f}}^{-1} \\ 0 & V_{\mathsf{f}}^{-1} \end{pmatrix} \begin{pmatrix} \mathsf{P}_\sigma & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} = \begin{pmatrix} \mathsf{P}_\sigma & 0 \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & -\mathsf{P}_\sigma^{-1} V_{\mathsf{uf},\mathsf{f}} V_{\mathsf{f}}^{-1} \\ 0 & V_{\mathsf{f}}^{-1} \end{pmatrix}.$$

So $\big(\mathsf{var}_{t'}^N\big)^{-1}$ is also of the form in (4.1). The other conditions in Definition 4.2 can be checked easily. ∎

Lemma 4.10. *Assume that* $\mathsf{var}^N_{t'} \colon N(t')_{\mathbb{Q}} \to N(t)_{\mathbb{Q}}$ *is a linear map. Let* var^M_t *denote its pullback. The following diagram commutes if and only if* $\mathsf{var}^N_{t'}$ *preserves* ω

$$
\begin{array}{ccc}
M^\circ(t)_{\mathbb{Q}} & \xrightarrow{\ \mathsf{var}^M_t\ } & M^\circ(t')_{\mathbb{Q}} \\
\uparrow p^* & & \uparrow p^* \\
N(t)_{\mathbb{Q}} & \xleftarrow{\ \mathsf{var}^N_{t'}\ } & N(t')_{\mathbb{Q}}.
\end{array}
\tag{4.4}
$$

Note that, by taking the matrix presentations, the commutative diagram (4.4) is represented by a quadratic equation on the entries of $\mathsf{var}^N_{t'}$.

Proof. Take any $i, j \in I$. On the one hand, we have $p^*\mathbf{e}'_i(\mathbf{e}'_j) = \omega(\mathbf{e}'_i, \mathbf{e}'_j)$. On the other hand, we have

$$
\mathsf{var}^M_t\, p^*\, \mathsf{var}^N_{t'}(\mathbf{e}'_i)(\mathbf{e}'_j) = p^*\, \mathsf{var}^N_{t'}(\mathbf{e}'_i)\big(\mathsf{var}^N_{t'}\, \mathbf{e}'_j\big) = \omega\big(\mathsf{var}^N_{t'}\, \mathbf{e}'_i, \mathsf{var}^N_{t'}\, \mathbf{e}'_j\big).
$$

Therefore, $p^* = \mathsf{var}^M_t\, p^*\, \mathsf{var}^N_{t'}$ if and only if $\mathsf{var}^N_{t'}$ preserves ω. $\blacksquare$

4.2 Change of seeds

Let $t', t \in \Delta^+$ denote seeds similar up to a permutation σ. For any $k \in I_{\mathsf{uf}}$, consider the seeds $s = \mu_k t$ and $s' = \mu_{\sigma k} t'$.

We define the linear map $\mathsf{var}^N_{s'} \colon N(s') \to N(s)$ to be $\big(\psi^N_{t,s}\big)^{-1}\, \mathsf{var}^N_{t'}\, \psi^N_{t',s'}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
N(t) & \xleftarrow{\ \mathsf{var}^N_{t'}\ } & N(t') \\
\uparrow \psi^N_{t,s} & & \uparrow \psi^N_{t',s'} \\
N(s) & \xleftarrow{\ \mathsf{var}^N_{s'}\ } & N(s').
\end{array}
\tag{4.5}
$$

By Proposition 2.9 (3), the pullback $\big(\psi^M_{t,s}\big)^*$ is represented by the matrix $D^{-1}\big(\mathsf{P}^M_{k,+}(t)\big)^{\mathrm{T}} D = \mathsf{P}^N_{k,+}(t)^{-1}$. So we have $\big(\psi^M_{t,s}\big)^* = \big(\psi^N_{t,s}\big)^{-1}$. Diagram (4.5) should be compared with the pullback of (3.6). We have the following result in analogous of Proposition 3.12.

Proposition 4.11.

(1) *If* $\mathsf{var}^N_{t'}$ *is a variation map, so is* $\mathsf{var}^N_{s'}$.

(2) *If* $\mathsf{var}^N_{t'}$ *is invertible, so is* $\mathsf{var}^N_{s'}$.

(3) *If* $\mathsf{var}^N_{t'}$ *is Poisson, so is* $\mathsf{var}^N_{s'}$.

Proof. (2) The statement is obvious.

(1)–(3) As in the proof of Proposition 3.12 (1), straightforward computation shows that $\mathsf{var}^N_{s'}$ is a block triangular matrix of the form in (4.1). By Lemma 2.15, $\psi^N_{s,t}$ and $\psi^N_{s',t'}$ preserve the Poisson structure ω. So if $\omega\big(\mathsf{var}^N_{t'}\, \mathbf{e}_i(t'), \mathsf{var}^N_{t'}\, \mathbf{e}_j(t')\big) = \omega(\mathbf{e}_i(t'), \mathbf{e}_j(t'))$ for some i, j, then $\omega\big(\mathsf{var}^N_{s'}\, \mathbf{e}_i(s'), \mathsf{var}^N_{s'}\, \mathbf{e}_j(s')\big) = \omega(\mathbf{e}_i(s'), \mathbf{e}_j(s'))$. The desired claims follow. $\blacksquare$

Let $\mathsf{var}^X_{t'}$ and $\mathsf{var}^X_{s'}$ be the monomial maps associated to the linear maps in Proposition 4.11. Assume that $\mathsf{var}^X_{t'}$ is a variation map. Combining Proposition 4.11 and Remark 4.7, we obtain that $\mathsf{var}^X_{s'}$ is also a variation map.

Let T denote the subalgebra $\Bbbk[X_k]_{k \in I_{\mathsf{uf}}}$ of the quantum torus algebra $\mathcal{LP}^X(t)$. Let $\mathcal{F}^X_{\mathsf{uf}}(t)$ denote the subalgebra of the fraction field $\mathcal{F}^X(t)$ such that its elements take the form $P * Q^{-1}$ with $P \in \mathcal{LP}^X(t)$, $Q \in T$. Then it is easy to check $\big(\mu^X_k\big)^* \mathcal{F}^X_{\mathsf{uf}}(t') = \mathcal{F}^X_{\mathsf{uf}}(t)$ by using the invertibility of μ^X_k.

We naturally extend the variation map $\mathsf{var}^X_{t'} \colon \mathcal{LP}^X(t') \to \mathcal{LP}^X(t)$ to an algebra homomorphism $\mathsf{var}^X_{t'}$ from $\mathcal{F}^X_{\mathsf{uf}}(t')$ to $\mathcal{F}^X_{\mathsf{uf}}(t)$.

Proposition 4.12. *We have* $\left(\mu_k^X\right)^* \mathsf{var}_{s'}^X = \mathsf{var}_{t'}^X \left(\mu_{\sigma k}^X\right)^*$, *i.e., the following diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{F}_{\mathsf{uf}}^X(t) & \xleftarrow{\ \mathsf{var}_{t'}^X\ } & \mathcal{F}_{\mathsf{uf}}^X(t') \\
\left(\mu_k^X\right)^* \uparrow & & \uparrow \left(\mu_{\sigma k}^X\right)^* \\
\mathcal{F}_{\mathsf{uf}}^X(s) & \xleftarrow{\ \mathsf{var}_{s'}^X\ } & \mathcal{F}_{\mathsf{uf}}^X(s').
\end{array}
$$

Proof. We will denote $\mu_k^* = \left(\mu_k^X\right)^*$ and $\psi = \psi^N$ below for simplicity. Relabeling the vertices for s', t' if necessary, we can assume $\sigma = \mathsf{Id}$. It suffices to show $\mathsf{var}_{t'}^X \mu_k^* X_i(s') = \mu_k^* \mathsf{var}_{s'}^X X_i(s')$, $\forall i$.

(i) Let us first consider classical cases. We have

$$
\begin{aligned}
\mu_k^* X_i(s') &= \begin{cases} X_i(t') \cdot X_k(t')^{[b_{ki}(t')]_+} \cdot \left(1 + X_k(t')\right)^{-b_{ki}(t')}, & i \neq k, \\ X_k(t')^{-1}, & i = k \end{cases} \\
&= X(t')^{\psi_{t',s'} \mathbf{e}_i(s')} \cdot \left(1 + X_k(t')\right)^{b_{ki}(s')} \\
&= X(t')^{\psi_{t',s'} \mathbf{e}_i(s')} \cdot \left(1 + X_k(t')\right)^{-\omega(\mathbf{e}_k(s'), \mathbf{e}_i(s'))d_k}.
\end{aligned}
\tag{4.6}
$$

We set $\varepsilon = 1$ if $-\omega(\mathbf{e}_k(s'), \mathbf{e}_i(s')) \geq 0$ and $\varepsilon = -1$ otherwise. Then we have

$$
\mathsf{var}_{t'}^X \mu_k^* X_i(s')^\varepsilon = X(t)^{\varepsilon\, \mathsf{var}_{t'}^N \psi_{t',s'} \mathbf{e}_i(s')} \cdot \left(1 + X_k(t)\right)^{-\varepsilon\omega(\mathbf{e}_k(s'), \mathbf{e}_i(s'))d_k}.
\tag{4.7}
$$

Similar to (4.6), we have

$$
\begin{aligned}
\mu_k^* \mathsf{var}_{s'}^X X_i(s')^{\varepsilon'} &= \mu_k^* X(s)^{\varepsilon'\, \mathsf{var}_{s'}^N \mathbf{e}_i(s')} \\
&= X(t)^{\varepsilon'\psi_{t,s} \mathsf{var}_{s'}^N \mathbf{e}_i(s')} \cdot \left(1 + X_k(t)\right)^{-\varepsilon'\omega(\mathbf{e}_k(s), \mathsf{var}_{s'}^N \mathbf{e}_i(s'))d_k},
\end{aligned}
\tag{4.8}
$$

where we set $\varepsilon' = 1$ if $-\omega\left(\mathbf{e}_k(s), \mathsf{var}_{s'}^N \mathbf{e}_i(s')\right) \geq 0$ and $\varepsilon' = -1$ otherwise.

Note that $\mathsf{var}_{t'}^N \psi_{t',s'} \mathbf{e}_i(s') = \psi_{t,s} \mathsf{var}_{s'}^N \mathbf{e}_i(s')$ by the definition of $\mathsf{var}_{s'}^N$. Moreover, since $\mathsf{var}_{s'}^N$ is a variation map by Proposition 4.11, we have

$$
\omega\left(\mathbf{e}_k(s), \mathsf{var}_{s'}^N \mathbf{e}_i(s')\right) = \omega\left(\mathsf{var}_{s'}^N \mathbf{e}_k(s'), \mathsf{var}_{s'}^N \mathbf{e}_i(s')\right) = \omega(\mathbf{e}_k(s'), \mathbf{e}_i(s')).
$$

It follows that $\varepsilon = \varepsilon'$ and $\mathsf{var}_t^X \mu_k^* X_i(s')^\varepsilon = \mu_k^* \mathsf{var}_{s'}^X X_i(s')^{\varepsilon'}$.

(ii) In the quantum case, we can obtain $\mathsf{var}_t^X \mu_k^* X_i(s')^\varepsilon$ and $\mu_k^* \mathsf{var}_{s'}^X X_i(s')^{\varepsilon'}$ by replacing the binomial coefficients $\binom{a}{b}$ in polynomial expansions of the right hand sides of (4.7) and (4.8) by the quantum numbers $\binom{a}{b}_{v_k}$. Therefore, we still have

$$
\mathsf{var}_t^X \mu_k^* X_i(s') = \mu_k^* \mathsf{var}_{s'}^X X_i(s'). \qquad \blacksquare
$$

The following statement follows from Proposition 4.12 by tracking the degrees of X-variables. It is an analog of Lemma 3.13.

Lemma 4.13. *The variation map* $\mathsf{var}_{s'}^N \colon N(s') \to N(s)$ *is the linear map sending* $\mathbf{e}_i(s')$ *to* $\deg_s \left(\left(\mu_k^X\right)^*\right)^{-1} \mathsf{var}_{t'}^X \left(\mu_{\sigma k}^X\right)^* X_i(s')$.

Take any mutation sequence $\boldsymbol{\mu}$ and denote $s = \boldsymbol{\mu}t$, $s' = (\sigma\boldsymbol{\mu})t'$. By using Proposition 4.12 recursively, $\mathsf{var}_{t'}^X$ and $\boldsymbol{\mu}$ uniquely determine a variation map $\mathsf{var}_{s'}^X$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{F}_{\mathsf{uf}}^X(t) & \xleftarrow{\ \mathsf{var}_{t'}^X\ } & \mathcal{F}_{\mathsf{uf}}^X(t') \\
\uparrow \boldsymbol{\mu}^* & & \uparrow (\sigma\boldsymbol{\mu})^* \\
\mathcal{F}_{\mathsf{uf}}^X(s) & \xleftarrow{\ \mathsf{var}_{s'}^X\ } & \mathcal{F}_{\mathsf{uf}}^X(s').
\end{array}
\tag{4.9}
$$

By tracking the degrees of cluster variables, we see that linear the variation map $\mathsf{var}_{s'}^N$ associated to $\mathsf{var}_{s'}^X$ is determined by $\mathsf{var}_{t'}^N$ and $\boldsymbol{\mu}$. It still makes (4.5) commutative.

Note that, by the commutative diagrams (4.9), the variation maps $\mathsf{var}_{s'}^X$, $s' \in \Delta_{s'}^+$, defined using $\mathsf{var}_{t'}^X$, are identified via the mutation maps.

We have the following result in analogous to [44, Lemma 4.2.2 (ii)].

Lemma 4.14. *For any $i \in I$, the variation map $\mathsf{var}_{t'}^X$ sends the cluster X-variable $(\sigma\boldsymbol{\mu})^* X_i(s')$ to a Laurent monomial of the form $(\boldsymbol{\mu})^* X(s)^n$, $n \in \mathbb{Z}^I$.*

Proof. By (4.9), we have $\boldsymbol{\mu}^* \mathsf{var}_{s'}^X = \mathsf{var}_{t'}^X (\sigma\boldsymbol{\mu})^*$, where $\mathsf{var}_{s'}^X$ is also a variation map. In the seed s', the variation map $\mathsf{var}_{s'}^X$ sends the X-variable $X_i(s')$ to some Laurent monomial $X(s)^n$ by definition. The desired statement follows by applying mutations $(\sigma\boldsymbol{\mu})^*$ and $\boldsymbol{\mu}^*$. ∎

We have the following result in analogous to Lemma 3.10.

Lemma 4.15. *A variation map $\mathsf{var}_{t'}^X$ restricts to an algebra homomorphism from $\mathcal{U}^X(t')$ to $\mathcal{U}^X(t)$.*

Proof. The proof is similar to Lemma 3.10, where we replace [44, Lemma 4.2.2 (ii)] by Lemma 4.14. ∎

We have extended the variation map $\mathsf{var}_{t'}^X$ to a homomorphism from $\mathcal{F}_{\mathsf{uf}}^X(t')$ to $\mathcal{F}_{\mathsf{uf}}^X(t)$. We can also extend it to a homomorphism from $\widehat{\mathcal{LP}}^X(t')$ to $\widehat{\mathcal{LP}}^X(t)$. Let $\mathcal{F}_{\mathsf{uf}}^X(t') \cap \widehat{\mathcal{LP}}^X(t')$ denote the set of formal Laurent series $Z \in \widehat{\mathcal{LP}}^X(t')$ such that $Z * Q = P$ for some $P \in \Bbbk[(X')_i]_{i \in I}$, $Q \in \Bbbk[(X')_k]_{k \in I_{\mathsf{uf}}}$. Then Z is identified with the element $P * Q^{-1}$ in $\mathcal{F}_{\mathsf{uf}}^X(t')$.

Proposition 4.16. *The above two extended maps restrict to the same homomorphism from $\mathcal{F}_{\mathsf{uf}}^X(t') \cap \widehat{\mathcal{LP}}^X(t')$ to $\mathcal{F}_{\mathsf{uf}}^X(t) \cap \widehat{\mathcal{LP}}^X(t)$.*

Proof. We denote the above two extended maps by $f \colon \mathcal{F}_{\mathsf{uf}}^X(t') \to \mathcal{F}_{\mathsf{uf}}^X(t)$ and $g \colon \widehat{\mathcal{LP}}^X(t') \to \widehat{\mathcal{LP}}^X(t)$, respectively. Take any element $Z \in \mathcal{F}_{\mathsf{uf}}^X(t') \cap \widehat{\mathcal{LP}}^X(t')$ described as above. Then we have $g(Z) * g(Q) = g(P)$. In addition, we have $f(P) = \mathsf{var}_{t'}^X(P) = g(P) \in \Bbbk[X_i]_{i \in I}$ and

$$f(Q) = \mathsf{var}_{t'}^X(Q) = g(Q) \in \Bbbk[X_k]_{k \in I_{\mathsf{uf}}}.$$

It follows that $g(Z) * f(Q) = f(P)$, i.e, $g(Z)$ is contained in $\mathcal{F}_{\mathsf{uf}}^X(t) \cap \widehat{\mathcal{LP}}^X(t)$ and it is identified with $f(P * Q^{-1})$. ∎

4.3 Twist endomorphisms

Let $t' = \boldsymbol{\mu}t$ be two seeds similar up to σ as before. Note that the canonical Poisson structure on $\mathcal{LP}^X(t)$ naturally gives rise to a Poisson structure on $\mathcal{U}^X(t)$.

Definition 4.17. Let $\mathsf{var}_{t'}^X \colon \mathcal{LP}^X(t') \to \mathcal{LP}^X(t)$ be any variation map.[4] Then the composition $\mathsf{tw}_{t'}^X = (\boldsymbol{\mu}^{-1})^* \circ \mathsf{var}_{t'}^X$ is called a twist endomorphism on $\mathcal{U}^X(t')$ passing through t.

For classical cases, if $\mathsf{var}_{t'}^X$ is Poisson, then $\mathsf{tw}_{t'}^X$ is said to be Poisson.

Note that, by Proposition 4.12, when $\mathsf{tw}_{t'}^X$ is a twist endomorphism, it gives rise to twist endomorphisms $\mathsf{tw}_{s'}^X$, $s' \in \Delta^+$. They are identified via mutations. So we can simply denote them by tw^X.

Assume that $\mathsf{var}_{t'}^X$ is an invertible variation map. We have a twist automorphism $\mathsf{tw}_{t'}^X := (\boldsymbol{\mu}^{-1})^* \mathsf{var}_{t'}^X$ on $\mathcal{U}^X(t')$. Lemma 4.9 implies that $(\mathsf{var}_{t'}^X)^{-1}$ is still a variation map. So $\eta := \boldsymbol{\mu}^* (\mathsf{var}_{t'}^X)^{-1}$ is a twist automorphism on $\mathcal{U}^X(t)$.

---

[4]If we work with $\mathsf{var}_{t'}^N \colon N(t')_{\mathbb{Q}} \to N(t)_{\mathbb{Q}}$, the corresponding twist map should be defined on an algebra such that we allow roots of all the cluster X-variables, which are roots of rational functions. We do not consider this situation here.

Lemma 4.18. *We have* $\boldsymbol{\mu}^*\big(\mathsf{tw}_{t'}^X\big)^{-1}(\boldsymbol{\mu}^*)^{-1} = \eta$. *Namely, by identifying the fraction fields by mutations, the twist automorphisms associated to invertible variation maps* $\mathsf{var}_{t'}^X \colon \mathcal{LP}^X(t') \to \mathcal{LP}^X(t)$ *and* $(\mathsf{var}_{t'}^X)^{-1} \colon \mathcal{LP}^X(t) \to \mathcal{LP}^X(t')$ *are inverse to each other.*

Proof. The proof is similar to that of Lemma 3.18. ∎

Theorem 4.19. *Consider the classical case* $\Bbbk = \mathbb{Z}$. *Assume that we have a linear map* $\mathsf{var}_{t'}^N \colon N(t') \to N(t)$. *Let* var_t^M *denote its pullback from* $M^\circ(t)_{\mathbb{Q}}$ *to* $M^\circ(t')_{\mathbb{Q}}$ *and* $\mathsf{var}_{t'}^X$, var_t^A *denote the associated monomial maps. Define* $\mathsf{tw}_{t'}^X = \big(\boldsymbol{\mu}^{-1}\big)^* \circ \mathsf{var}_{t'}^X$ *and* $\mathsf{tw}_t^A = \boldsymbol{\mu}^* \circ \mathsf{var}_t^A$.

(1) $\mathsf{tw}_{t'}^X$ *is a twist endomorphism on* $\mathcal{U}^X(t')$ *if and only if* tw_t^A *is a twist endomorphism on* $\mathcal{U}^A(t)_{\mathcal{Z}\mathbb{Q}}$.

(2) *If* $\mathsf{tw}_{t'}^X$ *is a twist automorphism, then* $\big(\mathsf{tw}_{t'}^X\big)^{-1}$ *is also a twist automorphism by Lemma 4.9. Let* $\big(\mathsf{tw}_t^X\big)^{-1} := \boldsymbol{\mu}^*\big(\mathsf{tw}_{t'}^X\big)^{-1}(\boldsymbol{\mu}^*)^{-1}$ *denote the corresponding twist automorphism on* $\mathcal{U}^X(t)$. *Then* $\big(\mathsf{tw}_t^X\big)^{-1}$ *preserves the Poisson structure if and only if* $p^*\big(\mathsf{tw}_t^X\big)^{-1} = \mathsf{tw}_t^A\, p^*$.

Because we can quantized a Poisson variation map to a quantum variation map Remark 4.7, by Theorem 4.19 (2), the equality $p^*\big(\mathsf{tw}_t^X\big)^{-1} = \mathsf{tw}_t^A\, p^*$ provides a criterion for $\big(\mathsf{tw}_t^X\big)^{\pm}$ being quantum twist automorphisms for quantum cases. This equality can be presented by the quadratic equation in Lemma 4.10.

Proof of Theorem 4.19. (1) The statement follows from Proposition 4.8.

(2) On the one hand, we have $\mathsf{tw}_t^A\, p^* = \boldsymbol{\mu}^* \mathsf{var}_t^A\, p^*$. On the other hand, we have

$$p^*\big(\mathsf{tw}_t^X\big)^{-1} = p^* \boldsymbol{\mu}^*\big(\mathsf{tw}_{t'}^X\big)^{-1}(\boldsymbol{\mu}^*)^{-1} = p^* \boldsymbol{\mu}^*\big(\mathsf{var}_{t'}^X\big)^{-1}$$

by construction, which equals $\boldsymbol{\mu}^* p^*\big(\mathsf{var}_{t'}^X\big)^{-1}$. Moreover, Lemma 4.10 implies that $\mathsf{var}_t^A\, p^* = p^*\big(\mathsf{var}_{t'}^X\big)^{-1}$ if and only if $\mathsf{var}_{t'}^N$ preserves ω. The claim follows. ∎

Remark 4.20 (cluster Poisson algebras without coefficients). Let $t' = \boldsymbol{\mu} t$ be seeds similar up to σ. Let us restrict to the unfrozen cluster Poisson algebra $\mathcal{U}_{\mathsf{uf}}^X$, which consists of $P * Q^{-1}$ with $P, Q \in \Bbbk[X_k]_{k \in I_{\mathsf{uf}}}$. Then there is only one twist automorphism tw_t^X on $\mathcal{U}_{\mathsf{uf}}^X(t)$ passing through t'. It sends $X_k(t)$ to $\boldsymbol{\mu}^* X_{\sigma k}(t')$ and it preserves the Poisson structure when $\Bbbk = \mathbb{Z}$.

Assume $I = I_{\mathsf{uf}}$. Take any twist endomorphism tw_t^A on $\mathcal{U}^A(t)$ passing through t'. Then we have $\mathsf{tw}_t^A\, p^* X_k(t) = \boldsymbol{\mu}^* p^* X_{\sigma k}(t')$ for any $k \in I_{\mathsf{uf}}$ by (3.2). Since $\boldsymbol{\mu}^* p^* = p^* \boldsymbol{\mu}^*$, it follows that

$$\mathsf{tw}_t^A\, p^* X_k(t) = \boldsymbol{\mu}^* p^* X_{\sigma k}(t') = p^* \boldsymbol{\mu}^* X_{\sigma k}(t') = p^* \mathsf{tw}_t^X X_k(t),$$

i.e., Theorem 4.19 (2) always hold true.

Remark 4.21 (existence of tw^X). If all skew-symmetrizers d_i are equal, we can work with lattices N and M° instead of $\mathbb{Q}$-vector spaces in Lemma 4.1 and Proposition 4.8.

By using Proposition 4.8, we can describe the set of all linear variation maps $\mathsf{var}_{t'}^N \colon N(t')_{\mathbb{Q}} \to N(t)_{\mathbb{Q}}$ as in Proposition 3.11 (1)–(2).

Recall that W denotes the matrix of ω. If $\begin{pmatrix} W_{\mathsf{uf}} \\ W_{\mathsf{f},\mathsf{uf}} \end{pmatrix}$ is $\mathbb{Z}$-valued and an order $|I_{\mathsf{uf}}|$ minor of it equals ± 1, we can describe the set of all linear variation maps $\mathsf{var}_{t'}^N \colon N(t') \to N(t)$ as in Proposition 3.11 (3). In the classical case $\Bbbk = \mathbb{Z}$, these linear variation maps give rise to twist endomorphisms tw^X on $\mathcal{U}^X$ in analogous to Theorem 3.19 (2).

5   Twist automorphisms in special cases

5.1   Twist automorphism of Donaldson–Thomas type

Assume that t is injective-reachable such that $t' = t[1]$ is similar to t up to a permutation σ. Then a twist automorphism on t passing through $t[1]$ is called a twist automorphism of Donaldson–Thomas type[5] (DT-type for short).

Proposition 5.1. *There exists an invertible Poisson variation map* var_t^N *and an invertible variation map* var_t^M *passing through* $t[1]$*. Moreover, when* $\mathcal{T}^A(t)$ *has a compatible Poisson structure, we can choose* var_t^M *to be Poisson.*

Proof. Relabel the vertices for $t[1]$ so that we can assume $\sigma = \mathsf{Id}$. By (2.5), we have $\psi_{t,t[1]}^N \underline{\mathbf{e}}' = \underline{\mathbf{e}} \begin{pmatrix} -\mathsf{Id}_{\mathsf{uf}} & E_{\mathsf{uf},\mathsf{f}} \\ 0 & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} =: \underline{\mathbf{e}} E$ and, correspondingly, $\psi_{t,t[1]}^M \underline{\mathbf{f}}' = \underline{\mathbf{f}} \begin{pmatrix} -\mathsf{Id}_{\mathsf{uf}} & 0 \\ F_{\mathsf{f},\mathsf{uf}} & \mathsf{Id}_{\mathsf{f}} \end{pmatrix} =: \underline{\mathbf{f}} F$. Note that $\psi_{t,t[1]}^M$ is the identity on M_{f}°.

(1) Let us define the linear isomorphism $\mathsf{var}_t^M : M^{\circ}(t) \to M^{\circ}(t')$ such that

$$\mathsf{var}_t^M(\underline{\mathbf{f}}) = \underline{\mathbf{f}}' \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ -F_{\mathsf{f},\mathsf{uf}} & -\mathsf{Id}_{\mathsf{f}} \end{pmatrix}.$$

Then we have $\psi_{t,t[1]}^M \mathsf{var}_t^M(\underline{\mathbf{f}}) = -\underline{\mathbf{f}}$.

On the one hand, for any $k \in I$, we have

$$\psi_{t,t[1]}^M \mathsf{var}_t^M(p^* \mathbf{e}_k) = \psi_{t,t[1]}^M \mathsf{var}_t^M \left(\sum_{i \in I} b_{ik} \mathbf{f}_i \right) = -\sum_{i \in I} b_{ik} \mathbf{f}_i = -p^* \mathbf{e}_k.$$

On the other hand, by Lemma 2.15, for any $k \in I_{\mathsf{uf}}$, we have

$$\psi_{t,t[1]}^M p^*(\mathbf{e}_k') = p^*\left(\psi_{t,t[1]}^N \mathbf{e}_k' \right) = p^*(-\mathbf{e}_k) = -p^* \mathbf{e}_k.$$

It follows that $\mathsf{var}_t^M(p^* \mathbf{e}_k) = p^*(\mathbf{e}_k')$ for any $k \in I_{\mathsf{uf}}$. In particular, var_t^M is a variation map.

Assume there is a compatible Poisson structure λ. For any $i, j \in I$, we have

$$\lambda\left(\mathsf{var}_t^M \mathbf{f}_i, \mathsf{var}_t^M \mathbf{f}_j \right) = \lambda\left(\psi_{t,t[1]}^M \mathsf{var}_t^M \mathbf{f}_i, \psi_{t,t[1]}^M \mathsf{var}_t^M \mathbf{f}_j \right) = \lambda(-\mathbf{f}_i, -\mathbf{f}_j) = \lambda(\mathbf{f}_i, \mathbf{f}_j).$$

Therefore, var_t^M preserves λ.

(2) Define $\mathsf{var}_t^N : N(t) \to N(t[1])$ such that $\mathsf{var}_t^N(\underline{\mathbf{e}}) = \underline{\mathbf{e}}' \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & -E_{\mathsf{uf},\mathsf{f}} \\ 0 & -\mathsf{Id}_{\mathsf{f}} \end{pmatrix}$. It follows that

$$\psi_{t[1],t}^N \mathsf{var}_t^N(\underline{\mathbf{e}}) = -\underline{\mathbf{e}}.$$

Then, for any $i, j \in I$, we have

$$\omega\left(\mathsf{var}_t^N \mathbf{e}_i, \mathsf{var}_t^N \mathbf{e}_j \right) = \omega\left(\psi_{t,t[1]}^N \mathsf{var}_t^N \mathbf{e}_i, \psi_{t,t[1]}^N \mathsf{var}_t^N \mathbf{e}_j \right) = \omega(-\mathbf{e}_i, -\mathbf{e}_j) = \omega(\mathbf{e}_i, \mathbf{e}_j).$$

Therefore, var_t^N preserves ω. ∎

Theorem 5.2. *Assume that* t *is injective-reachable such that* $t' = t[1]$ *is similar to* t *up to a permutation* σ.

(1) *There exists a twist automorphism* tw_t^A *on* $\mathcal{U}^A(t)$ *of Donaldson–Thomas type.*

(2) *There exists a twist automorphism* tw_t^X *on* $\mathcal{U}^X(t)$ *of Donaldson–Thomas type.*

(3) *We can choose* tw_t^A *and* tw_t^X *in* (1)–(2) *such that* $p^* \mathsf{tw}_t^X = \mathsf{tw}_t^A p^*$.

---

[5]The mutation maps $\boldsymbol{\mu}^*$ from $\mathcal{F}^A(t[1])$ to $\mathcal{F}^A(t)$ is often called a Donaldson–Thomas transformation in cluster theory. See [40] for the relation between Donaldson–Thomas theory and cluster algebras.

Proof. (1)–(2) The statements follow from Proposition 5.1.

(3) We claim that the two variation maps in the proof of Proposition 5.1 are related by the pullback construction. The variation map var_t^M is represented by $\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ -F_{\mathsf{f,uf}} & -\mathsf{Id}_{\mathsf{f}} \end{pmatrix}$. Then its pullback $\left(\mathsf{var}_t^M\right)^*$ satisfies

$$\left(\mathsf{var}_t^M\right)^*(\underline{e}') = \underline{e}D^{-1} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & -F_{\mathsf{f,uf}}^{\mathrm{T}} \\ 0 & -\mathsf{Id}_{\mathsf{f}} \end{pmatrix} D$$

by Lemma 4.1. By (2.4), $E = D^{-1}F^{-\mathrm{T}}D$. It follows that $E_{\mathsf{uf,f}} = D_{\mathsf{uf}}^{-1}F_{\mathsf{f,uf}}^{\mathrm{T}}D_{\mathsf{f}}$. Define $\mathsf{var}_t^N \colon N(t) \to N(t[1])$ to be the inverse of $\left(\mathsf{var}_t^M\right)^*$. We obtain

$$\mathsf{var}_t^N(\underline{e}) = \underline{e}'D^{-1} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & -F_{\mathsf{f,uf}}^{\mathrm{T}} \\ 0 & -\mathsf{Id}_{\mathsf{f}} \end{pmatrix} D = \underline{e}' \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & -E_{\mathsf{uf,f}} \\ 0 & -\mathsf{Id}_{\mathsf{f}} \end{pmatrix}$$

as in the previous proof.

The desired claim follows from Theorem 4.19 (2). ∎

Example 5.3. Let us continue Example 2.13. Then we have $t' = t[1]$.

We can choose the variation map $\mathsf{var}_t^M \colon M^\circ(t) \to M^\circ(t[1])$ such that it is represented by the matrix $\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$. Then $\psi_{t,t[1]}^M \mathsf{var}_t^M$ is $-\mathsf{Id}$ on $M^\circ(t)$.

We can choose the variation map $\mathsf{var}_t^N \colon N(t) \to N(t[1])$ such that it is represented by the matrix $\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$. Then $\psi_{t,t[1]}^N \mathsf{var}_t^N$ is $-\mathsf{Id}$ on $N(t)$.

We refer the reader to Example 7.1 for an alternative example.

5.2   Twist automorphism for principal coefficients

Let t_0 denote an initial seed with principal coefficients, see Section 2.10. We denote its B-matrix by $B = \begin{pmatrix} B_{\mathsf{uf}} & -\mathsf{Id}_{\mathsf{uf}} \\ \mathsf{Id}_{\mathsf{uf}} & 0 \end{pmatrix}$. Recall that $D = \begin{pmatrix} D_{\mathsf{uf}} & \\ & D_{\mathsf{uf}} \end{pmatrix}$ in this case. We endow t_0 with the canonical Poisson structure whose Λ-matrix is

$$\Lambda := B^{-\mathrm{T}}D = \begin{pmatrix} 0 & -\mathsf{Id}_{\mathsf{uf}} \\ \mathsf{Id}_{\mathsf{uf}} & B_{\mathsf{uf}}^{\mathrm{T}} \end{pmatrix} D.$$

Then, for any seed $t' \in \Delta^+$, we can check that $\Lambda(t') = B(t')^{-\mathrm{T}}D$ by using Proposition 2.9 recursively.

Assume that $t \in \Delta_{t_0}^+$ is similar to the principal coefficients seed t_0 up to a permutation σ.

Theorem 5.4.

(1) *There exists a twist automorphism* $\mathsf{tw}_{t_0}^A$ *on the (quantum) upper cluster algebra on* $\mathcal{U}^A(t_0)$ *passing through* t.

(2) *There exists a twist automorphism* $\mathsf{tw}_{t_0}^X$ *on* $\mathcal{U}^X(t_0)$ *passing through* t.

(3) *We can choose* $\mathsf{tw}_{t_0}^A$ *and* $\mathsf{tw}_{t_0}^X$ *such that* $p^* \mathsf{tw}_{t_0}^X = \mathsf{tw}_{t_0}^A p^*$.

Proof. Relabeling the vertices in t if necessary, we can assume $\sigma = \mathsf{Id}$. Since t_0 has principal coefficients, we have $B(t) = \begin{pmatrix} B_{\mathsf{uf}} & -G^{-1} \\ C & 0 \end{pmatrix}$, where C and G denote the C-matrix and the G-matrix of the seed t with respect to the initial seed t_0, respectively, and we have $C^{\mathrm{T}}D_{\mathsf{uf}} = D_{\mathsf{uf}}G^{-1}$ (see Section 2.10).

(1) Define the linear map $\mathsf{var}_{t_0}^M \colon M^\circ(t_0) \to M^\circ(t)$ such that

$$\mathsf{var}_{t_0}^M(\underline{f}(t_0)) = \underline{f}(t) \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ 0 & C \end{pmatrix}.$$

It is clear that $\begin{pmatrix} 0 & C \end{pmatrix} \begin{pmatrix} B_{\mathsf{uf}} \\ \mathsf{Id} \end{pmatrix} = C$. So var_t^M is a variation map by Lemma 3.8.

We have $\Lambda(t) = B(t)^{-\mathrm{T}}D = \begin{pmatrix} 0 & -G^{\mathrm{T}} \\ C^{-\mathrm{T}} & C^{-\mathrm{T}}B_{\mathsf{uf}}^{\mathrm{T}}G^{\mathrm{T}} \end{pmatrix} D$. Recall that $\mathsf{var}_{t_0}^M$ is represented by $\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ 0 & C \end{pmatrix}$. We have

$$\begin{pmatrix} \mathsf{Id} & 0 \\ 0 & C \end{pmatrix}^{\mathrm{T}} \Lambda(t) \begin{pmatrix} \mathsf{Id} & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} \mathsf{Id} & 0 \\ 0 & C^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 0 & -G^{\mathrm{T}} \\ C^{-\mathrm{T}} & C^{-\mathrm{T}}B_{\mathsf{uf}}^{\mathrm{T}}G^{\mathrm{T}} \end{pmatrix} D \begin{pmatrix} \mathsf{Id} & 0 \\ 0 & C \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -G^{\mathrm{T}}D_{\mathsf{uf}}C \\ D_{\mathsf{uf}} & B_{\mathsf{uf}}^{\mathrm{T}}G^{\mathrm{T}}D_{\mathsf{uf}}C \end{pmatrix} = \begin{pmatrix} 0 & -D_{\mathsf{uf}} \\ D_{\mathsf{uf}} & B_{\mathsf{uf}}^{\mathrm{T}}D_{\mathsf{uf}} \end{pmatrix} = \Lambda.$$

Therefore, $\mathsf{var}_{t_0}^M$ preserves λ. The associated twist automorphism provides a desired solution.

(2) Let $(\mathsf{var}_{t_0}^M)^*: N(t) \to N(t_0)$ denote the pullback of $\mathsf{var}_{t_0}^M$ in the proof of (1). Then $(\mathsf{var}_{t_0}^M)^*(\underline{\mathbf{e}}(t)) = \underline{\mathbf{e}}(t_0)D^{-1}\begin{pmatrix} I & 0 \\ 0 & C^{\mathrm{T}} \end{pmatrix} D$. Define $\mathsf{var}_{t_0}^N$ as the inverse:

$$\mathsf{var}_{t_0}^N(\underline{\mathbf{e}}(t_0)) = \underline{\mathbf{e}}(t)D^{-1}\begin{pmatrix} I & 0 \\ 0 & C^{\mathrm{T}} \end{pmatrix}^{-1} D = \underline{\mathbf{e}}(t)\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & \\ & G \end{pmatrix}.$$

We have $W = B^{\mathrm{T}}D = \begin{pmatrix} B_{\mathsf{uf}}^{\mathrm{T}} & \mathsf{Id} \\ -\mathsf{Id} & 0 \end{pmatrix} D$ and $W(t) = B(t)^{\mathrm{T}}D = \begin{pmatrix} B_{\mathsf{uf}}^{\mathrm{T}} & C^{\mathrm{T}} \\ -G^{-\mathrm{T}} & 0 \end{pmatrix} D$. Using $D_{\mathsf{uf}}G = C^{-\mathrm{T}}D_{\mathsf{uf}}$, we obtain

$$\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ 0 & G \end{pmatrix}^{\mathrm{T}} W(t) \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & \\ & G \end{pmatrix} = \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ 0 & G^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} B_{\mathsf{uf}}^{\mathrm{T}} & C^{\mathrm{T}} \\ -G^{-\mathrm{T}} & 0 \end{pmatrix} D \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & \\ & G \end{pmatrix}$$
$$= \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ 0 & G^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} B_{\mathsf{uf}}^{\mathrm{T}} & C^{\mathrm{T}} \\ -G^{-\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & \\ & C^{-\mathrm{T}} \end{pmatrix} D$$
$$= \begin{pmatrix} B_{\mathsf{uf}}^{\mathrm{T}} & \mathsf{Id} \\ -\mathsf{Id} & 0 \end{pmatrix} D = W.$$

Therefore, $\mathsf{var}_{t_0}^N$ preserves ω. The associated twist automorphism provides a desired solution.

(3) We choose twist automorphisms as in the proof of (1)–(2). Then the claim follows from Theorem 4.19 (2). ∎

Example 5.5. We can verify that $\mathsf{var}_{t_0}^N$ and $\mathsf{var}_{t_0}^M$ constructed in the proof of Theorem 5.4 make the diagram (4.4) commutative, which is equivalent to the following:

$$
\begin{array}{ccc}
M^\circ(t_0) & \xrightarrow{\mathsf{var}_{t_0}^M} & M^\circ(t) \\
\uparrow p^* & & \uparrow p^* \\
N(t_0) & \xrightarrow{\mathsf{var}_{t_0}^N} & N(t).
\end{array}
\tag{5.1}
$$

More precisely, $\mathsf{var}_{t_0}^M p^*: N(t_0) \to M^\circ(t)$ is represented by the matrix $\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ 0 & C \end{pmatrix} B(t_0) = \begin{pmatrix} B_{\mathsf{uf}} & -\mathsf{Id} \\ C & 0 \end{pmatrix}$, and $p^* \mathsf{var}_{t_0}^N$ is also represented by the matrix $B(t)\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & 0 \\ 0 & G \end{pmatrix} = \begin{pmatrix} B_{\mathsf{uf}} & -\mathsf{Id} \\ C & 0 \end{pmatrix}$.

Equivalently, $\mathsf{var}_{t_0}^N$ and $\mathsf{var}_{t_0}^M$ are determined by each other such that the diagram (5.1) is commutative.

6 Permutations on bases

6.1 Bases for $\mathcal{U}^A$

For any seed $t \in \Delta^+$, assume that we have chosen a collection of polynomials for $m \in \mathbb{Z}^I$:

$$F_m(t) = \sum_{n \in \mathbb{N}^{I_{\mathsf{uf}}}} c_n(m;t)X(t)^n$$

with $c_0(m;t) = 1$, $c_n(m;t) \in \Bbbk$. Define the corresponding subset $\mathcal{S}(t) = \{S_m(t) \mid m \in \mathbb{Z}^I\}$ of $\mathcal{LP}^A(t)$, such that the elements S_m are given by

$$S_m(t) = A(t)^m \cdot F_m(t)|_{X(t)^n \mapsto A(t)^{p^*(t)n}}.$$

We often omit the symbol t for simplicity. $F_m(t)$ are called the F-polynomials of $S_m(t)$. Note that we use the commutative product $\cdot$ here, so that the F-polynomials are defined as in standard literature.

Note that, if the full rank assumption (see Assumption 2.4) holds, $S_m(t)$ are m-pointed and $\mathcal{S}(t)$ is be $\Bbbk$-linearly independent. Otherwise, $\mathcal{S}(t)$ is not necessarily linearly independent and different choices of $F_m(t)$ might produce the same $S_m(t)$.

Assumption 6.1. We assume the following conditions hold.

(1) The sets $\mathcal{S}(t)$, $t \in \Delta^+$, are identified by mutations: $\forall t' = \boldsymbol{\mu}t$, we have $\boldsymbol{\mu}^* \mathcal{S}(t') = \mathcal{S}(t)$.

(2) The chosen functions $F_m(t)$ only depend on $\mathsf{pr}_{I_{\mathsf{uf}}} m$ and $B(t)_{\mathsf{uf}}$:

- If $m' = m + u$ for some $u \in M_{\mathsf{f}}^\circ$, then $F_{m'}(t) = F_m(t)$.
- If t' and t are similar up to σ and $\mathsf{pr}_{I_{\mathsf{uf}}} m' = \sigma \, \mathsf{pr}_{I_{\mathsf{uf}}} m$ for some $m, m' \in \mathbb{Z}^I$, we have $c_{\sigma n}(m';t') = c_n(m;t)$, $\forall n$.

Note that Assumption 6.1 implies $\mathcal{S}(t) \subset \mathcal{U}^A(t)$.

We have the following important examples of $\mathcal{S}(t)$ satisfying Assumption 6.1:

- The set of generic cluster characters: When $B(t)_{\mathsf{uf}}$ is skew-symmetric and the corresponding cluster category satisfies some finiteness condition, we can construct the set of generic cluster characters $\mathbb{L}(t) = \{\mathbb{L}_m(t) | m \in M^\circ(t)\}$ in $\mathcal{U}^A$ using algebra representation theory. By [42], they satisfy the above assumptions.

  - By [45], when t is injective-reachable, $\mathbb{L}(t)$ is a basis of $\mathcal{U}^A$, called the generic basis in the sense of [8]. This family of bases includes the dual semicanonical basis [36] of a unipotent cell with symmetric Cartan datum [16].

- The common triangular basis [44]: this basis is known to satisfy the above assumptions [44]. Its existence is known for some cluster algebras with a Lie theoretic background. This family of bases includes the dual canonical basis of a quantum unipotent cell [29, 44, 47].

- The set of theta functions [25]: the set of the theta functions, under the assumption that they are Laurent polynomials,[6] satisfies the above assumption. They are known to form a basis of $\mathcal{U}^A(t)$ under appropriate conditions, for example, when t is injective-reachable.

Theorem 6.2. *Let* tw^A *denote any twist endomorphism on the (quantum) upper cluster algebra* $\mathcal{U}^A$. *Assume the sets* $\mathcal{S}(t)$, $t \in \Delta^+$, *satisfy Assumption* 6.1, *then the following statements are true.*

(1) *We have* $\mathsf{tw}^A \, \mathcal{S}(t) \subset \mathcal{S}(t)$.

(2) *If* tw^A *is a twist automorphism, we have* $\mathsf{tw}^A \, \mathcal{S}(t) = \mathcal{S}(t)$.

Proof. Recall that $\mathsf{tw}^A = \boldsymbol{\mu}^* \mathsf{var}_t^A$ for some $t' = \boldsymbol{\mu}t$ and some variation map var_t^A from $\mathcal{LP}^A(t)$ to $\mathcal{LP}^A(t')$.

---

[6] In general, the theta functions are formal Laurent series, see [45] for the formal completion. Our arguments for variation maps remain valid for formal Laurent series. For simplicity, we only treat Laurent polynomials in this paper.

(1) Denote $\mathcal{S}(t) = \{S_m \mid m \in \mathbb{Z}^I\}$ and $\mathcal{S}(t') = \{S'_{m'} \mid m' \in \mathbb{Z}^I\}$ for simplicity. Take any $S_m = A^m \cdot F_m|_{X^n \mapsto A^{p^*n}}$ from $\mathcal{S}(t)$, where $F_m = \sum c_n(m;t)X^n$. Then $\mathsf{var}_t^A(S_m) = A^{\mathsf{var}_t^M m} \cdot F'|_{(X')^n \mapsto (A')^{p^*(t')n}}$, where $F' = \sum c_n(m;t)(X')^{\sigma n}$. Note that $\mathsf{pr}_{I_{\mathsf{uf}}} \mathsf{var}_t^M m = \sigma\, \mathsf{pr}_{I_{\mathsf{uf}}} m$. By Assumption 6.1 (2), we have $F' = F_{\mathsf{var}_t^M m}(t')$. So $\mathsf{var}_t^A(S_m) = S'_{\mathsf{var}_t^M m}$.

By Assumption 6.1 (1), we have $\boldsymbol{\mu}^* S'_{\mathsf{var}_t^M m} \in \mathcal{S}(t)$. Therefore, $\mathsf{tw}^A S_m \in \mathcal{S}(t)$.

(2) Recall that the inverse of tw_t^A is also a twist automorphism, the claim follows. ∎

6.2 Bases for $\mathcal{U}^X$

Similarly, for any t, assume that we have a subset $\mathcal{Z}(t) = \{Z_n(t) \mid n \in \mathbb{Z}^I\}$ in $\mathcal{LP}^X(t)$, such that its elements $Z_n(t)$ are n-pointed. Then we can denote $Z_n(t) = X(t)^n \cdot F_n(t)$, whose F-polynomials take the form $F_n(t) = \sum_{e \in \mathbb{N}^{I_{\mathsf{uf}}}} c_e(n;t)X(t)^e$ with $c_0(n;t) = 1$, $c_e(n;t) \in \Bbbk$.

Note that $\mathcal{Z}(t)$ must be $\Bbbk$-linearly independent.

Assumption 6.3. We assume the following conditions hold.

(1) The sets $\mathcal{Z}(t)$, $\forall t \in \Delta^+$, are identified by mutations: for any $t' = \boldsymbol{\mu}t$, we have $\boldsymbol{\mu}^* \mathcal{Z}(t') = \mathcal{Z}(t)$.

(2) The F-polynomials $F_n(t)$ only depend on $\mathsf{pr}_{I_{\mathsf{uf}}} p^* n$ and $B(t)_{\mathsf{uf}}$:

- If $p^* n' = p^* n + u$ for some $u \in M_{\mathsf{f}}^{\circ}$, then $F_{n'}(t) = F_n(t)$.
- If t' and t are similar up to σ and $\mathsf{pr}_{I_{\mathsf{uf}}} B(t')n' = \sigma\, \mathsf{pr}_{I_{\mathsf{uf}}} B(t)n$ for some $n, n' \in \mathbb{Z}^I$, we have $c_{\sigma e}(n';t') = c_e(n;t)$, $\forall e$.

Assumption 6.3 (1) implies $\mathcal{Z}(t) \subset \mathcal{U}^X(t)$.

Theorem 6.4. *Let tw^X denote any twist endomorphism on $\mathcal{U}^X$. Assume the sets $\mathcal{Z}(t)$, $\forall t \in \Delta^+$, satisfy Assumption 6.3. Then the following statements are true.*

(1) *We have $\mathsf{tw}^X \mathcal{Z}(t) \subset \mathcal{Z}(t)$.*

(2) *If tw^X is a twist automorphism, we have $\mathsf{tw}^X \mathcal{Z}(t) = \mathcal{Z}(t)$.*

Proof. The proof is similar to that of Theorem 6.4. ∎

6.3 Construction of bases for $\mathcal{U}^X$

There is few literature on the bases for $\mathcal{U}^X$. Let us explain how to construct a basis for $\mathcal{U}^X$ from a basis for $\mathcal{U}^A$ following the idea of [25].

Let t^{prin} denote the principal coefficient seed constructed from t by adding a copy $I' = \{i' \mid i \in I\}$ of I as extra frozen variables, such that $B(t^{\mathsf{prin}}) = \begin{pmatrix} B(t) & -\mathsf{Id} \\ \mathsf{Id} & 0 \end{pmatrix}$. Assume that a set $\mathcal{S}(t^{\mathsf{prin}}) \subset \mathcal{U}^A(t^{\mathsf{prin}})$ as in Section 6.1 has been given. Then we can construct a set $\mathcal{Z}(t) \subset \mathcal{LP}^X(t)$ such that $Z_n(t) = X(t)^n \cdot F_n$ where F_n is the F-polynomial of $S_m(t^{\mathsf{prin}})$ for $m = B(t^{\mathsf{prin}})\begin{pmatrix} n \\ 0 \end{pmatrix} = \begin{pmatrix} B(t)n \\ n \end{pmatrix}$.

Lemma 6.5. *If the sets $(\boldsymbol{\mu}^A)^* \mathcal{S}(t^{\mathsf{prin}}) \subset \mathcal{U}^A(\boldsymbol{\mu}^{-1} t^{\mathsf{prin}})$, for all mutation sequences $\boldsymbol{\mu}$, satisfy Assumption 6.1, then the sets $\mathcal{Z}(\boldsymbol{\mu}^{-1} t) := (\boldsymbol{\mu}^X)^* \mathcal{Z}(t)$, $\forall \boldsymbol{\mu}$, are contained in $\mathcal{U}^X(t)$ and satisfy Assumption 6.3.*

Proof. We claim that $\boldsymbol{\mu}^* \mathcal{Z}(t)$ is contained in $\mathcal{LP}^X(\boldsymbol{\mu}^{-1} t)$ for any mutation sequence $\boldsymbol{\mu}$. If so, we deduce that the $\mathcal{Z}(t) \subset \cap_{\forall \boldsymbol{\mu}}(\boldsymbol{\mu}^*)^{-1} \mathcal{LP}^X(\boldsymbol{\mu}^{-1} t) = \mathcal{U}^X(t)$. Then the sets $\mathcal{Z}(\boldsymbol{\mu}^{-1} t)$, $\forall \boldsymbol{\mu}$, satisfy Assumption 6.3 (1) by definition. In addition, they satisfy Assumption 6.3 (2) because the sets $\mathcal{S}(\boldsymbol{\mu}^{-1} t^{\mathsf{prin}})$, for all $\boldsymbol{\mu}$, satisfy Assumption 6.1 (2).

Denote $s = t^{\mathsf{prin}}$, $t' = (\boldsymbol{\mu})^{-1} t$, and $s' = (\boldsymbol{\mu})^{-1} s$. Then we have the natural inclusions $\mathcal{LP}^X(t) \subset \mathcal{LP}^X(s)$ and $\mathcal{LP}^X(t') \subset \mathcal{LP}^X(s')$.

Note that the monomial maps $p^*(s)\colon \mathcal{LP}^X(s) \to \mathcal{LP}^A(s)$ and $p^*(s')\colon \widehat{\mathcal{LP}}^X(s) \to \widehat{\mathcal{LP}}^A(s)$ are injective. We also have injective maps $\iota^\bullet\colon \mathcal{LP}^\bullet(s) \to \widehat{\mathcal{LP}}^\bullet(s')$ taking the formal Laurent series after mutations (see Section 2.4).

Take any $Z \in \mathcal{Z}(t)$. Then we have $p^*(s)Z = S_m \in \mathcal{S}(s)$ for some m. So we get $(\boldsymbol{\mu}^A)^* p^*(s)Z = S'_m \in \mathcal{S}(s')$ for some m'. Therefore, $\iota^A(s)p^*(s)Z = S_{m'}$. But we also have $\iota^A(s)p^*(s)Z = p^*(s')\iota^X(s)Z$. So the formal series $\iota^X(s)Z$ is sent to a Laurent polynomial $S_{m'}$ by the injective map $p^*(s')$. It follows that $\iota^X(s)Z$ is a Laurent polynomial. Therefore, $(\boldsymbol{\mu}^X)^* Z = \iota^X Z$ is contained in $\mathcal{LP}^X(s')$. ∎

Using this construction, by taking $\mathcal{S}(t^{\mathsf{prin}})$ as the set of generic cluster characters, the common triangular basis, or the set of the theta functions which are assumed to be Laurent polynomials, we obtain the corresponding set $\mathcal{Z}(t) \subset \mathcal{U}^X(t)$.

Finally, let us discuss when $\mathcal{Z}(t)$ becomes a basis of $\mathcal{U}^X(t)$.

Theorem 6.6. *Under the assumption of Lemma 6.5, if $p^*\mathcal{U}^X(t^{\mathsf{prin}})$ is contained in the free $\Bbbk$-module $\oplus_m \Bbbk S_m(t^{\mathsf{prin}}) \subset \mathcal{U}^A(t^{\mathsf{prin}})$, then $\mathcal{Z}(t)$ is a $\Bbbk$-basis of $\mathcal{U}^X(t)$. In particular, when $\mathcal{S}(t^{\mathsf{prin}})$ is a basis of $\mathcal{U}^A(t^{\mathsf{prin}})$, $\mathcal{Z}(t)$ is a basis of $\mathcal{U}^X(t)$.*

Proof. On the one hand, we know that $\{S_m(t^{\mathsf{prin}}) \mid \exists n,\ m = \binom{B(t)n}{n}\}$ is contained in $p^*\mathcal{U}^X(t^{\mathsf{prin}})$.

On the other hand, let us endow $\mathcal{LP}(t^{\mathsf{prin}})$ with the $M^\circ(t)$-grading such that, $\forall i \in I$, $\deg A_i(t) = \mathbf{f}_i$ and $\deg A_{i'}(t) = -\sum_{j\in I} b_{ji}\mathbf{f}_j$. Then $S_m(t^{\mathsf{prin}})$ are homogeneous elements, and they have degree 0 if and only if $m = \binom{B(t)n}{n}$ for some n. So $\{S_m(t^{\mathsf{prin}}) \mid \exists n,\ m = \binom{B(t)n}{n}\}$ is a basis for the degree-0 subspace of $\oplus_m \Bbbk S_m(t^{\mathsf{prin}})$. Moreover, since $p^*X_i(t^{\mathsf{prin}})$ has degree 0 for any $i \in I$, $p^*\mathcal{U}^X(t^{\mathsf{prin}})$ consists of degree 0 elements.

Therefore, $\{S_m(t^{\mathsf{prin}}) \mid \exists n,\ m = \binom{B(t)n}{n}\}$ is a basis of $p^*\mathcal{U}^X(t^{\mathsf{prin}})$. It implies the desired claim. ∎

7 Examples

We work at the classical level $\Bbbk = \mathbb{Z}$ in this section for simplicity.

7.1 Twist automorphisms on unipotent cells

[2, 4] introduced an automorphism η_w on the unipotent cell $\mathbb{C}[N^w_-]$, which was called a twist automorphism. The quantized twist automorphism η_w was introduced by [33]. We refer the reader to loc. cit. for further details.

Moreover, it is known that (quantized) $\mathbb{C}[N^w_-]$ is a (quantum) upper cluster algebra $\mathcal{U}^A$ by [15, 17, 21, 23]. By [47], the (quantum) twist automorphism η is a twist automorphism in our sense.

Let us illustrate the twist automorphism η_w by an example.

Example 7.1. Consider the algebraic group $G = SL_3(\Bbbk)$ where $\Bbbk$ is a field of characteristic 0. Let B_+ (resp. B_-) denote its Borel subgroup consisting of the upper (resp. lower) triangular matrices in G. Let $N_\pm$ denote the subgroups of $B_\pm$ whose diagonal entries are 1, respectively.

We take w to be the longest element w_0 in the permutation group of 3 elements. It can be represented by the matrix

$$\mathsf{P}_{w_0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The corresponding (negative) unipotent cell is $N^{w_0}_- := B_+ \mathsf{P}_{w_0} B_+ \cap N_-$.

Let us denote an element $x \in N_-$ by

$$x = \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_1' & 1 \end{pmatrix}.$$

Let x_3 denote the minor $\left| \begin{pmatrix} x_1 & 1 \\ x_2 & x_1' \end{pmatrix} \right| = x_1 x_1' - x_2$. By [4], there exists a unique element y in $N_- \cap B_+ P_{w_0} x^T$ when $x \in N_-^{w_0}$. The twist automorphism $\widetilde{\eta}_{w_0}$ on N_- is defined such that $\widetilde{\eta}_{w_0}(x) = y$.

Let us compute y explicitly. Take any element

$$z = \begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix} \in B_+.$$

Note that

$$P_{w_0} x^T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & x_1' \\ 1 & x_1 & x_2 \end{pmatrix}.$$

We have

$$z\, P_{w_0} x^T = \begin{pmatrix} c & b + cx_1 & a + bx_1' + cx_2 \\ e & d + ex_1 & dx_1' + ex_2 \\ f & fx_1 & fx_2 \end{pmatrix}.$$

If $z\, P_{w_0} x^T \in N_-$, then we must have $c = 1$, $f = x_2^{-1}$, $b = -cx_1$, $a = -bx_1' - cx_2$, and

$$\begin{cases} d + ex_1 = 1, \\ dx_1' + ex_2 = 0. \end{cases}$$

The last linear system gives us $d = \frac{x_2}{x_2 - x_1 x_1'} = \frac{x_2}{x_3}$, $e = \frac{-x_1'}{x_2 - x_1 x_1'} = \frac{x_1'}{x_3}$. We deduce that $N_- \cap B_+ P_{w_0} x^T$ consists of a single element

$$y = \begin{pmatrix} 1 & 0 & 0 \\ x_1' x_3^{-1} & 1 & 0 \\ x_2^{-1} & x_1 x_2^{-1} & 1 \end{pmatrix}$$

if $x_2, x_3 \neq 0$, and it is empty if not.

It is easy to check that $\{ N_- \cap B_+ P_{w_0} x^T \mid \forall x \in N_- \} = N_- \cap B_+ P_{w_0} B_+ =: N_-^{w_0}$. Then we can deduce from the above computation of y that $N_-^{w_0} = \{ x \in N_- \mid x_2 \neq 0, x_3 \neq 0 \}$.

Define A_1 to be the function on $N_-^{w_0}$ sending a point x to x_1. Similarly define A_1', A_2, A_3. Then the coordinate ring $\Bbbk[N_-^{w_0}]$ is an (upper) cluster algebra with four cluster variables: A_1, A_1', A_2, A_3, see [46, Example 2.2]. Then $\widetilde{\eta}_{w_0}$ gives rise to an automorphism η_w such that

$$\eta_{w_0}(A_1) = A_1' A_3^{-1} = \frac{A_2 + A_3}{A_1} A_3^{-1},$$

$$\eta_{w_0}(A_1') = A_1 A_2^{-1} = \frac{A_2 + A_3}{A_1'} A_2^{-1},$$

$$\eta_{w_0}(X_i) = A_i^{-1}, \qquad i = 2, 3.$$

7.2 Dehn twists for surface cases

Let Σ denote a triangulable surface S with finitely many marked point. For any of its (tagged) triangulation Δ, one can construct a seed t_Δ, whose frozen vertices are contributed from curves on the boundary ∂S. We refer the reader to [12, 39] for details.

The punctures are the marked point in the interior S° of S. By [39], the seed t_Δ satisfies the full rank condition when Σ has no punctures.

Let L denote any closed loop in the interior of S which does not pass a marked point or is contractible to a marked point. A Dehn twist tw_L around L produces a new (tagged) triangulation $\mathsf{tw}_L \Delta$ from Δ. By [12], we have $t_{\mathsf{tw}_L \Delta} = \boldsymbol{\mu} t_\Delta$ for some mutation sequence $\boldsymbol{\mu}$. By construction, $B(t_\Delta) = B(t_{\mathsf{tw}_L \Delta})$. In particular, t_Δ and $t_{\mathsf{tw}_L \Delta}$ are similar.

Then one can construct the variation maps $\mathsf{var}_{t_\Delta}^M \colon M^\circ(t_\Delta) \to M^\circ(t_{\mathsf{tw}_L \Delta})$ and $\mathsf{var}_{t_\Delta}^N \colon N(t_\Delta) \to N(t_{\mathsf{tw}_L \Delta})$ represented by the identity matrices, respectively. The corresponding twist automorphisms are determined by

$$\mathsf{var}_{t_\Delta}^A(A_i(t_\Delta)) = \boldsymbol{\mu}^* A_i(t_{\mathsf{tw}_L \Delta}), \qquad \mathsf{var}_{t_\Delta}^X(X_i(t_\Delta)) = \boldsymbol{\mu}^* X_i(t_{\mathsf{tw}_L \Delta}).$$

They give rise to automorphisms on $\mathcal{U}^A(t_\Delta)$ and $\mathcal{U}^X(t_\Delta)$ associated to the Dehn twist tw_L.

7.3 Once-punctured digon

Example 7.2. The following cluster Poisson algebra arises from PGL_3-local systems on a once-punctured digon [49].

Take $I = \{1, 2, 3, 4\}$, with $I_{\mathsf{uf}} = \{2, 4\}$ and $I_{\mathsf{f}} = \{1, 3\}$. Define the initial seed t such that its B-matrix is

$$B = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

Note that $B_{\mathsf{uf}} = 0$. So we cannot endow a compatible Poisson structure λ for $\mathcal{LP}^A(t)$. We have $W = -B$.

Let t' denote the seed $\boldsymbol{\mu} t$ where $\boldsymbol{\mu} = \mu_2 \mu_4$. Its B-matrix is $B' = -B$. We have $W' = -W$. We can check the following mutation rule:

$$\boldsymbol{\mu}^* X_2' = X_2^{-1}, \qquad \boldsymbol{\mu}^* X_4' = X_4^{-1}, \qquad \boldsymbol{\mu}^* X_1' = X_1 \cdot X_2 \cdot (1 + X_2)^{-1} \cdot (1 + X_4),$$
$$\boldsymbol{\mu}^* X_3' = X_3 \cdot X_4 \cdot (1 + X_2) \cdot (1 + X_4)^{-1}.$$

Then $t' = t[1]$ with $\sigma = \mathsf{Id}$, $W' = -W$.

Let us construct Poisson automorphisms on $\mathcal{U}^X(t)$. Note that we have $W = \begin{pmatrix} 0 & -W_{\mathsf{f},\mathsf{uf}}^{\mathrm{T}} \\ W_{\mathsf{f},\mathsf{uf}} & 0 \end{pmatrix}$ with row and column indices $(2, 4, 1, 3)$, where $W_{\mathsf{f},\mathsf{uf}} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ with row indices $\{1, 3\}$ and column indices $\{2, 4\}$. We want to construct a linear isomorphism $\mathsf{var}_t^N \colon\!= N(t) \to N(t')$ represented by an $\mathbb{Z}$-valued invertible matrix $\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & V_{\mathsf{uf},\mathsf{f}} \\ 0 & V_{\mathsf{f}} \end{pmatrix}$, such that

$$\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & V_{\mathsf{uf},\mathsf{f}} \\ 0 & V_{\mathsf{f}} \end{pmatrix}^{\mathrm{T}} W' \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & V_{\mathsf{uf},\mathsf{f}} \\ 0 & V_{\mathsf{f}} \end{pmatrix} = W.$$

The desired equation is equivalent to

$$-\begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & V_{\mathsf{uf},\mathsf{f}} \\ 0 & V_{\mathsf{f}} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & -W_{\mathsf{f},\mathsf{uf}}^{\mathrm{T}} \\ W_{\mathsf{f},\mathsf{uf}} & 0 \end{pmatrix} \begin{pmatrix} \mathsf{Id}_{\mathsf{uf}} & V_{\mathsf{uf},\mathsf{f}} \\ 0 & V_{\mathsf{f}} \end{pmatrix} = \begin{pmatrix} 0 & -W_{\mathsf{f},\mathsf{uf}}^{\mathrm{T}} \\ W_{\mathsf{f},\mathsf{uf}} & 0 \end{pmatrix}.$$

By computing its block submatrices, we reduce the equation to the equations

$$-V_{\mathsf{f}}^{\mathrm{T}} W_{\mathsf{f},\mathsf{uf}} = W_{\mathsf{f},\mathsf{uf}}, \qquad V_{\mathsf{uf},\mathsf{f}}^{\mathrm{T}} W_{\mathsf{f},\mathsf{uf}}^{\mathrm{T}} V_{\mathsf{f}} - V_{\mathsf{f}}^{\mathrm{T}} W_{\mathsf{f},\mathsf{uf}} V_{\mathsf{uf},\mathsf{f}} = 0.$$

For the first equation (for var_t^N to be a variation map), the solution takes the form $V_{\mathsf{f}} = \begin{pmatrix} \lambda^{-1} & \mu \\ \lambda & \mu^{-1} \end{pmatrix}$ for any λ, μ. Since we are looking for a bijection var_t^N between lattices, we must have $\lambda + \mu = 0, 2$.

The second equation (for var_t^N to further be Poisson) is equivalent to

$$-V_{\mathsf{uf},\mathsf{f}}^{\mathrm{T}} W_{\mathsf{f},\mathsf{uf}}^{\mathrm{T}} + W_{\mathsf{f},\mathsf{uf}} V_{\mathsf{uf},\mathsf{f}} = 0$$

by the first equation. Then its solution takes the form $V_{\mathsf{uf},\mathsf{f}} = \begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix}$ for any α, β.

The resulting twist automorphism on the fraction field takes the following form:

$$\mathsf{tw}\, X_2 = X_2^{-1}, \qquad \mathsf{tw}\, X_4 = X_4^{-1},$$
$$\mathsf{tw}\, X_1 = \boldsymbol{\mu}^*((X')^{(\lambda-1,\alpha,\lambda,\alpha)}) = X^{(\lambda-1,\lambda-1-\alpha,\lambda,\lambda-\alpha)}(1 + X_2)(1 + X_4)^{-1},$$
$$\mathsf{tw}\, X_3 = \boldsymbol{\mu}^*((X')^{(\mu,\beta,\mu-1,\beta)}) = X^{(\mu,\mu-\beta,\mu-1,\mu-1-\beta)}(1 + X_2)^{-1}(1 + X_4).$$

Note that $\mathsf{tw}(X_1 X_3) = (X_1 X_3)^{\lambda+\mu-1}(X_2 X_4)^{\lambda+\mu-1-\alpha-\beta}$.

Let us consider elements $E = X_1 \cdot (1 + X_4)$, $F = X_3 \cdot (1 + X_2)$, $K = X_1 \cdot X_3 \cdot X_4$ and $K' = X_1 \cdot X_2 \cdot X_3$ (the Chevalley generators, see [49]). It is straightforward to check that

$$(\boldsymbol{\mu}^{-1})^* E = X_1' \cdot (1 + X_2'), \qquad (\boldsymbol{\mu}^{-1})^* F = X_3' \cdot (1 + X_2'), \qquad (\boldsymbol{\mu}^{-1})^* K = X_1' X_2' X_3',$$
$$(\boldsymbol{\mu}^{-1})^* K' = X_1' X_3' X_4',$$

so they are elements in the cluster Poisson algebra $\mathcal{U}^X$.

We check that

$$\mathsf{tw}\, E = X^{(\lambda-1,\lambda-1-\alpha,\lambda,\lambda-\alpha)}(1 + X_2)(1 + X_4)^{-1}\big(1 + X_4^{-1}\big) = (X_1 X_3)^{\lambda-1}(X_2 X_4)^{\lambda-1-\alpha} F,$$
$$\mathsf{tw}\, F = X^{(\mu,\mu-\beta,\mu-1,\mu-1-\beta)}(1 + X_4)(1 + X_2)^{-1}\big(1 + X_2^{-1}\big) = (X_1 X_3)^{\mu-1}(X_2 X_4)^{\mu-1-\beta} E,$$
$$\mathsf{tw}\, K = (X_1 X_3)^{\lambda+\mu-1}(X_2 X_4)^{\lambda+\mu-1-\alpha-\beta} X_4^{-1},$$
$$\mathsf{tw}\, K' = (X_1 X_3)^{\lambda+\mu-1}(X_2 X_4)^{\lambda+\mu-1-\alpha\beta} X_2^{-1}.$$

For the special case $\lambda = \mu = 1$, $\alpha = \beta = 0$, we recover the following automorphism in (Drinfeld double) quantum groups:

$$\mathsf{tw}\, E = F, \qquad \mathsf{tw}\, F = E, \qquad \mathsf{tw}\, K = K', \qquad \mathsf{tw}\, K' = K.$$

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