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Vector Fields and Flows on Subcartesian Spaces

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Abstract. This paper is part of a series of papers on differential geometry of C^∞-ringed spaces. In this paper, we study vector fields and their flows on a class of singular spaces. Our class includes arbitrary subspaces of manifolds, as well as symplectic and contact quotients by actions of compact Lie groups. We show that derivations of the C^∞-ring of global smooth functions integrate to smooth flows.

Key words: differential space; C^∞-ring; subcartesian; flow

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1 Introduction

This paper is one in a series of papers on differential geometry of C^∞-ringed spaces. Two other papers in the series are [9] and [10].

In this paper, we study vector fields and flows on subcartesian spaces.[1]

Singular spaces, that is, spaces that are not manifolds, arise naturally in differential geometry and in its applications to physics and engineering. There are many approaches to differential geometry on singular spaces, and there is a vast literature which we will not attempt to survey. In this paper we use differential spaces in the sense of Sikorski [15] as our model of singular spaces.

Śniatycki's book [18] contains a number of geometric tools that apply to differential spaces. Śniatycki is particularly interested in stratified spaces that arise through symplectic reduction (see [16]); he provides a new perspective by viewing these spaces as differential spaces. In this paper, we respond to, and elaborate on, Śniatycki's treatment of vector fields on differential spaces. Specifically, for a derivation of (the ring of global smooth functions on) a subcartesian space M, Śniatycki proves the existence and uniqueness of smooth maximal integral curves (see [18, Theorem 3.2.1] and [17, Theorem 1]), but he does not discuss the flow as a map from a subset of $M \times \mathbb{R}$ to M.

The main result of this paper is Theorem 3.14, which roughly says the following:

Theorem. *Let M be a differential space embeddable in some Euclidean space $\mathbb{R}^N$ and v a vector field on M. Assemble the maximal integral curves of v into a flow $\Phi\colon \mathcal{W} \to M$, where $\mathcal{W}$ is a subset $M \times \mathbb{R}$. Then the flow $\Phi\colon \mathcal{W} \to M$ is smooth.*

[1]For us a vector field is a derivation of the C^∞-ring of global smooth functions. This is different from Śniatycki's definition; see Remark 3.11. To reduce ambiguity, in our formal statements we say "derivation" rather than "vector field".

Thanks to an analogue of the Whitney embedding theorem for differential spaces, embeddability in a Euclidean space is a fairly mild assumption on a differential space that is *locally* embeddable in a Euclidean space, i.e., the space that is *subcartesian* (see Definition 2.37). See [1, 11] or [6] for various versions of the Whitney embedding theorem for subcartesian spaces. On the other hand, if a differential space is not subcartesian, then the flow of a vector field may not exist at all, see [7, Example 2, Section 32.12].

Theorem 3.14 relies on the existence and uniqueness of maximal integral curves. A few years ago Śniatycki gave a proof of existence and uniqueness of integral curves of vector fields on arbitrary subcartesian differential spaces (see [18, Theorem 3.2.1]). In a later paper [3], Cushman and Śniatycki have a similar theorem, Theorem 5.3, and they say that "Theorem 5.3 replaces [5, Theorem 3.2.1], which is incorrect" (their [5] is our [18]). However, it seems to us that there is nothing wrong with Śniatycki's Theorem 3.2.1, certainly not with its statement. To make sure, we provide a self-contained proof of existence and uniqueness of integral curves, under the mild assumptions that imply embeddability, see Corollary 3.22.

In a later paper [5], we remove the mild assumptions that imply embeddability. These assumptions are indeed mild: "reasonable" subcartesian spaces are embeddable. And removing these assumptions has a price; the proof becomes more involved: for embeddable spaces, we can rely on the integration of vector fields on open subsets of Euclidean spaces; for not-necessarily-embeddable spaces, we need to imitate the proof of integration of vector fields on manifolds.

Organization of the paper

In Section 2, we recall the definition and some properties of differential spaces. This material is standard. One novelty is that we explicitly mention C^∞-rings. In Section 3, we prove the existence and uniqueness of integral curves of derivations on embeddable subcartesian spaces and use this result to prove the main theorem of the paper. In Appendix A, we provide a proof of a special case of a theorem of Yamashita [19, Theorem 3.1]. Namely, we prove that any $\mathbb{R}$-algebra derivation of a point-determined C^∞-ring is automatically a C^∞-ring derivation. This fact is used in our proof of the existence and uniqueness of integral curves of derivations.

Assumptions

Throughout the paper, "manifold" means "smooth (i.e., C^∞) manifold". All manifolds are assumed to be second countable and Hausdorff.

2 Differential spaces

In this section, we recall the definition and some properties of differential spaces in the sense of Sikorski. It will be convenient to recall the notion of a C^∞-ring first. The definition below is not standard, but it is easier to understand on the first pass. It is equivalent to Lawvere's original definition; see [4].

Definition 2.1. A C^∞*-ring* is a set $\mathscr{C}$, equipped with operations

$$g_{\mathscr{C}}\colon\ \mathscr{C}^m \to \mathscr{C}$$

for all $m \in \mathbb{Z}_{\geq 0}$ and all $g \in C^\infty(\mathbb{R}^m)$, such that the following holds:

- For all $n, m \in \mathbb{Z}_{\geq 0}$, all $f_1, \dots, f_m \in C^\infty(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^m)$,

$$(g \circ (f_1, \dots, f_m))_{\mathscr{C}}(c_1, \dots, c_n) = g_{\mathscr{C}}((f_1)_{\mathscr{C}}(c_1, \dots, c_n), \dots, (f_m)_{\mathscr{C}}(c_1, \dots, c_n))$$

  for all $(c_1, \dots, c_n) \in \mathscr{C}^n$.

- For every $m > 0$ and for every coordinate function $x_j \colon \mathbb{R}^m \to \mathbb{R}$, $1 \leq j \leq m$,

$$(x_j)_{\mathscr{C}}(c_1, \dots, c_m) = c_j.$$

If $m = 0$, then $\mathscr{C}^0$ is a singleton $\{*\}$. Similarly, $C^\infty(\mathbb{R}^0) \simeq C^\infty(0) \simeq \mathbb{R}$. Thus 0-ary operations on $\mathscr{C}$ are maps $g_{\mathscr{C}} \colon \{*\} \to \mathscr{C}$, one for every $g \in \mathbb{R}$. Since any map $h \colon \{*\} \to \mathscr{C}$ can be identified with $h(*) \in \mathscr{C}$, we identify the 0-ary operation corresponding to $g \in \mathbb{R}$ with an element of $\mathscr{C}$, which we denote by $g_{\mathscr{C}}$.

Example 2.2. Let M be a C^∞-manifold and $C^\infty(M)$ the set of smooth (real-valued) functions. Then $C^\infty(M)$, equipped with the usual composition operations

$$g_{C^\infty(M)}(a_1, \dots, a_m) := g \circ (a_1, \dots, a_m),$$

is a C^∞ ring.

Example 2.3. Let M be a topological space and $C^0(M)$ the set of continuous real-valued functions. Then $C^0(M)$, equipped with the usual composition operations

$$g_{C^\infty(M)}(a_1, \dots, a_m) := g \circ (a_1, \dots, a_m),$$

is also a C^∞ ring.

Definition 2.4. A nonempty subset $\mathscr{C}$ of a C^∞-ring $\mathscr{A}$ is a C^∞*-subring* if $\mathscr{C}$ is closed under the operations of $\mathscr{A}$.

Example 2.5. If M is a manifold, then $C^\infty(M)$ is a C^∞-subring of $C^0(M)$.

We also need to recall the notion of an initial topology.

Definition 2.6. Let X be a set and $\mathscr{F}$ a set of maps from X to various topological spaces. The smallest topology on X making all functions in $\mathscr{F}$ continuous is called *initial*.

In particular, a collection of real-valued functions $\mathscr{F}$ on a set X uniquely defines an initial topology on X (we give the real line $\mathbb{R}$ the standard topology, of course).

Next we define differential spaces in the sense of Sikorski. The definition below agrees with the one in [18]. Some papers define differential spaces as ringed spaces; see [14], for example.

Definition 2.7. A *differential space* (in the sense of Sikorski) is a pair $(M, \mathscr{F})$, where M is a topological space and $\mathscr{F}$ is a (nonempty) set of real-valued functions on M, subject to the following three conditions:

(1) The topology on M is the smallest topology making every function in $\mathscr{F}$ continuous, i.e., it is the initial topology defined by the set $\mathscr{F}$.

(2) For any nonnegative integer m, any smooth function $g \in C^\infty(\mathbb{R}^m)$, and any m-tuple $f_1, \dots, f_m \in \mathscr{F}$, the composite $g \circ (f_1, \dots, f_m)$ is in $\mathscr{F}$.

(3) Let $g \colon M \to \mathbb{R}$ be a function. Suppose that for each point p of M there exist a neighborhood U of p and a function $a \in \mathscr{F}$ such that $g|_U = a|_U$. Then the function g is in $\mathscr{F}$.

We refer to $\mathscr{F}$ as a *differential structure* on M.

Remark 2.8.

- We think of the set of functions $\mathscr{F}$ on a differential space $(M, \mathscr{F})$ as "smooth functions by fiat". (Also, see Remark 2.17.)

- We may refer to a differential space $(M, \mathscr{F})$ simply as M.
- Condition (2) says that $\mathscr{F}$ is a C^∞-ring with the operations $g_{\mathscr{F}}\colon \mathscr{F}^m \to \mathscr{F}$ given by composition

  $$g_{\mathscr{F}}(f_1, \dots, f_m) := g \circ (f_1, \dots, f_m).$$

  Note that since $C^\infty(\mathbb{R}^1)$ includes constant functions, (2) implies that all constant functions are in $\mathscr{F}$. Recall that 0-ary operations on a C^∞-ring are indexed by constants $g \in C^\infty(\mathbb{R}^0) \simeq \mathbb{R}$. Given $g \in C^\infty(\mathbb{R}^0)$ we define the operation $g_{\mathscr{F}}\colon \mathscr{F}^0 = \{*\} \to \mathscr{F}$ by setting $g_{\mathscr{F}}(*)$ to be the constant function on M taking the value g everywhere. We know that such a constant function has to be in $\mathscr{F}$.

Remark 2.9. In the literature, the term "differential space" is used for a variety of mathematical objects, some of which are related to Sikorski's differential spaces, and some that are not related at all.

Example 2.10. Let M be a manifold (second countable and Hausdorff). Then the pair $(M, C^\infty(M))$, where $C^\infty(M)$ is the set of C^∞ functions, is a differential space in the sense of Definition 2.7. The main point to check is that the topology on M coincides with the smallest topology making all the functions in $C^\infty(M)$ continuous. This follows from the existence of bump functions on manifolds and from Lemma 2.15 below. Alternatively, it follows from a theorem of Whitney by which any closed subset of a manifold M is the zero set of a smooth function. See, for example, [8, Theorem 2.29].

Definition 2.11. Given a manifold M we refer to the C^∞-ring $C^\infty(M)$ of smooth functions on M as the *standard* differential structure.

Example 2.12. Let M be a manifold. Then the set $C^0(M)$ of *continuous* function on M is also a differential structure. Unless M is discrete, the C^∞-ring $C^0(M)$ is bigger than $C^\infty(M)$.

Definition 2.13. Let $(M, \mathscr{T})$ be a topological space, $C \subset M$ a closed set and $x \in M \setminus C$ a point. A *bump function* (relative to C and x) is a continuous function $\rho\colon M \to [0,1]$ so that $(\operatorname{supp} \rho) \cap C = \varnothing$ and ρ is identically 1 on a neighborhood of x.

Definition 2.14. Let $(M, \mathscr{T})$ be a topological space and $\mathscr{F} \subseteq C^0(M, \mathbb{R})$ a collection of continuous real-valued functions on M. The topology $\mathscr{T}$ on M is *$\mathscr{F}$-regular* iff for any closed subset C of M and any point $x \in M \setminus C$ there is a bump function $\rho \in \mathscr{F}$ with $\operatorname{supp} \rho \subset M \setminus C$ and ρ identically 1 on a neighborhood of x.

Lemma 2.15. *Let $(M, \mathscr{T})$ be a topological space and $\mathscr{F} \subset C^0(M, \mathbb{R})$ a C^∞-subring. Then $\mathscr{T}$ is the smallest topology making all the functions in $\mathscr{F}$ continuous if and only if the topology $\mathscr{T}$ is $\mathscr{F}$-regular.*

Proof. Let $\mathscr{T}_{\mathscr{F}}$ denote the smallest topology making all the functions in $\mathscr{F}$ continuous. The set

$$\mathscr{S} := \left\{ f^{-1}(I) \mid f \in \mathscr{F},\ I \text{ is an open interval} \right\}$$

is a sub-basis for $\mathscr{T}_{\mathscr{F}}$. Since all the functions in $\mathscr{F}$ are continuous with respect to $\mathscr{T}$, $\mathscr{T}_{\mathscr{F}} \subseteq \mathscr{T}$. Therefore, it is enough to argue that $\mathscr{T} \subseteq \mathscr{T}_{\mathscr{F}}$ if and only if $\mathscr{T}$ is $\mathscr{F}$-regular.

($\Rightarrow$) Suppose $\mathscr{T} \subseteq \mathscr{T}_{\mathscr{F}}$. Let $C \subset M$ be $\mathscr{T}$-closed and x a point in M which is not in C. Then $M \setminus C$ is $\mathscr{T}$-open. Since $\mathscr{T} \subseteq \mathscr{T}_{\mathscr{F}}$ by assumption, $M \setminus C$ is in $\mathscr{T}_{\mathscr{F}}$. Then there exist functions $h_1, \dots, h_k \in \mathscr{F}$ and open intervals $I_1, \dots, I_k$ such that $x \in \cap_{i=1}^k h_i^{-1}(I_i) \subset M \setminus C$. There is a C^∞ function $\rho\colon \mathbb{R}^k \to [0,1]$ with $\operatorname{supp} \rho \subset I_1 \times \cdots \times I_k$ and the property that $\rho = 1$

on a neighborhood of $(h_1(x), \dots, h_k(x))$ in $\mathbb{R}^k$. Then $\tau := \rho \circ (h_1, \dots, h_k)$ is in $\mathscr{F}$, since $\mathscr{F}$ is a C^∞-subring of $C^0(M)$. The function τ is a desired bump function.

($\Leftarrow$) Suppose the topology $\mathscr{T}$ is $\mathscr{F}$-regular. Let $U \in \mathscr{T}$ be an open set. Then $C = M \setminus U$ is closed. Since $\mathscr{T}$ is $\mathscr{F}$-regular, for any point $x \in U$ there is a bump function $\rho_x \in \mathscr{F}$ with $\operatorname{supp} \rho_x \subset U$ and ρ_x is identically 1 in a neighborhood of x. Then $\rho_x^{-1}((0,\infty)) \subset U$ and $\rho_x^{-1}((0,\infty)) \in \mathscr{T}_{\mathscr{F}}$. It follows that

$$U = \bigcup_{x \in U} \rho_x^{-1}((0,\infty)) \in \mathscr{T}_{\mathscr{F}}.$$

Since U is an arbitrary element of $\mathscr{T}$, $\mathscr{T} \subseteq \mathscr{T}_{\mathscr{F}}$. ■

Definition 2.16. A *smooth map* from a differential space $(M, \mathscr{F}_M)$ to a differential space $(N, \mathscr{F}_N)$ is a function $\varphi\colon M \to N$ such that for every $f \in \mathscr{F}_N$ the composite $f \circ \varphi$ is in $\mathscr{F}_M$.

Remark 2.17. Given a differential space $(M, \mathscr{F})$, the set $\mathscr{F}$ coincides with the set of smooth maps $(M, \mathscr{F}) \to (\mathbb{R}, C^\infty(\mathbb{R}))$.

Remark 2.18. A map between two manifolds is smooth in the usual sense if and only if it is a smooth map between the corresponding differential spaces (when both manifolds are given the standard differential structures).

Remark 2.19. It is easy to see that the composite of two smooth maps between differential spaces is again smooth. It is even easier to see that the identity map on a differential space is smooth. Consequently, differential spaces form a category.

Definition 2.20. A smooth map between two differential spaces is a *diffeomorphism* if it is invertible and the inverse is smooth.

Equivalently a smooth map is a diffeomorphism iff it is an isomorphism in the category of differential spaces.

Remark 2.21. Every smooth map of differential spaces is continuous; this follows from Definition 2.7 (1).

Remark 2.22. Any differential structure $\mathscr{F}$ is an $\mathbb{R}$-subalgebra of $C^0(M)$: for any $f_1, f_2 \in \mathscr{F}$, $\lambda, \mu \in \mathbb{R}$

$$\lambda f_1 + \mu f_2 = g \circ (f_1, f_2), \quad \text{where} \quad g(x,y) := \lambda x + \mu y \in C^\infty\big(\mathbb{R}^2\big),$$
$$f_1 f_2 = h \circ (f_1, f_2), \quad \text{where} \quad h(x,y) := xy \in C^\infty\big(\mathbb{R}^2\big).$$

Remark 2.23. Any C^∞-ring is an $\mathbb{R}$-algebra (more precisely: has an underlying $\mathbb{R}$-algebra structure). The binary operations $+$ and $\cdot$ come from the functions $h(x,y) = x + y$ and $g(x,y) = xy$ respectively. The scalars come from the 0-ary operations.

We will not notationally distinguish between a C^∞-ring and the corresponding (underlying) $\mathbb{R}$-algebra.

Definition 2.24. A differential structure $\mathscr{F}$ on a set M is *generated* by a subset $A \subseteq \mathscr{F}$ if $\mathscr{F}$ is the smallest differential structure containing the set A. That is, if $\mathscr{G}$ is a differential structure on M containing A, then $\mathscr{F} \subseteq \mathscr{G}$.

Lemma 2.25. *Given a collection A of real-valued functions on a set M there is a differential structure $\mathscr{F}$ on M generated by A. The initial topology for $\mathscr{F}$ is the initial topology for the set A.*

Proof. See [18, Theorem 2.1.7]. ■

Notation 2.26. *We write $\mathscr{F} = \langle A\rangle$ if the differential structure $\mathscr{F}$ is generated by the set A.*

Definition 2.27. Let $(M, \mathscr{F})$ be a differential space and $N \subseteq M$ a subset. The *subspace differential structure* $\mathscr{F}_N$ on N, also known as the *induced differential structure*, is the differential structure on N generated by the set A of restrictions to N of the functions in $\mathscr{F}$:

$$A = \{g\colon\ N \to \mathbb{R} \mid g = f|_N \text{ for some } f \in \mathscr{F}\}.$$

Definition 2.28. A smooth map $f\colon (M, \mathscr{F}_M) \to (N, \mathscr{F}_N)$ between two differential spaces is an *embedding* if f is injective and the induced map $f\colon (M, \mathscr{F}_M) \to (f(M), \langle \mathscr{F}_N|_{f(M)}\rangle)$ from M to its image (with the subspace differential structure) is a diffeomorphism.

Lemma 2.29. *Let $(M, \mathscr{F})$ be a differential space and $(N, \mathscr{F}_N)$ a subset of M with the induced/subspace differential structure. Then the smallest topology on N making all the functions of $\mathscr{F}_N$ continuous agrees with the subspace topology on N coming from the inclusion $i\colon N \hookrightarrow M$.*

Proof. The initial topology for the set $\mathscr{F}|_N$ of generators of $\mathscr{F}_N$ is the subspace topology. Consequently, the initial topology for $\mathscr{F}_N = \langle \mathscr{F}|_N\rangle$ is also the subspace topology (cf. Lemma 2.25). ■

Remark 2.30. The subspace differential structure $\mathscr{F}_N$ can be given a fairly explicit description:

$$\begin{aligned}\mathscr{F}_N = \big\{f\colon N \to \mathbb{R} \mid\ &\text{there is a collection of sets } \{U_i\}_{i\in I}, \text{ open in } M, \text{ with } \textstyle\bigcup_i U_i \supset N \\ &\text{and a collection } \{g_i\}_{i\in I} \subseteq \mathscr{F} \text{ such that } f|_{N\cap U_i} = g_i|_{N\cap U_i} \text{ for all indices } i\big\}.\end{aligned}$$

Remark 2.31. Let $(M, \mathscr{F})$ be a differential space and $(N, \mathscr{F}_N)$ a subset of M with the induced/subspace differential structure. Then the inclusion map $i\colon N \hookrightarrow M$ is smooth since for any $f \in \mathscr{F}$, $f \circ i = f|_N \in \mathscr{F}_N$ by definition of $\mathscr{F}_N$.

The subspace differential structure $\mathscr{F}_N$ is the *smallest* differential structure on N making the inclusion $i\colon N \to M$ smooth. This is because any differential structure $\mathscr{G}$ on N making $i\colon (N, \mathscr{G}) \to (M, \mathscr{F})$ smooth must contain the set $\mathscr{F}|_N$.

Lemma 2.32. *Let $(M, \mathscr{F})$ be a differential space and $(N, \mathscr{F}_N)$ a subset of M with the induced/subspace differential structure. For any differential space $(Y, \mathscr{G})$ and for any smooth map $\varphi\colon Y \to M$ that factors through the inclusion $i\colon N \to M$ (i.e., $\varphi(Y) \subset N$), the map $\varphi\colon (Y, \mathscr{G}) \to (N, \mathscr{F}_N)$ is smooth.*

Proof. We need to show that $\varphi^* h \equiv h \circ \varphi \in \mathscr{G}$ for any $h \in \mathscr{F}_N$. For any $f \in \mathscr{F}$,

$$\mathscr{G} \ni f \circ \varphi = f \circ i \circ \varphi = \varphi^*(f|_N).$$

Consequently, $\varphi^*(\mathscr{F}|_N) \subseteq \mathscr{G}$. Since $\mathscr{F}|_N$ generates $\mathscr{F}_N$ and since $\mathscr{G}$ is a differential structure, we must have $\varphi^*\mathscr{F}_N \subseteq \mathscr{G}$ as well. ■

Corollary 2.33. *Let $(M, \mathscr{F})$ be a differential space and $K \subseteq N \subseteq M$ subsets. Then*

$$\langle \mathscr{F}|_K\rangle = \langle \langle \mathscr{F}|_N\rangle|_K\rangle,$$

that is, the differential structure on K induced by the inclusion $K \hookrightarrow M$ agrees with the differential structure on K successively induced by the pair of the inclusions $K \hookrightarrow N \hookrightarrow M$.

Proof. Since for any $f \in \mathscr{F}$, $f|_K = (f|_N)|_K \in \langle \mathscr{F}|_N\rangle|_K$, we have $\mathscr{F}|_K \subseteq \langle\langle \mathscr{F}|_N\rangle|_K\rangle$. Therefore, $\langle \mathscr{F}|_K\rangle \subseteq \langle\langle \mathscr{F}|_N\rangle|_K\rangle$.

On the other hand, the inclusion $K \hookrightarrow M$ factors through the inclusion $K \hookrightarrow N$. Hence by Lemma 2.32, the map $j\colon (K, \langle \mathscr{F}|_K\rangle) \hookrightarrow (N, \langle \mathscr{F}|_N\rangle)$ is smooth. But the image of j lands in K. Hence the identity map $\mathrm{id}\colon (K, \langle \mathscr{F}|_K\rangle) \to (K, \langle\langle \mathscr{F}|_N\rangle|_K\rangle)$ is smooth. Consequently, $\langle\langle \mathscr{F}|_N\rangle|_K\rangle = \mathrm{id}^*\langle\langle \mathscr{F}|_N\rangle|_K\rangle \subseteq \langle \mathscr{F}|_K\rangle$. ■

In the case where the differential space $(M,\mathscr{F})$ is a manifold and N is a subset of M the subspace differential structure $\mathscr{F}_N$ has a simple description.

Lemma 2.34. *Let M be a manifold and N a subset of M, Then $f\colon N \to \mathbb{R}$ is in $C^\infty(M)_N := \langle C^\infty(M)|_N\rangle$ (the subspace differential structure on N) if and only if there is an open neighbourhood U of N in M and a smooth function $g\colon U \to \mathbb{R}$ such that $f = g|_N$. Moreover, if N is closed in M, we may take $U = M$.*

Remark 2.35. Let M be a manifold and $U \subset M$ an open subset. Then $C^\infty(U) = \langle C^\infty(M)|_U\rangle$. This follows from the existence of bump functions.

Proof of Lemma 2.34. Let $U \subset M$ be an open set with $N \subset U$, and let $g \in C^\infty(U)$.

By Remark 2.35 and Corollary 2.33, for any open set $U \subset M$ with $N \subset U$ and any $g \in C^\infty(U)$, the restriction $g|_N$ is in $C^\infty(M)_N$.

Conversely, suppose $f \in C^\infty(M)_N$. Then there is an collection of open sets $\{U_i\}_{i\in I}$ with $N \subset \bigcup_i U_i$ and $\{g_i\}_{i\in I} \subset C^\infty(M)$ so that $f|_{U_i\cap N} = g_i|_{U_i\cap N}$ for all i. Let $U = \bigcup_i U_i$. There is a partition of unity $\{\rho_i\}_{i\in I}$ on U subordinate to the cover $\{U\}_{i\in I}$. Consider $g := \sum \rho_i g_i \in C^\infty(U)$. Then $g|_N = f$.

If N is closed, then $\{U_i\}_{i\in I} \cup \{M \setminus N\}$ is an open cover of M. Choose a partition of unity $\{\rho_i\}_{i\in I} \cup \{\rho_0\}$ subordinate to this cover of M (with $\operatorname{supp}\rho_0 \subset M \setminus N$), and again set $g := \sum_{i\in I} \rho_i g_i$. Then g is a smooth function on all of M, and $g|_N = f$. ∎

Remark 2.36. Lemma 2.34 holds in greater generality. The proof does not really used the fact that M is a manifold, it only needs the existence of the partition of unity. These do exist for second countable Hausdorff locally compact differential spaces; see [18].

Definition 2.37. A differential space $(M,\mathscr{F})$ is *subcartesian* iff it is locally isomorphic to a subset of a Euclidean (a.k.a. Cartesian) space: for every point $p \in M$, there is an open neighborhood U of p in M and an embedding $\varphi\colon (U,\mathscr{F}_U) \to (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ (n depends on the point p).

Products

The domain of a flow of a vector field on a manifold M is a subset of the product $M \times \mathbb{R}$. Therefore, in order to define and understand flows of derivations on differential spaces we need to understand finite products in the category of differential spaces.

Given two differential spaces $(M_1,\mathscr{F}_1)$ and $(M_2,\mathscr{F}_2)$ there are many differential structures on their product $M_1 \times M_2$ so that the projections $\pi_i\colon M_1 \times M_2 \to M_i$, $i = 1,2$ are smooth. The smallest such structure is the one generated by the set $\pi_1^*\mathscr{F}_1 \cup \pi_2^*\mathscr{F}_2$. We denote this structure by $\mathscr{F}_{\mathrm{prod}}$. That is,

$$\mathscr{F}_{prod} := \langle \pi_1^*\mathscr{F}_1 \cup \pi_2^*\mathscr{F}_2\rangle.$$

Since the initial topology for $\pi_1^*\mathscr{F}_1 \cup \pi_2^*\mathscr{F}_2$ is the product topology, the initial topology for $\mathscr{F}_{\mathrm{prod}}$ is also the product topology (cf. Remark 2.25).

We next check that $(M_1 \times M_2, \mathscr{F}_{\mathrm{prod}})$ together with the projections π_1, π_2 has the universal properties of the product in the category of differential spaces. Note that the projections π_1, π_2 are smooth.

Lemma 2.38. *Let $(M_1,\mathscr{F}_1)$, $(M_2,\mathscr{F}_2)$ be two differential spaces, $(Y,\mathscr{G})$ another differential space, $\varphi_i\colon Y \to M_i$, $i = 1,2$ two smooth maps. Then there exists a unique smooth map $\varphi\colon (Y,\mathscr{G}) \to (M_1 \times M_2, \mathscr{F}_{\mathrm{prod}})$ with $\pi_i \circ \varphi = \varphi_i$, $i = 1,2$.*

Proof. Clearly there is a unique map of *sets* $\varphi\colon Y \to M_1 \times M_2$ with $\pi_i \circ \varphi = \varphi_i$, $i = 1, 2$. Moreover since φ_i ($i = 1, 2$) are smooth

$$\mathscr{G} \supseteq \varphi_i^* \mathscr{F}_i = \varphi^*(\pi^* \mathscr{F}_i).$$

Therefore, $\varphi^*(\pi_1^* \mathscr{F}_1 \cup \pi_2^* \mathscr{F}_2) \subseteq \mathscr{G}$. Since $\mathscr{F}_{\text{prod}} = \langle \pi_1^* \mathscr{F}_1 \cup \pi_2^* \mathscr{F}_2 \rangle$, $\varphi^* \mathscr{F}_{\text{prod}} \subseteq \mathscr{G}$ as well. Thus φ is smooth. ∎

Remark 2.39. Let M_1, M_2 be two manifolds with the usual differential structures (i.e., $C^\infty(M_1)$ and $C^\infty(M_2)$). Then $C^\infty(M_1 \times M_2)$ is the product differential structure on $M_1 \times M_2$.

We end the section by proving that taking products commutes with taking subspaces.

Lemma 2.40. *Let $(M_1, \mathscr{F}_1)$, $(M_2, \mathscr{F}_2)$ be two differential spaces, $N_1 \subseteq M_1$, $N_2 \subseteq M_2$ two subspaces, $\mathscr{G}_1$, $\mathscr{G}_2$ the subspace differential structures on N_1, N_2 respectively. Then the product differential structure $\mathscr{G}_{\text{prod}}$ on $N_1 \times N_2$ is the subspace differential structure $(\mathscr{F}_{\text{prod}})_{N_1 \times N_2}$.*

Proof. It is enough to check that $(N_1 \times N_2, (\mathscr{F}_{\text{prod}})_{N_1 \times N_2})$ together with the projections

$$\mathsf{pr}_i\colon\ (N_1 \times N_2, (\mathscr{F}_{\text{prod}})_{N_1 \times N_2}) \to (N_i, \mathscr{G}_i), \qquad i = 1, 2,$$

has the universal properties of the product (in the category of differential spaces).

We first argue that the projections pr_1, pr_2 are smooth. The projections

$$\pi_i\colon\ (M_1 \times M_2, \mathscr{F}_{\text{prod}}) \to (M_i, \mathscr{F}_i)$$

are smooth by definition of the product differential structure $\mathscr{F}_{\text{prod}}$. Hence their restrictions $\pi_i|_{N_1 \times N_2}\colon N_1 \times N_2 \to M_i$ are smooth as well. Since $\pi_i(N_1 \times N_2) \subseteq N_i$, the maps

$$\mathsf{pr}_i = \pi_i|_{N_1 \times N_2}\colon\ (N_1 \times N_2, (\mathscr{F}_{\text{prod}})_{N_1 \times N_2}) \to (N_i, \mathscr{G}_i)$$

are also smooth (by Lemma 2.32).

Now let $(Y, \mathscr{A})$ be a differential space and $\varphi_i\colon Y \to N_i$, $i = 1, 2$ be a pair of smooth maps. Since the inclusions $\jmath_i\colon N_i \to M_i$ are smooth, the composites $\jmath_i \circ \varphi_i\colon Y \to M_i$, $i = 1, 2$, are smooth. By the universal property of the product, there is a unique smooth map $\varphi\colon (Y, \mathscr{A}) \to (M_1 \times M_2, \mathscr{F}_{\text{prod}})$ so that $\pi_i \circ \varphi = \jmath_i \circ \varphi_i$. Consequently, $(\pi_i \circ \varphi)(Y) \subseteq N_i$. Hence $\varphi(Y) \subseteq N_1 \times N_2$. Since $N_1 \times N_2$ is a subspace of $(M_1 \times M_2, \mathscr{F}_{\text{prod}})$ the map $\varphi\colon (Y, \mathscr{A}) \to (N_1 \times N_2, (\mathscr{F}_{\text{prod}})_{N_1 \times N_2})$ is smooth (see Lemma 2.32). Therefore, $(N_1 \times N_2, (\mathscr{F}_{\text{prod}})_{N_1 \times N_2})$ together with the projections pr_1, pr_2 is the product of $(N_1, \mathscr{G}_1)$ and $(N_2, \mathscr{G}_2)$. We conclude that $(\mathscr{F}_{\text{prod}})_{N_1 \times N_2} = \mathscr{G}_{\text{prod}}$. ∎

3 Derivations and their flows

A vector field v on a manifold M can be defined as a derivation $v\colon C^\infty(M) \to C^\infty(M)$ of the $\mathbb{R}$-algebra of smooth functions on M: v is $\mathbb{R}$-linear and for any two functions $f, g \in C^\infty(M)$ the product rule holds:

$$v(fg) = v(f)g + fv(g).$$

One then proves that the chain rule also holds: for any $n \geq 1$, any $g \in C^\infty(\mathbb{R}^n)$ and any $f_1, \dots, f_n \in C^\infty(M)$

$$v(g \circ (f_1, \dots, f_n)) = \sum_{i=1}^{n} ((\partial_i g) \circ (f_1, \dots, f_n)) \cdot v(f_i). \tag{3.1}$$

Note that (3.1), appropriately interpreted, makes sense for any C^∞-ring, since any C^∞-ring is an $\mathbb{R}$-algebra (cf. Remark 2.23) and therefore carries the addition and multiplication operations. Namely, we have the following definition.

Definition 3.1. Let $\mathscr{A}$ be a C^∞-ring. A *C^∞-ring derivation* of $\mathscr{A}$ is a function $v\colon \mathscr{A} \to \mathscr{A}$ so that for any $n \geq 1$, any $g \in C^\infty(\mathbb{R}^n)$ and any $f_1, \ldots, f_n \in \mathscr{A}$

$$v(g_{\mathscr{A}}(f_1, \ldots, f_n)) = \sum_{i=1}^{n} ((\partial_i g)_{\mathscr{A}}(f_1, \ldots, f_n)) \cdot v(f_i).$$

It turns out, thanks to a theorem of Yamashita [19, Theorem 3.1], that any $\mathbb{R}$-algebra derivation of a jet-determined C^∞-ring is automatically a C^∞-ring derivation. We will not explain what "jet-determined" means, since this will take us too far afield. Suffices to say that all C^∞-rings arising as differential structures are jet-determined C^∞-rings. In fact, there is a class of C^∞-rings that includes differential structures (namely for point-determined C^∞-rings, see Definition A.2) for which Yamashita's theorem has a short proof. We present the proof in Appendix A. *From now on when talking about derivations of differential structures we will not distinguish between $\mathbb{R}$-algebra derivations and C^∞-ring derivations since they are one and the same.*

Remark 3.2. Given a differential space $(M, \mathscr{F})$ we view a derivation $v\colon \mathscr{F} \to \mathscr{F}$ as the correct analogue of a vector field on $(M, \mathscr{F})$. Thus "vector fields" in the title of the paper are derivations of differential structures. See also Remark 3.12.

We now define integral curves of a derivation.

Definition 3.3. An *interval* is a connected subset of the real line $\mathbb{R}$.

Remark 3.4.

- By Definition 3.3 a single point is an interval.
- The induced differential structure on an interval $I \subset \mathbb{R}$ is the set of smooth functions $C^\infty(I)$ on I (note that $C^\infty(I)$ makes sense in all cases: I is open, closed, half-closed or a single point).
- Unless the interval I is a singleton, there is a canonical derivation $\frac{\mathrm{d}}{\mathrm{d}x}\colon C^\infty(I) \to C^\infty(I)$ since we can differentiate smooth functions on an interval.

Definition 3.5. Let $v\colon \mathscr{F} \to \mathscr{F}$ be a derivation on a differential space $(M, \mathscr{F})$. An *integral curve γ of v* is either a map $\gamma\colon \{*\} \to M$ from a 1-point interval or a smooth map $\gamma\colon (J, C^\infty(J)) \to (M, \mathscr{F})$ from an interval $J \subset \mathbb{R}$ so that

$$\frac{\mathrm{d}}{\mathrm{d}x}(f \circ \gamma) = v(f) \circ \gamma$$

for all functions $f \in \mathscr{F}$.

The curve γ *starts at a point* $p \in M$ if $0 \in J$ and $\gamma(0) = p$.

Remark 3.6. We tacitly assume that all integral curves contain zero in their domain of definition. Thus any integral curve γ of a derivation starts at $\gamma(0)$.

Definition 3.7. An integral curve $\gamma\colon I \to M$ of a derivation v on a differential space $(M, \mathscr{F})$ is *maximal* if for any other integral curve $\tau\colon K \to M$ of v with $\tau(0) = \gamma(0)$ we have $K \subseteq I$ and $\gamma|_K = \tau$.

Remark 3.8. Note that maximal integral curves are necessarily unique.

The following examples are meant to illustrate two points:

(1) curves that only exist for time 0 should be allowed as integral curves of derivations and

(2) we should not require that the domain of an integral curve is an open interval.

Example 3.9. Consider the derivation $v = \frac{\mathrm{d}}{\mathrm{d}x}$ on the interval $[0,1]$. Then $\gamma\colon [-1/2, 1/2] \to [0,1]$, $\gamma(t) = t + 1/2$ is an integral curve of v. The curve γ is a maximal integral curve of v and its domain is a closed interval.

Example 3.10. Let M be the standard closed disk in $\mathbb{R}^2$: $M = \{(x,y) \mid x^2+y^2 \leq 1\}$. Then M is a manifold with boundary and a differential subspace of $\mathbb{R}^2$ (the two spaces of smooth functions are the same!). Consider the vector field $v = \frac{\partial}{\partial x}$ on M. The curve $\gamma\colon \{0\} \to M$, $\gamma(0) = (0,1)$ is an integral curve of v; it only exists for zero time. The derivation v does have a flow in the sense of Definition 3.13 below. The flow is

$$\Phi\colon\ U \equiv \{((x,y),t) \in \mathbb{R}^2 \times \mathbb{R} \mid x^2+y^2 \leq 1, (x+t)^2 + y^2 \leq 1\} \to M,$$
$$\Phi((x,y),t) = (x+t, y).$$

Note that while M is a manifold with boundary, the flow domain U is not a manifold with boundary nor a manifold with corners. The domain U is a differential space, and the flow Φ is smooth since it is the restriction to U of the smooth map $\Psi\colon \mathbb{R}^3 \to \mathbb{R}^2$, $\Psi((x,y),t) = (x+t,y)$.

Remark 3.11. Śniatycki defines a vector field on a differential space $(M, \mathscr{F})$ to be a derivation of $\mathscr{F}$ that integrates to local diffeomorphisms of M; see [18, Definition 3.2.2]. In particular, the domains of its maximal integral curves include neighbourhoods of 0. By this definition, a vector field on a manifold with boundary must be tangent to the boundary.

Remark 3.12. We were tempted to call all derivations on differential spaces "vector fields". In the end we decided against it to avoid the clash with Śniatycki's terminology.

Definition 3.13. Let $v\colon \mathscr{F} \to \mathscr{F}$ be a derivation on a differential space $(M, \mathscr{F})$. A *flow* of v is a smooth map $\Phi\colon W \to M$ from a subspace W of $M \times \mathbb{R}$ with $M \times \{0\} \subset W$ such that for all $x \in M$

(1) $\Phi(x,0) = x$;

(2) the set $I_x := \{t \in \mathbb{R} \mid (x,t) \in W\}$ is connected;

(3) the map $\Phi(x,\cdot)\colon I_x \to M$ is a maximal integral curve for v (see Definition 3.7).

We are now in position to state the main result of the paper.

Theorem 3.14. *Let $(M,\mathscr{F})$ be a differential space which is diffeomorphic to a subset of some $\mathbb{R}^n$, and $v\colon \mathscr{F} \to \mathscr{F}$ a derivation. Then v has a unique flow (see Definition 3.13).*

Remark 3.15. The conditions of Theorem 3.14 are not as restrictive as they may seem at the first glance since there a version of Whitney embedding theorem for subcartesian spaces [1, 6, 11].[2]

[2]To precisely state the embedding theorem of Breuer, Marshall, Kowalczyk and Motreanu we need to recall the definition of the structural dimension of a subcartesian space. It proceeds as follows: given a subcartesian space M its *structural dimension at a point* $x \in M$ is the smallest integer n_x so that a neighborhood of x can be embedded in $\mathbb{R}^{n_x}$. The *structural dimension* of a subcartesian space M is the supremum of the set of structural dimensions of points of M. The embedding theorem (see [1, Theorem 2.2]) then says:

Theorem 3.16. *Any second countable subcartesian space M of finite structural dimension can be embedded in some Euclidean space.*

Remark 3.17. If M is a subset of $\mathbb{R}^n$ (with the subset differential structure), then M is subcartesian and second countable, and its structural dimension is bounded above by n. So the conditions of Theorem 3.16 are necessary.

A subset M of $\mathbb{R}^n$ need not be locally compact. Note that the conditions of Theorem 3.16 do not require local compactness.

Remark 3.18. The disjoint union $\bigsqcup_{n\geq 0} \mathbb{R}^n$ is an example of a subcartesian space that is not embeddable in $\mathbb{R}^N$ for any N: its structural dimension is infinite.

On the other hand if the differential space is not subcartesian, then a derivation may have infinitely many integral curves starting at a given point: see [7, Example 2, Section 32.12]. Such a derivations does not have a flow. We are grateful to Wilmer Smilde for bringing this example to our attention.

We start the proof of Theorem 3.14 by proving existence and uniqueness of maximal integral curves. This, in turn, needs a lemma.

Lemma 3.19. *Let $\mathscr{A}$ be a C^∞-ring, $w\colon \mathscr{A} \to \mathscr{A}$ a derivation and $a, b \in \mathscr{A}$ two elements with $ab = 1$ (i.e., a is invertible and b is the inverse of a). Then*

$$w(b) = -b^2 w(a).$$

Proof. Since w is a derivation, $w(1) = 0$. Hence $0 = w(ab) = w(a)b + aw(b)$ and the result follows. ∎

Lemma 3.20. *Let $M \subset \mathbb{R}^n$ be a subset with the induced differential structure $\mathscr{F}$, $W \subset \mathbb{R}^n$ an open neighborhood of M and $v\colon \mathscr{F} \to \mathscr{F}$ a derivation. Then for any function $h \in C^\infty(W)$*

$$v(h|_M) = \sum_{i=1}^{n} (\partial_i h)|_M \cdot v(x_i|_M), \tag{3.2}$$

where $x_1, \ldots, x_n\colon \mathbb{R}^n \to \mathbb{R}$ are the standard coordinate functions.

Proof. If $h = k|_W$ for some function $k \in C^\infty(\mathbb{R}^n)$, then

$$h|_M = k|_M = (k \circ (x_1, \ldots, x_n))|_M = k_{\mathscr{F}}(x_1|_M, \ldots x_n|_M).$$

Hence, since v is a C^∞-ring derivation,

$$v(h|_M) = v(k_{\mathscr{F}}(x_1|_M, \ldots, x_n|_M)) = \sum_{i=1}^{n} (\partial_i k)|_M \cdot v(x_i|_M) = \sum_{i=1}^{n} (\partial_i h)|_M \cdot v(x_i|_M)$$

and (3.2) holds for such a function h.

Otherwise by the localization theorem [12][3] there exist functions $k, \ell \in C^\infty(\mathbb{R}^n)$ with $\ell|_W$ invertible in $C^\infty(W)$ so that $h = \frac{k|_W}{\ell|_W}$. Then $h|_M = k|_M(\ell|_M)^{-1}$ and therefore,

$$v(h|_M) = v\big(k|_M(\ell|_M)^{-1}\big) = v(k|_M) \cdot (\ell|_M)^{-1} - k|_M(\ell|_M)^{-2} v(\ell|_M)$$

(by Lemma 3.19)

$$= \sum_i \big((\partial_i k)|_M(\ell|_M)^{-1} - k|_M(\ell|_M)^{-2}(\partial_i \ell)|_M\big) \cdot v(x_i|_M)$$

(since (3.2) holds for $k|M$ and $\ell|_M$)

$$= \sum_i \partial_i \left(\frac{k|_W}{\ell|_W}\right)\bigg|_M \cdot v(x_i|_M) = \sum_{i=1}^{n} (\partial_i h)|_M \cdot v(x_i|_M).$$ ∎

Lemma 3.21. *Let $M \subset \mathbb{R}^n$ be a subset, $\mathscr{F}$ the induced differential structure on M and $v\colon \mathscr{F} \to \mathscr{F}$ a derivation. For any point $p \in M$, there exists a unique maximal integral curve $\gamma\colon I \to M$ of v with $\gamma(0) = p$.*

[3] For an exposition of this proof in English, see [13].

Proof. By definition of the induced differential structure $\mathscr{F}$ on M, the restrictions $x_i|_M$, $1 \le i \le n$, are in $\mathscr{F}$. Here as before $x_1, \dots, x_n \colon \mathbb{R}^n \to \mathbb{R}$ are the standard coordinate functions. Then the functions $v(x_i|_M)$ are also in $\mathscr{F}$, so there are open neighborhoods U_i of M in $\mathbb{R}^n$ and $b_i \in C^\infty(U_i)$ with $b_i|_M = v(x_i|_M)$ (cf. Lemma 2.34). Let $U = \bigcap_{1\le i\le n} U_i$, $V := \sum_i^n b_i \frac{\partial}{\partial x_i}$. Then V is a vector field on U. Let $\tilde{\gamma} \colon J \to \mathbb{R}^n$ be the unique maximal integral curve of the vector field V with $\gamma(0) = p$. Let I denote the connected component of the set $(\tilde{\gamma})^{-1}(M)$ that contains 0. We now argue that $\gamma := \tilde{\gamma}|_I$ is the desired maximal integral curve of the derivation v. Note that since the image of γ lands in M, the map γ is smooth as a map from $(I, C^\infty(I))$ into the differential subspace $(M, \mathscr{F})$ (see Lemma 2.32).

If I is the singleton $\{0\}$, there is nothing to prove. So suppose $I \neq \{0\}$. Given $f \in \mathscr{F}$ there is an open neighborhood W of M in $\mathbb{R}^n$ and a smooth function $h \in C^\infty(W)$ with $f = h|_M$ (see Lemma 2.34). By replacing W with $W \cap U$ if necessary, we may assume that $W \subset U$. Note that for any $t \in I$, $\gamma(t) = \tilde{\gamma}(t)$. We now compute

$$\frac{\mathrm{d}}{\mathrm{d}t}(f\circ\gamma)(t) = \frac{\mathrm{d}}{\mathrm{d}t}(h\circ\tilde{\gamma})(t) = V(h)(\gamma(t))$$

(since $\tilde{\gamma}$ is an integral curve of V)

$$= \sum_i (\partial_i h)(\tilde{\gamma}(t)) \cdot b_i(\tilde{\gamma}(t))$$

(by definition of V)

$$= \sum_i (\partial_i h)(\gamma(t)) \cdot v(x_i|_M)(\gamma(t))$$

(by definition of b_i's)

$$= v(h|_M)(\gamma(t))$$

(by (3.2))

$$= v(f)(\gamma(t)).$$

Since $f \in \mathscr{F}$ is arbitrary, the curve γ is an integral curve of the derivation v.

We now argue that γ is a *maximal* integral curve of v. Let $\sigma \colon K \to M$ be another integral curve of v with $\sigma(0) = p$. We first check that σ is an integral curve of the vector field V on W. Note that since the inclusion $M \hookrightarrow W$ is smooth, $\sigma \colon K \to W$ is smooth. Consider $h \in C^\infty(W)$. Then for any $t \in K$

$$\frac{\mathrm{d}}{\mathrm{d}t}(h\circ\sigma)(t) = \frac{\mathrm{d}}{\mathrm{d}t}((h|_M)\circ\sigma)(t) = v(h|_M)(\sigma(t))$$

(since σ is an integral curve of v)

$$= \sum_i (\partial_i h)|_M(\sigma(t)) \cdot v(x_i|_M)(\sigma(t))$$

(by (3.2))

$$= \sum_i (\partial_i h)(\sigma(t)) \cdot b_i(\sigma(t))$$

(by definition of b_i's)

$$= V(h)(\sigma(t)).$$

Hence σ is an integral curve of the vector field V as claimed.

Since $\tilde{\gamma}$ is the maximal integral curve of V, $\sigma = \tilde{\gamma}|_K$ and $K \subset \gamma^{-1}(M)$. Since $0 \in K$, K is connected and since I is the connected component of 0 in $\gamma^{-1}(M)$, $K \subset I$. It follows that $\sigma = (\tilde{\gamma}|_I)|_K = \gamma|_K$ and therefore $\gamma = \tilde{\gamma}|_I$ is the maximal integral curve of the derivation v. ■

We record two corollaries.

Corollary 3.22. *Let $(M, \mathscr{F})$ be a second countable subcartesian space of bounded dimension (so that the assumptions of the Whitney embedding theorem for subcartesian spaces apply, see Remark 3.15 and the footnote). Then for any derivation $v\colon \mathscr{F} \to \mathscr{F}$ and for any point $p \in M$ there is a unique maximal integral curve γ_p of v with $\gamma_p(0) = p$.*

The second corollary is really the corollary of the *proof* of Lemma 3.21.

Corollary 3.23. *Let $M \subset \mathbb{R}^n$ be a subset, $\mathscr{F}$ induced differential structure on M and $v\colon \mathscr{F} \to \mathscr{F}$ a derivation. There exists an open neighborhood U of M in $\mathbb{R}^n$ and a vector field V on U so that for any $p \in M$ the maximal integral curve $\gamma_p\colon I_p \to M$ of v with $\gamma_p(0) = p$ is of the form $\tilde{\gamma}_p|_{I_p}$ for the maximal integral curve $\tilde{\gamma}_p\colon J_p \to U$ of V with $\tilde{\gamma}(0) = p$.*

Proof of Theorem 3.14. It is no loss of generality to assume that $M \subset \mathbb{R}^n$ and that the differential structure $\mathscr{F}$ on M is the subspace differential structure: $\mathscr{F} = \langle C^\infty(\mathbb{R}^n)|_M\rangle$. By Corollary 3.23, there is an open neighborhood U of M in $\mathbb{R}^n$ and a vector field V on U so that for any $p \in M$ the maximal integral curve $\gamma_p\colon I_p \to M$ of v with $\gamma_p(0) = p$ is of the form $\tilde{\gamma}_p|_{I_p}$ for the maximal integral curve $\tilde{\gamma}_p\colon J_p \to U$ of V. Let

$$W = \bigcup_{p \in M} \{p\} \times I_p \subset M \times \mathbb{R} \subset U \times \mathbb{R}.$$

Note that by definition $M \times \{0\} \subset W$. Define the map $\Phi\colon W \to M$ by $\Phi(p, t) = \gamma_p(t)$ for all $(p, t) \in W$. Then Φ is a flow of v modulo the issue of smoothness which we now address.

The vector field V on U has the flow

$$\Psi\colon \widetilde{W} \to U, \qquad \Psi(x, t) := \tilde{\gamma}_x(t),$$

where, as above, $\tilde{\gamma}_x\colon J_x \to U$ is the maximal integral curve of V with $\tilde{\gamma}(0) = x$ and

$$\widetilde{W} := \bigcup_{x \in U} \{x\} \times J_x.$$

For any point $p \in M$,

$$\Psi|_{\{p\}\times I_x} = \Phi|_{\{p\}\times I_x}$$

(since $\tilde{\gamma}_p|_{I_p} = \gamma_p$). Therefore, $\Phi = \Psi|_W$, hence smooth with respect to the differential structure $\langle C^\infty(U \times \mathbb{R})|_W\rangle$ on W induced by the inclusion $W \hookrightarrow U \times \mathbb{R}$.

It remains to show that $\langle C^\infty(U \times \mathbb{R})|_W\rangle = \langle \mathscr{F}_{\mathrm{prod}}|_W\rangle$, where $\mathscr{F}_{\mathrm{prod}}$ is the product differential structure on $M \times \mathbb{R}$. Note that $\mathscr{F}_{\mathrm{prod}}$ depends only on the differential structures on M and $\mathbb{R}$.

By Lemma 2.40, $\mathscr{F}_{\mathrm{prod}} = \langle C^\infty(U \times \mathbb{R})|_{M\times\mathbb{R}}\rangle$. By Corollary 2.33,

$$\langle\langle C^\infty(U \times \mathbb{R})|_{M\times\mathbb{R}}\rangle|_W\rangle = \langle C^\infty(U \times \mathbb{R})|_W\rangle.$$

Therefore, $\langle \mathscr{F}_{\mathrm{prod}}|_W\rangle = \langle C^\infty(U \times \mathbb{R})|_W\rangle$. ■

Example 3.24. Let (M,ω) be a symplectic manifold with a Hamiltonian action of a compact Lie group G and let $\mu\colon M \to \mathfrak{g}^*$ denote the corresponding equivariant moment map. Assume that the action of G on M has only finitely many orbit types (this is the case, for example, when M is the cotangent bundle of a compact manifold or when M itself is compact). Recall that the symplectic quotient at $0 \in \mathfrak{g}^*$ is defined to be the subquotient

$$M_0 := \mu^{-1}(0)/G.$$

Let $\pi\colon \mu^{-1}(0) \to M_0$ denote the quotient map. The symplectic quotient M_0 can be given the structure of a differential space. Namely let $C^\infty(M)^G$ denote the space of G-invariant functions. It is easily seen to be a C^∞-subring of $C^\infty(M)$. We define

$$\mathscr{F} := \big\{f\colon M_0 \to \mathbb{R} \mid f\circ\pi = \tilde{f}|_{\mu^{-1}(0)} \text{ for some } \tilde{f} \in C^\infty(M)^G\big\}.$$

This idea goes back to the work of Cushman [2]. It is not hard to check that $\mathscr{F}$ is a differential structure on M_0. For instance, this follows from the existence of the desired bump functions. See also [18].

By [16, Example 6.6], the differential space $(M_0,\mathscr{F})$ is embeddable. Consequently, any derivation of $\mathscr{F}$ has a unique smooth flow.

A $\mathbb{R}$-algebra and C^∞-ring derivations of differential structures

The goal of this appendix is to prove that for any point-determined C^∞-ring $\mathscr{C}$ (see Definition A.2) any $\mathbb{R}$-algebra derivation $v\colon \mathscr{C} \to \mathscr{C}$ is automatically a C^∞-ring derivation. We start by defining $\mathbb{R}$-points of C^∞-rings.

Definition A.1. An $\mathbb{R}$-*point* of a C^∞-ring $\mathscr{C}$ is a nonzero homomorphism $\varphi\colon \mathscr{C} \to \mathbb{R}$ of C^∞-rings.

Definition A.2. A C^∞-ring $\mathscr{C}$ is *point-determined* if $\mathbb{R}$-points separate elements of the ring. That is for any $a \in \mathscr{C}$, $a \neq 0$ there is an $\mathbb{R}$-point $\varphi\colon \mathscr{C} \to \mathbb{R}$ with $\varphi(a) \neq 0$.

Example A.3. Let $(M,\mathscr{F})$ be a differential space and $x \in M$ a point. Then the evaluation map

$$\mathrm{ev}_x\colon\ \mathscr{F} \to \mathbb{R}, \qquad \mathrm{ev}_x(f) := f(x)$$

is an $\mathbb{R}$-point of $\mathscr{F}$. The C^∞-ring $\mathscr{F}$ is point-determined since for any nonzero function $f \in \mathscr{F}$ there is a point $x \in M$ with $0 \neq f(x) = \mathrm{ev}_x(f)$.

We next recall Hadamard's lemma.

Lemma A.4 (Hadamard's lemma)**.** *For any smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, there exist (non-unique) smooth functions $g_1,\dots,g_n \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$f(x) - f(y) = \sum_{i=1}^n (x_i - y_i) g_i(x,y)$$

for any pair of points $x, y \in \mathbb{R}^n$.

Moreover, for n-tuple of functions $h_1,\dots,h_n \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with the property that

$$f(x) - f(y) = \sum_{i=1}^n (x_i - y_i) h_i(x,y)$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$h_i(b,b) = (\partial_i f)(b)$$

for all $b \in \mathbb{R}^n$.

Proof.

$$f(x)-f(y)=\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}f(tx+(1-t)y)\mathrm{d}t=\int_0^1\sum_{i=1}^n \partial_i f(tx+(1-t)y)(x_i-y_i)\mathrm{d}t$$
$$=\sum_{i=1}^n (x_i-y_i)\int_0^1 \partial_i f(tx+(1-t)y)\mathrm{d}t.$$

Define

$$g_i(x,y)=\int_0^1 \partial_i f(tx+(1-t)y)\mathrm{d}t.$$

This proves existence of the desired functions $g_1,\dots,g_n$. To prove the second part of the lemma, note that

$$(\partial_i f)(b)=\lim_{s\to 0}\frac{1}{s}(f(b+se_i)-f(b)),$$

where e_i is the i^{th} standard basis vector. Therefore if $f(x)-f(y)=\sum_{i=1}^n (x_i-y_i)h_i(x,y)$, then

$$(\partial_i f)(b)=\lim_{s\to 0}\frac{1}{s}\sum_{j=1}^n((b+se_i)_j-b_j)h_j(b+se_i,b)=\lim_{s\to 0}\frac{1}{s}sh_i(b+se_i,b)=h_i(b,b). \qquad \blacksquare$$

We are now in position to prove the main result of the appendix.

Theorem A.5. *Let $\mathscr{A}$ be a point-determined C^∞ ring and $v\colon \mathscr{A}\to\mathscr{A}$ an $\mathbb{R}$-algebra derivation. Then v is a C^∞-derivation.*

Proof. Recall that if $\mathscr{A}$ is a unital $\mathbb{R}$-algebra and $v\colon \mathscr{A}\to\mathscr{A}$ is an $\mathbb{R}$-algebra derivation, then $v(1_{\mathscr{A}})=0_{\mathscr{A}}$ since $v(1_{\mathscr{A}})=v\big(1_{\mathscr{A}}^2\big)=1_{\mathscr{A}}v(1_{\mathscr{A}})+v(1_{\mathscr{A}})1_{\mathscr{A}}=v(1_{\mathscr{A}})+v(1_{\mathscr{A}})$.

Let $h\in C^\infty\big(\mathbb{R}^k\big)$ be a smooth function and $a_1,\dots,a_k\in\mathscr{A}$. Let $x\colon \mathscr{A}\to\mathbb{R}$ be an $\mathbb{R}$-point. Then $b=(b_1,\dots,b_k):=(x(a_1),\dots,x(a_k))$ is a point in $\mathbb{R}^k$. By Hadamard's lemma (see Lemma A.4), there are smooth functions $g_1,\dots,g_k\in C^\infty\big(\mathbb{R}^{2k}\big)$ such that

$$h(y)=h(b)+\sum_{j=1}^k (y_j-b_j)g_j(y,b)$$

for all $y\in\mathbb{R}^k$, and $g_j(b,b)=\partial_j h(b)$. Let $\hat{g}_j(y):=g_j(y,b)$. Then, for any $(a_1,\dots,a_k)\in\mathscr{A}^k$,

$$h_{\mathscr{A}}(a_1,\dots,a_k)=h(b)_{\mathscr{A}}+\sum_{j=1}^k (a_j-(b_j)_{\mathscr{A}})\cdot(\hat{g}_j)_{\mathscr{A}}(a_1,\dots,a_k).$$

Applying the algebraic derivation v to both sides and using the fact that v applied to a scalar is zero, we get

$$v(h_{\mathscr{A}}(a_1,\dots,a_k))=\sum_{j=1}^k v(a_j)(\hat{g}_j)_{\mathscr{A}}(a_1,\dots,a_k)+\sum_{j=1}^k (a_j-(b_j)_{\mathscr{A}})v((\hat{g}_j)_{\mathscr{A}}(a_1,\dots,a_k).$$

Now we apply the $\mathbb{R}$-point x to both sides and use the fact that $x(a_j-(b_j)_{\mathscr{A}})=x(a_j)-b_j=0$ for all j. We get

$$x(v(h_{\mathscr{A}}(a_1,\dots,a_k))=\sum_{j=1}^k x(v(a_j))\cdot x\big((\hat{g}_j)_{\mathscr{A}}(a_1,\dots,a_k)\big).$$

Finally, note that for each j,

$$x\big((\hat{g}_j)_{\mathscr{A}}(a_1, \dots, a_k)\big) = (\hat{g}_j)_{C^\infty(R)}\big(x(a_1), \dots, x(a_k)\big)$$

(since x is a homomorphism of C^∞-rings)

$$\begin{aligned} &= g_j(b_1, \dots, b_k, b_1, \dots, b_k) \\ &= (\partial_j h)(b_1, \dots, b_k) = (\partial_j h)(x(a_1), \dots, x(a_k)) \\ &= x\big((\partial_j h)_{\mathscr{A}}(a_1, \dots, a_k)\big) \end{aligned}$$

(since x is a homomorphism of C^∞-rings). Therefore

$$\begin{aligned} x\big(v(h_{\mathscr{A}}(a_1, \dots, a_k))\big) &= \sum_{j=1}^{k} x\big(v(a_j)\big) x\big((\partial_j h)_{\mathscr{A}}(a_1, \dots, a_k)\big) \\ &= x\Big(\sum_{j=1}^{k} (\partial_j h)_{\mathscr{A}}(a_1, \dots, a_k) v(a_j)\Big). \end{aligned}$$

Since $\mathscr{A}$ is point determined and since the $\mathbb{R}$-point x is arbitrary,

$$v\big(h_{\mathscr{A}}(a_1, \dots, a_k)\big) = \sum_{j=1}^{k} (\partial_j h)_{\mathscr{A}}(a_1, \dots, a_k) v(a_j),$$

i.e., v is a C^∞-ring derivation. ■

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