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Moduli Space of Factorized Ramified Connections and Generalized Isomonodromic Deformation

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Abstract. We introduce the notion of factorized ramified structure on a generic ramified irregular singular connection on a smooth projective curve. By using the deformation theory of connections with factorized ramified structure, we construct a canonical 2-form on the moduli space of ramified connections. Since the factorized ramified structure provides a duality on the tangent space of the moduli space, the 2-form becomes nondegenerate. We prove that the 2-form on the moduli space of ramified connections is d-closed via constructing an unfolding of the moduli space. Based on the Stokes data, we introduce the notion of local generalized isomonodromic deformation for generic unramified irregular singular connections on a unit disk. Applying the Jimbo–Miwa–Ueno theory to generic unramified connections, the local generalized isomonodromic deformation is equivalent to the extendability of the family of connections to an integrable connection. We give the same statement for ramified connections. Based on this principle of Jimbo–Miwa–Ueno theory, we construct a global generalized isomonodromic deformation on the moduli space of generic ramified connections by constructing a horizontal lift of a universal family of connections. As a consequence of the global generalized isomonodromic deformation, we can lift the relative symplectic form on the moduli space to a total closed form, which is called a generalized isomonodromic 2-form.

Key words: moduli; ramified connection; isomonodromic deformation; symplectic structure

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Contents

1 Introduction

Let C be a complex smooth projective curve and D be an effective divisor on C. Consider an algebraic vector bundle E on C of rank r and a rational connection $\nabla\colon E \longrightarrow E\otimes\Omega_C(D)$ admitting poles along D. The connection ∇ is said to be logarithmic at $x\in D$ if it has at most a simple pole at x. The notion of logarithmic connection is well formulated in [24] by adding parabolic structure on the underlying vector bundle. In [24], C.T. Simpson established the Riemann–Hilbert correspondence as an isomorphism between parabolic logarithmic connections and filtered local systems. The most important point of [24] is the non-abelian Hodge theory, which connects parabolic logarithmic connections with parabolic Higgs bundles through a harmonic metric. Its effect on the geometry of the corresponding two algebraic moduli spaces seems mysterious to the author.

The connection ∇ is said to be irregular singular at $x\in D$, if it cannot be reduced to a logarithmic connection via a meromorphic transform around x. So the order of pole of ∇ at x is at least two. An irregular singular connection ∇ is locally written $\nabla|_U = \mathrm{d}+A(z)\mathrm{d}z/z^m$ for a matrix $A(z)$ of holomorphic functions in z, where m is the order of pole of ∇ at x and z is a local holomorphic coordinate on a neighborhood U of x. We say that ∇ is generic unramified at x if the leading term $A(0)$ has r distinct eigenvalues. Among the irregular singular connections, a generic unramified connection is of most generic type. The next generic irregular singular connections are generic ramified connections. In this paper, we say that a connection (E,∇) is generic ν-ramified at x if the formal completion $\big(\widehat{E},\widehat{\nabla}\big)$ at x is isomorphic to $(\mathbb{C}[[w]],\nabla_\nu)$, where $w=z^{\frac{1}{r}}$, $\nu(w)\in\sum_{l=0}^{mr-r}\mathbb{C}w^l\mathrm{d}w/w^{mr-r+1}$, the formal connection ∇_ν is defined by

$$\nabla_{\nu(w)}\colon\ \mathbb{C}[[w]]\ni f(w)\mapsto \mathrm{d}f(w)+f(w)\nu(w)\in\mathbb{C}[[w]]\otimes\frac{\mathrm{d}z}{z^m} \tag{1.1}$$

and the $w\mathrm{d}w/w^{mr-r+1}$-coefficient of $\nu(w)$ does not vanish.

The moduli space of logarithmic connections is well formulated by adding the parabolic structure and it is smooth and has a symplectic structure. It is constructed in the work with K. Iwasaki and M.-H. Saito in [11, 14]. For unramified irregular singular connections, the moduli space is analytically constructed by O. Biquard and P. Boalch in [2] together with establishing the non-abelian Hodge theory. The algebraic construction of the moduli space of generic unramified irregular singular connections was done in the work with Masa-Hiko Saito in [15] by using the same method as in the logarithmic case. Compared with the unramified connections, it is a more difficult task to construct the moduli space of ramified connections. Over the trivial bundle on $\mathbb{P}^1$, Bremer and Sage construct, in [9], the moduli space of ramified connections via a careful consideration of the formal ramified structure from a viewpoint of representation theory. In a higher genus case, the moduli space of ramified connections of generic ramified type is constructed by the author in [13]. T. Pantev and B. Toën introduce in [22] the derived geometric approach to the moduli space of connections in a general abstract setting.

Both in logarithmic and unramified irregular singular cases, the moduli space of connections has a natural symplectic structure. Roughly speaking, the moduli space of parabolic logarithmic connections is a torsor over the moduli space of parabolic bundles, which is locally isomorphic to the cotangent bundle. So the moduli space has a natural symplectic structure, though we precisely need a more careful consideration to the locus of non-simple underlying parabolic bundles. The method of parabolic structure is also valid for the construction of symplectic form on the moduli space of unramified irregular singular connections. However, in the case of ramified irregular singular connections, the method of parabolic structure does not go well with the construction of symplectic form. In [13, Theorem 4.1], we proved the existence of a symplectic form on the moduli space of ramified connections, but the proof of nondegeneracy was not given directly. It is reduced to the nondegeneracy in the case of logarithmic or unramified irregular singular connections by using an argument of codimension. So, in [13], we could not find a duality on the tangent space like in logarithmic or unramified irregular singular case. In this paper, we introduce the notion of factorized ramified structure, which supplies the place of parabolic structure. It induces a canonical duality on the tangent space of the moduli space of ramified connections which was not done in [13]. In order to see it easily, we adopt a simpler setting than [13], while we follow almost the same formulation of the moduli space constructed in [13].

Let us see a rough idea of factorized ramified structure. Assume that a rank r irregular singular connection (E,∇) is formally isomorphic to $(\mathbb{C}[[w]],\nabla_{\nu(w)})$ at x for $\nabla_{\nu(w)}$ defined in (1.1). Let N be the endomorphism of $E|_{mx}$, which corresponds to the action of w on $\mathbb{C}[w]/(w^{mr})$. Then we can consider the $\mathcal{O}_{mx}[T]$-module structure on $E|_{mx}$ defined by $P(T)v := P(N)v$ for a polynomial $P(T)$ in $\mathcal{O}_{mx}[T]$. By the elementary linear algebra, we can see that there is an isomorphism $\mathcal{O}_{mx}[T]/(T^r-z)\xrightarrow{\sim} E|_{mx}$ of $\mathcal{O}_{mx}[T]$-modules. The dual $E|_{mx}^\vee$ also has the $\mathcal{O}_{mx}[T]$-module structure via the map ${}^tN$ and we have an isomorphism $E|_{mx}^\vee\xrightarrow{\sim}\mathcal{O}_{mx}[T]/(T^r-z)$ of $\mathcal{O}_{mx}[T]$-modules. Composing these isomorphisms, we get an isomorphism $\theta\colon E|_{mx}^\vee\xrightarrow{\sim}E|_{mx}$ of $\mathcal{O}_{mx}[T]$-modules. Set $\kappa:=\theta^{-1}\circ N$. Then θ induces a perfect pairing $\vartheta\colon E|_{mx}^\vee\times E|_{mx}^\vee\longrightarrow\mathcal{O}_{mx}$ which becomes symmetric and κ induces a pairing $\varkappa\colon E|_{mx}\times E|_{mx}\longrightarrow\mathcal{O}_{mx}$ which is also symmetric. Roughly speaking, a factorized ramified structure on (E,∇) at x is given by (θ,κ) or $(\vartheta,\varkappa)$.

The purpose of introducing factorized ramified structure is to construct a duality on the tangent space of the moduli space. So we require it to go well with the deformation theory. In that context, all the conditions for the connection (E,∇) should be given only by the restriction $(E,\nabla)|_{mx}$ to the divisor mx and the rational one form $\nu(w)$ should be considered modulo holomorphic forms in w. Under such setting, the endomorphism N on $E|_{mx}$ in fact has an ambiguity in z^{m-1}-term, while the restriction $N|_{(m-1)x}$ is uniquely determined from $\nabla|_{mx}$ and $\nu(w)$. We take account of this ambiguity in the precise formulation of factorized ν-ramified structure in Definition 3.1.

In Section 2, we introduce the notion of logarithmic λ-parabolic structure and that of generic unramified μ-parabolic structure, which locally characterize the parabolic connections introduced in [11] and the unramified parabolic connections introduced in [15], respectively. We also recall the notion of generic ν-ramified structure given in [13]. In Section 3, we introduce the notion of factorized ν-ramified structure and prove that it is equivalent to the generic ν-ramified structure given in Section 2. In Section 4, we see that a generic ν-ramified structure enables us to recover a formal isomorphism to the connection ∇_ν in (1.1). In Section 5, we give a construction of the moduli space of connections with (λ,μ,ν)-structure (Theorem 5.1) using an embedding to the moduli space of parabolic triples constructed in [14]. It is a variant of the standard method of the GIT-construction of the moduli space established by C.T. Simpson in [25, 26]. The following is an important property of the moduli space (see Theorem 8.1 in a precise setting).

Theorem 1.1. *There exists a canonical symplectic form on the moduli space of connections with (λ,μ,ν)-structure.*

For the construction of the canonical 2-form in Theorem 1.1 (or Theorem 8.1 precisely), we describe the tangent space of the moduli space using the hypercohomology of a complex defined in Section 6. In Section 7, we see that this tangent space has a canonical duality (Proposition 7.2) coming from the factorized ramified structure, which gives a canonical nondegenerate 2-form. This duality is also of benefit to prove the smoothness of the moduli space. We also need to prove that the canonical 2-form is d-closed. For its proof, we construct an unfolding of the moduli space of connections with (λ,μ,ν)-structure in Section 8. An unfolding means a deformation of the moduli space to logarithmic moduli spaces. A factorized ramified structure enables us to construct such an unfolding in an easy way. By reducing to the fact that the canonical 2-form on the logarithmic moduli space is d-closed, we can complete the proof of Theorem 1.1.

The main aim of considering the moduli space of connections with (λ, μ, ν)-structure is to construct the generalized isomonodromic deformation that fits in with our setting of the moduli space. In the logarithmic case, the isomonodromic deformation naively means that the monodromy representation corresponding to the connection is constant. Over the trivial bundle on $\mathbb{P}^1$, the isomonodromic deformation is classically known as the Schlesinger equation. The formulation of isomonodromic deformation in a higher genus case requires an appropriate setting of the moduli space of connections, which is done in the work with K. Iwasaki and M.-H. Saito in [14] and in [11]. A cohomological description of the isomonodromic deformation on the moduli space is also established by I. Biswas, V. Heu, J. Hurtubise and A. Komyo in [3, 4, 18]. Conceptually, the isomonodromic deformation is obtained by pulling back, via the Riemann–Hilbert morphism, the local trivial foliation on the family of character varieties.

For irregular singular connections, we cannot recover a meromorphic connection from the naive monodromy data and we need to consider the Stokes data. By virtue of the theorem of Deligne, Malgrange and Sibuya [1, Theorems 4.5.1 and 4.7.3], there is a bijective correspondence between the local meromorphic connections and the Stokes data on a punctured disk. The generalized isomonodromic deformation means a family of irregular singular connections, whose corresponding monodromy representation equipped with the Stokes data is locally constant. In [16], M. Jimbo, T. Miwa and K. Ueno established the formulation of generalized isomonodromic deformation of generic unramified irregular singular connections over the trivial bundle on $\mathbb{P}^1$ and described its differential equation completely. The generalized isomonodromic deformation was also introduced by B. Malgrange in [20]. The purpose of this paper is to extend this theory to higher genus case including generic ramified connections. In order to realize the formulation of generalized isomonodromic deformation in such a general setting, we need the moduli space of connections with (λ, μ, ν)-structure constructed in Section 5.

In [5], P. Boalch constructs the moduli space of unramified connections over the trivial bundle on $\mathbb{P}^1$ and describes the generalized isomonodromic deformation in [16] through the correspondence with the wild character variety which is the moduli space of monodromy Stokes data. P. Boalch extends the framework of wild character variety to the higher genus case in [6]. In [27], M. van der Put and M.-H. Saito gives another construction of the moduli space of monodromy Stokes data, which includes all possible singularities, and provides the explicit descriptions of the moduli spaces in the case of Painlevé equations. I.Krichever also extends the argument by Jimbo, Miwa and Ueno in [16] to the higher genus case and describes the generalized isomonodromic 2-form in [19]. Placing importance on the Simpson's framework of Betti and de Rham correspondence in [26], the generalized isomonodromic deformation is formulated via the full moduli space of generic unramified connections on curves of general genus in the work with M.-H. Saito in [15] and in [12]. C. Bremer and D. Sage establish the generalized isomonodromic deformation of ramified connections over the trivial bundle on $\mathbb{P}^1$ in [8] and they prove the integrability condition of the generalized isomonodromic deformation via examining a property of the corresponding differential ideal. Their work is based on the construction of the moduli space in [9], which partially uses the method by P. Boalch in [5].

In Section 9, we recall a brief sketch of the local analytic theory of ramified irregular singular connections. First we consider the pullback of a generic ramified connection to a local analytic ramified cover. After applying an elementary transform of vector bundle to the pullback of the ramified connection, we get an unramified irregular connection. Such a process is called a shearing transformation method [28, Section 19.3]. Its description is given by K. Diarra, F. Loray and A. Komyo in [10, 17] for rank 2 ramified connections on $\mathbb{P}^1$. On the other hand, we give a brief idea of producing the Stokes data corresponding to the unramified connection on the local analytic ramified cover. Then we give a definition of local generalized isomonodromic deformation of generic unramified irregular singular connections on a unit disk in Definition 9.4.

Applying the Jimbo–Miwa–Ueno theory in [16] to the local setting, we get the following theorem (see Theorem 9.7 precisely).

Theorem 1.2 (Jimbo, Miwa and Ueno)**.** *A family of generic unramified irregular singular connections on a unit disk is a local generalized isomonodromic deformation if and only if it can be extended to an integrable connection.*

Precisely, there are ambiguities in the asymptotic solutions in our setting and our proof of Theorem 1.2 (Theorem 9.7 precisely) follows from the asymptotic property of flat solutions, which is essentially the result by T. Mochizuki in [21, Chapter 20]. Using Theorem 1.2 (precisely Theorem 9.7), we get a similar statement for local ramified connections in Corollary 9.11, which is a main consequence of Section 9.

Based on the viewpoint of Theorem 1.2 (precisely Theorem 9.7 and its consequence Corollary 9.11), we formulate the generalized isomonodromic deformation on the moduli space of ramified connections in Section 11. For the construction, we introduce in Section 10 the notion of horizontal lift (Definition 10.1) of the universal family of connections on the moduli space. The horizontal lift is locally a restriction of the family of integrable connections, given in Theorem 1.2 (precisely Corollary 9.11), to a first order infinitesimal neighborhood of the base parameter space. Nevertheless, it is defined purely algebraically. In the case of logarithmic or unramified irregular singular connections, the notion of horizontal lift is introduced in [11, 12, 15]. We can prove the existence and the uniqueness of the horizontal lift in Propositions 10.7 and 10.10, whose proof needs an isomorphism $(E,\nabla)|_{qx} \cong (\mathbb{C}[[w]],\nabla_\nu)|_{qx}$ in deep order (for $q = 2m-1$ or $q = 3m-1$), that is proved in Proposition 4.1. The existence of horizontal lift in Proposition 10.7 produces a tangent splitting $\Psi\colon \pi^*_{\mathcal{T}} T_{\mathcal{T}} \longrightarrow T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ in Section 11, equation (11.2), where $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$ is a family of moduli spaces of $\boldsymbol{\alpha}$-stable connections with $(\lambda,\tilde{\mu},\tilde{\nu})$-structure and $\mathcal{T}$ is the space of time variables parameterizing local exponents and curves with divisors. We call the subbundle $\operatorname{Im}\Psi \subset T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ the generalized isomonodromic subbundle (Definition 11.3). The main purpose of this paper is the following theorem (see Theorem 11.6 precisely).

Theorem 1.3. *The generalized isomonodromic subbundle* $\operatorname{Im}\Psi$ *of* $T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ *satisfies the integrability condition* $[\operatorname{Im}\Psi,\operatorname{Im}\Psi] \subset \operatorname{Im}\Psi$*.*

In the proof of the above theorem, we need the uniqueness of the horizontal lift with respect to two deformation parameters ϵ_1, ϵ_2, which is proved in Proposition 10.10. We can prove the integrability condition of Theorem 1.3 by looking at the $\epsilon_1\epsilon_2$-term of the horizontal lift.

By Theorem 1.3 (or Theorem 11.6), the generalized isomonodromic subbundle $\operatorname{Im}\Psi$ determines a foliation on the moduli space $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$, which we call the generalized isomonodromic foliation (Definition 11.7). We regard the generalized isomonodromic subbundle or the induced foliation as the generalized isomonodromic deformation. However, our construction of generalized isomonodromic deformation is not complete, because we do not establish the generalized Riemann–Hilbert correspondence between the moduli space of connections and the wild character variety. The construction of wild character variety in [7] will be a key work in that framework.

The generalized isomonodromic deformation is known to be characterized by a canonical 2-form, which is introduced in [16] and extended to higher genus case in [19]. The works [5, 8] are also based on this principle. By means of the generalized isomonodromic subbundle $\operatorname{Im}\Psi$ constructed in Theorem 1.3, we can extend the relative symplectic form given in Theorem 1.1 to a total 2-form (Definition 11.4), which we call the generalized isomonodromic 2-form. Using the generalized isomonodromic foliation produced by Theorem 1.3, we can prove in Corollary 11.8 that the generalized isomonodromic 2-form is d-closed.

2 Logarithmic, unramified irregular singular or ramified irregular singular structure on a connection

Let C be a complex smooth projective curve of genus g. We consider an effective divisor $D = D_{\log} + D_{\mathrm{un}} + D_{\mathrm{ram}}$ on C, where $D_{\log}$, D_{un} and D_{ram} are mutually disjoint, $D_{\log}$ is a reduced divisor, $D_{\mathrm{un}} = \sum_{x\in D_{\mathrm{un}}} m_x x$ and $D_{\mathrm{ram}} = \sum_{x\in D_{\mathrm{ram}}} m_x x$ are multiple divisors with $m_x \geq 2$ for $x \in D_{\mathrm{un}} \cup D_{\mathrm{ram}}$.

For each point $x \in D_{\log}$, we fix a tuple $(\lambda^x_0, \dots, \lambda^x_{r-1}) \in \mathbb{C}^r$ and put $\lambda^x := (\lambda^x_k)_{0\leq k\leq r-1}$ and $\lambda := (\lambda^x)_{x\in D_{\log}}$.

For $x \in D_{\mathrm{un}}$, we take $\mu^x_0, \dots, \mu^x_{r-1} \in \Omega^1_C(m_x x)\big|_{m_x x}$ whose leading terms are mutually distinct. In other words, $\mu^x_k - \mu^x_{k'}$ is a generator of the $\mathcal{O}_{m_x x}$-module $\Omega^1_C(m_x x)\big|_{m_x x}$ for $k \neq k'$. We write $\mu^x := (\mu^x_k)_{0\leq k\leq r-1}$ and $\mu := (\mu^x)_{x\in D_{\mathrm{un}}}$.

Let E be an algebraic vector bundle on C of rank r and let $\nabla\colon E \longrightarrow E \otimes \Omega^1_C(D)$ be an algebraic connection admitting poles along D.

Definition 2.1. We say that l^x is a logarithmic λ^x-parabolic structure on (E, ∇) at $x \in D_{\log}$, if it is a filtration $E|_x = l^x_0 \supset \cdots \supset l^x_{r-1} \supset l^x_r = 0$ satisfying $(\mathsf{res}_x(\nabla) - \lambda^x_k \mathrm{id})(l^x_k) \subset l^x_{k+1}$ for $k = 0, \dots, r-1$, where $\mathsf{res}_x(\nabla)\colon E|_x \longrightarrow E|_x$ is the linear map determined by taking the residue at x.

Definition 2.2. We say that ℓ^x is a generic unramified μ^x-parabolic structure on (E, ∇) at $x \in D_{\mathrm{un}}$, if it is a filtration $E|_{m_x x} = \ell^x_0 \supset \cdots \supset \ell^x_{r-1} \supset \ell^x_r = 0$ satisfying $\ell^x_k/\ell^x_{k+1} \cong \mathcal{O}_{m_x x}$ and $(\nabla|_{m_x x} - \mu^x_k \mathrm{id})(\ell^x_k) \subset \ell^x_{k+1}$ for $k = 0, \dots, r-1$, where $\nabla|_{m_x x}\colon E|_{m_x x} \longrightarrow E \otimes \Omega^1_X(D)|_{m_x x}$ is the $\mathcal{O}_{m_x x}$-homomorphism given by the restriction of ∇ to the finite subscheme $m_x x \subset X$.

For each $x \in D_{\mathrm{ram}}$, we take a generator z of the maximal ideal of the local ring $\mathcal{O}_{C,x}$. Assume that

$$\nu^x_0(z) \in \Omega^1_C(D_{\mathrm{ram}})|_{m_x x}, \qquad \nu^x_1(z), \dots, \nu^x_{r-1}(z) \in \Omega^1_C(D_{\mathrm{ram}})|_{(m_x-1)x}$$

are given and that the leading term of $\nu^x_1(z)$ does not vanish. In other words, $\nu^x_1(z)$ is a generator of the $\mathcal{O}_{C,x}$-module $\Omega^1_C(D_{\mathrm{ram}})|_{(m_x-1)x}$. We take a variable w with $w^r = z$ and put

$$\nu^x(w) = \nu^x_0(z) + \nu^x_1(z)w + \cdots + \nu^x_{r-1}(z)w^{r-1}.$$

We write $\nu = (\nu^x(w))_{x\in D_{\mathrm{ram}}}$. Furthermore, we assume the following

Assumption 2.3. We assume that

$$d \ := \ -\sum_{x\in D_{\log}} \sum_{k=0}^{r-1} \lambda^x_k - \sum_{x\in D_{\mathrm{un}}} \sum_{k=0}^{r-1} \mathsf{res}_x(\mu^x_k) - \sum_{x\in D_{\mathrm{ram}}} \left(r\, \mathsf{res}_x(\nu^x_0) + \frac{r-1}{2} \right)$$

is an integer.

Next we recall the formulation of ramified connection given in [13]. In this paper, we give a simplified version, since the formulation in [13] is somewhat complicated. Before stating the precise definition, we will see the reason why we introduce a filtration on $E|_{m_x x}$. What we want to consider is a connection (E, ∇) on C with a formal isomorphism $(E, \nabla) \otimes \widehat{\mathcal{O}}_{C,x} \cong (\mathbb{C}[[w]], \nabla_{\nu^x})$. However, it is difficult to treat the formal isomorphism in the construction of the moduli space and also in the deformation theory. It is rather convenient to formulate the ramified condition only by the data of the restriction $(E,\nabla)|_{m_x x}$. With respect to the frame of $E|_{m_x x}$ corresponding to $1, w, \dots, w^{r-1}$, the representation matrix of $\nabla|_{mx}$ is

$$\begin{pmatrix} \nu_0^x(z) & z\nu_{r-1}^x(z) & \cdots & z\nu_1^x(z) \\ \nu_1^x(z) & \nu_0^x(z) + \frac{\mathrm{d}z}{rz} & \cdots & z\nu_2^x(z) \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{r-1}^x(z) & \nu_{r-2}^x(z) & \cdots & \nu_0^x(z) + \frac{(r-1)\mathrm{d}z}{r} \end{pmatrix}.$$

However, the assumption on $\nabla|_{m_x x}$ by the above matrix is too strict and that does not go well with the formulation of the moduli space. It is rather better to allow ambiguities in $\nabla|_{m_x x}$ which is given by

$$\begin{pmatrix} \nu_0^x(z) & z\nu_{r-1}^x(z) & \cdots & z\nu_1^x(z) \\ \nu_1^x(z) + a_{1,0}\frac{\mathrm{d}z}{z} & \nu_0^x(z) + \frac{\mathrm{d}z}{rz} & \cdots & z\nu_2^x(z) \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{r-1}^x(z) + a_{r-1,0}\frac{\mathrm{d}z}{z} & \nu_{r-2}^x(z) + a_{r-1,1}\frac{\mathrm{d}z}{z} & \cdots & \nu_0^x(z) + \frac{(r-1)\mathrm{d}z}{r} \end{pmatrix} \tag{2.1}$$

where $a_{i,j} \in \mathbb{C}$ for $r-1 \geq i > j \geq 0$. Indeed, if $\nabla|_{m_x x}$ is given by the above matrix with ambiguities, there is a formal isomorphism $\big(\widehat{E}, \widehat{\nabla}\big) \cong (\mathbb{C}[[w]], \nabla_{\nu^x})$ (which will follow from [13, Proposition 1.3] or Corollary 4.3 later). In order to allow the ambiguities of $\nabla|_{m_x x}$ as above, we introduce a filtration $E|_{m_x x} = V_0^x \supset V_1^x \supset \cdots \supset V_{r-1}^x \supset zV_0^x$. If we identify $E|_{m_x x}$ with $\mathbb{C}[[w]]/(w^{m_x r})$ via the formal isomorphism, then we set $V_k^x := (w^k)/(w^{m_x r})$ and $L_k^x := (w^k)/\big(w^{m_x r - r + k + 1}\big)$, where (w^k) the ideal of $\mathbb{C}[w]$ generated by w^k. Then all the conditions in the following definition will be obvious.

Definition 2.4 ([13, Definitions 1.2 and 2.1])**.** Let (E,∇) be a pair of an algebraic vector bundle E of rank r on C and an algebraic connection ∇ on E. We say that a tuple $\mathcal{V}^x = \big(\big(V_k^x, L_k^x, \pi_k^x\big)_{0\leq k\leq r-1}, \big(\phi_k^x\big)_{1\leq k\leq r}\big)$ is a generic ν-ramified structure on (E,∇) at $x \in D_{\mathrm{ram}}$, if

(i) $E|_{m_x x} = V_0^x \supset V_1^x \supset \cdots \supset V_{r-1}^x \supset V_r^x = zV_0^x$ is a filtration by $\mathcal{O}_{m_x x}$-submodules which satisfies $\operatorname{length}(V_k^x/V_{k+1}^x) = 1$ and $\nabla|_{m_x x}(V_k^x) \subset V_k^x \otimes \Omega_C^1(D)|_{m_x x}$ for $0 \leq k \leq r-1$,

(ii) $\pi_k^x \colon V_k^x \otimes \mathbb{C}[w]/(w^{m_x r - r + 1}) \longrightarrow L_k^x$ is a quotient free $\mathbb{C}[w]/(w^{m_x r - r+1})$-module of rank one for $0 \leq k \leq r-1$ such that the restrictions $\pi_k^x|_{V_k^x} \colon V_k^x \hookrightarrow V_k^x \otimes \mathbb{C}[w]/(w^{m_x r - r+1}) \xrightarrow{\pi_k^x} L_k^x$ are surjective and that the diagrams

$$\begin{CD} V_k^x \otimes \mathbb{C}[w]/\big(w^{m_x r - r+1}\big) @>{\pi_k^x}>> L_k^x \\ @V{\nabla|_{m_x x}}VV @VV{\nu^x(w) + \frac{k}{r}\frac{\mathrm{d}z}{z}}V \\ V_k^x \otimes \Omega_C^1(D) \otimes \mathbb{C}[w]/\big(w^{m_x r - r+1}\big) @>{\pi_k^x \otimes 1}>> L_k^x \otimes \Omega_C^1(D) \end{CD}$$

are commutative for $0 \leq k \leq r-1$,

(iii) $\phi_k \colon L_k^x \longrightarrow wL_{k-1}^x$ for $1 \leq k \leq r-1$ and $\phi_r^x \colon (z)/\big(z^{m_x+1}\big) \otimes L_0^x \longrightarrow wL_{r-1}^x$ are surjective $\mathbb{C}[w]$-homomorphisms such that the diagrams

$$\begin{CD} V_k^x \otimes \mathbb{C}[w]/\big(w^{m_x r - r+1}\big) @>{\pi_k^x}>> L_k^x \\ @VVV @VV{\phi_k^x}V \\ V_{k-1}^x \otimes \mathbb{C}[w]/\big(w^{m_x r - r+1}\big) @>{\pi_{k-1}^x}>> L_{k-1}^x \end{CD}$$

are commutative for $1 \leq k \leq r-1$ and that the diagram

$$\begin{tikzcd}
(z)/\big(z^{m_x+1}\big) \otimes V_0^x \otimes \mathbb{C}[w]/\big(w^{m_x r-r+1}\big) \arrow[r, "1\otimes\pi_0^x"] \arrow[d] & (z)/\big(w^{m_x r+1}\big) \otimes L_0^x \arrow[d, "\phi_r^x"] \\
V_{r-1}^x \otimes \mathbb{C}[w]/\big(w^{m_x r-r+1}\big) \arrow[r, "\pi_{r-1}^x"] & L_{r-1}^x
\end{tikzcd}$$

is commutative,

(iv) there are isomorphisms $\psi_k^x \colon L_k^x \xrightarrow{\sim} (w)/\big(w^{m_x r-r+2}\big) \otimes L_{k-1}^x$ of $\mathbb{C}[w]$-modules for $1 \leq k \leq r-1$ such that the composition $L_k^x \xrightarrow{\psi_k^x} (w)/\big(w^{m_x r-r+2}\big) \otimes L_{k-1}^x \longrightarrow wL_{k-1}^x$ coincides with ϕ_k^x and that the composition

$$\begin{aligned}
&(z)/\big(w^{m_x r+1}\big) \otimes L_0^x \xrightarrow{\phi_r^x} L_{r-1}^x \xrightarrow[\sim]{\psi_{r-1}^x} (w)/\big(w^{m_x r-r+2}\big) \otimes L_{r-2}^x \\
&\xrightarrow[\sim]{\psi_{r-2}^x} \cdots \xrightarrow[\sim]{\psi_1^x} \big((w)/\big(w^{m_x r-r+2}\big)\big)^{\otimes r-1} \otimes L_0^x \xrightarrow{\sim} \big(w^{r-1}\big)/\big(w^{m_x r}\big) \otimes L_0^x
\end{aligned}$$

coincides with the $\mathbb{C}[w]$-homomorphism obtained by tensoring L_0^x to the canonical map $(z)/\big(w^{m_x r+1}\big) \longrightarrow \big(w^{r-1}\big)/\big(w^{m_x r}\big)$.

Two ramified structures $\big(V_k^x, L_k^x, \pi_k^x, \phi_k^x\big)$ and $\big(V_k'^x, L_k'^x, \pi_k'^x, \phi_k'^x\big)$ on (E, ∇) at $x \in D_{\mathrm{ram}}$ are equivalent if $V_k^x = V_k'^x$ for $0 \leq k \leq r$, there are isomorphisms $\sigma_k \colon L_k^x \xrightarrow{\sim} L_k'^x$ of $\mathbb{C}[w]$-modules for $0 \leq k \leq r-1$ such that the diagrams

$$\begin{tikzcd}
V_k^x \arrow[r, "\pi_k^x|_{V_k^x}"] \arrow[d, equal] & L_k^x \arrow[d, "\cong"', "\sigma_k"] \\
V_k^x \arrow[r, "\pi_k'^x|_{V_k^x}"] & L_k'^x
\end{tikzcd} \quad (0 \leq k \leq r-1) \qquad
\begin{tikzcd}
L_k^x \arrow[r, "\phi_k^x"] \arrow[d, "\sigma_k"', "\cong"] & L_{k-1}^x \arrow[d, "\cong"', "\sigma_{k-1}"] \\
L_k'^x \arrow[r, "\phi_k'^x"] & L_{k-1}'^x
\end{tikzcd} \quad (1 \leq k \leq r-1)$$

and the diagram

$$\begin{tikzcd}
(z)/\big(w^{m_x r+1}\big) \otimes L_0^x \arrow[r, "\phi_r^x"] \arrow[d, "\mathrm{id}\otimes\sigma_0"', "\cong"] & L_{r-1}^x \arrow[d, "\cong"', "\sigma_{r-1}"] \\
(z)/\big(w^{m_x r+1}\big) \otimes L_0'^x \arrow[r, "\phi_r'^x"] & L_{r-1}'^x
\end{tikzcd}$$

are commutative.

Remark 2.5. In the condition (iv) of Definition 2.4, the composition $\psi_1^x \circ \cdots \circ \psi_{r-1}^x \circ \phi_r^x$ is independent of the choices of the lifts ψ_k of ϕ_k taken for $1 \leq k \leq r-1$. In particular, the condition (iv) is independent of the choices of ψ_k.

Example 2.6. Let us consider the typical case $(E, \nabla) \otimes \widehat{\mathcal{O}}_{C,x} = (\mathbb{C}[[w]], \nabla_\nu)$, where $z \in \mathcal{O}_{C,x}$ is a generator of the maximal ideal, $w = z^{\frac{1}{r}}$ and the connection ∇_ν is given by

$$\nabla_\nu \colon\ \mathbb{C}[[w]] \ \ni\ f(w) \ \mapsto\ df(w) + f(w)\nu \ \in \mathbb{C}[[w]] \otimes \frac{\mathrm{d}z}{z^m}.$$

In this case, a generic ν-ramified structure in Definition 2.4 is given in the following way. We consider the filtration $\mathbb{C}[[w]]/z^m\mathbb{C}[[w]] \supset (w)/\big(w^{mr}\big) \supset \cdots \supset \big(w^{r-1}\big)/\big(w^{mr}\big) \supset z\mathbb{C}[[w]]/z^m\mathbb{C}[[w]]$

and put $V_k := \big(w^k\big)/\big(w^{mr}\big)$ for $0 \le k \le r-1$. We put $L_k := \big(w^k\big)/\big(w^{mr-r+k+1}\big)$ and regard it as a $\mathbb{C}[w]/\big(w^{mr-r+1}\big)$-module. The canonical surjection

$$V_k = \big(w^k\big)/\big(w^{mr}\big) \longrightarrow \big(w^k\big)/\big(w^{mr-r+k+1}\big) = L_k$$

induces a surjective homomorphism

$$\pi_k \colon\; V_k \otimes_{\mathbb{C}[z]/(z^m)} \mathbb{C}[w]/\big(w^{mr-r+1}\big) \longrightarrow L_k$$

of $\mathbb{C}[w]/\big(w^{mr-r+1}\big)$-modules. Then the conditions (i), (iii), (iv) of Definition 2.4 are obvious for such data. Since the restriction

$$\nabla_\nu \colon\; w^k\mathbb{C}[[w]] \longrightarrow w^k\mathbb{C}[[w]] \otimes \frac{\mathrm{d}z}{z^m}$$

satisfies the equality

$$\begin{aligned}\nabla_\nu\big(w^k f(w)\big) &= k w^{k-1}\mathrm{d}w\, f(w) + w^k \mathrm{d}f(w) + w^k f(w)\nu \\ &= w^k f(w)\frac{k}{r}\frac{\mathrm{d}z}{z} + w^k(\mathrm{d}f(w) + f(w)\nu),\end{aligned}$$

we can also see the commutativity of the diagrams in Definition 2.4 (ii).

We will see later in Corollary 4.3 that any connection with generic ν-ramified structure at x is in fact isomorphic to the one given in this example.

Definition 2.7. We say that $(E, \nabla, l, \ell, \mathcal{V})$ is a connection with (λ, μ, ν)-structure, if

(i) E is an algebraic vector bundle of rank r on C of degree d,

(ii) $\nabla \colon E \longrightarrow E \otimes \Omega^1_C(D)$ is an algebraic connection admitting poles along D,

(iii) $l = (l^x)_{x\in D_{\log}}$ is a tuple of logarithmic λ^x-parabolic structures l^x on (E,∇) at $x \in D_{\log}$,

(iii) $\ell = (\ell^x)_{x\in D_{\mathrm{un}}}$ is a tuple of generic unramified μ^x-parabolic structures ℓ^x on (E,∇) at $x \in D_{\mathrm{un}}$,

(iv) $\mathcal{V} = (\mathcal{V}^x)_{x\in D_{\mathrm{ram}}}$ is a tuple of generic ν^x-ramified structures $\mathcal{V}^x$ on (E,∇) at $x \in D_{\mathrm{ram}}$.

We take a tuple $\boldsymbol{\alpha} = (\alpha^x_k)^{x\in D}_{1\le k\le r}$ of positive rational numbers such that $0 < \alpha^x_1 < \cdots < \alpha^x_r < 1$ for any $x \in D$ and that $\alpha^x_k \ne \alpha^{x'}_{k'}$ for $(x,k) \ne (x',k')$.

For a non-zero subbundle F of E, we write

$$\begin{aligned}\operatorname{pardeg}^{\boldsymbol{\alpha}}(F) = \deg F &+ \sum_{x\in D_{\log}}\sum_{k=1}^{r} \alpha^x_k \operatorname{length}((F|_x \cap l^x_{k-1})/(F|_x \cap l^x_k)) \\ &+ \sum_{x\in D_{\mathrm{un}}}\sum_{k=1}^{r} \alpha^x_k \operatorname{length}((F|_{n_x x} \cap \ell^x_{k-1})/(F|_{n_x x} \cap \ell^x_k)) \\ &+ \sum_{x\in D_{\mathrm{ram}}}\sum_{k=1}^{r} \alpha^x_k \operatorname{length}\big((F|_{m_x x} \cap V^x_{k-1})/(F|_{m_x x} \cap V^x_k)\big)\,.\end{aligned}$$

Definition 2.8. We say that a connection $(E, \nabla, l, \ell, \mathcal{V})$ with (λ, μ, ν)-structure is $\boldsymbol{\alpha}$-stable (resp. $\boldsymbol{\alpha}$-semistable) if the inequality

$$\frac{\operatorname{pardeg}^{\boldsymbol{\alpha}}(F)}{\operatorname{rank} F} < \frac{\operatorname{pardeg}^{\boldsymbol{\alpha}}(E)}{\operatorname{rank} E} \qquad \left(\text{resp. } \frac{\operatorname{pardeg}^{\boldsymbol{\alpha}}(F)}{\operatorname{rank} F} \le \frac{\operatorname{pardeg}^{\boldsymbol{\alpha}}(E)}{\operatorname{rank} E}\right)$$

holds for any subbundle $0 \ne F \subsetneq E$ satisfying $\nabla(F) \subset F \otimes \Omega^1_C(D)$.

Remark 2.9. If $D_{\mathrm{ram}} \ne \varnothing$, then we can see $(E,\nabla)\otimes\widehat{\mathcal{O}}_{C,x} \cong (\mathbb{C}[[w]], \nabla_\nu)$ by Corollary 4.3, which will be proved later. Since $(\mathbb{C}[[w]], \nabla_\nu)$ is irreducible, (E,∇) is also irreducible and $(E, \nabla, l, \ell, \mathcal{V})$ is automatically $\boldsymbol{\alpha}$-stable for any parabolic weight $\boldsymbol{\alpha}$ in this case.

3 Factorized ramified structure

In this section, we introduce the notion of factorized ramified structure which is a rephrasing of generic ν-ramified structure in Definition 2.4. This notion is useful for the description of symplectic form later. In the Introduction, we saw a rough idea of factorized ramified structure. Before giving the precise definition of factorized ramified structure, we will see another aspect of the ambiguity in (2.1), which affects the definition of factorized ramified structure.

Let (E,∇) be a connection on C with a formal isomorphism $(E,\nabla)\otimes\widehat{\mathcal{O}}_{C,x}\cong(\mathbb{C}[[w]],\nabla_\nu)$, where z is a generator of the maximal ideal of $\mathcal{O}_{C,x}$, $w^r=z$ and ∇_ν is the connection defined in (1.1). Write $\nu(w)=\sum_{k=0}^{r-1}c_k(z)w^k\mathrm{d}z/z^{m_x x}$ with $c_0(z),\dots,c_{r-1}(z)\in\mathcal{O}_{m_x x}$ and $c_1(z)\in\mathcal{O}_{m_x x}^{\times}$. There is in fact an ambiguity coming from the choice of z, but it can be expressed by a modification of ν and we do not pursue this point any more. Recall that the endomorphism N on $E|_{m_x x}$ corresponds to the action of w via the isomorphism $E|_{m_x x}\xrightarrow{\sim}\mathbb{C}[w]/\big(w^{rm_x}\big)$. Since $c_1(z)$ is invertible, there is a polynomial $P(T)\in\mathcal{O}_{m_x x}[T]$ satisfying the equality

$$w=P\big(c_0(z)+c_1(z)w+\dots+c_{r-1}w^{r-1}\big)$$

in the ring $\mathcal{O}_{m_x x}[w]/(w^r-z)$. Since the equality $\nabla|_{(m_x-1)x}=\nu(N)|_{(m_x-1)x}$ holds, $N|_{(m_x-1)x}$ is uniquely determined from $\nabla|_{m_x x}$ by substitution to $P(T)$. However, N always has an ambiguity in the z^{m_x-1}-coefficients. This ambiguity causes the ambiguity in the matrix (2.1) of $\nabla|_{m_x x}$.

In order to see more precisely, consider the filtration $E|_{m_x}=V_0\supset V_1\supset\dots\supset V_{r-1}\supset V_r=zV_0$ given in Definition 2.4 (i). Since this filtration is determined by $N(V_i)=V_{i+1}$, it is uniquely determined from $\nabla|_{m_x x}$. Then the restriction $N|_{V_i}$ induces an endomorphism on $V_i/z^{m_x-1}V_{i+1}$, which is uniquely determined from $\nabla|_{m_x}$. So the factorization $N=\theta\circ\kappa$ will be justified when we replace it with the induced maps on $V_i/z^{m_x-1}V_{i+1}$ or on its dual. Although we need a careful consideration for the expression of these induced maps, all the conditions in the following definition will be natural.

Let C, $D_{\log}$, $D_{\rm un}$, $D_{\rm ram}$, ν, z, w be as in Section 1 and let (E,∇) be a pair of an algebraic vector bundle E of rank r on C and an algebraic connection ∇ on E with poles along D.

Definition 3.1. We say that a tuple $(V_k,\vartheta_k,\varkappa_k)_{0\le k\le r-1}$ is a factorized ν-ramified structure on (E,∇) at $x\in D_{\rm ram}$, if

(i) $E|_{m_x x}=V_0\supset V_1\supset\dots\supset V_{r-1}\supset V_r=zV_0$ is a filtration by $\mathcal{O}_{m_x x}$-submodules satisfying $\nabla|_{m_x x}(V_k)\subset V_k\otimes\Omega^1_C(D)$ and $\mathrm{length}(V_k/V_{k+1})=1$ for $0\le k\le r-1$,

(ii) for $\overline{V}_k:=V_k/z^{m_x-1}V_{k+1}$ and $\overline{W}_k:=\big(\overline{V}_{r-k-1}\big)^\vee=\mathrm{Hom}_{\mathcal{O}_{m_x x}}\big(\overline{V}_{r-k-1},\mathcal{O}_{m_x x}\big)$,

$$\vartheta_k\colon\ \overline{W}_k\times\overline{W}_{r-k-1}\longrightarrow\mathcal{O}_{m_x x}$$

is an $\mathcal{O}_{m_x x}$-bilinear pairing for $0\le k\le r-1$ such that the equality $\vartheta_k(v,v')=\vartheta_{r-k-1}(v',v)$ holds for $v\in\overline{W}_k$ and $v'\in\overline{W}_{r-k-1}$ and that the induced homomorphisms

$$\theta_k\colon\ \overline{W}_k\xrightarrow{\sim}\mathrm{Hom}(\overline{W}_{r-k-1},\mathcal{O}_{m_x x})=\overline{V}_k\qquad(0\le k\le r-1)$$

are isomorphisms, which make the diagrams

$$\begin{array}{ccc}\overline{W}_k & \longrightarrow & \overline{W}_{k-1}\\ {\scriptstyle\theta_k}\downarrow\cong & & {\scriptstyle\theta_{k-1}}\downarrow\cong\\ \overline{V}_k & \longrightarrow & \overline{V}_{k-1}\end{array}\quad(1\le k\le r-1)\qquad\begin{array}{ccc}(z)/\big(z^{m_x+1}\big)\otimes\overline{W}_0 & \longrightarrow & \overline{W}_{r-1}\\ {\scriptstyle 1\otimes\theta_0}\downarrow\cong & & {\scriptstyle\theta_{r-1}}\downarrow\cong\\ (z)/\big(z^{m_x+1}\big)\otimes\overline{V}_0 & \xrightarrow[\sim]{} & \overline{V}_{r-1}\end{array}$$

commutative, where the horizontal arrow $\overline{W}_k\longrightarrow\overline{W}_{k-1}$ is the dual of $\overline{V}_{r-k}\longrightarrow\overline{V}_{r-k-1}$ and the horizontal arrow $(z)/\big(z^{m_x+1}\big)\otimes\overline{W}_0\longrightarrow\overline{W}_{r-1}$ is induced by tensoring $(z)/\big(z^{m_x+1}\big)$ to $\overline{W}_0=\mathrm{Hom}(\overline{V}_{r-1},\mathcal{O}_{m_x x})\longrightarrow\mathrm{Hom}((z)/\big(z^{m_x+1}\big)\otimes\overline{V}_0,\mathcal{O}_{m_x x})=\big((z)/\big(z^{m_x+1}\big)\big)^\vee\otimes\overline{W}_{r-1}$,

(iii) for $0 \le k \le r-1$,

$$\varkappa_k \colon \ \overline{V}_k \times \overline{V}_{r-k-1} \longrightarrow \mathcal{O}_{m_x x}$$

is an $\mathcal{O}_{m_x x}$-bilinear pairing such that the equality $\varkappa_k(v,v') = \varkappa_{r-k-1}(v',v)$ holds for $v \in \overline{V}_k$, $v' \in \overline{V}_{r-k-1}$ and that the induced homomorphisms

$$\kappa_k \colon \ \overline{V}_k \longrightarrow \mathrm{Hom}_{\mathcal{O}_{m_x x}}\left(\overline{V}_{r-k-1}, \mathcal{O}_{m_x x}\right) = \overline{W}_k \qquad (0 \le k \le r-1)$$

make the diagrams

$$\begin{array}{ccc} \overline{V}_k & \longrightarrow & \overline{V}_{k-1} \\ {\scriptstyle \kappa_k}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_{k-1}} \\ \overline{W}_k & \longrightarrow & \overline{W}_{k-1} \end{array} \quad (1 \le k \le r-1) \qquad \begin{array}{ccc} (z)/\left(z^{m_x+1}\right) \otimes \overline{V}_0 & \longrightarrow & \overline{V}_{r-1} \\ {\scriptstyle 1\otimes\kappa_0}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_{r-1}} \\ (z)/\left(z^{m_x+1}\right) \otimes \overline{W}_0 & \longrightarrow & \overline{W}_{r-1} \end{array}$$

commutative,

(iv) the composition $N_k := \langle \vartheta_k, \varkappa_k \rangle = \theta_k \circ \kappa_k \colon \overline{V}_k \longrightarrow \overline{V}_k$ satisfies the equalities $(N_k)^r = z\,\mathrm{id}_{\bar{V}_k}$ and $(N_k)^{m_x r - r+1} = 0$, from which the injective ring homomorphism

$$\mathbb{C}[w]/\left(w^{m_x r-r+1}\right) \ni \overline{f(w)} \mapsto f(N_k) \in \mathrm{End}_{\mathcal{O}_{m_x x}}\left(\overline{V}_k\right) \tag{3.1}$$

is induced and the diagrams

$$\begin{array}{ccc} V_k & \xrightarrow{\nabla|_{m_x x}} & V_k \otimes \Omega^1_C(D) \\ \big\downarrow & & \big\downarrow \\ \overline{V}_k & \xrightarrow{\nu(N_k) + \frac{k}{r}\frac{\mathrm{d}z}{z}} & \overline{V}_k \otimes \Omega^1_C(D) \end{array}$$

are commutative for $k = 0, 1, \ldots, r-1$,

(v) with respect to the $\mathbb{C}[w]$-module structure on $\overline{V}_k$ defined by the ring homomorphism (3.1), there are $\mathbb{C}[w]$-isomorphisms $\psi_k \colon \overline{V}_k \xrightarrow{\sim} (w)/\left(w^{m_x r-r+2}\right) \otimes \overline{V}_{k-1}$ such that the composition

$$\overline{V}_k \underset{\sim}{\xrightarrow{\psi_k}} (w)/\left(w^{m_x r - r+2}\right) \otimes \overline{V}_{k-1} \longrightarrow w\overline{V}_{k-1} \hookrightarrow \overline{V}_{k-1}$$

coincides with the homomorphism $\overline{V}_k \longrightarrow \overline{V}_{k-1}$ induced by the inclusion $V_k \hookrightarrow V_{k-1}$ and that the composition

$$\begin{aligned} &(z)/\left(z^{m_x+1}\right) \otimes \overline{V}_0 \to \overline{V}_{r-1} \underset{\sim}{\xrightarrow{\psi_{r-1}}} (w)/\left(w^{m_x r-r+2}\right) \otimes \overline{V}_{r-2} \\ &\qquad \underset{\sim}{\xrightarrow{\psi_{r-2}}} \cdots \underset{\sim}{\xrightarrow{\psi_1}} \left(w^{r-1}\right)/\left(w^{m_x r}\right) \otimes \overline{V}_0 \end{aligned}$$

coincides with the homomorphism $(z)/\left(z^{m_x+1}\right) \otimes \overline{V}_0 \longrightarrow \left(w^{r-1}\right)/\left(w^{m_x r}\right) \otimes \overline{V}_0$ obtained by tensoring $\overline{V}_0$ to the canonical homomorphism $(z)/\left(z^{m_x+1}\right) \longrightarrow \left(w^{r-1}\right)/\left(w^{m_x r}\right)$.

Two factorized ramified structures $(V_k, \vartheta_k, \varkappa_k)$ and $(V'_k, \vartheta'_k, \varkappa'_k)$ are equivalent if $V_k = V'_k$ and $\langle \vartheta_k, \varkappa_k \rangle = N_k = N'_k = \langle \vartheta'_k, \varkappa'_k \rangle$ for any k and there are isomorphisms $\varsigma_k \colon \overline{W}_k \xrightarrow{\sim} \overline{W}_k$ satisfying ${}^tN_{r-k-1} \circ \varsigma_k = \varsigma_k \circ {}^tN_{r-k-1}$, $\theta'_k = \theta_k \circ \varsigma_k$, $\kappa'_k = \varsigma_k^{-1} \circ \kappa_k$ and the commutative diagrams

$$\begin{array}{ccc} (z)/\left(z^{m_x+1}\right) \otimes \overline{W}_0 & \longrightarrow & \overline{W}_{r-1} \\ {\scriptstyle 1\otimes\varsigma_0}\big\downarrow\cong & & {\scriptstyle \varsigma_{r-1}}\big\downarrow\cong \\ (z)/\left(z^{m_x+1}\right) \otimes \overline{W}_0 & \longrightarrow & \overline{W}_{r-1} \end{array} \qquad \begin{array}{ccc} \overline{W}_k & \longrightarrow & \overline{W}_{k-1} \\ {\scriptstyle \varsigma_k}\big\downarrow\cong & & {\scriptstyle \varsigma_{k-1}}\big\downarrow\cong \\ \overline{W}_k & \longrightarrow & \overline{W}_{k-1} \end{array} \quad (1 \le k \le r-1).$$

Remark 3.2. The condition $\vartheta_k(v,v') = \vartheta_{r-k-1}(v',v)$ for $v \in \overline{W}_k$, $v' \in \overline{W}_{r-k-1}$ in Definition 3.1 (ii) is equivalent to the condition ${}^t(\theta_k) = \theta_{r-k-1}$ under the identifications $\overline{W}_{r-k-1} = (\overline{V}_k)^\vee$ and $(\overline{W}_k)^\vee = \overline{V}_{r-k-1}$ for $0 \le k \le r-1$. Similarly, the condition $\varkappa_k(v,w) = \varkappa_{r-k-1}(w,v)$ for $v \in \overline{V}_k$, $w \in \overline{V}_{r-k-1}$ in Definition 3.1 (ii) is equivalent to the condition ${}^t\kappa_k = \kappa_{r-k-1}$ under the identifications $(\overline{W}_k)^\vee = \overline{V}_{r-k-1}$, $(\overline{V}_k)^\vee = \overline{W}_{r-k}$,

For a factorized ν-ramified structure $(V_k, \vartheta_k, \varkappa_k)$ on (E,∇), we can regard the $\mathcal{O}_{m_x x}$-module $\overline{V}_k = V_k/z^{m_x-1}V_{k+1}$ as a $\mathbb{C}[w]$-module by using the ring homomorphism in Definition 3.1 (iv), (3.1) and we have $\overline{V}_k \cong \mathbb{C}[w]/\big(w^{m_x r-r+1}\big)$. The canonical surjection $V_k \longrightarrow \overline{V}_k$ induces a surjection $\pi_k\colon V_k \otimes_{\mathbb{C}[z]/(z^{m_x})} \mathbb{C}[w]/\big(w^{m_x r-r+1}\big) \longrightarrow \overline{V}_k$ of $\mathbb{C}[w]/\big(w^{m_x r-r+1}\big)$-modules. For $1 \le k \le r-1$, the canonical inclusion $\iota_k\colon V_k \hookrightarrow V_{k-1}$ induces a homomorphism $\bar{\iota}_k\colon \overline{V}_k \longrightarrow \overline{V}_{k-1}$ and the canonical homomorphism $(z)/\big(z^{m_x+1}\big) \otimes V_0 \to zV_0 \hookrightarrow V_{r-1}$ induces a homomorphism $\bar{\iota}_r\colon (z)/\big(z^{m_x+1}\big) \otimes \overline{V}_0 \longrightarrow \overline{V}_{r-1}$. Then $\big(V_k, \overline{V}_k, \pi_k, \bar{\iota}_k\big)$ becomes a generic ν-ramified structure on (E,∇) at $x \in D_{\mathrm{ram}}$ in the sense of Definition 2.4.

Proposition 3.3. *The correspondence $(V_k, \vartheta_k, \varkappa_k) \mapsto \big(V_k, \overline{V}_k, \pi_k, \bar{\iota}_k\big)$ gives a bijection between the set of equivalence classes of factorized ν-ramified structures on (E,∇) at $x \in D_{\mathrm{ram}}$ and the set of isomorphism classes of generic ν-ramified structures on (E,∇) at $x \in D_{\mathrm{ram}}$.*

Proof. We will construct the inverse correspondence. Let $(V_k, L_k, \pi_k, \phi_k)$ be a generic ν-ramified structure on (E,∇) at $x \in D_{\mathrm{ram}}$. By Definition 2.4 (ii), the restriction $\pi_k|_{V_k}\colon V_k \longrightarrow L_k$ is a surjection, which induces the isomorphism $\overline{V}_k = V_k/z^{m_x-1}V_{k+1} \xrightarrow{\sim} L_k$. Take a generator $\bar{e}_0$ of L_0 as a $\mathbb{C}[w]$-module. Let $\bar{e}_k$ be the element of L_k which corresponds to $w^k \otimes \bar{e}_0$ via the isomorphism

$$L_k \underset{\sim}{\xrightarrow{\psi_k}} (w) \otimes L_{k-1} \underset{\sim}{\xrightarrow{\psi_{k-1}}} \cdots \underset{\sim}{\xrightarrow{\psi_1}} \big(w^k\big) \otimes L_0.$$

Since $\pi_k|_{V_k}$ is surjective, we can take an element $e_k \in V_k$ satisfying $\pi_k(e_k) = \bar{e}_k$. Then $e_0, e_1, \dots, e_{r-1}$ is a basis of the free $\mathcal{O}_{m_x x}$-module $E|_{m_x x}$ and we have

$$\begin{aligned} &\pi_k(e_l) = (\phi_{k+1} \circ \cdots \circ \phi_l)(\pi_l(e_l)) = w^{l-k}\pi_k(e_k) \qquad \text{if } k \le l \le r-1, \\ &\pi_k(ze_l) = (\phi_{k+1} \circ \cdots \circ \phi_r)(z \otimes \pi_0(e_l)) = w^{r-k+l}\pi_k(e_k) \qquad \text{if } 0 \le l < k. \end{aligned}$$

Furthermore, V_k is generated by $e_k, e_{k+1}, \dots, e_{r-1}, ze_0, \dots, ze_{k-1}$. If we define a homomorphism $N\colon E|_{m_x x} \longrightarrow E|_{m_x x}$ by

$$N(e_k) = \begin{cases} e_{k+1} & \text{if } 0 \le k \le r-2, \\ ze_0 & \text{if } k = r-1, \end{cases}$$

then N preserves V_k and the diagram

$$\begin{array}{ccc} V_k & \xrightarrow{\pi_k|_{V_k}} & L_k \\ {\scriptstyle N|_{V_k}}\big\downarrow & & \big\downarrow{\scriptstyle w} \\ V_k & \xrightarrow{\pi_k|_{V_k}} & L_k \end{array}$$

is commutative. By the definition, we have the equality $N^r = z \cdot \mathrm{id}_{E|_{m_x x}}$. The induced ring homomorphism

$$\mathcal{O}_{m_x x}[w]/\big(w^r - z\big) \ni f(w) \mapsto f(N) \in \mathrm{End}_{\mathcal{O}_{m_x x}}(E|_{m_x x})$$

endows $E|_{m_x x}$ with a structure of $\mathcal{O}_{m_x x}[w]$-module. Since the minimal polynomial of $N|_x$ is w^r whose degree is r, we can see $E|_x \cong \mathbb{C}[w]/(w^r)$ by elementary linear algebra. By Nakayama's lemma, we can extend it to an isomorphism

$$E|_{m_x x} \cong \mathcal{O}_{m_x x}[w]/\big(w^r - z\big) \tag{3.2}$$

of $\mathcal{O}_{m_x x}[w]$-modules. Similarly, the endomorphism ${}^tN$ on $E|_{m_x x}^\vee$ induces a structure of $\mathcal{O}_{m_x x}[w]$-module and we have an isomorphism

$$E|_{m_x x}^\vee \cong \mathcal{O}_{m_x x}[w]/\big(w^r - z\big). \tag{3.3}$$

Combining (3.2) and (3.3), we get an isomorphism

$$\theta\colon\ E|_{m_x x}^\vee \xrightarrow{\sim} E|_{m_x x}$$

of $\mathcal{O}_{m_x x}[w]$-modules. Let

$$\vartheta\colon\ E|_{m_x x}^\vee \times E|_{m_x x}^\vee \longrightarrow \mathcal{O}_{m_x x} \tag{3.4}$$

be the corresponding bilinear pairing defined by $\vartheta(v^*, w^*) = w^*(\theta(v^*))$ for $v^*, w^* \in E|_{m_x x}^\vee$. Take a generator e^* of $E|_{m_x x}^\vee$ as an $\mathcal{O}_{m_x x}[w]$-module. Then any element $v^*, w^* \in E|_{m_x x}^\vee$ can be written $v^* = P\big({}^tN\big)e^*$, $w^* = Q\big({}^tN\big)e^*$ for polynomials $P(w), Q(w) \in \mathcal{O}_{m_x x}[w]$ in w. So we have

$$\begin{aligned}
\vartheta(v^*, w^*) = w^*(\theta(v^*)) &= \big(Q\big({}^tN\big)e^*\big)\big(\theta(P\big({}^tN\big)e^*)\big) \\
&= (e^* \circ Q(N))(P(N)(\theta(e^*))) \\
&= (e^* \circ Q(N) \circ P(N) \circ \theta)(e^*) \\
&= (e^* \circ P(N) \circ Q(N) \circ \theta)(e^*) = \vartheta(w^*, v^*).
\end{aligned} \tag{3.5}$$

In other words, the pairing ϑ defined in (3.4) is symmetric, which is also equivalent to ${}^t\theta = \theta$. If we put

$$\kappa := \theta^{-1} \circ N\colon\ E|_{m_x x} \longrightarrow E|_{m_x x}^\vee,$$

then we have $\theta \circ \kappa = N$. By the similar calculation to (3.5), we can see that the bilinear pairing

$$\varkappa\colon\ E|_{m_x x} \times E|_{m_x x} \longrightarrow \mathcal{O}_{m_x x},$$

determined by $\varkappa(v, w) = \kappa(v)(w)$ is also symmetric, which is equivalent to ${}^t\kappa = \kappa$.

Now we put

$$W_k := \left\{v^* \in E|_{m_x x}^\vee \;\middle|\; v^*\big(z^{m_x - 1}V_{r-k}\big) = 0\right\} = \ker\left(z^{m_x-1}\big({}^tN\big)^{r-k}\right)$$

for $0 \le k \le r$. Then we get the exact commutative diagram

$$\begin{tikzcd}
 & z^{m_x-1}W_{k+1} \arrow[r, phantom, "="] \arrow[d] & z^{m_x-1}W_{k+1} \arrow[d] & \\
0 \arrow[r] & W_k \arrow[r] \arrow[d] & E|_{m_x x}^\vee \arrow[r] \arrow[d] & (z^{m_x-1}V_{r-k})^\vee \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \overline{V}_{r-k-1}^\vee \arrow[r] \arrow[d] & V_{r-k-1}^\vee \arrow[r] \arrow[d] & (z^{m_x-1}V_{r-k})^\vee \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

So we have an isomorphism

$$W_k/z^{m_x-1}W_{k+1} \xrightarrow{\sim} \overline{V}^{\vee}_{r-k-1} = \overline{W}_k.$$

Using $W_k = \ker\left(z^{m_x-1}\left({}^tN\right)^{r-k}\right)$, we can see

$$\theta(W_k) = \theta\left(\ker\left(z^{m_x-1}\left({}^tN\right)^{r-k}\right)\right) = \ker\left(z^{m_x-1}N^{r-k}\right) = V_k.$$

So $\theta|_{W_k}$ induces an isomorphism $\theta_k \colon \overline{W}_k \xrightarrow{\sim} \overline{V}_k$ which makes the diagram

$$\begin{tikzcd} W_k \arrow[r, "\theta|_{W_k}", "\sim"'] \arrow[d] & V_k \arrow[d] \\ \overline{W}_k \arrow[r, "\theta_k", "\sim"'] & \overline{V}_k \end{tikzcd}$$

commutative. By the equality $\kappa = \theta^{-1}N$, we have $\kappa(V_k) \subset W_k$ for $0 \le k \le r$ and get the commutative diagram

$$\begin{tikzcd} V_k \arrow[r, "\kappa|_{V_k}", "\sim"'] \arrow[d] & W_k \arrow[d] \\ \overline{V}_k \arrow[r, "\kappa_k", "\sim"'] & \overline{W}_k. \end{tikzcd}$$

We can associate $(\vartheta_k, \varkappa_k)$ to (θ_k, κ_k) and the conditions (ii) and (iii) of Definition 3.1 follow from the properties of θ, κ. The other conditions (i), (iv) and (v) of Definition 3.1 are satisfied by that of $(V_k, L_k, \pi_k, \phi_k)$. So we get a factorized $\nu(w)$-ramified structure $(V_k, \vartheta_k, \varkappa_k)$.

Assume that there is another factorized ramified structure $(V_k, \vartheta'_k, \varkappa'_k)$ which gives the same generic ν-ramified structure $(V_k, L_k, \pi_k, \phi_k)$. Recall that $\overline{V}_k \xrightarrow{\sim} L_k$. So we have $\theta'_k \circ \kappa'_k = N_k = \theta_k \circ \kappa_k$, because both sides correspond to the multiplication by w on L_k. Since the diagram

$$\begin{tikzcd} \overline{W}_k \arrow[r, "\theta'_k"] \arrow[d, "{}^tN_k = "', "\kappa'_k\circ\theta'_k"] & \overline{V}_k \arrow[d, "N_k"', "=\theta'_k\circ\kappa'_k"] \\ \overline{W}_k \arrow[r, "\theta'_k"] & \overline{V}_k \end{tikzcd}$$

is commutative, $\theta'_k \colon \overline{W}_k \xrightarrow{\sim} \overline{V}_k$ is an isomorphism of free $\mathbb{C}[w]/\left(w^{m_xr-r+1}\right)$-modules of rank one. So there is an element $\beta_k(w) \in \mathbb{C}[w]/\left(w^{m_xr-r+1}\right)^{\times}$ such that $\theta_k = \theta'_k \circ \beta_k({}^tN_k)$. Then we also have $\kappa_k = \beta_k\left({}^tN_k\right)^{-1} \circ \kappa'_k$. Taking account of the compatibility of (θ'_k, κ'_k) with $(\theta'_{k-1}, \kappa'_{k-1})$, we can see $\beta_k(w) \equiv \beta_{k-1}(w) \pmod{w^{mr-r}}$ for $k = 1, \ldots, r-1$. Thus we have $(V_k, \vartheta'_k, \varkappa'_k) \sim (V_k, \vartheta_k, \varkappa_k)$. In other words, the equivalence class of factorized ν-ramified structure $(V_k, \theta_k, \kappa_k)$ is uniquely determined by the generic ν-ramified structure $(V_k, L_k, \pi_k, \phi_k)$. So we can define a correspondence

$$(V_k, L_k, \pi_k, \phi_k) \mapsto (V_k, \theta_k, \kappa_k)$$

and it is the inverse to the correspondence stated in the proposition. ■

Example 3.4. We will see what the factorized ramified structure is in the typical case explained in Example 2.6. We have $(E, \nabla) \otimes \widehat{\mathcal{O}}_{C,x} = (\mathbb{C}[[w]], \nabla_\nu)$ in that case and the filtration in Definition 3.1 (i) is given by $V_k = \left(w^k\right)/\left(w^{mr}\right)$ for $0 \le k \le r$. Consider the trace map

$$\mathrm{Tr}\colon \ \mathbb{C}[[w]] \longrightarrow \mathbb{C}[[z]].$$

For $f(w)\in\mathbb{C}[[w]]$, $\mathrm{Tr}(f(w))$ is defined as the trace of the endomorphism $\mathbb{C}[[w]]\xrightarrow{f(w)}\mathbb{C}[[w]]$ on the free $\mathbb{C}[[z]]$-module $\mathbb{C}[[w]]$ of rank r. By construction, we have $\mathrm{Tr}\left(z^l\right)=rz^l$ and $\mathrm{Tr}\left(w^kz^l\right)=0$ for $1\le k\le r-1$. So the above map induces a homomorphism $\mathrm{Tr}\colon \mathbb{C}[w]/\left(w^{mr-r+1}\right)\to\mathbb{C}[z]/(z^m)$, which also induces

$$\mathbf{Tr}\colon\ \left(w^{r-1}\right)/\left(w^{mr}\right)\otimes\Omega^1_{\mathbb{C}[[w]]/\mathbb{C}}=\mathbb{C}[w]/\left(w^{mr-r+1}\right)\otimes\Omega^1_{\mathbb{C}[[z]]/\mathbb{C}}\xrightarrow{\mathrm{Tr}\otimes\mathrm{id}}\mathbb{C}[z]/\left(z^m\right)\otimes\Omega^1_{\mathbb{C}[[z]]/\mathbb{C}}.$$

Then we can define a pairing

$$\Theta_k\colon\ \left(w^k\right)/\left(w^{mr-r+k+1}\right)\times\left(w^{r-k-1}\right)/\left(w^{mr-k}\right)\longrightarrow\mathbb{C}[z]/\left(z^m\right)$$

by setting

$$\Theta_k\big(f(w),g(w)\big)\mathrm{d}z=\mathbf{Tr}\big(f(w)g(w)\mathrm{d}w\big)$$

for $f(w)\in\left(w^k\right)/\left(w^{mr-r+k+1}\right)$ and $g(w)\in\left(w^{r-k-1}\right)/\left(w^{mr-k}\right)$. By the construction, the induced $\mathbb{C}[z]/(z^m)$-homomorphism $\left(w^k\right)/\left(w^{mr-r+k+1}\right)\to\left(\left(w^{r-k-1}\right)/\left(w^{mr-k}\right)\right)^\vee$ is an isomorphism. If we denote the inverse of this homomorphism by

$$\theta_k\colon\ \left(\left(w^{r-k-1}\right)/\left(w^{mr-k}\right)\right)^\vee\xrightarrow{\sim}\left(w^k\right)/\left(w^{mr-r+k+1}\right),$$

then θ_k induces a pairing

$$\vartheta_k\colon\ \left(\left(w^{r-k-1}\right)/\left(w^{mr-k}\right)\right)^\vee\times\left(\left(w^k\right)/\left(w^{mr-r+k+1}\right)\right)^\vee\longrightarrow\mathbb{C}[z]/\left(z^m\right)$$

satisfying $\vartheta_k(v,v')=\vartheta_{r-k-1}(v',v)$ for $v\in\left(\left(w^{r-k-1}\right)/\left(w^{mr-k}\right)\right)^\vee$ and $v'\in\left(\left(w^k\right)/\left(w^{mr-r+k+1}\right)\right)^\vee$. We can also define a pairing

$$\varkappa_k\colon\ \left(w^k\right)/\left(w^{mr-r+k+1}\right)\times\left(w^{r-k-1}\right)/\left(w^{mr-k}\right)\longrightarrow\mathbb{C}[z]/\left(z^m\right)$$

by setting

$$\varkappa_k(f(w),g(w))=\Theta_k(wf(w)g(w))$$

for $f(w)\in\left(w^k\right)/\left(w^{mr-r+k+1}\right)$ and $g(w)\in\left(w^{r-k-1}\right)/\left(w^{mr-k}\right)$. We can see that the filtration $\mathbb{C}[[w]]/z^m\mathbb{C}[[w]]\supset(w)/\left(w^{mr}\right)\supset\left(w^2\right)/\left(w^{mr}\right)\supset\cdots\supset\left(w^{r-1}\right)/\left(w^{mr}\right)\supset z\mathbb{C}[[w]]/z^m\mathbb{C}[[w]]$ together with $(\vartheta_k,\varkappa_k)_{0\le k\le r-1}$ gives a factorized ν-ramified structure on (E,∇) at x.

Remark 3.5. We can extend the notion of generic ν-ramified structure or that of factorized ν-ramified structure in a relative setting. So, if S is a noetherian scheme (or a noetherian ring) and if (E,∇) is a pair of a vector bundle E on $C\times S$ and a connection ∇ on E, we can mention about a generic ν-ramified structure on (E,∇).

4 Recovery of formal structure from a generic ramified structure

In this section, we will see in Corollary 4.4 that the generic ramified condition given in the Introduction is equivalent to the generic ramified structure (Definition 2.4) or the factorized ν-ramified structure (Definition 3.1). The most essential point is to recover a formal isomorphism from a generic ramified structure or a factorized ν-ramified structure (in Corollary 4.3). In fact, we proved it in [13, Proposition 1.3] by using the Hukuhara–Levert–Turrittin theorem (see [1, Proposition 1.4.1] or [23, Theorem 6.8.1] for example). In this paper, we will examine it by a direct computation only by using regular formal transforms rather than formal Laurent transforms in the Hukuhara–Levert–Turrittin theorem. It has the advantage of applying to (9.5) or (9.10) later. For such applications, we actually require a formal isomorphism in a relative setting in Corollary 4.3.

Let A be a noetherian ring over $\mathbb{C}$. Take a flat family $U \longrightarrow \operatorname{Spec} A$ of smooth affine curves over $\operatorname{Spec} A$ and let $\tilde{x}$ be a section of U over $\operatorname{Spec} A$. We can take a local defining equation $z \in \mathcal{O}_U$ of $\tilde{x}$. Let w be a variable satisfying $w^r = z$. We take an integer m with $m \geq 2$. Choose

$$\big(a_0^{(0)}, a_1^{(0)}, \dots, a_{m-1}^{(0)}\big) \in A^m, \qquad \big(a_0^{(k)}, a_1^{(k)}, \dots, a_{m-2}^{(k)}\big) \in A^{m-1} \qquad (k = 1, \dots, r-1) \tag{4.1}$$

with the condition $a_0^{(1)} \in A^\times$. Using the data (4.1), we put

$$\nu_0(z) = \sum_{l=0}^{m-1} a_l^{(0)} z^l \frac{\mathrm{d}z}{z^m}, \qquad \nu_k(z) = \sum_{l=0}^{m-2} a_l^{(k)} z^l \frac{\mathrm{d}z}{z^m} \qquad (k = 1, \dots, r-1) \tag{4.2}$$

and set

$$\nu(w) := \nu_0(z) + \nu_1(z) w + \cdots + \nu_{r-1}(z) w^{r-1}. \tag{4.3}$$

For an integer q with $q \geq m$, we can regard $A[w]/(w^{qr})$ as a free $A[z]/(z^q)$-module of rank r. Define the A-linear homomorphism

$$\nabla_\nu|_{q\tilde{x}} \colon\ A[w]/(w^{qr}) \longrightarrow A[w]/(w^{qr}) \otimes \Omega^1_{U/A}(m\tilde{t})|_{q\tilde{x}}$$

by setting $\nabla_\nu|_{q\tilde{x}}(f(w)) = df(w) + f(w)\nu(w)$ for $f(w) \in A[w]/(w^{qr})$.

We need the following proposition in the construction of generalized isomonodromic deformation later in Sections 10 and 11.

Proposition 4.1. *Let the notations be as in (4.1), (4.2) and (4.3) with the assumption that the leading coefficient $a_0^{(1)}$ of $\nu_1(z)$ is invertible in A. Take a vector bundle E on U of rank r and a connection $\nabla \colon E \longrightarrow E \otimes \Omega^1_{U/A}(m\tilde{x})$ with a generic ν-ramified structure*

$$\big((V_k, \pi_k, , L_k)_{0 \leq k \leq r-1}, (\phi_k)_{1 \leq k \leq r}\big)$$

at $\tilde{x}$. Then, for any integer q with $q \geq m$, there is an isomorphism

$$\sigma \colon E|_{q\tilde{x}} \xrightarrow{\sim} (A[z]/(z^q))[w]/(w^r - z) \cong A[w]/(w^{qr})$$

which makes the diagram

$$\begin{array}{ccc} E|_{q\tilde{x}} & \xrightarrow[\sim]{\sigma} & A[w]/(w^{qr}) \\ {\scriptstyle \nabla|_{q\tilde{x}}}\big\downarrow & & {\scriptstyle \nabla_\nu|_{q\tilde{x}}}\big\downarrow \\ E|_{q\tilde{x}} \otimes \Omega^1_{U/A}(m\tilde{x})|_{q\tilde{x}} & \xrightarrow[\sim]{\sigma\otimes 1} & A[w]/(w^{qr}) \otimes \frac{\mathrm{d}z}{z^m} \end{array}$$

commutative.

Proof. Let $\tilde{V}_k$ be the pullback of V_k via the canonical surjection $E|_{q\tilde{x}} \longrightarrow E|_{m\tilde{x}}$ for $0 \leq k \leq r-1$. We take a generator $e'_0 \in L_0$ as an $A[w]/\big(w^{mr-r+1}\big)$-module. By the condition (iv) of Definition 2.4, there is a composition of isomorphisms

$$L_k \xrightarrow[\sim]{\psi_k} (w) \otimes L_{k-1} \xrightarrow[\sim]{\psi_{k-1}} \cdots \xrightarrow[\sim]{\psi_1} \big(w^k\big) \otimes L_0.$$

Let $e'_k \in L_k$ be the element corresponding to $w^k \otimes e'_0$ via this isomorphism. Since $\pi_k|_{V_k} : V_k \xrightarrow{\pi_k|_{V_k}} L_k$ is surjective, we can take $\bar{e}_k \in V_k$ satisfying $\pi_k(\bar{e}_k) = e'_k$. Then we have

$$\pi_k(\bar{e}_{k+l}) = w^l \pi_k(\bar{e}_k) \qquad \text{for } 0 \le l \le r-k-1,$$
$$\pi_k(z\bar{e}_l) = w^{r-k+l} \pi_k(\bar{e}_k) \qquad \text{for } 0 \le l \le k-1.$$

We take lifts $e_0, e_1, \ldots, e_{r-1} \in E|_{q\tilde{x}}$ of $\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{r-1} \in E|_{m\tilde{x}}$. The commutativity of the diagram in Definition 2.4 (ii) yields the equality

$$\nabla|_{q\tilde{x}}(e_k) \equiv \left(\nu_0(z) + \frac{k\,\mathrm{d}z}{rz}\right) e_k + \sum_{l=k+1}^{r-1} \nu_{l-k}(z) e_l + \sum_{l=0}^{k-1} z\nu_{r+l-k}(z) e_l \qquad \left(\mathrm{mod}\ z^{m-1}\tilde{V}_{k+1}\frac{\mathrm{d}z}{z^m}\right)$$

for $k = 0, 1, \ldots, r-1$. Applying the following lemma to the cases

$$(q', s) = (m,1), (m,2), \ldots, (m,r), (m+1,1), (m+1,2), \ldots, (q,1), \ldots, (q, r-1)$$

successively, we get the proposition. ∎

Lemma 4.2. *Let q', s be integers with $m \le q' \le q$ and $1 \le s \le r$. Assume that the equalities*

$$\nabla|_{q\tilde{x}}(e_k) \equiv \left(\nu_0 + \frac{k\,\mathrm{d}z}{rz}\right) e_k + \sum_{l=k+1}^{r-1} \nu_{l-k} e_l + \sum_{l=0}^{k-1} \nu_{r+l-k} z e_l \qquad \left(\mathrm{mod}\ z^{q'-1}\tilde{V}_{k+s}\frac{\mathrm{d}z}{z^m}\right) \tag{4.4}$$

hold for $0 \le k < r-s$ and the equalities

$$\nabla|_{q\tilde{x}}(e_k) \equiv \left(\nu_0 + \frac{k\,\mathrm{d}z}{r\,z}\right) e_k + \sum_{l=k+1}^{r-1} \nu_{l-k} e_l + \sum_{l=0}^{k-1} \nu_{r+l-k} z e_l \qquad \left(\mathrm{mod}\ z^{q'}\tilde{V}_{k+s-r}\frac{\mathrm{d}z}{z^m}\right) \tag{4.5}$$

hold for $r-s \le k \le r-1$. Then there exist $c, b_1, \ldots, b_{r-1} \in A$ such that the replacement

$$\tilde{e}_0 = \begin{cases} e_0 + cz^{q'-m} e_s & \text{if } 1 \le s \le r-1, \\ e_0 + cz^{q'-m+1} e_0 & \text{if } s = r, \end{cases}$$
$$\tilde{e}_k = \begin{cases} e_k + cz^{q'-m} e_{k+s} + b_k z^{q'-1} e_{k+s-1} & \text{if } k+s < r \text{ and } 1 \le k \le r-1, \\ e_k + cz^{q'-m+1} e_{k+s-r} + b_k z^{q'-1} e_{k+s-1} & \text{if } k+s = r \text{ and } 1 \le k \le r-1, \\ e_k + cz^{q'-m+1} e_{k+s-r} + b_k z^{q'} e_{k+s-1-r} & \text{if } k+s > r \text{ and } 1 \le k \le r-1 \end{cases} \tag{4.6}$$

leads to the equalities

$$\nabla|_{q\tilde{x}}(\tilde{e}_k) \equiv \left(\nu_0 + \frac{k\,\mathrm{d}z}{rz}\right) \tilde{e}_k + \sum_{l=k+1}^{r-1} \nu_{l-k} \tilde{e}_l + \sum_{l=0}^{k-1} \nu_{r+l-k} z \tilde{e}_l \qquad \left(\mathrm{mod}\ z^{q'-1}\tilde{V}_{k+s+1}\frac{\mathrm{d}z}{z^m}\right) \tag{4.7}$$

for $0 \le k < r-s-1$ and the equalities

$$\nabla|_{q\tilde{x}}(\tilde{e}_k) \equiv \left(\nu_0 + \frac{k\,\mathrm{d}z}{rz}\right) \tilde{e}_k + \sum_{l=k+1}^{r-1} \nu_{l-k} \tilde{e}_l + \sum_{l=0}^{k-1} \nu_{r+l-k} z \tilde{e}_l \qquad \left(\mathrm{mod}\ z^{q'}\tilde{V}_{k+s+1-r}\frac{\mathrm{d}z}{z^m}\right) \tag{4.8}$$

for $r-s-1 \le k \le r-1$.

Proof. By the assumption (4.4), we can find $\eta_0,\dots,\eta_{r-s-1}\in z^{q'-1}\Omega^1_{U/A}(D)|_{q\tilde{x}}$ satisfying the equalities

$$\begin{aligned}\nabla|_{q\tilde{x}}(e_k)&\equiv\left(\nu_0+\frac{k\,\mathrm{d}z}{rz}\right)e_k+\sum_{l=k+1}^{r-1}\nu_{l-k}e_l\\&\quad+\sum_{l=0}^{k-1}\nu_{r+l-k}ze_l+\eta_k e_{k+s}\qquad\left(\mathrm{mod}\ z^{q'-1}\tilde{V}_{k+s+1}\frac{\mathrm{d}z}{z^m}\right)\end{aligned}$$

for $0\le k<r-s-1$ and the equality

$$\nabla|_{q\tilde{x}}(e_k)\equiv\left(\nu_0+\frac{k\,\mathrm{d}z}{rz}\right)e_k+\sum_{l=k+1}^{r-1}\nu_{l-k}e_l+\sum_{l=0}^{k-1}\nu_{r+l-k}ze_l+\eta_k e_{r-1}\qquad\left(\mathrm{mod}\ z^{q'}\tilde{V}_0\frac{dz}{z^m}\right)$$

for $k+s=r-1$. By the assumption (4.5), we can find $\eta_{r-s},\dots,\eta_{r-1}\in z^{q'-1}\Omega^1_{U/A}(D)|_{q\tilde{x}}$ satisfying the equalities

$$\begin{aligned}\nabla|_{q\tilde{x}}(e_k)&\equiv\left(\nu_0+\frac{k\,\mathrm{d}z}{rz}\right)e_k+\sum_{l=k+1}^{r-1}\nu_{l-k}e_l\\&\quad+\sum_{l=0}^{k-1}\nu_{r+l-k}ze_l+\eta_k ze_{k+s-r}\qquad\left(\mathrm{mod}\ z^{q'}\tilde{V}_{k+s+1-r}\frac{\mathrm{d}z}{z^m}\right)\end{aligned}$$

for $r-s\le k\le r-1$. We will determine $c,b_1,\dots,b_{r-1}\in A$ so that the substitution of (4.6) enables the equalities (4.7) and (4.8) to hold.

Consider the substitution of $\tilde{e}_k$ for $0\le k<r-s$. In that case, we have

$$\begin{aligned}\nabla|_{q\tilde{x}}(\tilde{e}_k)&=\nabla|_{q\tilde{x}}(e_k)+(q'-m)cz^{q'-m-1}\mathrm{d}ze_{k+s}+cz^{q'-m}\nabla(e_{k+s})\\&\quad+(q'-1)b_kz^{q'-2}\mathrm{d}ze_{k+s-1}+b_kz^{q'-1}\nabla(e_{k+s-1}).\end{aligned}$$

If we put $b_0:=0$ and $b_r:=0$, then we can calculate the above substitution in the following, while using $b_kz^{q'-1}\nu_{l-k-s+1}e_l\equiv 0\ (\mathrm{mod}\ z^{q'-1}\tilde{V}_{k+s+1}\mathrm{d}z/z^m)$ for $l\ge k+s+1$ in the second equality;

$$\begin{aligned}\nabla|_{q\tilde{x}}(\tilde{e}_k)&\equiv(q'-m)cz^{q'-m-1}\mathrm{d}ze_{k+s}+(q'-1)b_kz^{q'-2}dze_{k+s-1}+\left(\nu_0+\frac{k\,\mathrm{d}z}{rz}\right)e_k+\nu_1e_{k+1}\\&\quad+\sum_{l=k+2}^{r-1}\nu_{l-k}e_l+\sum_{l=0}^{k-1}\nu_{r+l-k}ze_l+cz^{q'-m}\left(\nu_0+\frac{(k+s)}{r}\frac{\mathrm{d}z}{z}\right)e_{k+s}\\&\quad+cz^{q'-m}\nu_1e_{k+s+1}+\sum_{l=k+s+2}^{r-1}cz^{q'-m}\nu_{l-k-s}e_l+\sum_{l=0}^{k+s-1}cz^{q'-m+1}\nu_{r+l-k-s}e_l\\&\quad+b_kz^{q'-1}\nu_0e_{k+s-1}+b_kz^{q'-1}\nu_1e_{k+s}+\sum_{l=k+s+1}^{r-1}b_kz^{q'-1}\nu_{l-k-s+1}e_l\\&\quad+\sum_{l=0}^{k+s-2}b_kz^{q'}\nu_{r+l-k-s+1}e_l+\eta_ke_{k+s}\\&\equiv\left(\nu_0+\frac{k\,\mathrm{d}z}{rz}\right)\tilde{e}_k+\left(\nu_s+\eta_k+\frac{((q'-m)r+s)z^{q'-1}c}{rz^m}\mathrm{d}z+(b_k-b_{k+1})z^{q'-1}\nu_1\right)\tilde{e}_{k+s}\\&\quad+\nu_1\tilde{e}_{k+1}+\sum_{k+2\le l\le r-1,l\ne k+s}\nu_{l-k}\tilde{e}_l\ +\sum_{l=0}^{k-1}\nu_{r+l-k}z\tilde{e}_l\qquad\left(\mathrm{mod}\ z^{q'-1}\tilde{V}_{k+s+1}\frac{\mathrm{d}z}{z^m}\right).\end{aligned}$$

We can similarly calculate the substitution of $\tilde{e}_k$ for $r-s \leq k \leq r-1$ and we have

$$\begin{aligned}\nabla|_{q\tilde{x}}(\tilde{e}_k) \equiv & \left(\nu_0 + \frac{k\,\mathrm{d}z}{rz}\right)\tilde{e}_k \\ & + \left(\nu_s + \eta_k + \frac{((q'-m)r+s)z^{q'-1}c}{rz^m}\mathrm{d}z + (b_k - b_{k+1})z^{q'-1}\nu_1\right) z\tilde{e}_{k+s-r} \\ & + \sum_{k+1\leq l \leq r-1, l\neq k+s} \nu_{l-k}\tilde{e}_l \; + \sum_{l=0}^{k-1} \nu_{r+l-k} z\tilde{e}_l \quad \left(\mathrm{mod}\ z^{q'}\tilde{V}_{k+s+1-r}\frac{\mathrm{d}z}{z^m}\right).\end{aligned}$$

So it is sufficient to solve the equation

$$\begin{pmatrix} \frac{((q'-m)r+s)z^{q'-1}}{rz^m}\mathrm{d}z & -z^{q'-1}\nu_1 & 0 & \cdots & 0 \\ \frac{((q'-m)r+s)z^{N'-1}}{rz^m}\mathrm{d}z & z^{q'-1}\nu_1 & -z^{q'-1}\nu_1 & \ddots & 0 \\ \frac{((q'-m)r+s)z^{q'-1}}{rz^m}\mathrm{d}z & 0 & z^{q'-1}\nu_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -z^{q'-1}\nu_1 \\ \frac{((q'-m)r+s)z^{q'-1}}{rz^m}\mathrm{d}z & 0 & \cdots & 0 & z^{q'-1}\nu_1 \end{pmatrix} \begin{pmatrix} c \\ b_1 \\ b_2 \\ \vdots \\ b_{r-2} \\ b_{r-1} \end{pmatrix} = \begin{pmatrix} -\eta_0 \\ -\eta_1 \\ -\eta_2 \\ \vdots \\ -\eta_{r-2} \\ -\eta_{r-1} \end{pmatrix},$$

which is possible because the $r \times r$ matrix of the left hand side is invertible. ■

Under the setting (4.1), (4.2) and (4.3), let $\nabla_\nu \colon A[[w]] \longrightarrow A[[w]] \otimes \Omega^1_{U/A}(m\tilde{t})$ be the relative formal connection defined by $\nabla_\nu(f(w)) = \mathrm{d}f(w) + f(w)\nu$ for $f(w) \in A[[w]]$. If we take the inverse limit of the isomorphisms $(E,\nabla) \otimes A[z]/(z^q) \xrightarrow{\sim} (A[[w]]/(w^{qr}), \nabla_\nu|_{q\tilde{x}})$ constructed in Proposition 4.1, we get the following corollary.

Corollary 4.3. *Under the same assumption as Proposition 4.1, there is an isomorphism*

$$(E,\nabla) \otimes A[[z]] \cong (A[[w]], \nabla_\nu).$$

If a connection (E,∇) has a formal isomorphism $(E,\nabla) \otimes \widehat{\mathcal{O}}_{C,x} \cong (\mathbb{C}[[w]], \nabla_\nu)$ at x, then it induces a generic ν-ramified structure as in Example 2.6. Conversely, the above corollary enables us to recover a formal isomorphism from a ν-ramified structure in Definition 2.4 or a factorized ν-ramified structure in Definition 3.1. So we have the following corollary.

Corollary 4.4. *Let (E,∇) be a pair of a vector bundle E of rank r on a curve C and a connection $\nabla \colon E \longrightarrow E \otimes \Omega^1_C(D)$ with poles along the divisor D whose multiplicity at x is m. Take a generator z of the maximal ideal of $\mathcal{O}_{X,x}$ and a variable w with $w^r = z$. Consider a rational one form $\nu(w) = \nu_0(z) + \nu_1(z)w + \cdots + \nu_{r-1}(z)w^{r-1}$ such that $\nu_0(z) \in \sum_{i=0}^{m-1} \mathbb{C}z^{i-m}\mathrm{d}z$, $\nu_k(z) \in \sum_{i=0}^{m-2} \mathbb{C}z^{i-m}\mathrm{d}z$ for $1 \leq k \leq r-1$ and that the leading term of $\nu_1(z)$ does not vanish. Then the following conditions are equivalent.*

1. *(E,∇) is generic ν-ramified at x, that is, $(\widehat{E},\widehat{\nabla}) \cong (\mathbb{C}[[w]], \nabla_\nu)$.*
2. *There is a generic ν-ramified structure on (E,∇) at x in the sense of Definition 2.4.*
3. *There is a factorized ν-ramified structure on (E,∇) at x in the sense of Definition 3.1.*

5 Construction of the moduli space of connections

The moduli space of ramified connections is constructed in [13]. Since some notations in this paper are different from those in [13], we recall the construction of the moduli space in our setting.

Let $n_{\log}$, n_{un}, n_{ram} be non-negative integers and put $n = n_{\log} + n_{\mathrm{un}} + n_{\mathrm{ram}}$. Consider the moduli stack $\mathcal{M}_{g,n}$ of n-pointed curves $\big(C, x_1^{(\log)}, \dots, x_{n_{\log}}^{(\log)}, x_1^{(\mathrm{un})}, \dots, x_{n_{\mathrm{un}}}^{(\mathrm{un})}, x_1^{(\mathrm{ram})}, \dots, x_{n_{\mathrm{ram}}}^{(\mathrm{ram})}\big)$ of genus g over $\operatorname{Spec}\mathbb{C}$. We can take a smooth algebraic scheme $\mathcal{H}$ over $\operatorname{Spec}\mathbb{C}$ with a smooth surjective morphism $H \longrightarrow \mathcal{M}_{g,n}$. Indeed, we can take a subscheme H' of $\mathrm{Hilb}_{\mathbb{P}^L}$ parameterizing the l-th canonical embeddings $C \hookrightarrow \mathbb{P}(H^0(\omega_C^l))$ of smooth projective curves C of genus g for a fixed large l if $g \geq 2$. If $g = 1$, we take H' as the open subset of $\mathbb{P}_*(H^0(\mathcal{O}_{\mathbb{P}^2}(3)))$ parameterizing the smooth cubic curves in $\mathbb{P}^2$. If $g = 0$, we take H' as a point. In any case, there is a universal family $\mathcal{Z} \subset \mathbb{P}^L \times H'$ of curves over H'. Then the open subscheme $\mathcal{H}$ of the fiber product of n copies of $\mathcal{Z}$ over H' parameterizing the distinct n points on the curves satisfies our request. We can take a universal family $\big(\mathcal{C} \times \mathcal{H}, (\tilde{x}_i^{\log})_{1 \leq i \leq n_{\log}}, (\tilde{x}_i^{\mathrm{un}})_{1 \leq i \leq n_{\mathrm{un}}}, (\tilde{x}_i^{\mathrm{ram}})_{1 \leq i \leq n_{\mathrm{ram}}}\big)$ consisting of flat family of curves of genus g over $\mathcal{H}$ and sections $\tilde{x}_i^{\log}$ $(1 \leq i \leq n_{\log})$, $\tilde{x}_i^{\mathrm{un}}$ $(1 \leq i \leq n_{\mathrm{un}})$, $\tilde{x}_i^{\mathrm{ram}}$ $(1 \leq i \leq n_{\mathrm{ram}})$ of $\mathcal{C}$ over $\mathcal{H}$. We denote the ideal sheaf of $\tilde{x}_i^{\mathrm{un}}$ (resp. $\tilde{x}_j^{\mathrm{ram}}$) by $I_{\tilde{x}_i^{\mathrm{un}}}$ (resp. $I_{\tilde{x}_j^{\mathrm{ram}}}$).

Assume that integers $m_i^{\mathrm{un}} \geq 2$ are given for $1 \leq i \leq n_{\mathrm{un}}$ and integers $m_i^{\mathrm{ram}} \geq 2$ are given for $1 \leq i \leq n_{\mathrm{ram}}$. We put

$$\mathcal{D}_{\log} := \sum_{i=1}^{n_{\log}} \tilde{x}_i^{\log}, \qquad \mathcal{D}_{\mathrm{un}} := \sum_{i=1}^{n_{\mathrm{un}}} m_i^{\mathrm{un}} \tilde{x}_i^{\mathrm{un}}, \qquad \mathcal{D}_{\mathrm{ram}} := \sum_{i=1}^{n_{\mathrm{ram}}} m_i^{\mathrm{ram}} \tilde{x}_i^{\mathrm{ram}},$$
$$\mathcal{D} := \mathcal{D}_{\log} + \mathcal{D}_{\mathrm{un}} + \mathcal{D}_{\mathrm{ram}}.$$

Let $\mathcal{X}$ be the maximal open subset of

$$\operatorname{Spec}\operatorname{Sym}_{\mathcal{O}_{\mathcal{H}}}\left(\mathcal{H}om_{\mathcal{O}_{\mathcal{H}}}\left(\bigoplus_{i=1}^{n_{\mathrm{un}}} I_{\tilde{x}_i^{\mathrm{un}}}/(I_{\tilde{x}_i^{\mathrm{un}}})^{m_i^{\mathrm{un}}+1} \oplus \bigoplus_{j=1}^{n_{\mathrm{ram}}} I_{\tilde{x}_j^{\mathrm{ram}}}/(I_{\tilde{x}_j^{\mathrm{ram}}})^{m_j^{\mathrm{ram}}+1}, \, \mathcal{O}_{\mathcal{H}} \right)\right)$$

such that the restriction $\bar{z}$ of the universal section to $\mathcal{X}$ gives a generator of $\big(I_{\tilde{x}_i^{\mathrm{un}}}/I_{\tilde{x}_i^{\mathrm{un}}}^{m_i^{\mathrm{un}}+1}\big) \otimes_{\mathcal{O}_{\mathcal{H}}} \mathcal{O}_{\mathcal{X}}$ at each $\tilde{x}_i^{\mathrm{un}}$ (resp. a generator of $\big(I_{\tilde{x}_j^{\mathrm{ram}}}/I_{\tilde{x}_j^{\mathrm{ram}}}^{m_j^{\mathrm{ram}}+1}\big) \otimes_{\mathcal{O}_{\mathcal{H}}} \mathcal{O}_{\mathcal{X}}$ at each $\tilde{x}_j^{\mathrm{ram}}$).

Fix complex numbers

$$\lambda = \big(\lambda_k^{(i)}\big)_{1 \leq i \leq n_{\log},\, 0 \leq k \leq r-1} \in \mathbb{C}^{r n_{\log}},$$
$$c^{\mathrm{un}} = \big(c_{i,k}^{\mathrm{un}}\big)_{1 \leq i \leq n_{\mathrm{un}},\, 0 \leq k \leq r-1} \in \mathbb{C}^{r n_{\mathrm{un}}},$$
$$c^{\mathrm{ram}} = \big(c_i^{\mathrm{ram}}\big)_{1 \leq i \leq n_{\mathrm{ram}}} \in \mathbb{C}^{n_{\mathrm{ram}}},$$

which satisfy the equality

$$d + \sum_{i=1}^{n_{\log}} \sum_{k=0}^{r-1} \lambda_k^{(i)} + \sum_{i=1}^{n_{\mathrm{un}}} \sum_{k=0}^{r-1} c_{i,k}^{\mathrm{un}} + \sum_{i=1}^{n_{\mathrm{ram}}} \left(r c_i^{\mathrm{ram}} + \frac{r-1}{2} \right) = 0$$

for an integer d. We set

$$\mathbf{V} = \operatorname{Spec}\left(\operatorname{Sym}_{\mathcal{O}_{\mathcal{X}}}\left(\bigoplus_{i=1}^{n_{\mathrm{un}}} \mathcal{O}_{\mathcal{X}}^{\oplus (m_i^{\mathrm{un}}-1)r} \oplus \bigoplus_{j=1}^{n_{\mathrm{ram}}} \mathcal{O}_{\mathcal{X}}^{\oplus (m_j^{\mathrm{ram}}-1)r} \right)\right)$$

and take universal sections

$$\big(\tilde{a}_{i,k,j}^{\mathrm{un}}\big)_{1 \leq i \leq n_{\mathrm{un}}, 0 \leq k \leq r-1, 0 \leq j \leq m_i^{\mathrm{un}}-2} \in \bigoplus_{i=1}^{n_{\mathrm{un}}} \bigoplus_{k=0}^{r-1} \bigoplus_{j=0}^{m_i^{\mathrm{un}}-2} \mathcal{O}_{\mathbf{V}},$$
$$\big(\tilde{a}_{i,k,j}^{\mathrm{ram}}\big)_{1 \leq i \leq n_{\mathrm{ram}}, 0 \leq k \leq r-1, 0 \leq j \leq m_i^{\mathrm{ram}}-2} \in \bigoplus_{i=1}^{n_{\mathrm{ram}}} \bigoplus_{k=0}^{r-1} \bigoplus_{j=0}^{m_i^{\mathrm{ram}}-2} \mathcal{O}_{\mathbf{V}}.$$

Let $\mathcal{T}$ be the Zariski open subset of $\mathbf{V}$ defined by

$$\mathcal{T} = \left\{ t \in \mathbf{V} \,\middle|\, \begin{array}{l} \text{for each } 1 \leq i \leq n_{\mathrm{un}},\ \tilde{a}^{\mathrm{un}}_{i,k,0}(t) \neq \tilde{a}^{\mathrm{un}}_{i,k',0}(t) \text{ for } k \neq k', \\ \text{and } \tilde{a}^{\mathrm{ram}}_{i,1,0}(t) \neq 0 \text{ for any } 1 \leq i \leq n_{\mathrm{ram}} \end{array} \right\}.$$

We take a lift z of $\overline{z}$ as a local algebraic function in a neighborhood of $\mathcal{D}$ and rephrase the above universal sections by setting

$$\tilde{\mu}_k(\bar{z}) = \sum_{i=1}^{n_{\mathrm{un}}} \left(\tilde{a}^{\mathrm{un}}_{i,k,0} + \cdots + \tilde{a}^{\mathrm{un}}_{i,k,m^{\mathrm{un}}_i-2}\, \bar{z}^{m^{\mathrm{un}}_i-2} + c^{\mathrm{un}}_{i,k}\, \bar{z}^{m^{\mathrm{un}}_i-1}\right) \frac{\mathrm{d}z}{z^{m^{\mathrm{un}}_i}}\bigg|_{m^{\mathrm{un}}_i(\tilde{x}_i)_{\mathcal{T}}} \quad (0 \leq k \leq r-1),$$

$$\tilde{\nu}_0(\bar{z}) = \sum_{i=1}^{n_{\mathrm{ram}}} \left(\tilde{a}^{\mathrm{ram}}_{i,0,0} + \tilde{a}^{\mathrm{ram}}_{i,0,1}\, \bar{z} + \cdots + \tilde{a}^{\mathrm{ram}}_{i,0,m^{\mathrm{ram}}_i-2}\, \bar{z}^{m^{\mathrm{ram}}_i-2} + c^{\mathrm{ram}}_{i}\, \bar{z}^{m^{\mathrm{ram}}_i-1}\right) \frac{\mathrm{d}z}{z^{m^{\mathrm{ram}}_i}}\bigg|_{m^{\mathrm{ram}}_i(\tilde{x}_i)_{\mathcal{T}}},$$

$$\tilde{\nu}_k(\bar{z}) = \sum_{i=1}^{n_{\mathrm{ram}}} \left(\tilde{a}^{\mathrm{ram}}_{i,k,0} + \tilde{a}^{\mathrm{ram}}_{i,k,1}\, \bar{z} + \cdots + \tilde{a}^{\mathrm{ram}}_{i,k,m^{\mathrm{ram}}_i-2}\, \bar{z}^{m^{\mathrm{ram}}_i-2}\right) \frac{\mathrm{d}z}{z^{m^{\mathrm{ram}}_i}}\bigg|_{m^{\mathrm{ram}}_i(\tilde{x}_i)_{\mathcal{T}}} \quad (1 \leq k \leq r-1),$$

$$\tilde{\nu}(w) = \tilde{\nu}_0(\bar{z}) + \tilde{\nu}_1(\bar{z})w + \cdots + \tilde{\nu}_{r-1}(\bar{z})w^{r-1}$$

and we write $\tilde{\mu} := (\tilde{\mu}_k)_{0\leq k\leq r-1}$ and $\tilde{\nu} := \tilde{\nu}(w)$. Note that the restriction of the differential forms $\frac{\mathrm{d}z}{z^{m^{\mathrm{un}}_i}}\big|_{m^{\mathrm{un}}_i(\tilde{x}_i)_{\mathcal{T}}}$, $\frac{\mathrm{d}z}{z^{m^{\mathrm{ram}}_i}}\big|_{m^{\mathrm{ram}}_i(\tilde{x}_i)_{\mathcal{T}}}$ are independent of the choice of the representative z of $\overline{z}$ and are uniquely determined by $\overline{z}$.

We fix a parabolic weight $\boldsymbol{\alpha} = \left((\alpha^{\log}_k)^{1\leq i\leq n_{\log}}_{1\leq k\leq r}, (\alpha^{\mathrm{un}}_{i,k})^{1\leq i\leq n_{\mathrm{un}}}_{1\leq k\leq r}, (\alpha^{\mathrm{ram}}_{i,k})^{1\leq i\leq n_{\mathrm{ram}}}_{1\leq k\leq r}\right)$ as in Definition 2.8.

Theorem 5.1 ([13, Theorem 2.1])**.** *There exists a relative coarse moduli space $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu}) \longrightarrow \mathcal{T}$ of $\boldsymbol{\alpha}$-stable connections with $(\lambda, \tilde{\mu}, \tilde{\nu})$-structure on $(\mathcal{C}, \mathcal{D})$. Furthermore, $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu}) \longrightarrow \mathcal{T}$ is a quasi-projective morphism.*

Proof. We use the same argument as in the proof of [15, Theorem 2.1] and [13, Theorem 2.1]. Consider the moduli functor $\mathcal{M}$ of tuples $(E, \nabla, l, \ell, (V_k))$ consisting of rank r vector bundles E, connections ∇ admitting poles along $\mathcal{D}$ and parabolic structure $\ell, l, (V_k)$ along $\mathcal{D}$ satisfying $\boldsymbol{\alpha}$-stability. Then we can embed $\mathcal{M}$ to a locally closed subfunctor of the moduli functor of stable parabolic triples $(E_1, E_2, \phi, \nabla, F_*(E_1))$, whose existence is proved in [14, Theorem 5.1]. So we can get a moduli space M which represents the étale sheafification of $\mathcal{M}$ and M is quasi-projective over $\mathcal{T}$. We can construct a quasi-projective scheme $M_{\lambda,\tilde{\mu}}$ over M which parameterizes $(\lambda, \tilde{\mu})$-structure on (E, ∇) compatible with l, ℓ as in the proof of [11, Theorem 2.1] and [15, Theorem 2.1].

We only have to construct a parameter space of $\tilde{\nu}$-ramified structure over $M_{\lambda,\tilde{\mu}}$ such that the filtration in Definition 2.4 (i) coincides with the given filtration (V_k). There is an étale surjective morphism $M' \longrightarrow M_{\lambda,\tilde{\mu}}$ with a universal family $\left(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, (\tilde{V}_k)\right)$ on $\mathcal{C}_{M'}$ over M'. We set

$$A_w := \prod_{i=1}^{n_{\mathrm{ram}}} \mathcal{O}_{M'}[w]/\left(w^{m^{\mathrm{ram}}_i r - r + 1}\right).$$

Since A_w is a finite module over M', we can construct a locally closed subscheme Q of a product of Quot-schemes over M' such that the set of S-valued points of Q is

$$Q(S) = \left\{ \left(\tilde{V}_k \otimes (A_w)_S \xrightarrow{\pi_k} L_k\right)_{0\leq k\leq r-1} \,\middle|\, \begin{array}{l} L_k \text{ is a quotient } A_w \text{ module of } \tilde{V}_k \otimes (A_w)_S \text{ and} \\ L_k \text{ is a locally free } (A_w)_S\text{-module of rank one} \end{array} \right\}.$$

Let $\pi_k \colon \tilde{V}_k \otimes (A_w)_Q \longrightarrow \tilde{L}_k$ be the universal quotient sheaf. There exists a maximal locally closed subscheme $\Sigma \subset Q$ such that the restrictions $(\pi_k)_\Sigma\big|_{(\tilde{V}_k)_\Sigma} \colon \left(\tilde{V}_k\right)_\Sigma \longrightarrow \left(\tilde{L}_k\right)_\Sigma$ are surjective, the diagrams

$$\begin{array}{ccc}
\big(\tilde{V}_k\big)_\Sigma & \xrightarrow{\tilde{\nabla}|_{(\mathcal{D}_{\mathrm{ram}})_\Sigma}} & (\tilde{V}_k)_\Sigma \otimes \Omega^1_{\mathcal{C}_\Sigma/\Sigma}((\mathcal{D}_{\mathrm{ram}})_\Sigma) \\
{\scriptstyle \pi_k}\big\downarrow & & \big\downarrow{\scriptstyle \pi_k \otimes \mathrm{id}} \\
\big(\tilde{L}_k\big)_\Sigma & \xrightarrow{\nu(w)+\frac{kdz}{rz}} & \big(\tilde{L}_k\big)_\Sigma \otimes \Omega^1_{\mathcal{C}_\Sigma/\Sigma}((\mathcal{D}_{\mathrm{ram}})_\Sigma)
\end{array}$$

are commutative for $0 \leq k \leq r-1$, each composition $\big(\tilde{V}_k\big)_\Sigma \longrightarrow \big(\tilde{V}_{k-1}\big)_\Sigma \xrightarrow{\pi_{k-1}} \big(\tilde{L}_{k-1}\big)_\Sigma$ factors through an $(A_w)_\Sigma$-homomorphism $\tilde{\phi}_k \colon \big(\tilde{L}_k\big)_\Sigma \longrightarrow \big(\tilde{L}_{k-1}\big)_\Sigma$ whose image is $w\big(\tilde{L}_{k-1}\big)_\Sigma$ for $1 \leq k \leq r-1$ and the composition $(z) \otimes (V_0)_\Sigma \longrightarrow (V_{r-1})_\Sigma \xrightarrow{\pi_{r-1}} \big(\tilde{L}_{r-1}\big)_\Sigma$ factors thorough an $(A_w)_\Sigma$-homomorphism $\tilde{\phi}_r \colon (z) \otimes \big(\tilde{L}_0\big)_\Sigma \longrightarrow (\tilde{L}_{r-1})_\Sigma$ whose image is $w\big(\tilde{L}_{r-1}\big)_\Sigma$. We denote the free $(A_w)_\Sigma$-module $\bigoplus_{i=1}^{n_{\mathrm{ram}}} \big(w^k\big)/\big(w^{k+m_i^{\mathrm{ram}}r-r+1}\big)$ simply by $\big(w^k\big)$. Consider the affine space bundle

$$\mathbf{V}_k = \operatorname{Spec}\operatorname{Sym}_{\mathcal{O}_\Sigma}\big(\mathcal{H}\mathrm{om}_{\mathcal{O}_\Sigma}\big(\mathcal{H}\mathrm{om}_{(A_w)_\Sigma}\big(\big(\tilde{L}_k\big)_\Sigma, ((w)\otimes A_w \otimes \tilde{L}_{k-1})_\Sigma\big), \mathcal{O}_\Sigma\big)\big) \longrightarrow \Sigma$$

for $k = 1, \ldots, r-1$ and take a universal section

$$\psi_k \colon \ \big(\tilde{L}_k\big)_{\mathbf{V}_k} \longrightarrow \big((w)\otimes A_w \otimes \tilde{L}_{k-1}\big)_{\mathbf{V}_k}.$$

There is a morphism

$$c_k \colon \ \mathbf{V}_k \longrightarrow \operatorname{Spec}\operatorname{Sym}_{\mathcal{O}_\Sigma}\big(\mathcal{H}\mathrm{om}_{\mathcal{O}_\Sigma}\big(\mathcal{H}\mathrm{om}_{(A_w)_\Sigma}\big(\big(\tilde{L}_k\big)_\Sigma, \big(w\tilde{L}_{k-1}\big)_\Sigma\big), \mathcal{O}_\Sigma\big)\big)$$

over Σ defined by the composition

$$\big(\tilde{L}_k\big)_{\mathbf{V}_k} \xrightarrow{\psi_k} \big((w)\otimes A_w \otimes \tilde{L}_{k-1}\big)_{\mathbf{V}_k} \longrightarrow \big(w\tilde{L}_{k-1}\big)_{\mathbf{V}_k}.$$

Over the fiber $c_k^{-1}\big(\tilde{\phi}_k\big) \subset \mathbf{V}_k$, the composition

$$\big(\tilde{L}_k\big)_{c_k^{-1}(\tilde{\phi}_k)} \xrightarrow{\psi_k} \big((w)\otimes A_w \otimes \tilde{L}_{k-1}\big)_{c_k^{-1}(\tilde{\phi}_k)} \longrightarrow \big(w\tilde{L}_{k-1}\big)_{c_k^{-1}(\tilde{\phi}_k)}$$

coincides with $\big(\tilde{\phi}_k\big)_{c_k^{-1}(\tilde{\phi}_k)} \colon \big(\tilde{L}_k\big)_{c_k^{-1}(\tilde{\phi}_k)} \longrightarrow \big(w\tilde{L}_{k-1}\big)_{c_k^{-1}(\tilde{\phi}_k)}$, which is surjective. So, we can see by the Nakayama's lemma, that $(\psi_k)_{c_k^{-1}(\tilde{\phi}_k)} \colon \big(\tilde{L}_k\big)_{c_k^{-1}(\tilde{\phi}_k)} \longrightarrow (w)\otimes\big(\tilde{L}_{k-1}\big)_{c_k^{-1}(\tilde{\phi}_k)}$ is surjective and then $(\psi_k)_{c_k^{-1}(\tilde{\phi}_k)}$ is isomorphic because it is a surjection between locally free $(A_w)_{c_k^{-1}(\tilde{\phi}_k)}$-modules of rank one. Consider the group scheme G over Σ whose set of S-valued points is

$$G(S) = \prod_{i=1}^{n_{\mathrm{ram}}} \big(1 + H^0(\mathcal{O}_S) z^{m_i^{\mathrm{ram}}-1}\big),$$

where each component $\big(1+H^0(\mathcal{O}_S)z^{m_i^{\mathrm{ram}}-1}\big)$ is regarded as a subgroup of the group of invertible elements of $H^0((A_w)_S)$. Then there is a canonical action of G on the product $Y := \prod_{k=1}^{r-1} c_k^{-1}\big(\tilde{\phi}_k\big)$ and

$$Y = \prod_{k=1}^{r-1} c_k^{-1}\big(\tilde{\phi}_k\big) \longrightarrow \Sigma$$

is a G-torsor. Consider the composition

$$\begin{aligned}
\psi_1 \circ \cdots \circ \psi_{r-1} \circ \phi_r \colon \ (z)\otimes\big(\tilde{L}_0\big)_Y \xrightarrow{\tilde{\phi}_r} \big(\tilde{L}_{r-1}\big)_Y &\xrightarrow[\sim]{\psi_{r-1}} (w)\otimes\big(\tilde{L}_{r-2}\big)_Y \\
&\xrightarrow[\sim]{\psi_{r-2}} \cdots \xrightarrow[\sim]{\psi_1} \big(w^{r-1}\big)\otimes\big(\tilde{L}_0\big)_Y.
\end{aligned}$$

Then there exists a maximal closed subscheme $Z \subset Y$ such that the composition $\big(\psi_1 \circ \cdots \circ \psi_{r-1} \circ \tilde{\phi}_r\big)_Y$ coincides with the canonical homomorphism $(z) \otimes \big(\tilde{L}_0\big)_Y \longrightarrow \big(w^{r-1}\big) \otimes \big(\tilde{L}_0\big)_Y$ induced by the inclusion $(z) \hookrightarrow \big(w^{r-1}\big)$. By the construction, Z is invariant under the action of G. So Z descends to a closed subscheme $\Sigma_{\tilde{\nu}} \subset \Sigma$. We can see that the quasi-projective scheme $\Sigma_{\tilde{\nu}}$ over M' descends to a quasi-projective scheme $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})$ over $M_{\lambda,\tilde{\mu}}$ which is the desired moduli space. ∎

6 Tangent space of the moduli space using factorized ramified structure

The aim of introducing the factorized ramified structure is to construct a duality on the tangent space of the moduli space, which was not achieved in [13]. We will first describe the tangent space of the moduli space by the infinitesimal deformation of factorized ramified structure.

Let the notation be as in Section 5. Take a point $t \in \mathcal{T}$. We will describe the tangent space of the fiber $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})_t$ of the moduli space over t. We write $C := \mathcal{C}_t$, $D := \mathcal{D}_t$, $D_{\mathrm{log}} = (\mathcal{D}_{\mathrm{log}})_t$, $D_{\mathrm{un}} = (\mathcal{D}_{\mathrm{un}})_t$, $D_{\mathrm{ram}} = (\mathcal{D}_{\mathrm{ram}})_t$ and $(\mu, \nu) := (\tilde{\mu}, \tilde{\nu})_t$. We put $m_x := m^{\mathrm{un}}_i$ for $x = \tilde{x}^{\mathrm{un}}_i|_t$ and $m_x := m^{\mathrm{ram}}_i$ for $x = \tilde{x}^{\mathrm{ram}}_i|_t$.

Let $(E, \nabla, l, \ell, \mathcal{V})$ be a connection on (C, D) with (λ, μ, ν)-structure. If we put

$$l_k := \bigoplus_{x \in D_{\mathrm{log}}} l^x_k, \qquad \ell_k := \bigoplus_{x \in D_{\mathrm{un}}} \ell^x_k,$$

then we get filtrations $E|_{D_{\mathrm{log}}} = l_0 \supset l_1 \supset \cdots \supset l_{r-1} \supset l_r = 0$, $E|_{D_{\mathrm{un}}} = \ell_0 \supset \ell_1 \supset \cdots \supset \ell_{r-1} \supset l_r = 0$ such that $l_k/l_{k+1} \cong \mathcal{O}_{D_{\mathrm{log}}}$ and $\ell_k/\ell_{k+1} \cong \mathcal{O}_{D_{\mathrm{un}}}$ for $0 \leq k \leq r-1$. If we put

$$V_k := \bigoplus_{x \in D_{\mathrm{ram}}} V^x_k, \qquad \overline{V}_k := \bigoplus_{x \in D_{\mathrm{ram}}} \overline{V}^x_k, \qquad \overline{W}_k := \bigoplus_{x \in D_{\mathrm{ram}}} \overline{W}^x_k,$$

then we get a filtration $E|_{D_{\mathrm{ram}}} = V_0 \supset V_1 \supset \cdots \supset V_{r-1} \supset V_r = zV_0$ with surjections $V_k \longrightarrow \overline{V}_k$ and isomorphisms $\overline{W}_k \cong \mathrm{Hom}_{\mathcal{O}_{D_{\mathrm{ram}}}}(\overline{V}_{r-k-1}, \mathcal{O}_{D_{\mathrm{ram}}})$ for $0 \leq k \leq r-1$.

Define a complex $\mathcal{G}^\bullet$ of sheaves on C by setting

$$\mathcal{G}^0 = \left\{ u \in \mathcal{E}\mathrm{nd}(E) \,\middle|\, \begin{array}{l} u|_{D_{\mathrm{log}}}(l_k) \subset l_k,\ u|_{D_{\mathrm{un}}}(\ell_k) \subset \ell_k \text{ and } u|_{D_{\mathrm{ram}}}(V_k) \subset V_k \\ \text{for } 0 \leq k \leq r-1 \end{array} \right\}, \tag{6.1}$$

$$\mathcal{G}^1 = \left\{ v \in \mathcal{E}\mathrm{nd}(E) \otimes \Omega^1_C(D) \,\middle|\, \begin{array}{l} v|_{D_{\mathrm{log}}}(l_k) \subset l_{k+1} \otimes \Omega^1_C(D),\ v|_{D_{\mathrm{un}}}(\ell_k) \subset \ell_{k+1} \otimes \Omega^1_C(D) \\ \text{and } v|_{D_{\mathrm{ram}}}(V_k) \subset V_k \otimes \Omega^1_C(D) \text{ for } 0 \leq k \leq r-1 \end{array} \right\},$$

and by defining the homomorphism

$$d^0_{\mathcal{G}^\bullet} \colon\ \mathcal{G}^0 \ni u \mapsto \nabla \circ u - (u \otimes 1) \circ \nabla \in \mathcal{G}^1. \tag{6.2}$$

The meaning of the hypercohomology $\mathbb{H}^1(\mathcal{G}^\bullet)$ is the tangent space of the moduli space of connections (E, ∇) on C equipped with logarithmic λ-parabolic structure along D_{log}, generic unramified μ-parabolic structure along D_{un} and a filtration $E|_{D_{\mathrm{ram}}} = V_0 \supset V_1 \supset \cdots \supset V_{r-1} \supset V_r = zV_0$ preserved by ∇. For the description of the tangent space of the moduli space $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})_t$, we will construct the data of deformation of factorized ramified structure in addition to $\mathbb{H}^1(\mathcal{G}^\bullet)$.

For $(v_k) \in \bigoplus_{k=0}^{r-1} \mathrm{Hom}\big(\overline{V}_k, \overline{V}_k \otimes \Omega^1_C(D)\big)$, consider the diagrams

$$\begin{array}{ccc} z\mathcal{O}_{D_{\mathrm{ram}}} \otimes \overline{V}_0 & \xrightarrow{v_k} & z\mathcal{O}_{D_{\mathrm{ram}}} \otimes \overline{V}_0 \otimes \Omega^1_C(D) \\ \big\downarrow & & \big\downarrow \\ \overline{V}_{r-1} & \xrightarrow{v_{k-1}} & \overline{V}_{r-1} \otimes \Omega^1_C(D), \end{array}$$

$$\begin{array}{ccc} \overline{V}_k & \xrightarrow{v_k} & \overline{V}_k \otimes \Omega^1_C(D) \\ \downarrow & & \downarrow \\ \overline{V}_{k-1} & \xrightarrow{v_{k-1}} & \overline{V}_{k-1} \otimes \Omega^1_C(D) \end{array} \qquad (1 \le k \le r-1). \tag{6.3}$$

If we put

$$G^1 = \left\{ (v_k) \in \bigoplus_{k=0}^{r-1} \mathrm{Hom}(\overline{V}_k, \overline{V}_k \otimes \Omega^1_C(D)) \,\middle|\, \begin{array}{l}\text{all the diagrams in (6.3)} \\ \text{are commutative}\end{array} \right\},$$

then there is a canonical homomorphism

$$\varpi_G \colon \ \mathcal{G}^1 \longrightarrow G^1$$

defined by $\varpi_G(v) = \left(\overline{v|_{D_{\mathrm{ram}}}}\right)_k$, where $\overline{v|_{D_{\mathrm{ram}}}} \colon \overline{V}_k \longrightarrow \overline{V}_k \otimes \Omega^1_C(D)$ is the homomorphism induced by $v|_{D_{\mathrm{ram}}}$. We can see the surjectivity of ϖ_G by the following lemma, which is often used later.

Lemma 6.1. *For any tuple $(h_k) \in \prod_{k=0}^{r-1} \mathrm{End}_{\mathcal{O}_{D_{\mathrm{ram}}}}\left(\overline{V}_k\right)$ of endomorphisms satisfying the commutative diagrams*

$$\begin{array}{ccc} z\mathcal{O}_{D_{\mathrm{ram}}} \otimes \overline{V}_0 & \xrightarrow{\mathrm{id} \otimes h_0} & z\mathcal{O}_{D_{\mathrm{ram}}} \otimes \overline{V}_0 \\ \downarrow & & \downarrow \\ \overline{V}_{r-1} & \xrightarrow{h_{r-1}} & \overline{V}_{r-1}, \end{array} \qquad \begin{array}{ccc} \overline{V}_k & \xrightarrow{h_k} & \overline{V}_k \\ \downarrow & & \downarrow \\ \overline{V}_{k-1} & \xrightarrow{h_{k-1}} & \overline{V}_{k-1} \end{array} \qquad (1 \le k \le r-1),$$

there exists an endomorphism $h \in \mathrm{End}_{\mathcal{O}_{D_{\mathrm{ram}}}}(E|_{D_{\mathrm{ram}}})$ satisfying $h(V_k) \subset V_k$ and the commutative diagrams

$$\begin{array}{ccc} V_k & \xrightarrow{\pi_k} & \overline{V}_k \\ {\scriptstyle h|_{V_k}}\downarrow & & \downarrow{\scriptstyle h_k} \\ V_k & \xrightarrow{\pi_k} & \overline{V}_k \end{array}$$

for $0 \le k \le r-1$. Moreover, $\mathrm{Tr}(h) \in \mathcal{O}_{D_{\mathrm{ram}}}$ is uniquely determined by (h_k) and independent of the choice of h.

Proof. Let $e_0, \dots, e_{r-1}$ be the basis of $E|_{D_{\mathrm{ram}}}$ taken in the proof of Proposition 3.3. Then we can write

$$h_k(e_k) = a_{k,k}e_k + \overline{a}_{k+1,k}e_{k+1} + \cdots + \overline{a}_{r-1,k}e_{r-1} + za_{0,k}e_1 + \cdots + za_{k-1,k}e_{k-1}$$

for $a_{k,k} \in \mathcal{O}_{D_{\mathrm{ram}}}$ and $\overline{a}_{l,k} \in \mathcal{O}_{D'_{\mathrm{ram}}}$ for $l \neq k$, where we put $D'_{\mathrm{ram}} := \sum_{x \in D_{\mathrm{ram}}} (m_x - 1)x$ and $za_{l,k}$ is the image of $z \otimes \overline{a}_{l,k}$ under the isomorphism $(z) \otimes \mathcal{O}_{D'_{\mathrm{ram}}} \xrightarrow{\sim} z\mathcal{O}_{D_{\mathrm{ram}}}$ for $l < k$ We can see that a lift $h \in \mathrm{End}_{\mathcal{O}_{D_{\mathrm{ram}}}}(E|_{D_{\mathrm{ram}}})$ of (h_k) desired in the lemma is given by the matrix

$$\begin{pmatrix} a_{0,0} & za_{0,1} & \cdots & za_{0,r-1} \\ a_{1,0} & a_{1,1} & \cdots & z_{1,r-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,0} & a_{r-1,1} & \cdots & a_{r-1,r-1} \end{pmatrix}$$

with respect to the basis $e_0, \dots, e_{r-1}$, where $a_{l,k} \in \mathcal{O}_{D_{\mathrm{ram}}}$ are lifts of $\overline{a}_{k,l}$ for $l > k$. In particular, we obtain the existence of h. The ambiguities of h are the lower triangular entries $a_{i,j}$ with $i > j$. So its trace $\mathrm{Tr}(h) = a_{0,0} + \cdots + a_{r-1,r-1}$ is independent of the choice of h. ∎

The trace pairing $\mathrm{Tr}\colon \ker(\varpi_G)\otimes\mathcal{G}^0 \ni v\otimes u \mapsto \mathrm{Tr}(v\circ u)\in\Omega^1_C$ induces an isomorphism

$$\ker\varpi_G \xrightarrow{\sim} \left(\mathcal{G}^0\right)^\vee\otimes\Omega^1_C.$$

For $(\tau_k)\in\bigoplus_{k=0}^{r-1}\mathrm{Hom}\left(\overline{W}_k,\overline{V}_k\right)$, consider the diagrams

$$\begin{tikzcd} \overline{W}_k \arrow[r] \arrow[d, "\tau_k"'] & \overline{W}_{k-1} \arrow[d, "\tau_{k-1}"'] \\ \overline{V}_k \arrow[r] & \overline{V}_{k-1} \end{tikzcd} \quad (1\le k\le r-1), \qquad \begin{tikzcd} z\mathcal{O}_{D_{\mathrm{ram}}}\otimes\overline{W}_0 \arrow[r] \arrow[d, "\mathrm{id}\otimes\tau_0"'] & \overline{W}_{r-1} \arrow[d, "\tau_{r-1}"'] \\ z\mathcal{O}_{D_{\mathrm{ram}}}\otimes\overline{V}_0 \arrow[r] & \overline{V}_{r-1} \end{tikzcd} \tag{6.4}$$

and for $(\xi_k)\in\bigoplus_{k=0}^{r-1}\mathrm{Hom}\left(\overline{V}_k,\overline{W}_k\right)$, consider the diagrams

$$\begin{tikzcd} \overline{V}_k \arrow[r] \arrow[d, "\xi_k"'] & \overline{V}_{k-1} \arrow[d, "\xi_{k-1}"'] \\ \overline{W}_k \arrow[r] & \overline{W}_{k-1} \end{tikzcd} \quad (1\le k\le r-1), \qquad \begin{tikzcd} z\mathcal{O}_{D_{\mathrm{ram}}}\otimes\overline{V}_0 \arrow[r] \arrow[d, "\mathrm{id}\otimes\xi_0"'] & \overline{V}_{r-1} \arrow[d, "\xi_{r-1}"'] \\ z\mathcal{O}_{D_{\mathrm{ram}}}\otimes\overline{W}_0 \arrow[r] & \overline{W}_{r-1}. \end{tikzcd} \tag{6.5}$$

Then we put

$$\mathrm{Sym}^2\left(\overline{W}\right) = \left\{ (\tau_k)\in\bigoplus_{k=0}^{r-1}\mathrm{Hom}\left(\overline{W}_k,\overline{V}_k\right) \,\middle|\, \begin{array}{l}\text{the diagrams (6.4) are commutative}\\ \text{and } {}^t\tau_{r-k-1}=\tau_k \text{ for } 0\le k\le r-1\end{array} \right\},$$
$$\mathrm{Sym}^2\left(\overline{V}\right) = \left\{ (\xi_k)\in\bigoplus_{k=0}^{r-1}\mathrm{Hom}\left(\overline{V}_k,\overline{W}_k\right) \,\middle|\, \begin{array}{l}\text{the diagrams (6.5) are commutative}\\ \text{and } {}^t\xi_{r-k-1}=\xi_k \text{ for } 0\le k\le r-1\end{array} \right\}$$

and put

$$A^0 = \left\{ (a_k(w)) \in \bigoplus_{x\in D_{\mathrm{ram}}}\prod_{k=0}^{r-1}\mathbb{C}[w]/\left(w^{m_x r-r+1}\right) \,\middle|\, \begin{array}{l} w\left(a_k(w)-a_{k+1}(w)\right)=0\\ \text{for } 0\le k\le r-2\end{array} \right\},$$
$$A^1 = \mathrm{Hom}_{\mathcal{O}_{D_{\mathrm{ram}}}}\left(A^0,\mathcal{O}_{D_{\mathrm{ram}}}\right).$$

We need the following lemma which is similar to Lemma 6.1.

Lemma 6.2. *Assume that $(\tau_k)\in\mathrm{Sym}^2\left(\overline{W}\right)$ and $(\xi_k)\in\mathrm{Sym}^2\left(\overline{V}\right)$ are given. Then there are homomorphisms $\tau\colon E|_{D_{\mathrm{ram}}}^\vee\longrightarrow E|_{D_{\mathrm{ram}}}$, $\xi\colon E|_{D_{\mathrm{ram}}}\longrightarrow E|_{D_{\mathrm{ram}}}^\vee$ satisfying ${}^t\tau=\tau$, ${}^t\xi=\xi$, $\tau(W_k)\subset V_k$, $\xi(V_k)\subset W_k$ and the commutative diagrams*

$$\begin{tikzcd} W_k \arrow[r, "\tau|_{W_k}"] \arrow[d] & V_k \arrow[d] \\ \overline{W}_k \arrow[r, "\tau_k"] & \overline{V}_k, \end{tikzcd} \qquad \begin{tikzcd} V_k \arrow[r, "\xi|_{V_k}"] \arrow[d] & W_k \arrow[d] \\ \overline{V}_k \arrow[r, "\xi_k"] & \overline{W}_k \end{tikzcd}$$

for $k=0,1\dots,r-1$, where $W_k=\bigoplus_{x\in D_{\mathrm{ram}}}\ker(z^{m_x-1}({}^tN)^{r-k})\subset E|_{D_{\mathrm{ram}}}^\vee$.

Proof. Choose the basis $e_0,\dots,e_{r-1}$ of $E|_{D_{\mathrm{ram}}}$ taken in the proof of Proposition 3.3 and its dual basis $e_0^*,\dots,e_{r-1}^*$. Since $\tau_k\left(\overline{W}_k\right)\subset\overline{V}_k$, we can write

$$\begin{aligned}\tau_k(e^*_{r-k-1}) &= zb_{0,r-k-1}e_0+\dots+zb_{k-1,r-k-1}e_{k-1}+b_{k,r-k-1}e_k\\ &\quad+\overline{b}_{k+1,r-k-1}e_{k+1}+\dots+\overline{b}_{r-1,r-k-1}e_{r-1},\end{aligned}$$

where $\overline{b}_{l,r-k-1} \in \mathcal{O}_{D'_{\mathrm{ram}}}$ for $l \geq k+1$ and $b_{l,r-k-1} \in \mathcal{O}_{D_{\mathrm{ram}}}$for $l \leq k$. Take a lift $b_{l,r-k-1} \in \mathcal{O}_{D_{\mathrm{ram}}}$ of $\overline{b}_{l,r-k-1}$ for $l \geq k+1$. Then we have

$$\begin{aligned} zb_{l,r-k-1} &= \tau_k(e^*_{r-k-1})(e^*_l) = \tau_{r-k-1}(e^*_l)(e^*_{r-k-1}) \\ &= \tau_{r-l-1}(e^*_l)(e^*_{r-k-1}) = zb_{r-k-1,l} \qquad (\text{for } l \leq k-1), \\ b_{k,r-k-1} &= \tau_k(e^*_{r-k-1})(e^*_k) = \tau_{r-k-1}(e^*_k)(e^*_{r-k-1}) = b_{r-k-1,k}, \\ zb_{l,r-k-1} &= \tau_k(e^*_{r-k-1})(ze^*_l) = \tau_{r-k-1}(ze^*_l)(e^*_{r-k-1}) \\ &= \tau_{r-l-1}(ze^*_l)(e^*_{r-k-1}) = zb_{r-k-1,l} \qquad (\text{for } l \geq k+1). \end{aligned}$$

After replacing $b_{r-k-1,l}$ for $l \geq k+1$, we may assume $b_{l,r-k-1} = b_{r-k-1,l}$ for $l \geq k+1$. Let $\tau\colon E|^{\vee}_{D_{\mathrm{ram}}} \longrightarrow E|_{D_{\mathrm{ram}}}$ be the homomorphism given by the matrix

$$\begin{pmatrix} zb_{00}(z) & & \cdots & & b_{0,r-1}(z) \\ \vdots & & & \iddots & \vdots \\ zb_{0,k}(z) & \cdots & b_{k,r-k-1}(z) & \cdots & b_{k,r-1}(z) \\ \vdots & \iddots & & & \vdots \\ b_{0,r-1}(z) & & \cdots & & b_{r-1,r-1}(z) \end{pmatrix}$$

with respect to the bases $(e^*_0,\dots,e^*_{r-1})$ and $(e_0,\dots,e_{r-1})$. Then we have ${}^t\tau = \tau$ and τ also satisfies the other required conditions of the lemma. The same statement holds for (ξ_k). ■

We define a complex $\mathcal{S}^{\bullet}_{\mathrm{ram}}$ by setting

$$\mathcal{S}^0_{\mathrm{ram}} = A^0, \qquad \mathcal{S}^1_{\mathrm{ram}} = \mathrm{Sym}^2\left(\overline{W}\right) \oplus \mathrm{Sym}^2\left(\overline{V}\right), \qquad \mathcal{S}^2_{\mathrm{ram}} = G^1 \oplus A^1 \tag{6.6}$$

and by setting the homomorphisms

$$\begin{aligned} d^0_{\mathcal{S}^\bullet}\colon\ & \mathcal{S}^0_{\mathrm{ram}} \ni (a_k(w)) \mapsto \big((\theta_k \circ a_k(\kappa_k \circ \theta_k)), (-a_k(\kappa_k \circ \theta_k) \circ \kappa_k)\big) \in \mathcal{S}^1_{\mathrm{ram}}, \\ d^1_{\mathcal{S}^\bullet}\colon\ & \mathcal{S}^1_{\mathrm{ram}} \ni ((\tau_k),(\xi_k)) \mapsto \big(-(\delta_{(\tau_k,\xi_k)}), \Theta_{(\tau_k,\xi_k)}\big) \in \mathcal{S}^2_{\mathrm{ram}}, \end{aligned} \tag{6.7}$$

where $\delta_{(\tau_k,\xi_k)} \in G^1$ and $\Theta_{(\tau_k,\xi_k)} \in A^1$ are defined by

$$\begin{aligned} \delta_{(\tau_k,\xi_k)} &= \left(\sum_{p=1}^{r-1}\sum_{l=1}^{p} \nu_p(z) N_k^{p-l} \circ \left(\theta_k \circ \xi_k + \tau_k \circ \kappa_k\right) \circ N_k^{l-1}\right), \\ \Theta_{(\tau_k,\xi_k)}\big(\big(\overline{f_k(w)}\big)\big) &= \mathrm{Tr}\left(f \circ (\theta \circ \xi + \tau \circ \kappa)\right), \end{aligned}$$

where θ, κ are lifts of $(\theta_k), (\kappa_k)$ chosen as in the proof of Proposition 3.3, τ, ξ are lifts of (τ_k), (ξ_k) given by Lemma 6.2 and $f \in \mathrm{End}(E|_{D_{\mathrm{ram}}})$ is a lift of $(f_k(\theta_k \circ \kappa_k))$ given by Lemma 6.1. By virtue of Lemma 6.1, we can see that $\Theta_{(\tau_k,\xi_k)}$ is independent of the choices of θ, κ, τ, ξ and f. We can also check $d^1_{\mathcal{S}^\bullet} \circ d^0_{\mathcal{S}^\bullet} = 0$. The meaning of the cohomology $H^1(\mathcal{S}^\bullet_{\mathrm{ram}})$ is the first order deformation of factorized ramified structure.

We define a homomorphism of complexes $\gamma^\bullet\colon \mathcal{G}^\bullet \longrightarrow \mathcal{S}^\bullet_{\mathrm{ram}}[1]$ by

$$\begin{aligned} \gamma^0\colon\ & \mathcal{G}^0 \ni u \mapsto \big((\overline{u|_{D_{\mathrm{ram}}}} \circ \theta_k + \theta_k \circ {}^t\overline{u|_{D_{\mathrm{ram}}}}), (-\kappa_k \circ \overline{u|_{D_{\mathrm{ram}}}} - {}^t\overline{u|_{D_{\mathrm{ram}}}} \circ \kappa_k)\big) \in \mathcal{S}^1_{\mathrm{ram}}, \\ \gamma^1\colon\ & \mathcal{G}^1 \ni v \mapsto (-\varpi_G(v), 0) \in G^1 \oplus A^1 = \mathcal{S}^2_{\mathrm{ram}}, \end{aligned} \tag{6.8}$$

where $\overline{u|_{D_{\mathrm{ram}}}}\colon \overline{V}_k \longrightarrow \overline{V}_k$ is the homomorphism induced by $u|_{D_{\mathrm{ram}}}$.

For $u \in \mathcal{G}^0$, we have

$$\delta_{\gamma^0(u)} = \left(\sum_{p=1}^{r-1}\sum_{l=1}^{p} \nu_p(z) N_k^{p-l} \big(\overline{u|_{D_{\mathrm{ram}}}} N_k - N_k \overline{u|_{D_{\mathrm{ram}}}}\big) N_k^{l-1}\right)$$

$$= \left(\overline{u|_{D_{\rm ram}}}\nu(N_k) - \nu(N_k)\overline{u|_{D_{\rm ram}}}\right).$$

On the other hand, the restriction $\nabla|_{D_{\rm ram}}$ induces the homomorphism $\nu(N_k) + \frac{k\,{\rm d}z}{rz}{\rm id}$ on $\overline{V}_k$. So we have $\delta_{\gamma^0(u)} = -\varpi_G(\nabla u - u\nabla)$. Thus we have $d^0_{\mathcal{S}^\bullet[1]}\gamma^0 = \gamma^1 d^0_{\mathcal{G}^\bullet}$, where $d^0_{\mathcal{S}^\bullet[1]} = -d^1_{\mathcal{S}^\bullet}$. Set

$$\mathcal{F}^\bullet := {\rm Cone}\big(\mathcal{G}^\bullet \xrightarrow{\gamma^\bullet} \mathcal{S}^\bullet_{\rm ram}[1]\big)[-1]. \tag{6.9}$$

So we have

$$\mathcal{F}^0 = \mathcal{G}^0 \oplus A^0, \qquad \mathcal{F}^1 = \mathcal{G}^1 \oplus {\rm Sym}^2\left(\overline{V}\right) \oplus {\rm Sym}^2\left(\overline{W}\right), \qquad \mathcal{F}^2 = G^1 \oplus A^1$$

and $d^0_{\mathcal{F}^\bullet}\colon \mathcal{F}^0 \longrightarrow \mathcal{F}^1$, $d^1_{\mathcal{F}^\bullet}\colon \mathcal{F}^1 \longrightarrow \mathcal{F}^2$ are defined by

$$d^0_{\mathcal{F}^\bullet}(u, (a_k(w))) = \left(\nabla \circ u - (u \otimes {\rm id}) \circ \nabla, -\gamma^0(u) + d^0_{\mathcal{S}^\bullet}((a_k(w)))\right)$$
$$d^1_{\mathcal{F}^\bullet}(v, ((\tau_k), (\xi_k))) = \left(\varpi_G(v) - (\delta_{(\tau_k,\xi_k)}), \Theta_{(\tau_k,\xi_k)}\right).$$

Consider the complexes $\mathcal{F}^\bullet_0 = \left[\mathcal{G}^0 \oplus \mathcal{S}^0_{\rm ram} \longrightarrow {\rm Sym}^2\left(\overline{W}\right)\right]$, $\mathcal{F}^\bullet_1 = \left[\mathcal{G}^1 \oplus {\rm Sym}^2\left(\overline{V}\right) \longrightarrow \mathcal{S}^2_{\rm ram}\right]$ defined by

$$d^0_{\mathcal{F}^\bullet_0}\colon\ \mathcal{G}^0 \oplus A^0 \ni (u, (\overline{a_k(w)})) \mapsto \left(-\overline{u|_{D_{\rm ram}}} \circ \theta_k - \theta_k \circ {}^t\overline{u|_{D_{\rm ram}}} + \theta_k \circ a_k(\kappa_k \circ \theta_k)\right) \in {\rm Sym}^2\left(\overline{W}\right),$$
$$d^0_{\mathcal{F}^\bullet_1}\colon\ \mathcal{G}^1 \oplus {\rm Sym}^2\left(\overline{V}\right) \ni (v, (\xi_k)) \mapsto (\varpi_G(v) - (\delta_{(0,\xi_k)}), (\Theta_{(0,\xi_k)})) \in G^1 \oplus A^1.$$

Then there is an exact sequence of complexes

$$0 \longrightarrow \mathcal{F}^\bullet_1[-1] \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet_0 \longrightarrow 0$$

which is expressed by the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{G}^0 \oplus A^0 \arrow[r] \arrow[d, "d^0_{\mathcal{F}^\bullet}"'] & \mathcal{G}^0 \oplus A^0 \arrow[d, "d^0_{\mathcal{F}^\bullet_0}"'] \\
\mathcal{G}^1 \oplus {\rm Sym}^2\left(\overline{V}\right) \arrow[r] \arrow[d, "d^0_{\mathcal{F}^\bullet_1}"'] & \mathcal{G}^1 \oplus {\rm Sym}^2\left(\overline{W}\right) \oplus {\rm Sym}^2\left(\overline{V}\right) \arrow[r] \arrow[d, "d^1_{\mathcal{F}^\bullet}"'] & {\rm Sym}^2(\overline{W}) \arrow[d] \\
G^1 \oplus A^1 \arrow[r] & G^1 \oplus A^1 \arrow[r] & 0.
\end{tikzcd}$$

So we get the following exact sequence of hyper cohomologies:

$$\begin{aligned}
0 &\to \mathbf{H}^0(\mathcal{F}^\bullet) \to \mathbf{H}^0(\mathcal{F}^\bullet_0) \to \mathbf{H}^0(\mathcal{F}^\bullet_1) \to \mathbf{H}^1(\mathcal{F}^\bullet) \to \mathbf{H}^1(\mathcal{F}^\bullet_0) \to \mathbf{H}^1(\mathcal{F}^\bullet_1) \\
&\to \mathbf{H}^2(\mathcal{F}^\bullet) \to 0.
\end{aligned} \tag{6.10}$$

Proposition 6.3. *The relative tangent space of $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$ over $\mathcal{T}$ at $(E,\nabla,\{l,\ell,\mathcal{V}\})$ is isomorphic to $\mathbf{H}^1(\mathcal{F}^\bullet)$.*

Proof. Take a point $t \in \mathcal{T}$ and a point $y \in M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$ over t corresponding to a connection $(E,\nabla,\{l,\ell,\mathcal{V}\})$ with (λ,μ,ν)-structure. Giving a tangent vector v of the fiber $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_t$ of the moduli space at y is equivalent to giving a flat family $(\tilde{E},\tilde{\nabla},\{\tilde{l},\tilde{\ell},\tilde{\mathcal{V}}\})$ of connections with (λ,μ,ν)-structure on $C \times {\rm Spec}\,\mathbb{C}[\epsilon]$ satisfying $\left(\tilde{E},\tilde{\nabla},\{\tilde{l},\tilde{\ell},\tilde{\mathcal{V}}\}\right) \otimes \mathbb{C}[\epsilon]/(\epsilon) \cong (E,\nabla,\{l,\ell,\mathcal{V}\})$, where $\mathbb{C}[\epsilon] = \mathbb{C}[\epsilon]/\left(\epsilon^2\right)$. Take an affine open covering $\{U_\alpha\}$ of C such that $E|_{U_\alpha} \cong \mathcal{O}^{\oplus r}_{U_\alpha}$ for any α. Put $U_\alpha[\epsilon] := U_\alpha \times {\rm Spec}\,\mathbb{C}[\epsilon]$. We may assume that for each $x \in D$, there exists exactly one index α satisfying $x \in U_\alpha$ and that each U_α contains at most one point in D. We can take a lift $\varphi_\alpha\colon E \otimes \mathbb{C}[\epsilon]|_{U_\alpha[\epsilon]} \xrightarrow{\sim} \tilde{E}|_{U_\alpha[\epsilon]}$ of the given isomorphism $E|_{U_\alpha} \xrightarrow{\sim} \tilde{E} \otimes \mathbb{C}[\epsilon]/(\epsilon)|_{U_\alpha}$. We may assume that φ_α preserves l if $\mathcal{D}_{\log}\cap U_\alpha\neq\varnothing$ and preserves ℓ if $\mathcal{D}_{\mathrm{un}}\cap U_\alpha\neq\varnothing$. If $\mathcal{D}_{\mathrm{ram}}\cap U_\alpha\neq\varnothing$, then we may assume that φ_α sends the filtration $\{V_k\otimes\mathbb{C}[\epsilon]\}$ to the filtration $\{\tilde{V}_k\}$. Set

$$\begin{aligned}
&\epsilon u_{\alpha\beta}=\varphi_\alpha^{-1}\circ\varphi_\beta-\mathrm{id},\\
&\epsilon v_\alpha=(\varphi_\alpha\otimes\mathrm{id})^{-1}\circ\tilde{\nabla}\circ\varphi_\alpha-\nabla\otimes\mathbb{C}[\epsilon],\\
&\epsilon\eta_\alpha=\left(\varphi_\alpha|_{D_{\mathrm{ram}}}^{-1}\circ\tilde{\theta}_k\circ{}^t\varphi_\alpha|_{D_{\mathrm{ram}}}^{-1}-\theta_k,{}^t\varphi_\alpha|_{D_{\mathrm{ram}}}\circ\tilde{\kappa}_k\circ\varphi_\alpha|_{D_{\mathrm{ram}}}-\kappa_k\right).
\end{aligned}$$

Then we get a cohomology class $[\{u_{\alpha\beta}\},\{v_\alpha,(\eta_\alpha)\}]\in\mathbf{H}^1(\mathcal{F}^\bullet)$, which can be checked to be independent of the choice of $\{U_\alpha,\varphi_\alpha\}$.

Conversely, assume that a cohomology class $[\{u_{\alpha\beta}\},\{v_\alpha\},\{\eta_\alpha\}]\in\mathbf{H}^1(\mathcal{F}^\bullet)$ is given. We define

$$\begin{aligned}
&\sigma_{\beta\alpha}=\mathrm{id}+\epsilon u_{\alpha\beta}\colon\ \mathcal{O}_{U_{\alpha\beta}[\epsilon]}^{\oplus r}\xrightarrow{\sim}\mathcal{O}_{U_{\alpha\beta}[\epsilon]}^{\oplus r},\\
&\nabla_\alpha=\nabla+\epsilon v_\alpha\colon\ \mathcal{O}_{U_\alpha[\epsilon]}^{\oplus r}\longrightarrow\mathcal{O}_{U_\alpha[\epsilon]}^{\oplus r}\otimes\Omega_C^1(D).
\end{aligned}\tag{6.11}$$

If $U_\alpha\cap D_{\log}\neq\varnothing$ we put $l_\alpha:=l|_{U_\alpha}\otimes\mathbb{C}[\epsilon]$ and we put $\ell_\alpha:=\ell|_{U_\alpha\otimes\mathbb{C}[\epsilon]}$ if $U_\alpha\cap D_{\mathrm{un}}\neq\varnothing$. Then we can see that ∇_α preserves l_α if $U_\alpha\cap D_{\log}\neq\varnothing$ and preserves ℓ_α if $U_\alpha\cap D_{\mathrm{un}}\neq\varnothing$, because v_α preserves $l|_{U_\alpha}$ and $\ell|_{U_\alpha}$ by the definition of $\mathcal{G}^1$.

Consider the case $U_\alpha\cap D_{\mathrm{ram}}=\{x\}$. We can write $\eta_\alpha=(\tau_k,\xi_k)_{0\leq k\leq r-1}$. By the choice of η_α, we have $\delta_{(\tau_k,\xi_k)}=\overline{v_\alpha|_{D_{\mathrm{ram}}}}$ and $\Theta_{(\tau_k,\xi_k)}=0$, which yield the equalities

$$\begin{aligned}
&\mathrm{Tr}\left((\theta\circ\xi+\tau\circ\kappa)\circ N^j\right)=0,\qquad 0\leq j\leq r-1,\\
&\sum_{p=1}^{r-1}\sum_{j=1}^{p}\nu_p(z)N_k^{p-j}(\theta_k\xi_k+\tau_k\kappa_k)N_k^{j-1}=\overline{v_\alpha|_{D_{\mathrm{ram}}}},\qquad 0\leq k\leq r-1,
\end{aligned}\tag{6.12}$$

where N, θ and κ are lifts of (N_k), (θ_k) and (κ_k) chosen as in the proof of Proposition 3.3 and τ, ξ are lifts of (τ_k), (ξ_k) given by Lemma 6.2.

Since the minimal polynomial w^r of $N|_x$ is of degree r, we can see from [12, Lemma 1.4]. that

$$\mathrm{Im}(\mathrm{ad}(N))=\left\{f\in\mathrm{End}_{\mathcal{O}_{m_xx}}(E|_{m_xx})\,\middle|\,\mathrm{Tr}\left(f\circ N^l\right)=0\text{ for any }l\geq 0\right\}.$$

So we can find an endomorphism $f\in\mathrm{End}(E|_{m_xx})$ satisfying $\theta\circ\xi+\tau\circ\kappa=f\circ N-N\circ f$.

Now we will construct a factorized ramified structure on $\left(\mathcal{O}_{U_\alpha[\epsilon]}^{\oplus r},\nabla_\alpha\right)$. We take $(V_k\otimes\mathbb{C}[\epsilon])_{0\leq k\leq r-1}$ as the relative version of the filtration in Definition 3.1 (i). The homomorphisms

$$\begin{aligned}
&\theta_{k,\epsilon}:=\theta_k+\epsilon\tau_k\colon\ \overline{W}_k\otimes\mathbb{C}[\epsilon]\longrightarrow\overline{V}_k\otimes\mathbb{C}[\epsilon],\\
&\kappa_{k,\epsilon}:=\kappa_k+\epsilon\xi_k\colon\ \overline{V}_k\otimes\mathbb{C}[\epsilon]\longrightarrow\overline{W}_k\otimes\mathbb{C}[\epsilon]
\end{aligned}$$

become lifts of θ_k and κ_k, respectively. They determine bilinear pairings

$$\begin{aligned}
&\vartheta_{k,\epsilon}\colon\ \left(\overline{W}_k\otimes\mathbb{C}[\epsilon]\right)\times\left(\overline{W}_{r-k-1}\otimes\mathbb{C}[\epsilon]\right)\longrightarrow\mathcal{O}_{m_xx}\otimes\mathbb{C}[\epsilon],\\
&\varkappa_{k,\epsilon}\colon\ \left(\overline{V}_k\otimes\mathbb{C}[\epsilon]\right)\times\left(\overline{V}_{r-k-1}\otimes\mathbb{C}[\epsilon]\right)\longrightarrow\mathcal{O}_{m_xx}\otimes\mathbb{C}[\epsilon],
\end{aligned}$$

which satisfy the commutative diagrams in (ii), (iii) of Definition 3.1. Since $N^r=z\cdot\mathrm{id}_{E|_{m_xx}}$, the equality

$$\begin{aligned}
(N+\epsilon(\theta\circ\xi+\tau\circ\kappa))^r&=N^r+\epsilon\sum_{j=0}^{r-1}N^j\circ(\theta\circ\xi+\tau\circ\kappa)\circ N^{r-j-1}\\
&=N^r+\epsilon\sum_{j=0}^{r-1}N^j\circ(f\circ N-N\circ f)\circ N^{r-j-1}
\end{aligned}$$

$$= N^r + f\circ N^r - N^r\circ f = z\,\mathrm{id}_{\mathcal{O}^{\oplus r}_{U_\alpha[\epsilon]}}$$

holds. By the equality (6.12),

$$\begin{aligned}&\nu(N_k+\epsilon(\theta_k\circ\xi_k+\tau_k\circ\kappa_k))+k\,\mathrm{d}z/rz\,\mathrm{id}\\ &\quad=\nu(N_k)+k\,\mathrm{d}z/rz\,\mathrm{id}+\epsilon\sum_{p=1}^{r-1}\sum_{j=1}^{p}\nu_p(z)N_k^{p-j}(\theta_k\circ\xi_k+\tau_k\circ\kappa_k)N_k^{j-1}\\ &\quad=\nu(N_k)+k\,\mathrm{d}z/rz\,\mathrm{id}+\epsilon\overline{v_\alpha|_{D_{\mathrm{ram}}}}\end{aligned}$$

coincides with the map induced by ∇_α. So the relative version of the condition (iv) of Definition 3.1 is satisfied. The endomorphism $N_k+\epsilon(\theta_k\circ\xi_k+\tau_k\circ\kappa_k)$ defines a $\mathbb{C}[w]\otimes_{\mathbb{C}}\mathbb{C}[\epsilon]$-module structure on $\overline{V}_k\otimes\mathbb{C}[\epsilon]$. Define an isomorphism

$$\psi_{k,\epsilon}\colon\ \overline{V}_k\otimes\mathbb{C}[\epsilon]\xrightarrow{\sim}(w)/\big(w^{m_x r-r+2}\big)\otimes\overline{V}_{k-1}\otimes\mathbb{C}[\epsilon]$$

of $\mathbb{C}[w]\otimes_{\mathbb{C}}\mathbb{C}[\epsilon]$-modules by setting

$$\psi_{k,\epsilon}\big(\pi_k\big((N+\epsilon(\theta\circ\xi+\tau\circ\kappa))^k e_0\big)\big)=w\otimes\pi_{k-1}\big((N+\epsilon(\theta\circ\xi+\tau\circ\kappa))^{k-1}e_0\big),$$

where π_k means $\pi_k\otimes\mathbb{C}[\epsilon]$. Then the image of $z\otimes\pi_0(e_0)$ via the composition

$$\begin{aligned}&(z)\otimes\overline{V}_0\otimes\mathbb{C}[\epsilon]\to\overline{V}_{r-1}\otimes\mathbb{C}[\epsilon]\xrightarrow[\sim]{\psi_{r-1,\epsilon}}(w)\otimes\overline{V}_{r-2}\otimes\mathbb{C}[\epsilon]\\ &\qquad\qquad\xrightarrow[\sim]{\psi_{r-2,\epsilon}}\cdots\xrightarrow[\sim]{\psi_{1,\epsilon}}(w^{r-1})\otimes\overline{V}_0\otimes\mathbb{C}[\epsilon]\end{aligned}\tag{6.13}$$

coincides with $(\psi_{1,\epsilon}\circ\cdots\circ\psi_{r-1,\epsilon})\big(\pi_{r-1}\big((N+\epsilon(\theta\circ\xi+\tau\circ\kappa))^r e_0\big)\big)=w^r\otimes\pi_0(e_0)$. Thus the composition (6.13) coincides with the homomorphism $(z)\otimes\overline{V}_0\longrightarrow\big(w^{r-1}\big)\otimes L_0$ obtained by tensoring $\overline{V}_0\otimes\mathbb{C}[\epsilon]$ to the canonical homomorphism $(z)\longrightarrow\big(w^{r-1}\big)$.

If we put $\mathcal{V}_\alpha:=\big(\overline{V}_k\otimes\mathbb{C}[\epsilon],\vartheta_{k,\epsilon},\kappa_{k,\epsilon}\big)_{0\le k\le r-1}$, then we can see from the above arguments that $\big(\mathcal{O}^{\oplus r}_{U_\alpha[\epsilon]},\nabla_\alpha,\mathcal{V}_\alpha\big)$ is a flat family of local connections with $\nu(w)_x$-ramified structure which is a lift of $(E,\nabla,\mathcal{V})|_{U_\alpha}$. We can patch all the local connections $\big(\mathcal{O}^{\oplus r}_{U_\alpha[\epsilon]},l_\alpha,\ell_\alpha\nabla_\alpha,\mathcal{V}_\alpha\big)$ with (λ,μ,ν)-structure via the isomorphisms $\sigma_{\beta\alpha}$ defined in (6.11). Then we obtain a global flat family of connections $\big(\tilde{E},\tilde{\nabla},\tilde{l},\tilde{\ell},\tilde{\mathcal{V}}\big)$ with (λ,μ,ν)-structure which gives a tangent vector $v\in T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})/\mathcal{T}}(y)$ at y. We can see from its construction that the map $[\{u_{\alpha\beta}\},\{v_\alpha,\eta_\alpha\}]\mapsto v$ gives the desired inverse. ∎

7 Smoothness of the moduli space

In this section, we assume the same notations as in Sections 5 and 6. Take a connection $(E,\nabla,\{l,\ell,\mathcal{V}\})\in M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_t$ with (λ,μ,ν)-structure. We define a pairing

$$\Xi_{\mathrm{ram}}\colon\ \mathcal{S}^1_{\mathrm{ram}}\times\mathcal{S}^1_{\mathrm{ram}}\longrightarrow\Omega^1_C(D)|_{D_{\mathrm{ram}}}$$

by setting

$$\begin{aligned}&\Xi_{\mathrm{ram}}((\tau_k,\xi_k),(\tau'_k,\xi'_k))\\ &\qquad:=\sum_{p=1}^{r-1}\sum_{j=1}^{p}\frac{\nu_p(z)}{2}\,\mathrm{Tr}\left(\tau'\circ{}^tN^{p-j}\circ\xi\circ N^{j-1}-N^{p-j}\circ\tau\circ{}^tN^{j-1}\circ\xi'\right)\end{aligned}\tag{7.1}$$

for $(\tau_k),(\tau'_k) \in \operatorname{Sym}^2\left(\overline{W}\right)$ and $(\xi_k),(\xi'_k) \in \operatorname{Sym}^2\left(\overline{V}\right)$, where $\tau,\tau' \in \operatorname{Hom}(E|_{D_{\mathrm{ram}}}^\vee, E|_{D_{\mathrm{ram}}})$ are lifts of $(\tau_k),(\tau'_k)$ and $\xi,\xi' \in \operatorname{Hom}\left(E|_{D_{\mathrm{ram}}}, E|_{D_{\mathrm{ram}}}^\vee\right)$ are lifts of $(\xi_k),(\xi'_k)$ given by Lemma 6.2, respectively.

Take an affine open covering $C = \bigcup_\alpha U_\alpha$ for the calculation of the hypercohomologies in Čech cohomology. We define a bilinear pairing

$$\omega_{(E,\nabla,\{l,\ell,\mathcal{V}\})}\colon\ \mathbf{H}^1(\mathcal{F}^\bullet)\times\mathbf{H}^1(\mathcal{F}^\bullet)\longrightarrow\mathbf{H}^2\big(\mathcal{O}_C\to\Omega^1_C(D_{\mathrm{ram}})\to\Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}}\big)=\mathbb{C} \tag{7.2}$$

by setting

$$\begin{aligned}&\omega_{(E,\nabla,\{l,\ell,\mathcal{V}\})}\big(\big[\{u_{\alpha\beta}\},\{v_\alpha,\eta_\alpha\}\big],\big[\{u'_{\alpha\beta}\},\{v'_\alpha,\eta'_\alpha\}\big]\big)\\ &\qquad=\big[\{\operatorname{Tr}(u_{\alpha\beta}\circ u'_{\beta\gamma})\},\{-\operatorname{Tr}(u_{\alpha\beta}\circ v'_\beta-v_\alpha\circ u'_{\alpha\beta})\},\{\Xi_{\mathrm{ram}}(\eta_\alpha,\eta'_\alpha)\}\big]\end{aligned} \tag{7.3}$$

for $u_{\alpha\beta},u'_{\alpha\beta}\in\mathcal{G}^0|_{U_{\alpha\beta}}$, $v_\alpha,v'_\alpha\in\mathcal{G}^1|_{U_\alpha}$, $\eta_\alpha,\eta'_\alpha\in\mathcal{S}^1_{\mathrm{ram}}|_{U_\alpha}$ satisfying the cocycle conditions

$$\begin{aligned}&\nabla u_{\alpha\beta}-u_{\alpha\beta}\nabla=v_\beta-v_\alpha, &&\gamma^1(v_\alpha)=d^1_{\mathcal{S}^\bullet}(\eta_\alpha),\\ &\nabla u'_{\alpha\beta}-u'_{\alpha\beta}\nabla=v'_\beta-v'_\alpha, &&\gamma^1(v'_\alpha)=d^1_{\mathcal{S}^\bullet}(\eta'_\alpha),\end{aligned}$$

where $d^1_{\mathcal{S}^\bullet}$ and γ^1 are defined in (6.7) and (6.8). From the following lemma, we can see that the pairing $\omega_{(E,\nabla,\{l,\ell,\mathcal{V}\})}\big(\big[\{(u_{\alpha\beta}\},\{v_\alpha,\eta_\alpha\}\big],\big[\{(u'_{\alpha\beta}\},\{v'_\alpha,\eta'_\alpha\}\big]\big)$ in (7.3) depends only on the cohomology classes $\big[\{(u_{\alpha\beta}\},\{v_\alpha,\eta_\alpha\}\big],\big[\{(u'_{\alpha\beta}\},\{v'_\alpha,\eta'_\alpha\}\big]\in\mathbf{H}^1(\mathcal{F}^\bullet)$.

Lemma 7.1. *The equality*

$$\omega_{(E,\nabla,\{l,\ell,\mathcal{V}\})}\big(\big[\{u_{\alpha\beta}\},\{v_\alpha,\eta_\alpha\}\big],\big[\{u'_{\alpha\beta}\},\{v'_\alpha,\eta'_\alpha\}\big]\big)=0$$

holds if there exists $\{u_\alpha,(a_{k,\alpha}(w))\}\in C^0\big(\{U_\alpha\},\mathcal{F}^0\big)$ *which satisfies the equalities*

$$\begin{aligned}&u_{\alpha\beta}=u_\beta-u_\alpha,\\ &v_\alpha=\nabla\circ u_\alpha-(u_\alpha\otimes\mathrm{id})\circ\nabla,\\ &\eta_\alpha=-\gamma^0(u_\alpha)+d^0_{\mathcal{S}^\bullet}((a_{k,\alpha}(w))),\end{aligned}$$

where $\gamma^0\colon\mathcal{G}^0\longrightarrow\mathcal{S}^1_{\mathrm{ram}}$ *is defined in* (6.8) *and* $d^0_{\mathcal{S}^\bullet}\colon\mathcal{S}^0_{\mathrm{ram}}\longrightarrow\mathcal{S}^1_{\mathrm{ram}}$ *is defined in* (6.7).

Proof. We put $c_{\alpha\beta}:=\operatorname{Tr}(u_\alpha\circ u'_{\alpha\beta})$ and $b_\alpha:=\operatorname{Tr}(u_\alpha\circ v'_\alpha)$. It is sufficient to prove the equality

$$d(\{c_{\alpha\beta}\},\{b_\alpha\})=\big(\{\operatorname{Tr}(u_{\alpha\beta}\circ u'_{\beta\gamma})\},\{-\operatorname{Tr}(u_{\alpha\beta}\circ v'_\beta-v_\alpha\circ u'_{\alpha\beta})\},\{\Xi_{\mathrm{ram}}(\eta_\alpha,\eta'_\alpha)\}\big).$$

We need a certain amount of calculations for checking the above equality, but we can do it in the same way as that of [12, pp. 37–39]. ■

Proposition 7.2. *The bilinear pairing* $\omega_{(E,\nabla,\{l,\ell,\mathcal{V}\})}\colon\mathbf{H}^1(\mathcal{F}^\bullet)\times\mathbf{H}^1(\mathcal{F}^\bullet)\longrightarrow\mathbb{C}$*, defined by the equality* (7.3) *in* (7.2)*, is a nondegenerate pairing.*

Proof. The bilinear pairing $\omega_{(E,\nabla,\{l,\ell,\mathcal{V}\})}$ corresponds to a homomorphism $\sigma\colon\mathbf{H}^1(\mathcal{F}^\bullet)\longrightarrow\mathbf{H}^1(\mathcal{F}^\bullet)^\vee$ which induces the exact commutative diagram

$$\begin{tikzcd}
\mathbf{H}^0(\mathcal{F}^\bullet_0) \arrow[r] \arrow[d, "\sigma_1"'] & \mathbf{H}^0(\mathcal{F}^\bullet_1) \arrow[r] \arrow[d, "\sigma_2"'] & \mathbf{H}^1(\mathcal{F}^\bullet) \arrow[r] \arrow[d, "\sigma"'] & \mathbf{H}^1(\mathcal{F}^\bullet_0) \arrow[r] \arrow[d, "\sigma_3"'] & \mathbf{H}^1(\mathcal{F}^\bullet_1) \arrow[d, "\sigma_4"'] \\
\mathbf{H}^1(\mathcal{F}^\bullet_1)^\vee \arrow[r] & \mathbf{H}^1(\mathcal{F}^\bullet_0)^\vee \arrow[r] & \mathbf{H}^1(\mathcal{F}^\bullet)^\vee \arrow[r] & \mathbf{H}^0(\mathcal{F}^\bullet_1)^\vee \arrow[r] & \mathbf{H}^0(\mathcal{F}^\bullet_0)^\vee.
\end{tikzcd}$$

The homomorphism $\sigma_2\colon\mathbf{H}^0(\mathcal{F}^\bullet_1)\longrightarrow\mathbf{H}^1(\mathcal{F}^\bullet_0)^\vee$ is given by the pairing

$$\mathbf{H}^0(\mathcal{F}^\bullet_1)\times\mathbf{H}^1(\mathcal{F}^\bullet_0)\longrightarrow\mathbf{H}^2(\mathcal{O}_C\to\Omega^1_C(D_{\mathrm{ram}})\to\Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}})\cong\mathbb{C},$$

$$\left([\{(v_\alpha,(\xi_{k,\alpha}))\}],[\{(u'_{\alpha\beta},(\tau'_{k,\alpha}))\}]\right)\mapsto\left[\{\mathrm{Tr}(v_\alpha\circ u'_{\alpha\beta})\},\{\Xi_{\mathrm{ram}}((0,\xi_{k,\alpha}),(\tau'_{k,\alpha},0))\}\right]$$

and σ_3 is defined similarly. There is an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0\big(\ker(\mathcal{G}^1\to G^1)\big) \arrow[r] \arrow[d, "\eta_1"'] & \mathbf{H}^0(\mathcal{F}_1^\bullet) \arrow[r] \arrow[d, "\sigma_2"'] & \ker\left(\mathrm{Sym}^2\left(\overline{V}\right)\to A^1\right) \arrow[r] \arrow[d, "\eta_2"'] & H^1\big(\ker(\mathcal{G}^1\to G^1)\big) \arrow[d, "\eta_3"'] \\
0 \arrow[r] & H^1\left(\mathcal{G}^0\right)^\vee \arrow[r] & \mathbf{H}^1(\mathcal{F}_0^\bullet)^\vee \arrow[r] & \mathrm{coker}\left(A^0\to\mathrm{Sym}^2\left(\overline{W}\right)\right)^\vee \arrow[r] & H^0\left(\mathcal{G}^0\right)^\vee
\end{tikzcd}$$

whose horizontal sequences are induced by the exact sequences

$$0\longrightarrow\left[\mathcal{G}^1\to G^1\right]\longrightarrow\mathcal{F}_1^\bullet\longrightarrow\left[\mathrm{Sym}^2\left(\overline{V}\right)\to A^1\right]\longrightarrow 0,$$
$$0\longrightarrow\left[A^0\to\mathrm{Sym}^2\left(\overline{W}\right)\right]\longrightarrow\mathcal{F}_0^\bullet\longrightarrow\mathcal{G}^0\longrightarrow 0.$$

Since the trace pairing induces an isomorphism $\ker\left(\mathcal{G}^1\to G^1\right)\xrightarrow{\sim}\left(\mathcal{G}^0\right)^\vee\otimes\Omega^1_C$, we can see by the Serre duality that η_1 and η_3 are isomorphisms.

The map η_2 is induced by the trace pairing

$$\begin{aligned}\ker\left(\mathrm{Sym}^2\left(\overline{V}\right)\to A^1\right)\times\mathrm{coker}\left(A^0\to\mathrm{Sym}^2\left(\overline{W}\right)\right)&\longrightarrow\Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}},\\ ((\xi_k),(\tau_k))&\mapsto\Xi_{\mathrm{ram}}\left((0,\xi_k),(\tau_k,0)\right)\end{aligned}\tag{7.4}$$

composed with $\Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}}\longrightarrow\mathbf{H}^2\big(\mathcal{O}_C\to\Omega^1_C(D_{\mathrm{ram}})\to\Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}}\big)$.

Assume that $(\xi_k)\in\ker\left(\mathrm{Sym}^2(\overline{V})\to A^1\right)$ satisfies $\Xi_{\mathrm{ram}}((0,\xi_k),(\tau_k,0))=0$ for any $(\tau_k)\in\mathrm{Sym}^2\left(\overline{W}\right)$. We can take a lift ξ of (ξ_k) given by Lemma 6.2. For any endomorphism $h\in\mathrm{End}(E|_{D_{\mathrm{ram}}})$, $\psi:=z\big(h\circ\theta+\theta\circ{}^th\big)\colon E|_{D_{\mathrm{ram}}}^\vee\longrightarrow E|_{D_{\mathrm{ram}}}$ is a homomorphism satisfying ${}^t\psi=\psi$ and $\psi(W_k)\subset V_k$. So ψ induces $(\psi_k)\in\mathrm{Sym}^2\left(\overline{W}\right)$ and the equality

$$0=2\Xi_{\mathrm{ram}}((0,\xi_k),(\psi_k,0))=\sum_{p=1}^{r-1}\sum_{j=1}^{p}\nu_p(z)\,\mathrm{Tr}\left(z\big(h\circ\theta+\theta\circ{}^th\big)\circ{}^tN^{p-j}\circ\xi\circ N^{j-1}\right).$$

holds by the assumption. Since

$$\begin{aligned}\sum_{j=1}^{p}\mathrm{Tr}\left(z\theta\circ{}^th\circ{}^tN^{p-j}\circ\xi\circ N^{j-1}\right)&=\sum_{j=1}^{p}\mathrm{Tr}\left(z\,{}^tN^{j-1}\circ\xi\circ N^{p-j}\circ h\circ\theta\right)\\&=\sum_{j=1}^{p}\mathrm{Tr}(zh\circ\theta\circ{}^tN^{p-j}\circ\xi\circ N^{j-1}),\end{aligned}$$

we can deduce $\mathrm{Tr}\left(h\circ z\sum_{p=1}^{r-1}\sum_{j=1}^{p}\nu_p(z)\theta\circ{}^tN^{p-j}\circ\xi\circ N^{j-1}\right)=0$. Since the usual trace pairing is nondegenerate, we have $z\sum_{p=1}^{r-1}\sum_{j=1}^{p}\nu_p(z){}^tN^{p-j}\circ\xi\circ N^{j-1}=0$. Let

$$U=\begin{pmatrix} z^{m-1}a_{0,0} & \cdots & z^{m-1}a_{0,r-1}\\ \vdots & & \vdots\\ z^{m-1}a_{0,r-1,0} & \cdots & z^{m-1}a_{r-1,r-1}\end{pmatrix}$$

be the symmetric matrix representing $\sum_{p=1}^{r-1}\sum_{j=1}^{p}\nu_p(z)\,{}^tN^{p-j}\circ\xi\circ N^j$ with respect to the bases $(e_0,\ldots,e_{r-1})$ and $(e_0^*,\ldots,e_{r-1}^*)$. Consider the trace pairing $\mathrm{Tr}(U(E_{ij}+E_{ji}))$ for $i+j>r-1$, where E_{ij} is the matrix whose (i,j) entry is 1 and the other entries are zero. Then $E_{ij}+E_{ji}$ becomes a lift of an element of $\mathrm{Sym}^2\left(\overline{W}\right)$. So we have $\mathrm{Tr}(U(E_{ij}+E_{ji}))=z^{m-1}(a_{ij}+a_{ji})=0$. Since U is symmetric, we have $z^{m-1}a_{ij}=0$ for $i+j\geq r$. So we have

$$\sum_{p=1}^{r-1}\sum_{j=1}^{p}\nu_p(z)N_k^{p-j}\circ\theta_k\circ\xi_k\circ N_k^{j-1}=0$$

for each k. By the way, $(\xi_k) \in \ker\big(\mathrm{Sym}^2\big(\overline{V}\big) \to A^1\big)$ implies $\mathrm{Tr}\big(\theta \circ \xi \circ N^l\big) = 0$ for any $0 \le l \le r-1$. So there is an endomorphism $f \in \mathrm{End}(E|_{D_{\mathrm{ram}}})$ satisfying $\theta \circ \xi = Nf - fN$. Moreover, we have $f(V_k) \subset V_k$ for any k. Thus we have

$$0 = \sum_{p=1}^{r-1}\sum_{j=1}^{p} \nu_p(z) N_k^{p-j} \circ (N_k \circ f_k - f_k \circ N_k) \circ N_k^{j-1} = \nu(N_k) f_k - f_k \nu(N_k)$$

for each $0 \le k \le r-1$, where f_k is the endomorphism of $\overline{V}_k$ induced by f. Since the $w\frac{\mathrm{d}z}{z^m}$-coefficient of $\nu(w)$ does not vanish, we can deduce $N_k \circ f_k - f_k \circ N_k = 0$ from the above equality. Thus we have $(\xi_k) = 0$. Hence the pairing (7.4) is a perfect pairing of $\mathcal{O}_{D_{\mathrm{ram}}}$-modules, since $\mathrm{length}\big(\ker\big(\mathrm{Sym}^2\big(\overline{V} \to A^1\big)\big)\big) = \mathrm{length}\big(\mathrm{coker}\big(A^0 \to \mathrm{Sym}^2\big(\overline{W}\big)\big)\big)$. Note that the map

$$\Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}} \longrightarrow \mathbf{H}^2(\mathcal{O}_C \to \Omega^1_C(D_{\mathrm{ram}}) \to \Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}}) \cong \mathbb{C}$$

is identified with the residue map. So we can see that the pairing

$$\ker\big(\mathrm{Sym}^2\big(\overline{V}\big) \to A^1\big) \times \mathrm{coker}\big(A^0 \to \mathrm{Sym}^2\big(\overline{W}\big)\big) \longrightarrow \mathbb{C}$$

induced by (7.4) is a perfect pairing of vector spaces, which means that η_2 is an isomorphism.

Since η_1, η_3 and η_2 are isomorphic, $\sigma_2 \colon \mathbf{H}^0(\mathcal{F}_1^\bullet) \xrightarrow{\sim} \mathbf{H}^1(\mathcal{F}_0^\bullet)^\vee$ is an isomorphism by the five lemma. Then $\sigma_3 \colon \mathbf{H}^1(\mathcal{F}_0^\bullet) \xrightarrow{\sim} \mathbf{H}^0(\mathcal{F}_1^\bullet)^\vee$ is also isomorphic because it is the dual of σ_2.

On the other hand, $\sigma_1 \colon \mathbf{H}^0(\mathcal{F}_0^\bullet) \longrightarrow \mathbf{H}^1(\mathcal{F}_1^\bullet)^\vee$ is given by the pairing

$$\begin{aligned}
&\mathbf{H}^0(\mathcal{F}_0^\bullet) \times \mathbf{H}^1(\mathcal{F}_1^\bullet) \longrightarrow \mathbf{H}^2\big(\mathcal{O}_C \to \Omega^1_C(D_{\mathrm{ram}}) \to \Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}}\big),\\
&\big([\{u_\alpha, (a_{k,\alpha}(w))\}], [\{v_{\alpha\beta}\}, \{(\overline{v}_{k,\alpha}, b_\alpha)\}]\big)\\
&\qquad \mapsto \big[\{\mathrm{Tr}(v_{\alpha\beta} \circ u_\beta)\}, \big\{\mathrm{Tr}(\overline{v}_\alpha \circ u_\alpha) + \tfrac{b_\alpha}{2}(\nu'(w) a_{k,\alpha}(w))\big\}\big],
\end{aligned}$$

where $\nu'(w) := \sum_{k=0}^{r-1} k\nu_k(z) w^{k-1}$ and $\overline{v}_\alpha \in \mathrm{End}(E|_{D_{\mathrm{ram}}})$ is a lift of $(\overline{v}_{k,\alpha})$ given by Lemma 6.1. We have the exact commutative diagram

$$\begin{array}{ccccccc}
0 = \ker\big(A^0 \to \mathrm{Sym}^2\big((\overline{W})\big)\big) & \to & \mathbf{H}^0(\mathcal{F}_0^\bullet) & \to & H^0(\mathcal{G}^0) & \to & \mathrm{coker}\big(A^0 \to \mathrm{Sym}^2\big((\overline{W})\big)\big)\\
 & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle {}^t\eta_3} & & \downarrow{\scriptstyle {}^t\eta_2}\\
0 = \mathrm{coker}\big(\mathrm{Sym}^2\big((\overline{V})\big) \to A^1\big)^\vee & \to & \mathbf{H}^1(\mathcal{F}_1^\bullet)^\vee & \to & H^1\big(\ker\big(\mathcal{G}^1 \to G^1\big)\big)^\vee & \to & \ker\big(\mathrm{Sym}^2\big((\overline{V})\big) \to A^1\big)^\vee
\end{array}$$

and the five lemma implies that $\sigma_1 \colon \mathbf{H}^0(\mathcal{F}_0^\bullet) \longrightarrow \mathbf{H}^1(\mathcal{F}_1^\bullet)^\vee$ is isomorphic because ${}^t\eta_3$ and ${}^t\eta_2$ are isomorphic.

We can see that $\sigma_4 \colon \mathbf{H}^1(\mathcal{F}_1^\bullet) \longrightarrow \mathbf{H}^0(\mathcal{F}_0^\bullet)^\vee$ is also isomorphic since it is the dual of σ_1. Since σ_1, σ_2, σ_3, σ_4 are all isomorphic, $\sigma \colon \mathbf{H}^1(\mathcal{F}^\bullet) \longrightarrow \mathbf{H}^1(\mathcal{F}^\bullet)^\vee$ is isomorphic by the five lemma. ■

We define a complex $\tilde{\Omega}^\bullet$ by setting $\tilde{\Omega}^0 = \mathcal{O}_C$, $\tilde{\Omega}^1 = \Omega^1_C(D_{\mathrm{ram}}) \oplus A^1$, $\tilde{\Omega}^2 = \Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}} \oplus A^1$ and

$$\begin{aligned}
d^0_{\tilde{\Omega}^\bullet} &\colon\ \mathcal{O}_C \ni f \mapsto (df, 0) \in \Omega^1_C(D_{\mathrm{ram}}) \oplus A^1,\\
d^1_{\tilde{\Omega}^\bullet} &\colon\ \Omega^1_C(D_{\mathrm{ram}}) \oplus A^1 \ni (\eta, b) \mapsto ((\eta|_{D_{\mathrm{ram}}} - b(\nu'(w))), b) \in \Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}} \oplus A^1,
\end{aligned}$$

where the k-th component of $(\nu'(w)) \in A^0 \otimes \Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}}$ is given by $\nu'(w) = \sum_{j=0}^{r-1} j\nu_j(z) w^{j-1}$. Then we can define a homomorphism of complexes $\mathrm{Tr}^\bullet \colon \mathcal{F}^\bullet \longrightarrow \tilde{\Omega}^\bullet$ by

$$\begin{aligned}
\mathrm{Tr}^0 &\colon\ \mathcal{F}^0 = \mathcal{G}^0 \oplus A^0 \ni (u, (f_k(w))) \mapsto \mathrm{Tr}(u) \in \mathcal{O}_C,\\
\mathrm{Tr}^1 &\colon\ \mathcal{F}^1 = \mathcal{G}^1 \oplus \mathrm{Sym}^2\big(\overline{W}\big) \oplus \mathrm{Sym}^2\big(\overline{V}\big) \ni (v, (\tau_k), (\xi_k))\\
&\qquad\qquad \mapsto (\mathrm{Tr}(v), (\Theta_{(\tau_k,\xi_k)})) \in \Omega^1_C(D_{\mathrm{ram}}) \oplus A^1,\\
\mathrm{Tr}^2 &\colon\ \mathcal{F}^2 = G^1 \oplus A^1 \ni ((\overline{v}_k), b)) \mapsto (\mathrm{Tr}(\overline{v}), b) \in \Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}} \oplus A^1,
\end{aligned}$$

where $\overline{v} \in \mathrm{Hom}\big(E|_{D_{\mathrm{ram}}}, E \otimes \Omega^1_D(D)|_{D_{\mathrm{ram}}}\big)$ is a lift of $(\overline{v}_k)$ given by Lemma 6.1.

Lemma 7.3. *Assume that the endomorphism ring of E, preserving l, ℓ, $\mathcal{V}$ and commuting with ∇, consists of the scalar multiplications $\mathbb{C}\mathrm{id}_E$. Then the map*

$$\mathbf{H}^2(\mathrm{Tr})\colon\ \mathbf{H}^2(\mathcal{F}^\bullet) \longrightarrow \mathbf{H}^2\big(\tilde{\Omega}^\bullet\big) \cong \mathbf{H}^2(\Omega^\bullet_C) \cong \mathbb{C}$$

is an isomorphism.

Proof. First note that $\mathbf{H}^0(\mathcal{F}^\bullet) = \mathbb{C}$ because there are only scalar endomorphisms of E commuting with ∇ and preserving the (λ,μ,ν)-structure. Under the identification $\mathbf{H}^0(\mathcal{F}^\bullet) \cong \mathbb{C} \cong \mathbf{H}^2\big(\tilde{\Omega}^\bullet\big)^\vee$, there is an exact commutative diagram

$$\begin{array}{ccccccc}
\mathbf{H}^1(\mathcal{F}_0^\bullet) & \longrightarrow & \mathbf{H}^1(\mathcal{F}_1^\bullet) & \longrightarrow & \mathbf{H}^2(\mathcal{F}^\bullet) & \longrightarrow & 0 \\
{\scriptstyle\sigma_3}\big\downarrow & & {\scriptstyle\sigma_4}\big\downarrow & & {\scriptstyle\mathbf{H}^2(\mathrm{Tr}^\bullet)}\big\downarrow & & \\
\mathbf{H}^0(\mathcal{F}_1^\bullet)^\vee & \longrightarrow & \mathbf{H}^0(\mathcal{F}_0^\bullet)^\vee & \longrightarrow & \mathbf{H}^0(\mathcal{F}^\bullet)^\vee & \longrightarrow & 0.
\end{array}$$

Since σ_3 and σ_4 are isomorphisms, $\mathbf{H}^2(\mathrm{Tr}^\bullet)$ is also an isomorphism. ∎

Remark 7.4. If $(E,\nabla,l,\ell,\mathcal{F})$ is $\boldsymbol{\alpha}$-stable, then the assumption of Lemma 7.3 holds.

Theorem 7.5. *The moduli space $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$ of connections with $(\lambda,\tilde{\mu},\tilde{\nu})$-structure is smooth over $\mathcal{T}$. The dimension of the fiber $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_t$ over $t\in\mathcal{T}$ is $2r^2(g(\mathcal{C}_t)-1)+2+r(r-1)\deg\mathcal{D}_t$ if it is non-empty.*

Proof. For the proof of the smoothness, take an Artinian local ring A over $\mathcal{T}$ with the maximal ideal $\mathfrak{m}$ and an ideal I of A satisfying $\mathfrak{m}I = 0$. Assume that a flat family $(E,\nabla,l,\ell,\mathcal{V})$ of connections on $\mathcal{C}\otimes A/I$ is given. Consider the complex $\mathcal{F}^\bullet$ determined from $(E,\nabla,l,\ell,\mathcal{V})\otimes A/\mathfrak{m}$ by (6.9). We take an affine open covering $\{U_\alpha\}$ of $\mathcal{C}\otimes A$ as in the proof of Proposition 6.3. If $U_\alpha\cap(\mathcal{D}_{\mathrm{ram}})_A=\varnothing$, we can easily take a lift $(E_\alpha,\nabla_\alpha,\{l_\alpha,\ell_\alpha,\mathcal{V}_\alpha\})$ of $(E,\nabla,\{l,\ell,\mathcal{V}\})|_{U_\alpha\otimes A/I}$. If $U_\alpha\cap(\mathcal{D}_{\mathrm{ram}})_A\neq\varnothing$, then we may assume that $\mathcal{V}\cap U_\alpha$ is given by a factorized $\tilde{\nu}$-ramified structure $(V_k,\vartheta_k,\varkappa_k)$. As in the proof of Proposition 3.3, we can choose an endomorphism N on $E|_{(\mathcal{D}_{\mathrm{ram}})_{A/I}}$ inducing $\theta_k\circ\kappa_k$ on $\overline{V}_k$ for $0\le k\le r-1$. The representation matrix of N is given by

$$\begin{pmatrix}
0 & 0 & \cdots & 0 & z \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}.$$

with respect to the basis $e_0,\dots,e_{r-1}$ chosen as in the proof of Proposition 3.3. Then we can give a factorization $N=\theta\circ\kappa$ by the matrix factorization

$$\begin{pmatrix}
0 & 0 & \cdots & 0 & z \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}
=
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & ⋰ & ⋰ & \vdots \\
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 1 & 0 & 0 \\
\vdots & \vdots & ⋰ & ⋰ & \vdots & \vdots \\
0 & 1 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & z
\end{pmatrix}$$

with respect to the basis $e_0,\dots,e_{r-1}$ of $E|_{(\mathcal{D}_{\mathrm{ram}})_{A/I}}$ and its dual basis $e_0^*,\dots,e_{r-1}^*$. Let E_α be a free $\mathcal{O}_{U_\alpha}$-module with $E_\alpha\otimes A/I = E|_{U_\alpha\otimes A/I}$. Define $\tilde{N}\colon E_\alpha|_{(\mathcal{D}_{\mathrm{ram}})_A\cap U_\alpha}\longrightarrow E_\alpha|_{(\mathcal{D}_{\mathrm{ram}})_A\cap U_\alpha}$,

$\tilde{\theta}\colon E_\alpha|^\vee_{(\mathcal{D}_{\mathrm{ram}})_A\cap U_\alpha} \longrightarrow E_\alpha|_{(\mathcal{D}_{\mathrm{ram}})_A\cap U_\alpha}$ and $\tilde{\kappa}\colon E_\alpha|_{(\mathcal{D}_{\mathrm{ram}})_A\cap U_\alpha} \longrightarrow E_\alpha|^\vee_{(\mathcal{D}_{\mathrm{ram}})_A\cap U_\alpha}$ by the same representation matrices as N, θ and κ respectively. Then $\tilde{N}$, $\tilde{\theta}$ and $\tilde{\kappa}$ are lifts of N, θ and κ and they induce a lift $\mathcal{V}_\alpha = (\tilde{V}_k, \tilde{\vartheta}_k.\tilde{\varkappa}_k)$ of $(V_k, \vartheta_k, \varkappa_k)$ over A. We can easily take a relative connection ∇_α on E_α which is a lift of $\nabla|_{U_\alpha}$ and which is compatible with $\mathcal{V}_\alpha$. So we obtain a lift $(E_\alpha, \nabla_\alpha, \{l_\alpha, \ell_\alpha, \mathcal{V}_\alpha\})$ of $(E, \nabla, \{l, \ell, \mathcal{V}\})|_{U_\alpha\otimes A/I}$ when $U_\alpha \cap (\mathcal{D}_{\mathrm{ram}})_A \neq \varnothing$.

Take an isomorphism $\theta_{\beta\alpha}\colon E_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} E_\beta|_{U_{\alpha\beta}}$, where $U_{\alpha\beta} = U_\alpha \cap U_\beta$. If we put

$$
\begin{aligned}
u_{\alpha\beta\gamma} &= \theta_{\gamma\alpha}^{-1} \circ \theta_{\gamma\beta} \circ \theta_{\beta\alpha} - \mathrm{id},\\
v_{\alpha\beta} &= (\theta_{\beta\alpha} \otimes \mathrm{id})^{-1} \circ \nabla_\beta \circ \theta_{\beta\alpha} - \nabla_\alpha,
\end{aligned}
$$

then the class $[\{u_{\alpha\beta\gamma}\}, \{v_{\alpha\beta}\}] \in \mathbf{H}^2(\mathcal{F}^\bullet) \otimes I$ is nothing but the obstruction for the lifting of $(E, \nabla, \{l, \ell, \mathcal{V}\})$ to a flat family of connections on $\mathcal{C} \otimes A$ over A. We can see that the image $\mathbf{H}^2(\mathrm{Tr}^\bullet)([\{u_{\alpha\beta\gamma}\}, \{v_{\alpha\beta}\}])$ under the isomorphism $\mathbf{H}^2(\mathrm{Tr}^\bullet)\colon \mathbf{H}^2(\mathcal{F}^\bullet) \xrightarrow{\sim} \mathbf{H}^2(\Omega^\bullet_{\mathcal{C}\otimes A/\mathfrak{m}})$ is nothing but the obstruction for the lifting of the determinant line bundle $\det(E, \nabla)$ with the induced connection. Consider the moduli space $M(\sum \lambda_k, \sum \mu_k, (r-1)dz/2 + r\nu_0)$ of pairs (L, ∇_L) of a line bundle L on the fibers of $\mathcal{C}$ over $\mathcal{T}$ and a connection ∇_L on L admitting poles along $\mathcal{D}$ whose residue along $\mathcal{D}_{\log}$ is $\sum_{1\leq k\leq r} \lambda_k$, whose restriction to $\mathcal{D}_{\mathrm{un}}$ is $\sum_{1\leq k\leq r} \mu_k$ and whose restriction to $\mathcal{D}_{\mathrm{ram}}$ is $\sum((r-1)dz/2 + r\nu_0)$. Then $M(\sum \lambda_k, \sum \mu_k, (r-1)dz/2 + r\nu_0)$ is smooth over $\mathcal{T}$, since it is an affine space bundle over the relative Jacobian of $\mathcal{C}$ over $\mathcal{T}$. In particular, we have $\mathbf{H}^2(\mathrm{Tr}^\bullet)([\{u_{\alpha\beta\gamma}\}, \{v_{\alpha\beta}\}]) = 0$ which is equivalent to $[\{u_{\alpha\beta\gamma}\}, \{v_{\alpha\beta}\}] = 0$. Thus $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})$ is smooth over $\mathcal{T}$.

By Proposition 6.3, the dimension of the moduli space at $(E, \nabla, l, \ell, \mathcal{V}) \otimes A/\mathfrak{m}$ is given by $\dim \mathbf{H}^1(\mathcal{F}^\bullet)$. We write $\mathcal{D} \otimes A/\mathfrak{m} = D$, $\mathcal{D}_{\log} \otimes A/\mathfrak{m} = D_{\log}$ and so on. Using the exact sequence (6.10) and the equality $\dim \mathbf{H}^0(\mathcal{F}^\bullet) = \dim \mathbf{H}^2(\mathcal{F}^\bullet) = 1$ by Lemma 7.3, we have

$$
\begin{aligned}
\dim \mathbf{H}^1(\mathcal{F}^\bullet) &= \chi(\mathcal{F}^\bullet_1) - \chi(\mathcal{F}^\bullet_0) + 2\\
&= \chi\big(\mathcal{G}^1\big) - \dim_{\mathbb{C}} G^1 + \dim_{\mathbb{C}} \mathrm{Sym}^2\big(\overline{V}\big) - \dim_{\mathbb{C}} A^1\\
&\quad - \chi\big(\mathcal{G}^0\big) - \dim_{\mathbb{C}} A^0 + \dim_{\mathbb{C}} \mathrm{Sym}^2\big(\overline{W}\big) + 2.
\end{aligned}
\tag{7.5}
$$

Since $\ker\big(\mathcal{G}^1 \to G^1\big) \cong \big(\mathcal{G}^0\big)^\vee \otimes \Omega^1_C$, we have

$$
\begin{aligned}
\chi\big(\mathcal{G}^1\big) - \dim_{\mathbb{C}} G^1 &= -\chi\big(\mathcal{G}^0\big)\\
&= r^2(g-1) + (\deg D_{\log} + \deg D_{\mathrm{un}})r(r-1)/2 + \sum_{x\in D_{\mathrm{ram}}} r(r-1)/2.
\end{aligned}
\tag{7.6}
$$

By the same method as in the proof of Lemma 6.2, we can see that the elements of $\mathrm{Sym}^2\big(\overline{V}\big)$ are given by the data

$$
\begin{aligned}
&(a_{r-k-1,k}(z))_{0\leq k\leq r-1} \in \big(\mathbb{C}[z]/(z^{m_x})\big)^r \quad \text{such that } za_{r-k-1,k} = za_{k,r-k-1},\\
&(\bar{a}_{ij}(z))_{0\leq i,j\leq r-1, i+j\neq r-1} \in \big(\mathbb{C}[z]/\big(z^{m_x-1}\big)\big)^{r^2-r} \quad \text{such that } \bar{a}_{ji} = \bar{a}_{ij} \quad (x \in D_{\mathrm{ram}})
\end{aligned}
$$

and each $\xi_k \in \mathrm{Hom}\big(\overline{V}_k, \overline{W}_k\big)|_{m_x x}$ is given by the matrix

$$
\begin{pmatrix}
\overline{a}_{00}(z) & & \cdots & & za_{0,r-1}(z)\\
\vdots & & & \iddots & \vdots\\
\overline{a}_{r-k-1,0}(z) & \cdots & a_{r-k-1,k}(z) & \cdots & za_{k,r-1}(z)\\
\vdots & \iddots & & & \vdots\\
\overline{a}_{r-1,0}(z) & & \cdots & & za_{r-1,r-1}(z)
\end{pmatrix},
$$

where $za_{i,j}$ is the image of $z \otimes \overline{a}_{i,j}$ by $(z) \otimes \mathcal{O}_{(m_x-1)x} \xrightarrow{\sim} z\mathcal{O}_{m_x x}$. So we can see that

$$\dim_{\mathbb{C}} \mathrm{Sym}^2\left(\overline{V}\right) = \dim_{\mathbb{C}} \mathrm{Sym}^2\left(\overline{W}\right) = \sum_{x \in D_{\mathrm{ram}}} \left(r + \frac{1}{2}(m_x - 1)r(r+1) \right). \tag{7.7}$$

Finally note that

$$\dim_{\mathbb{C}} A^0 = \dim_{\mathbb{C}} A^1 = \sum_{x \in D_{\mathrm{ram}}} m_x r. \tag{7.8}$$

Substituting (7.6), (7.7) and (7.8) to (7.5), we get the desired equality $\dim \mathbf{H}^1(\mathcal{F}^\bullet) = 2r^2(g-1) + 2 + r(r-1)\deg D$. ■

8 Symplectic structure on the moduli space

In this section, we assume again the same notations as in Section 5, Section 6 and Section 7.

There is an étale surjective morphism $M' \longrightarrow M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})$, such that there is a universal family of connections $\big(\tilde{E}, \tilde{\nabla}, \{\tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}}\}\big)$ on $\mathcal{C}_{M'}$ over M'. We can define a complex $\mathcal{G}^\bullet_{M'}$ on $\mathcal{C}_{M'}$ from $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}}\})$ in the same way as $\mathcal{G}^\bullet$ given by (6.1), (6.2). We can also define a complex $\mathcal{S}^\bullet_{\mathrm{ram},M'}$ on $\mathcal{C}_{M'}$ in the same way as $\mathcal{S}^\bullet_{\mathrm{ram}}$ given by (6.6), (6.7). Then we can define a complex

$$\tilde{\mathcal{F}}^\bullet_{M'} := \mathrm{Cone}\big(\mathcal{G}^\bullet_{M'} \to \mathcal{S}^\bullet_{\mathrm{ram},M'}[1]\big)[-1]$$

in the same way as $\mathcal{F}^\bullet$ defined in (6.9).

Let $p_{M'} \colon \mathcal{C}_{M'} \longrightarrow M'$ be the projection. Then we can see by Proposition 6.3 that the relative tangent bundle $T_{M'/\mathcal{T}}$ of M' over $\mathcal{T}$ is isomorphic to $\mathbf{R}^1 p_{M'*}\big(\tilde{\mathcal{F}}^\bullet_{M'}\big)$. We can define a pairing $\Xi_{\mathrm{ram}} \colon \mathcal{S}^1_{\mathrm{ram},M'} \times \mathcal{S}^1_{\mathrm{ram},M'} \longrightarrow \Omega^1_{\mathcal{C}_{M'}/M'}(\mathcal{D}_{M'})|_{(\mathcal{D}_{\mathrm{ram}})_{M'}}$ in the same way as (7.1). Consider the pairing

$$\begin{aligned} \omega_{M'} \colon\ & \mathbf{R}^1 p_{M'*}\big(\tilde{\mathcal{F}}^\bullet_{M'}\big) \times \mathbf{R}^1 p_{M'*}\big(\tilde{\mathcal{F}}^\bullet_{M'}\big) \longrightarrow \\ & \mathbf{R}^2 p_{M'*}\big[\mathcal{O}_{\mathcal{C}} \to \Omega^1_{\mathcal{C}/\mathcal{T}}(\mathcal{D}_{\mathrm{ram}}) \to \Omega^1_{\mathcal{C}/\mathcal{T}}(\mathcal{D}_{\mathrm{ram}})|_{\mathcal{D}_{\mathrm{ram}}}\big]_{M'} \cong \mathbf{R}^2 p_{M'*}\Omega^\bullet_{\mathcal{C}_{M'}/M'} \cong \mathcal{O}_{M'} \end{aligned} \tag{8.1}$$

defined by

$$\begin{aligned} & \omega_{M'}\big(\big[\{u_{\alpha\beta}\}, \{v_\alpha, \eta_\alpha\}\big], \big[\{u'_{\alpha\beta}\}, \{v'_\alpha, \eta'_\alpha\}\big]\big) \\ & \quad = \big[\{\mathrm{Tr}(u_{\alpha\beta} \circ u'_{\beta\gamma})\}, \{-\mathrm{Tr}(u_{\alpha\beta} \circ v'_\beta - v_\alpha \circ u'_{\alpha\beta})\}, \{\Xi_{\mathrm{ram}}(\eta_\alpha, \eta'_\alpha)\}\big] \end{aligned}$$

in the same way as (7.3). We can check $\omega_{M'}(v, v) = 0$ for $v \in \mathbf{R}^1 p_{M'*}(\tilde{\mathcal{F}}^\bullet_{M'})$ and $\omega_{M'}$ descends to a $\mathcal{T}$-relative 2-form $\omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ on $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})$.

Theorem 8.1. *The 2-form $\omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ defined by (8.1) is a $\mathcal{T}$-relative symplectic form on the moduli space $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})$ of $\boldsymbol{\alpha}$-stable connections on $(\mathcal{C}, \mathcal{D})$ with $(\lambda, \tilde{\mu}, \tilde{\nu})$-structure.*

The restriction $\omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}|_p$ at each point $p \in M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})$ is nondegenerate by Proposition 7.2. It remains to prove that $d\omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})} = 0$. Since $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})$ is smooth over $\mathcal{T}$, we only have to show the vanishing $d\omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_t} = 0$ of the restriction to the fiber $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})_t$ over $t \in \mathcal{T}$. For its proof we use a construction of an unfolding of the moduli space.

Put $\mathcal{C}_t = C$, $\mathcal{D}_t = D$, $(\mathcal{D}_{\mathrm{un}})_t = D_{\mathrm{un}}$, $(\mathcal{D}_{\mathrm{ram}})_t = D_{\mathrm{ram}}$ and $(\lambda, \mu, \nu) = (\lambda, \tilde{\mu}, \tilde{\nu})_t$. For each $x \in D$, choose a defining equation z of D_{red} on an affine open neighborhood of x, which is a lift of $\overline{z}$. Take distinct complex numbers $s^x_1, \ldots, s^x_{m_x-1}, s^x_{m_x} \in \mathbb{C}$. Let $D^x_{\mathrm{un},h}$ be the divisor on $C \times \mathrm{Spec}\,\mathbb{C}[h]$ defined by the equation $(z - hs^x_1) \cdots (z - hs^x_{m_x}) = 0$ and put $D_{\mathrm{un},h} = \sum_{x \in D_{\mathrm{un}}} D^x_{\mathrm{un},h}$.

For each $x \in D_{\mathrm{ram}}$, take distinct complex numbers $q^x_1, \dots, q^x_{m_x-1}, q^x_{m_x} \in \mathbb{C}$ with $q^x_{m_x} = 1$. Let $D^x_{\mathrm{ram},h}$ be the divisor on $C \times \operatorname{Spec}\mathbb{C}[h]$ defined by the equation $(z - h^r q^x_1) \cdots (z - h^r q^x_{m_x-1})(z - h^r) = 0$ and put $D_{\mathrm{ram},h} := \sum_{x \in D_{\mathrm{ram}}} D^x_{\mathrm{ram},h}$. We set

$$D_h := D_{\log} + D_{\mathrm{un},h} + D_{\mathrm{ram},h}.$$

Note that D_h is a reduced divisor for generic h and it coincides with D if $h = 0$. So we can take a Zariski open subset H° of $\operatorname{Spec}\mathbb{C}[h]$ containing 0 such that D_h is a reduced divisor for any $h \in H^\circ \setminus \{0\}$.

For $x \in D_{\mathrm{un}}$, we can write

$$\mu_k|_{m_x x} = \left(b_{k,0} + b_{k,1} z + \cdots + b_{k,m_x-1} z^{m_x-1}\right) \frac{\mathrm{d}z}{z^{m_x}}, \qquad k = 0, \dots, r-1.$$

We define $\mu_{k,h} \in \Omega^1_{C \times \operatorname{Spec}\mathbb{C}[h]/\operatorname{Spec}\mathbb{C}[h]}(D_{\mathrm{un},h})|_{D_{\mathrm{un},h}}$ by

$$\mu_{k,h}|_{D_{\mathrm{un},h}} = \frac{b_{k,0} + b_{k,1} z + \cdots + b_{k,m_x-1} z^{m_x-1}}{(z - h s^x_1) \cdots (z - h s^x_{m_x})} \mathrm{d}z, \qquad k = 0, \dots, r-1.$$

We can write

$$\nu^x_0(z) = \left(a^x_{0,0} + a^x_{0,1} z + \cdots + a^x_{0,m_x-2} z^{m_x-2} + a^x_{0,m_x-1} z^{m_x-1}\right) \frac{\mathrm{d}z}{z^{m_x}},$$
$$\nu^x_k(z) = \left(a^x_{k,0} + a^x_{k,1} z + \cdots + a^x_{k,m_x-2} z^{m_x-2}\right) \frac{\mathrm{d}z}{z^{m_x}}, \qquad k = 1, \dots, r-1.$$

Then we define $\nu_{k,h}(z) \in \Omega^1_{C \times \operatorname{Spec}\mathbb{C}[h]/\operatorname{Spec}\mathbb{C}[h]}(D_{\mathrm{ram},h})|_{D_{\mathrm{ram},h}}$ for $0 \le k \le r-1$ by

$$\nu_{0,h}(z)|_{D_{\mathrm{ram},h}} = \frac{a^x_{0,0} + a^x_{0,1} z + \cdots + a^x_{0,m_x-2} z^{m_x-2} + a^x_{0,m_x-1} z^{m_x-1}}{(z - h^r q^x_1) \cdots (z - h^r q^x_{m_x-1})(z - h^r)} \mathrm{d}z,$$
$$\nu_{k,h}(z)|_{D_{\mathrm{ram},h}} = \frac{a^x_{k,0} + a^x_{k,1} z + \cdots + a^x_{k,m_x-2} z^{m_x-2}}{(z - h^r q^x_1) \cdots (z - h^r q^x_{m_x-1})(z - h^r)} \mathrm{d}z, \qquad k = 1, \dots, r-1,$$

and we set

$$\nu_h(w) := \nu_{0,h}(z) + \nu_{1,h}(z) w + \cdots + \nu_{r-1,h}(z) w^{r-1}.$$

Consider the moduli space

$$\mathcal{M}_{H^\circ} = \{(E, \nabla, l, (\ell_k)_{0 \le k \le r-1}, (V_k, \vartheta_k, \varkappa_k)_{0 \le k \le r-1})\} \longrightarrow H^\circ,$$

where

(i) E is an algebraic vector bundle on C of rank r and degree d,

(ii) $\nabla \colon E \longrightarrow E \otimes \Omega^1_C(D_h)$ is a connection admitting poles along D_h,

(iii) l is a logarithmic λ-parabolic structure on (E, ∇) along $D_{\log}$,

(iv) $E|_{D_{\mathrm{un},h}} = \ell_0 \supset \cdots \supset \ell_{r-1} \supset \ell_r = 0$ is a filtration such that $\ell_k/\ell_{k+1} \cong \mathcal{O}_{D_{\mathrm{un},h}}$ for any k and that $(\nabla|_{D_{\mathrm{un},h}} - \mu_{k,h}\mathrm{id})(\ell_k) \subset \ell_{k+1} \otimes \Omega^1_C(D_{\mathrm{un},h})$ for any k,

(v) $E|_{D_{\mathrm{ram},h}} = V_0 \supset V_1 \supset \cdots \supset V_{r-1} \supset V_r = (z - h^r)V_0$ is a filtration by $\mathcal{O}_{D_{\mathrm{ram},h}}$-submodules such that $V_j/V_{j+1} \cong \mathcal{O}_{D_{\mathrm{ram},h}}/(z - h^r)$ and $\nabla|_{D^x_{\mathrm{ram},h}}(V_k) \subset V_k \otimes \Omega^1_C(D_{\mathrm{ram},h})$ for $0 \le k \le r-1$,

(vi) for $\overline{V}^x_k : V_k|_{D^x_{\mathrm{ram},h}} \big/ \prod_{j=1}^{m_x-1}(z-h^r q^x_j)V_{k+1}|_{D^x_{\mathrm{ram},h}}$ and $\overline{W}^x_k = \mathrm{Hom}_{\mathcal{O}_{D^x_{\mathrm{ram},h}}}(\overline{V}^x_{r-k-1}, \mathcal{O}_{D^x_{\mathrm{ram},h}})$,

$$\vartheta^x_k \colon\ \overline{W}^x_k \times \overline{W}^x_{r-k-1} \longrightarrow \mathcal{O}_{D^x_{\mathrm{ram}}}, \qquad 0 \le k \le r-1,$$

are $\mathcal{O}_{D^x_{\mathrm{ram},h}}$-bilinear pairings such that the homomorphisms $\theta^x_k \colon \overline{W}^x_k \longrightarrow \big(\overline{W}^x_{r-k-1}\big)^\vee = \overline{V}^x_k$ induced by ϑ^x_k are isomorphisms, the equalities $\vartheta^x_k(v,v') = \vartheta^x_{r-k-1}(v',v)$ hold for $v \in \overline{W}^x_k$, $v' \in \overline{W}^x_{r-k-1}$ and that the equalities $\vartheta^x_{k-1}\big(v_1|_{\overline{V}^x_{r-k}}, v_2\big) = \vartheta_k\big(v_1, v_2|_{\overline{V}^x_k}\big)$ hold for $v_1 \in \overline{W}^x_k = \mathrm{Hom}\big(\overline{V}^x_{r-k-1}, \mathcal{O}_{D^x_{\mathrm{ram},h}}\big)$, $v_2 \in \overline{W}^x_{r-k} = \mathrm{Hom}\big(\overline{V}^x_{k-1}, \mathcal{O}_{D^x_{\mathrm{ram},h}}\big)$ when $1 \le k \le r-1$ and the equality $\vartheta_{r-1}((z-h^r)v_1, v_2) = \vartheta_0(v_1, (z-h^r)v_2)$ holds for $v_1, v_2 \in \overline{W}^x_0$,

(vii) $\varkappa^x_k \colon \overline{V}^x_k \times \overline{V}^x_{r-k-1} \longrightarrow \mathcal{O}_{D^x_{\mathrm{ram},h}}$ are $\mathcal{O}_{D^x_{\mathrm{ram},h}}$-bilinear pairings for $0 \le k \le r-1$ such that the equalities $\varkappa^x_k(v,v') = \varkappa^x_{r-k-1}(v',v)$ hold for $v \in \overline{V}^x_k$, $v' \in \overline{V}^x_{r-k-1}$, the equalities $\varkappa^x_{k-1}(\overline{v_1}, v_2) = \varkappa^x_k(v_1, \overline{v_2})$ hold for $v_1 \in \overline{V}_k$, $v_2 \in \overline{V}_{r-k}$ and for the image $\overline{v_1}$ (resp. $\overline{v_2}$) of v_1 (resp. v_2) via the canonical map $\overline{V}^x_k \to \overline{V}^x_{k-1}$ (resp. $\overline{V}^x_{r-k} \to \overline{V}^x_{r-k-1}$), the equality $\varkappa_{r-1}((z-h^r)v_1, v_2) = \varkappa_0(v_1, (z-h^r)v_2)$ holds for $v_1, v_2 \in \overline{V}^x_0$ and that the equalities $(\theta^x_k \circ \kappa^x_k)^r = (z-h^r)\cdot \mathrm{id}_{\overline{V}^x_k}$ hold for the homomorphisms $\kappa^x_k \colon \overline{V}^x_k \longrightarrow \big(\overline{V}^x_{r-k-1}\big)^\vee = \overline{W}^x_k$ induced by $\varkappa^x_k$,

(viii) the homomorphism

$$\mathcal{O}_{D^x_{\mathrm{ram},h}}[w]\big/\big(w^r - z + h^r, (z-h^r q^x_1)\cdots(z-h^r q^x_{m_x-1})w\big) \longrightarrow \mathrm{End}_{\mathcal{O}_{D^x_{\mathrm{ram},h}}}\big(\overline{V}^x_k\big),$$
$$\overline{f(w)} \mapsto f\big(\theta^x_k \circ \kappa^x_k\big)$$

is injective and the diagrams

$$\begin{tikzcd}
V_k|_{D^x_{\mathrm{ram},h}} \arrow[r, "\nabla|_{D^x_{\mathrm{ram},h}}"] \arrow[d] & V_k|_{D^x_{\mathrm{ram},h}} \otimes \Omega^1_C(D_{\mathrm{ram},h}) \arrow[d] \\
\overline{V}^x_k \arrow[r, "\nu_h(\theta^x_k\circ\kappa^x_k) + \frac{k}{r}\frac{dz}{z-h^r}"] & \overline{V}^x_k \otimes \Omega^1_C(D_{\mathrm{ram},h})
\end{tikzcd}$$

are commutative for $k = 0, 1, \ldots, r-1$,

(ix) there is an isomorphism $\psi_k \colon \overline{V}^x_k \xrightarrow{\sim} (w)\big/\big(w^2(z-h^r q^x_1)\cdots(z-h^r q^x_{m_x-1})\big) \otimes \overline{V}^x_{k-1}$ which is a lift of $\overline{V}^x_k \longrightarrow \overline{V}^x_{k-1}$ such that the composition

$$(z-h^r)\big/\big(w(z-h^r q^x_1)\cdots(z-h^r q^x_{m_x-1})(z-h^r)\big) \otimes \overline{V}^x_0 \longrightarrow \overline{V}^x_{r-1},$$
$$\xrightarrow[\sim]{\psi_{r-1}} \cdots\cdots \xrightarrow[\sim]{\psi_1} (w^{r-1})\big/\big((z-h^r q^x_1)\cdots(z-h^r q^x_{m_x-1})(z-h^r)\big) \otimes \overline{V}^x_0$$

coincides with the homomorphism obtained by tensoring $\overline{V}^x_0$ to

$$(w^r)\big/\big(w(z-h^r q^x_1)\cdots(z-h^r q^x_{m_x-1})(z-h^r)\big)$$
$$\to (w^{r-1})\big/\big((z-h^r q^x_1)\cdots(z-h^r q^x_{m_x-1})(z-h^r)\big)$$

for $1 \le k \le r-1$ and

(x) the ring of endomorphisms of E preserving l, (ℓ_k), $(V_k, \vartheta_k, \varkappa_k)$ and commuting with ∇ consists of scalar endomorphisms $\mathbb{C}\mathrm{id}_E$.

We can prove that the moduli space $\mathcal{M}_{H^\circ}$ exists as an algebraic space, by modifying the proof of Theorem 5.1. The proof is rather easier because we do not need a GIT construction. So we omit the proof of the following proposition.

Proposition 8.2. *There exists a relative moduli space $\mathcal{M}_{H^\circ} \longrightarrow H^\circ$ as an algebraic space.*

Note that the fiber $\mathcal{M}_{H^\circ,0}$ of the moduli space $\mathcal{M}_{H^\circ}$ over $h=0$ is the moduli space of simple connections on (C,D) with (λ,μ,ν)-structure.

There is a scheme $\tilde{\mathcal{M}}_{H^\circ}$ of finite type over H° with an étale surjective morphism $\tilde{\mathcal{M}}_{H^\circ} \longrightarrow \mathcal{M}_{H^\circ}$ such that a universal family $\big(\tilde{E}_{\tilde{\mathcal{M}}_{H^\circ}}, \tilde{\nabla}_{\tilde{\mathcal{M}}_{H^\circ}}, \tilde{l}_{\tilde{\mathcal{M}}_{H^\circ}}, \tilde{\ell}_{\tilde{\mathcal{M}}_{H^\circ}}, \tilde{\mathcal{V}}_{\tilde{\mathcal{M}}_{H^\circ}}\big)$ exists over $\tilde{\mathcal{M}}_{H^\circ}$. We can define a complex

$$\begin{aligned}\mathcal{F}^\bullet_{\tilde{\mathcal{M}}_{H^\circ}} = \big[&\mathcal{G}^0_{\tilde{\mathcal{M}}_{H^\circ}} \oplus A^0_{\tilde{\mathcal{M}}_{H^\circ}} \to \mathcal{G}^1_{\tilde{\mathcal{M}}_{H^\circ}} \oplus \operatorname{Sym}^2\big((\overline{W}_{\tilde{\mathcal{M}}_{H^\circ}})\big) \oplus \operatorname{Sym}^2\big((\overline{V}_{\tilde{\mathcal{M}}_{H^\circ}})\big) \\ &\to G^1_{\tilde{\mathcal{M}}_{H^\circ}} \oplus A^1_{\tilde{\mathcal{M}}_{H^\circ}}\big]\end{aligned}$$

from $\big(\tilde{E}_{\tilde{\mathcal{M}}_{H^\circ}}, \tilde{\nabla}_{\tilde{\mathcal{M}}_{H^\circ}}, \tilde{l}_{\tilde{\mathcal{M}}_{H^\circ}}, \tilde{\ell}_{\tilde{\mathcal{M}}_{H^\circ}}, \tilde{\mathcal{V}}_{\tilde{\mathcal{M}}_{H^\circ}}\big)$ in a similar way to (6.9). We can see by the same argument as Proposition 6.3 and Theorem 7.5 that $\mathcal{M}_{H^\circ}$ is smooth over H° and $\mathbf{R}^1\big(p_{\tilde{\mathcal{M}}_{H^\circ}}\big)_*\big(\mathcal{F}^\bullet_{\tilde{\mathcal{M}}_{H^\circ}}\big)$ is the H°-relative tangent bundle of $\tilde{\mathcal{M}}_{H^\circ}$. We can define a pairing

$$\begin{aligned}\omega_{\tilde{\mathcal{M}}_{H^\circ}} \colon\ &\mathbf{R}^1\big(p_{\tilde{\mathcal{M}}_{H^\circ}}\big)_*\big(\mathcal{F}^\bullet_{\tilde{\mathcal{M}}_{H^\circ}}\big) \times \mathbf{R}^1\big(p_{\tilde{\mathcal{M}}_{H^\circ}}\big)_*\big(\mathcal{F}^\bullet_{\tilde{\mathcal{M}}_{H^\circ}}\big) \\ &\longrightarrow \mathbf{R}^2\big(p_{\tilde{\mathcal{M}}_{H^\circ}}\big)_*\big[\mathcal{O}_{C\times H^\circ} \to \Omega^1_{C\times H^\circ/H^\circ}(D_{\mathrm{ram},h}) \to \Omega^1_{C\times H^\circ/H^\circ}(D_{\mathrm{ram},h})\big|_{D_{\mathrm{ram},h}}\big]_{\tilde{\mathcal{M}}_{H^\circ}} \\ &\cong \mathbf{R}^2(p_{\tilde{\mathcal{M}}_{H^\circ}})_*\Omega^\bullet_{C\times\tilde{\mathcal{M}}_{H^\circ}/\tilde{\mathcal{M}}_{H^\circ}} \cong \mathcal{O}_{\tilde{\mathcal{M}}_{H^\circ}}\end{aligned} \tag{8.2}$$

by the same formula as (8.1). We can see that it defines a relative 2-form $\omega_{\mathcal{M}_{H^\circ}}$ on $\mathcal{M}_{H^\circ}$ over H°. The moduli space $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_t$ is a Zariski open subset of the fiber $(\mathcal{M}_{H^\circ})_0$ over $h=0$ and the restriction $\omega_{\mathcal{M}_{H^\circ}}\big|_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_t}$ is nothing but the 2-form $\omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_t}$ on $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_t$ defined by (8.1). So Theorem 8.1 follows from the following proposition.

Proposition 8.3. *The relative 2-form $\omega_{\mathcal{M}_{H^\circ}}$ on $\mathcal{M}_{H^\circ}$ defined by (8.2) is d-closed: $d\omega_{\mathcal{M}_{H^\circ}}=0$.*

Proof. Let $\mathcal{M}_{H^\circ,h}$ be the fiber of the moduli space $\mathcal{M}_{H^\circ}$ over generic $h\in H^\circ\setminus\{0\}$.

Consider the point $z=hs^x_j$ in $D^x_{\mathrm{un},h}$ for generic $h\in H^\circ$. Then ∇ is logarithmic at $z=hs^x_j$ and the filtration $\ell|_{z=hs^x_j}$ is a logarithmic $(\mathsf{res}_{z=hs^x_j}(\mu^x_{k,h})_{0\le k\le r-1})$-parabolic structure at the point $z=hs^x_j$.

Consider the point $z=h^rq^x_j$ in $D^x_{\mathrm{ram},h}$ for generic $h\in H^\circ$. Then the restriction of $\theta^x_k\circ\kappa^x_k$ to $\overline{V}^x_k|_{z=h^rq^x_j}=E|_{z=h^rq^x_j}$ satisfies the equalities $(\theta^x_k\circ\kappa^x_k|_{z=h^rq^x_j})^r-h^r(q^x_j-1)=0$ for $1\le j\le m_x-1$. So it has r distinct eigenvalues $\zeta_r^s h\sqrt[r]{q^x_j-1}$ $(s=0,1,\dots,r-1)$, where ζ_r is a primitive r-th root of unity. Then

$$\begin{aligned}\mathsf{res}_{z=h^rq^x_j}(\nabla) &= \mathsf{res}_{z=h^rq^x_j}(\nu_0(z)) + \mathsf{res}_{z=h^rq^x_j}(\nu_1(z))(\theta^x_k\circ\kappa^x_k)|_{z=h^rq^x_j} + \cdots \\ &\quad + \mathsf{res}_{z=h^rq^x_j}(\nu_{r-1}(z))((\theta^x_k\circ\kappa^x_k)|_{z=h^rq^x_j})^{r-1}\end{aligned}$$

also has r distinct eigenvalues if h is sufficiently generic. The data of filtration $\{V_k\}$ given in (v) is equivalent to the filtration $E|_{z=h^r}=V^x_0|_{z=h^r}\supset\cdots\supset V^x_{r-1}|_{z=h^r}\supset V^x_r|_{z=h^r}=0$ satisfying $\big(\mathsf{res}_{z=h^r}(\nabla)-\big(\mathsf{res}_{z=h^r}(\nu_0)+\frac{k}{r}\big)\mathrm{id}\big)\big(V^x_k\big|_{z=h^r}\big)\subset V^x_{k+1}\big|_{z=h^r}$ for $0\le k\le r-1$ at each x. So the restriction $(V^x_k|_{z=h^r})_{0\le k\le r-1}$ is a logarithmic parabolic structure on (E,∇).

For generic h, we define a complex $\mathcal{F}^{\mathrm{diag}\bullet}_{\tilde{\mathcal{M}}_{H^\circ,h}}$ on the fiber $\tilde{\mathcal{M}}_{H^\circ,h}$ by setting

$$\begin{aligned}\mathcal{F}^{\mathrm{diag},0}_{\tilde{\mathcal{M}}_{H^\circ,h}} &= \ker\big(\mathcal{G}^0_{\tilde{\mathcal{M}}_{H^\circ,h}} \longrightarrow \operatorname{coker}\big(A^0_{\tilde{\mathcal{M}}_{H^\circ,h}} \to \operatorname{Sym}^2\big((\overline{W}_{\tilde{\mathcal{M}}_{H^\circ,h}})\big) \oplus \operatorname{Sym}^2\big((\overline{V}_{\tilde{\mathcal{M}}_{H^\circ,h},h})\big)\big)\big), \\ \mathcal{F}^{\mathrm{diag},1}_{\tilde{\mathcal{M}}_{H^\circ,h}} &= \ker\big(\mathcal{G}^1_{\tilde{\mathcal{M}}_{H^\circ,h}} \to G^1_{\tilde{\mathcal{M}}_{H^\circ,h}}\big), \\ d^0_{\mathcal{F}^{\mathrm{diag},\bullet}_{\tilde{\mathcal{M}}_{H^\circ,h}}} &= d^0_{\mathcal{F}^\bullet_{\tilde{\mathcal{M}}_{H^\circ,h}}}\Big|_{\mathcal{F}^{\mathrm{diag},0}_{\tilde{\mathcal{M}}_{H^\circ,h}}} \colon\ \mathcal{F}^{\mathrm{diag},0}_{\tilde{\mathcal{M}}_{H^\circ,h}} \longrightarrow \mathcal{F}^{\mathrm{diag},1}_{\tilde{\mathcal{M}}_{H^\circ,h}}.\end{aligned}$$

Note that $\mathcal{F}^{\mathrm{diag},0}_{\tilde{\mathcal{M}}_{H^\circ,h}}$ is the sheaf of endomorphisms of E preserving the eigen decomposition of $\mathsf{res}_{z=h^r q^x_j}(\nabla)$ at $z = h^r q^x_j$ in $D^x_{\mathrm{ram},h}$ for $1 \le j \le m_x - 1$, preserving the parabolic structure l^x at each $x \in D_{\log}$, preserving the parabolic structure $\left(\ell^x_k|_{z=hs^x_j}\right)_{0\le k\le r-1}$ at $z = hs^x_j$ in $D^x_{\mathrm{un},h}$ for $1 \le j \le hs^x_{n_x}$ and preserving the parabolic structure $\left(\overline{V}^x_k|_{z=h^r}\right)_{0\le k\le r-1}$ at $z = h^r$ in $D^x_{\mathrm{ram},h}$. We can see that the canonical map

$$\mathcal{F}^{\mathrm{diag},\bullet}_{\tilde{\mathcal{M}}_{H^\circ,h}} \longrightarrow \mathcal{F}^{\bullet}_{\tilde{\mathcal{M}}_{H^\circ,h}}$$

is a quasi-isomorphism. On the other hand, we can define a complex $\mathcal{F}^\bullet_{\mathrm{par}}$ on $C \times \tilde{\mathcal{M}}_{H^\circ,h}$ in the same way as in the proof of [11, Proposition 7.2] by associating the parabolic structure induced by the eigen decomposition at each point defined by $z = h^r q^x_j$ in $D^x_{\mathrm{ram},h}$ for $1 \le j \le m^x - 1$. Then the canonical map

$$\mathcal{F}^{\mathrm{diag},\bullet}_{\tilde{\mathcal{M}}_{H^\circ,h}} \longrightarrow \mathcal{F}^{\bullet}_{\mathrm{par}}$$

is a quasi-isomorphism. We can see that the restriction $\omega_{\tilde{\mathcal{M}}_{H^\circ,h}}$ to a generic fiber $\tilde{\mathcal{M}}_{H^\circ,h}$ of the 2-form $\omega_{\tilde{\mathcal{M}}_{H^\circ}}$ coincides with the 2-form constructed in [11, Proposition 7.2], because it is expressed by the same formula as (8.1). Since the 2-form in [11, Proposition 7.2] is d-closed by [11, Proposition 7.3], we have $\mathrm{d}\omega_{\mathcal{M}_{H^\circ,h}} = 0$ for generic h. Thus we can deduce $\mathrm{d}\omega_{\mathcal{M}_{H^\circ}} = 0$, because $\mathcal{M}_{H^\circ}$ is smooth over H°. ∎

9 Local generalized isomonodromic deformation on a ramified covering

In this section, we will consider the pullback of a generic ramified connection via a local analytic ramified covering map. Furthermore, we will give a brief sketch of the Stokes data of the pullback and its generalized isomonodromic deformation established by Jimbo, Miwa and Ueno in [16].

Let Δ_z and Δ_w be unit disks equipped with the variables z and w, respectively. Consider the ramified covering map

$$p\colon\ \Delta_w \ni w \mapsto w^r = z \in \Delta_z. \tag{9.1}$$

There is a canonical action of the Galois group $\mathrm{Gal}(\Delta_w/\Delta_z) = \{\sigma^k \mid |0 \le k \le r-1\}$ which is generated by the automorphism $\sigma\colon \Delta_w \ni w \mapsto \zeta_r w \in \Delta_w$, where $\zeta_r = \exp\left(2\pi\sqrt{-1}/r\right)$ is a primitive root of unity.

Take $\nu_0(z) \in (\mathbb{C} + \mathbb{C}z + \cdots + \mathbb{C}z^{mr-r})\mathrm{d}z/z^m$, $\nu_1(z) \in (\mathbb{C}^\times + \mathbb{C}z + \cdots + \mathbb{C}z^{mr-r-1})\mathrm{d}z/z^m$ and $\nu_2(z), \dots, \nu_{r-1}(z) \in (\mathbb{C} + \mathbb{C}z + \cdots + \mathbb{C}z^{mr-r-1})\mathrm{d}z/z^m$. Then we put

$$\nu(w) := \nu_0(z) + \nu_1(z)w + \cdots + \nu_{r-1}(z)w^{r-1},$$

which is said to be a ramified exponent. We define a formal connection ∇_ν on $\mathbb{C}[[w]]$ by

$$\nabla_\nu\colon\ \mathbb{C}[[w]] \ni f(w) \mapsto \mathrm{d}f(w) + f(w)\nu(w) \in \mathbb{C}[[w]] \otimes \frac{\mathrm{d}z}{z^m}.$$

Let (E, ∇) be a meromorphic connection on Δ_z with a formal isomorphism

$$\left(\widehat{E}, \widehat{\nabla}\right) := (E, \nabla) \otimes \widehat{\mathcal{O}}_{\Delta_z,0} \xrightarrow{\sim} (\mathbb{C}[[w]], \nabla_\nu). \tag{9.2}$$

Consider the pullback $(p^*E, p^*\nabla)$ of the meromorphic connection (E, ∇) by the ramified cover p given in (9.1). The formal isomorphism (9.2) induces a canonical surjection

$$\pi\colon\ p^*E \otimes \widehat{\mathcal{O}}_{\Delta_w,0} = \widehat{E} \otimes_{\mathbb{C}[[z]]} \mathbb{C}[[w]] \longrightarrow \mathbb{C}[[w]]$$

which makes the diagram

$$\begin{array}{ccc} \widehat{E}\otimes\mathbb{C}[[w]] & \xrightarrow{\pi} & \mathbb{C}[[w]] \\ {\scriptstyle \widehat{\nabla}\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \nabla_\nu} \\ \widehat{E}\otimes\mathbb{C}[[w]]\otimes\frac{\mathrm{d}z}{z^m} & \xrightarrow{\pi\otimes\mathrm{id}} & \mathbb{C}[[w]]\otimes\frac{\mathrm{d}z}{z^m} \end{array}$$

commutative. The Galois transform of π by the element σ^k of $\mathrm{Gal}(\Delta_w/\Delta_z)$ is given by

$$\sigma^k\circ\pi\circ\sigma^{-k}\colon\ \widehat{E}\otimes\mathbb{C}[[w]]\xrightarrow[\sim]{\mathrm{id}_{\widehat{E}}\otimes\sigma^{-k}}\widehat{E}\otimes\mathbb{C}[[w]]\xrightarrow{\pi}\mathbb{C}[[w]]\xrightarrow[\sim]{\sigma^k}\mathbb{C}[[w]],$$

which makes the diagram

$$\begin{array}{ccc} \widehat{E}\otimes\mathbb{C}[[w]] & \xrightarrow{\sigma^k\circ\pi\circ\sigma^{-k}} & \mathbb{C}[[w]] \\ {\scriptstyle \widehat{\nabla}\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \nabla_{\sigma^k\nu}} \\ \widehat{E}\otimes\mathbb{C}[[w]]\otimes\frac{\mathrm{d}z}{z^m} & \xrightarrow{(\sigma^k\circ\pi\circ\sigma^{-k})\otimes\mathrm{id}} & \mathbb{C}[[w]]\otimes\frac{\mathrm{d}z}{z^m} \end{array}$$

commutative, where we put $\sigma^k\nu(w):=\nu\big(\zeta_r^k w\big)$. So we get a morphism

$$\varpi\colon\ \big(p^*\widehat{E},p^*\widehat{\nabla}\big)\xrightarrow{\bigoplus_{k=0}^{r-1}\sigma^k\circ\pi\circ\sigma^{-k}}\bigoplus_{k=0}^{r-1}\big(\mathbb{C}[[w]],\nabla_{\sigma^k\nu(w)}\big),\tag{9.3}$$

whose underlying homomorphism on vector bundles over $\mathbb{C}[[w]]$ is generically isomorphic. Choose a generator e_0 of the underlying bundle $\mathbb{C}[[w]]$ of $(\mathbb{C}[[w]],\nabla_\nu)$ (we may choose $e_0=1$). We denote the same element of the underlying bundle of $(\mathbb{C}[[w]],\nabla_{\sigma^k\nu})$ by $\sigma^k(e_0)$. Then we can define an action of $\mathrm{Gal}(\Delta_w/\Delta_z)$ on the right-hand side of (9.3) by setting

$$\sigma^l\cdot\sum_{k=0}^{r-1}f_k(w)\sigma^k(e_0):=\sum_{k=0}^{r-1}f_k\big(\zeta_r^l w\big)\sigma^{k+l}(e_0).$$

The connection $\bigoplus_{k=0}^{r-1}\nabla_{\sigma^k\nu}$ on the right-hand side of (9.3) commutes with the Galois action. The morphism ϖ in (9.3) is a $\mathbb{C}[[w]]$-homomorphism, which commutes with the connections and with the Galois actions on the both sides.

We can see that the image $\mathrm{Im}\,\varpi$ of the homomorphism (9.3) is generated by

$$\left\{\sum_{l=0}^{r-1}\zeta_r^{kl}w^k\sigma^l(e_0)\,\middle|\,k=0,1,\dots,r-1\right\}$$

as a $\mathbb{C}[[w]]$-module. Then we can check the inclusion $w^{r-1}\cdot\bigoplus_{k=0}^{r-1}\mathbb{C}[[w]]\sigma^k(e_0)\subset\mathrm{Im}\,\varpi$. Consider the restriction

$$\varpi|_{w^{r-1}=0}\colon\ \widehat{E}|_{w^{r-1}=0}\otimes\mathbb{C}[w]/\big(w^{r-1}\big)\xrightarrow{\varpi|_{w^{r-1}=0}}\mathrm{Im}(\varpi|_{w^{r-1}=0})\subset\bigoplus_{k=0}^{r-1}\mathbb{C}[w]/\big(w^{r-1}\big)\cdot\sigma^k(e_0)$$

of the morphism ϖ in (9.3) to the divisor on Δ_w defined by $w^{r-1}=0$. Then the composition

$$\varphi\colon\ p^*(E)\longrightarrow p^*(E)|_{w^{r-1}=0}=\widehat{E}\otimes\mathbb{C}[w]/\big(w^{r-1}\big)\xrightarrow{\varpi|_{w^{r-1}=0}}\mathrm{Im}(\varpi|_{w^{r-1}=0})$$

commutes with $p^*(\nabla)$ and $\bigoplus_{k=0}^{r-1} \nabla_{\sigma^k \nu}|_{w^{r-1}=0}$. So we have

$$(p^*\nabla)(\ker\varphi) \subset \ker\varphi \otimes \frac{\mathrm{d}w}{w^{mr-r+1}}.$$

Consider the line bundle $\mathcal{O}_{\Delta_w}\big((r-1)\cdot\{0\}\big)$ on Δ_w with the connection

$$\begin{aligned}\nabla_{-\nu_0(z)}\colon\ \mathcal{O}_{\Delta_w}\big((r-1)\cdot\{0\}\big) \ni f(w) \mapsto \mathrm{d}f(w) - f(w)\nu_0(z) \\ \in \mathcal{O}_{\Delta_w}\big((r-1)\cdot\{0\}\big) \otimes \frac{\mathrm{d}w}{w^{mr-r+1}}.\end{aligned}$$

If we modify $(\ker\varphi, p^*\nabla|_{\ker\varphi})$ by setting

$$(E',\nabla') := \big(\ker\varphi, (p^*\nabla)|_{\ker\varphi}\big) \otimes \big(\mathcal{O}_{\Delta_w}\big((r-1)\cdot\{0\}\big), \nabla_{-\nu_0(z)}\big), \tag{9.4}$$

then the order of pole of ∇' at $w=0$ is $mr-r$. Indeed, the morphism ϖ in (9.3) induces a formal isomorphism

$$\big(\widehat{E'},\widehat{\nabla'}\big) \xrightarrow{\sim} \bigoplus_{k=0}^{r-1} \big(\mathbb{C}[[w]], \nabla_{\nu(\zeta_r^k w)-\nu_0(z)}\big)$$

and the matrix of the connection $\nabla_{\nu(\zeta_r^k w)-\nu_0(z)}$ of the right-hand side is

$$\begin{pmatrix} \sum_{k=1}^{r-1}\nu_k(z)w^k & 0 & \cdots & 0 \\ 0 & \sum_{k=1}^{r-1}\nu_k(z)\zeta_r^k w^k & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=1}^{r-1}\nu_k(z)\zeta_r^{k(r-1)}w^k \end{pmatrix}.$$

Since the leading terms of the diagonal entries of the above matrix are distinct, (E',∇') is a generic unramified connection. Furthermore, there is a canonical action of $\mathrm{Gal}(\Delta_w/\Delta_z)$ on (E',∇'), since φ and $\otimes\big(\mathcal{O}_{\Delta_w}\big((r-1)\cdot\{0\}\big),\nabla_{-\nu_0(z)}\big)$ preserve the Galois action.

Proposition 9.1. *The correspondence $(E,\nabla)\mapsto(E',\nabla')$ given by the formula (9.4) is a bijection between the meromorphic ν-ramified connections (E,∇) on Δ_z equipped with a formal isomorphism $\big(\widehat{E},\widehat{\nabla}\big)\xrightarrow{\sim}(\mathbb{C}[[w]],\nabla_\nu)$ and the $\mathrm{Gal}(\Delta_w/\Delta_z)$-equivariant $\big(\nu\big(\zeta_r^k w\big)-\nu_0(z)\big)_{0\le k\le r-1}$-unramified meromorphic connections (E',∇') on Δ_w equipped with a Galois equivariant formal isomorphism $\big(\widehat{E'},\widehat{\nabla'}\big)\xrightarrow{\sim}\bigoplus(\mathbb{C}[[w]],\nabla_{\sigma^k\nu})$.*

Proof. We have to give the inverse correspondence. If (E',∇') is a $\big(\nu\big(\zeta_r^k w\big)-\nu_0(z)\big)_{0\le k\le r-1}$-unramified meromorphic connection on Δ_w compatible with an action of $\mathrm{Gal}(\Delta_w/\Delta_z)$, we put

$$\tilde{E}' := \ker\Big(E' \longrightarrow \mathrm{coker}\Big(\big(E'|_{w^{mr-r}=0}\big)^{\mathrm{Gal}(\Delta_w/\Delta_z)} \otimes \mathbb{C}[w]/(w^{mr-r}) \to E'|_{w^{mr-r}=0}\Big)\Big),$$

where $\big(E'|_{w^{mr-r}=0}\big)^{\mathrm{Gal}(\Delta_w/\Delta_z)}$ is the submodule of $E'|_{w^{mr-r}=0}$ consisting of the $\mathrm{Gal}(\Delta_w/\Delta_z)$-invariant sections. Let $\big(\tilde{E}'\big)^{\mathrm{Gal}(\Delta_w/\Delta_z)}$ be the subsheaf of $p_*\big(\tilde{E}'\big)$ consisting of $\mathrm{Gal}(\Delta_w/\Delta_z)$-invariant sections. Then $\big(\tilde{E}'\big)^{\mathrm{Gal}(\Delta_w/\Delta_z)}$ becomes a locally free sheaf on Δ_z of rank r and the connection $\nabla'|_{\tilde{E}'}\otimes\nabla_{\nu_0(z)}$ on $\tilde{E}'$ descends to a connection $\big(\nabla'|_{\tilde{E}'}\otimes\nabla_{\nu_0(z)}\big)^{\mathrm{Gal}(\Delta_w/\Delta_z)}$ on

$\big(\tilde{E}'\big)^{\mathrm{Gal}(\Delta_w/\Delta_z)}$. We can check that $\Big(\big(\tilde{E}'\big)^{\mathrm{Gal}(\Delta_w/\Delta_z)}, \big(\nabla'\big|_{\tilde{E}'} \otimes \nabla_{\nu_0(z)}\big)^{\mathrm{Gal}(\Delta_w/\Delta_z)}\Big)$ is a meromorphic ν-ramified connection on Δ_z. From the construction,

$$(E',\nabla') \mapsto \Big(\big(\tilde{E}'\big)^{\mathrm{Gal}(\Delta_w/\Delta_z)}, \big(\nabla'\big|_{\tilde{E}'} \otimes \nabla_{\nu_0(z)}\big)^{\mathrm{Gal}(\Delta_w/\Delta_z)}\Big)$$

gives the inverse to $(E,\nabla) \mapsto (E',\nabla')$. ■

Remark 9.2. The process of getting the vector bundle $\ker\varphi$ or E' from p^*E is called an elementary transform or a Hecke modification. The construction of (E',∇') from (E,∇) is known [28, Section 19.3] as a shearing transformation method.

We will apply Proposition 9.1 to a family of connections. From now on, let the notations $\mathcal{T}$, $\mathcal{C}$, λ, $\tilde{\mu}$, $\tilde{\nu}$ and $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$ be as in Section 5.

We take a point $x = (\tilde{x}_i)_t \in (\mathcal{D}_{\mathrm{ram}})_t$ in the fiber over $t \in \mathcal{T}$. We can take an analytic open neighborhood $\mathcal{T}^\circ$ of t such that $\bar{z}_{\mathcal{T}^\circ}$ can be extended to a local holomorphic function $z \in \mathcal{O}^{\mathrm{hol}}_{\mathcal{C}_{\mathcal{T}^\circ}}$ whose zero set coincides with the section $\tilde{x} = (\tilde{x}_i)_{\mathcal{T}^\circ}$. Precisely, there is an analytic open immersion

$$\Delta_z \times \mathcal{T}^\circ \hookrightarrow \mathcal{C}_{\mathcal{T}^\circ}$$

for a unit disk Δ_z, such that the coordinate of Δ_z corresponds to z. We can assume the existence of a universal family $\big(\tilde{E},\tilde{\nabla},\tilde{l},\tilde{\ell},\tilde{\mathcal{V}}\big)$ on some analytic open neighborhood $M^\circ \subset M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu}) \times_{\mathcal{T}} \mathcal{T}^\circ$. By Corollary 4.3, we may further assume that there is an isomorphism

$$\big(\tilde{E},\tilde{\nabla}\big) \otimes \widehat{\mathcal{O}}_{\tilde{C}_{M^\circ},\tilde{x}} \xrightarrow{\ \sim\ } \big(\mathcal{O}^{\mathrm{hol}}_{M^\circ}[[w]], \nabla_{\tilde{\nu}}\big), \tag{9.5}$$

where $\widehat{\mathcal{O}}_{\mathcal{C}_{M^\circ},\tilde{x}} = \varprojlim \mathcal{O}^{\mathrm{hol}}_{\mathcal{C}_{M^\circ}}/I^j_{\tilde{x}} \cong \mathcal{O}^{\mathrm{hol}}_{M^\circ}[[w]]$. Consider a family of ramified covering maps (9.1)

$$p_{M^\circ}\colon\ \Delta_w \times M^\circ \ni (w,y) \mapsto (w^r,y) \in \Delta_z \times M^\circ.$$

We write $m := m_i^{\mathrm{ram}}$ for simplicity. As in the former argument, the isomorphism (9.5) induces a canonical surjection

$$\pi_{M^\circ}\colon\ p^*_{M^\circ}\tilde{E} \otimes \widehat{\mathcal{O}}_{\mathcal{C}_{M^\circ},\tilde{x}} \longrightarrow \mathcal{O}^{\mathrm{hol}}_{M^\circ}[[w]],$$

which also induces a morphism

$$\varpi_{M^\circ}\colon\ \big(p^*_{M^\circ}\tilde{E}, p^*_{M^\circ}\tilde{\nabla}\big) \otimes \widehat{\mathcal{O}}_{\mathcal{C}_{M^\circ},\tilde{x}} \xrightarrow{\bigoplus_{k=0}^{r-1}\sigma^k\circ\pi_{M^\circ}\circ\sigma^{-k}} \bigoplus_{k=0}^{r-1} \big(\mathcal{O}^{\mathrm{hol}}_{M^\circ}[[w]], \nabla_{\sigma^k\tilde{\nu}}\big) \tag{9.6}$$

between rank r connections over $\mathcal{O}^{\mathrm{hol}}_{M^\circ}[[w]]$. Let $\tilde{x}'$ be the divisor on $\Delta_w \times M^\circ$ defined by the equation $w = 0$. The composition

$$\varphi_{M^\circ}\colon\ p^*_{M^\circ}\big(\tilde{E}|_{\Delta_z\times M^\circ}\big) \longrightarrow p^*_{M^\circ}\big(\tilde{E}|_{\Delta_z\times M^\circ}\big)|_{(r-1)\tilde{x}'} \longrightarrow \mathrm{Im}\,\big(\varpi_{M^\circ}|_{(r-1)\tilde{x}'}\big)$$

is a surjective homomorphism and we have $\big(p^*_{M^\circ}\tilde{\nabla}\big)(\ker\varphi) \subset \ker\varphi \otimes \Omega^1_{\Delta_w\times M^\circ/M^\circ}((mr-r+1)\tilde{x}')$. Setting

$$\big(\tilde{E}',\tilde{\nabla}'\big) := \big(\ker\varphi, p^*_{M^\circ}\tilde{\nabla}|_{\ker\varphi}\big) \otimes \big(\mathcal{O}^{\mathrm{hol}}_{\Delta_w\times M^\circ}((r-1)\tilde{x}'), \nabla_{-\tilde{\nu}_0}\big), \tag{9.7}$$

we get a connection

$$\tilde{\nabla}'\colon\ \tilde{E}' \longrightarrow \tilde{E}' \otimes \Omega^1_{\Delta_w\times M^\circ/M^\circ}\big((mr-r)\tilde{x}'\big).$$

The morphism ϖ_{M° in (9.6) induces an isomorphism

$$\big(\tilde{E}', \tilde{\nabla}'\big) \otimes \widehat{\mathcal{O}}_{\tilde{C}_{M^\circ}, \tilde{x}} \xrightarrow{\sim} \bigoplus_{k=0}^{r-1} \big(\mathcal{O}^{\mathrm{hol}}_{M^\circ}[[w]], \nabla_{\tilde{\nu}(\zeta_r^k w) - \tilde{\nu}_0(z)}\big). \tag{9.8}$$

The connection $\nabla_{\tilde{\nu}(\zeta_r^k w) - \tilde{\nu}_0(z)}$ of the right-hand side is given by $d + \Lambda(w,t)$ with

$$\Lambda(w,t) := \begin{pmatrix} \sum\limits_{k=1}^{r-1} \tilde{\nu}_k(z,t) w^k & 0 & \cdots & 0 \\ 0 & \sum\limits_{k=1}^{r-1} \tilde{\nu}_k(z,t)\zeta_r^k w^k & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sum\limits_{k=1}^{r-1} \tilde{\nu}_k(z,t)\zeta_r^{k(r-1)} w^k \end{pmatrix}. \tag{9.9}$$

Now we will see the corresponding Stokes data. We set $E'_0 := \big(\mathcal{O}^{\mathrm{hol}}_{\Delta_w \times \mathcal{T}^\circ}\big)^{\oplus r}$ and fix a connection $\nabla'_0 \colon E'_0 \longrightarrow E'_0 \otimes \Omega^1_{\Delta_w \times \mathcal{T}^\circ/\mathcal{T}^\circ}((mr - r)\tilde{x}')$ defined by

$$\begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{d}f_1 \\ \vdots \\ \mathrm{d}f_r \end{pmatrix} + \Lambda(w,t) \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}.$$

We call (E'_0, ∇'_0) a normal form.

It is a general fact [16, Proposition 2.2] that there is a matrix $P(w,t)$ of formal power series in w with coefficients in $\mathcal{O}^{\mathrm{hol}}_{M^\circ}$, which gives a formal isomorphism

$$(E'_0, \nabla'_0) \otimes \widehat{\mathcal{O}}_{\tilde{C}_{M^\circ}, \tilde{x}} \xrightarrow[\sim]{P(w,t)} \big(\tilde{E}', \tilde{\nabla}'\big) \otimes \widehat{\mathcal{O}}_{\tilde{C}_{M^\circ}, \tilde{x}}. \tag{9.10}$$

If $\tilde{\nabla}'$ is given by $\mathrm{d} + A'(w,t)\mathrm{d}w/w^{mr-r}$ for a matrix $A'(w,t)$ of holomorphic functions in w, t, then we have

$$P(w,t)^{-1}\mathrm{d}P(w,t) + P(w,t)^{-1}A'(w,t)\frac{\mathrm{d}w}{w^{mr-r}}P(w,t) = \Lambda(w,t).$$

In fact, we can give the formal transform $P(w,t)$ as the inverse of (9.8), which is induced by the formal transform (9.5) over $\mathcal{O}^{\mathrm{hol}}_{M^\circ}[[z]]$. Indeed, if we denote the inverse formal transform of (9.5) by

$$Q(z,t) \colon \ \big(\mathcal{O}^{\mathrm{hol}}_{M^\circ}[[w]], \nabla_{\tilde{\nu}}\big) \xrightarrow{\sim} \big(\tilde{E}, \tilde{\nabla}\big) \otimes \widehat{\mathcal{O}}_{\tilde{C}_{M^\circ}, \tilde{x}} \tag{9.11}$$

and if we denote the rational gauge transform $p^*_{M'}\big(\tilde{E}|_{\Delta_z \times M'}\big) \hookrightarrow \tilde{E}'$ by $S(w)$, then we can give $P(w)$ by

$$P(w,t) = S(w,t)Q(z,t) \begin{pmatrix} 1 & w & \cdots & w^{r-1} \\ 1 & \zeta_r w & \cdots & \zeta_r^{r-1} w^{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_r^{r-1} w & \cdots & \zeta_r^{(r-1)^2} w^{r-1} \end{pmatrix}^{-1}. \tag{9.12}$$

Remark 9.3. The above procedure is explained in [10, Proposition 10] for the explicit case of rank 2 connections on $\mathbb{P}^1$.

Take any point $u \in (\Delta_w \setminus \{0\}) \times M^\circ$. By the fundamental existence theorem [28, Theorem 12.1] of asymptotic solution, there are a sector $\Gamma_u = \{w \in \Delta_w \mid a < \arg(w) < b\}$ in $\Delta_w \setminus \{0\}$ for some $a, b \in \mathbb{R}$ and an open subset $M_u \subset M^\circ$ satisfying $u \in \Gamma_u \times M_u$ such that there exists a fundamental solution $Y_\Sigma(w,t) = (y_1(w,t), \ldots, y_r(w,t))$ of $\tilde{\nabla}'$ on $\Sigma = \Gamma_u \times M_u$ satisfying the asymptotic property

$$Y_\Sigma(w,t) \exp\left(\int \Lambda(w) \right) \sim P(w,t) \qquad \text{as } w \to 0 \text{ on } \Sigma = \Gamma_u \times M_u, \tag{9.13}$$

where the path integral of $\Lambda(w)$, which is defined in (9.9), is with respect to the w-variable. If we put $P(w,t) = \sum_{j=0}^{\infty} P_j(t) w^j$, the asymptotic relation (9.13) means

$$\lim_{w \to 0, w \in \Gamma_u} \frac{\left\| Y_\Sigma(w,t) \exp\left(\int \Lambda(w) \right) - \sum_{j=0}^{N} P_j(t) w^j \right\|}{|w|^N} = 0 \tag{9.14}$$

for any positive integer N and the convergence in (9.14) is uniform in $t \in M_u$.

Fix a point $t' \in M^\circ$. Taking a finite subcover of $\{\Sigma = \Gamma_u \times M_u\}$, we can choose an open neighborhood $U_{t'}$ of t' in M° and a covering $\{\Sigma\}$ of $(\Delta_w \setminus \{0\}) \times U_{t'}$ such that each Σ is of the form $\Sigma = \Gamma_u \times U_{t'}$ for a sector Γ_u in $\Delta_w \setminus \{0\}$.

If we take another $\Sigma' = \Gamma_{u'} \times U_{t'}$ in the above covering, and if we choose a fundamental solution $Y_{\Sigma'}(w,t)$ on Σ' with the same asymptotic property as (9.13) on Σ', we can write

$$Y_{\Sigma'}(w,t) = Y_\Sigma(w,t) C_{\Sigma,\Sigma'}(t) \tag{9.15}$$

for a matrix $C_{\Sigma,\Sigma'}(t)$ constant in w. We call $C_{\Sigma,\Sigma'}(t)$ a Stokes matrix.

Definition 9.4. We say that a family of connections $(\tilde{E}', \tilde{\nabla}')|_{\Delta_w \times \mathcal{L}}$ over a submanifold $\mathcal{L} \subset M^\circ$ is a local generalized isomonodromic deformation, if for each $t' \in \mathcal{L}$, we can take an open neighborhood $\mathcal{L}_{t'}$ of t' in $\mathcal{L}$, a replacement of the formal transform $P(w,t)$ in (9.10) and a covering $\{\Sigma = \Gamma_u \times \mathcal{L}_{t'}\}$ of $(\Delta_w \setminus \{0\}) \times \mathcal{L}_{t'}$ for sectors Γ_u in $\Delta_w \setminus \{0\}$ such that

(i) there is a fundamental solution $Y_\Sigma(w,t)$ of $\tilde{\nabla}'|_\Sigma$ with the asymptotic property (9.13) and

(ii) all the Stokes matrices $C_{\Sigma,\Sigma'}(t)$ defined by (9.15) are constant in $t \in \mathcal{L}_{t'}$.

Remark 9.5.

(1) The ambiguity of the path integral $\int \Lambda(w)$ in (9.13) is included in the replacement of the formal transform $P(w,t)$ in Definition 9.4.

(2) In our definition of Stokes matrices $C_{\Sigma,\Sigma'}(t)$, there is an ambiguity in the choice of the fundamental solution $Y_\Sigma(w,t)$. On the other hand, [16, Proposition 2.4] requires Σ to be taken sufficiently large so that there is no ambiguity in $Y_\Sigma(w,t)$. Due to this difference, we will need an additional argument later in Proposition 9.6.

Let us recall the argument in the proof of [16, Theorem 3.1]. Assume that $\mathcal{L} \subset M^\circ$ is a submanifold, $\{\Sigma\}$ is a covering of $(\Delta_w \setminus \{0\}) \times \mathcal{L}$ as in Definition 9.4 and that $Y_\Sigma(w,t)$ is a fundamental solution of $\tilde{\nabla}'|_{\Delta_w \times \mathcal{L}}$ on each Σ such that all the matrices $C_{\Sigma,\Sigma'}(t)$ are constant in $t \in \mathcal{L}$. We choose a local coordinate system $(t_1, \ldots, t_n)$ of $\mathcal{L}$ around $t' \in \mathcal{L}$. Rewriting (9.15), we have $Y_\Sigma(w,t)^{-1} Y_{\Sigma'}(w,t) = C_{\Sigma,\Sigma'}$, which is constant in t. Differentiate it in $t_1, \ldots, t_n$, we have

$$-Y_\Sigma(w,t)^{-1} \frac{\partial Y_\Sigma(w,t)}{\partial t_j} Y_\Sigma(w,t)^{-1} Y_{\Sigma'}(w,t) + Y_\Sigma(w,t)^{-1} \frac{\partial Y_{\Sigma'}(w,t)}{\partial t_j} = 0,$$

which is equivalent to the equality

$$-\frac{\partial Y_\Sigma(w,t)}{\partial t_j} Y_\Sigma(w,t)^{-1} = -\frac{\partial Y_{\Sigma'}(w,t)}{\partial t_j} Y_{\Sigma'}(w,t)^{-1} \tag{9.16}$$

in $\mathrm{End}(\mathcal{O}^{\oplus r}_{\Sigma\cap\Sigma'})\otimes\Omega^1_{\Sigma\cap\Sigma'}$. So we get a matrix $B_j(w,t)$ of single valued functions on $(\Delta_w\setminus\{0\})\times\mathcal{L}$ by patching the matrices (9.16).

On the other hand, since the convergence in (9.14) is uniform in $t\in\mathcal{L}$, the differentiation of (9.13) in t_j provides the asymptotic relation

$$\frac{\partial Y_\Sigma}{\partial t_j}\exp\left(\int\Lambda(w)\right)+Y_\Sigma\exp\left(\int\Lambda(w)\right)\int\frac{\partial\Lambda}{\partial t_j}\sim\frac{\partial P}{\partial t_j}\qquad\text{as } w\to 0 \text{ on } \Sigma.$$

Multiplying $w^{mr-r-1}P^{-1}\sim w^{mr-r-1}\exp\left(-\int\Lambda(w)\right)Y_\Sigma^{-1}$ from the right to the above, we get

$$-w^{mr-r-1}B_j=w^{mr-r-1}\frac{\partial Y_\Sigma}{\partial t_j}Y_\Sigma^{-1}\sim w^{mr-r-1}\left(\frac{\partial P}{\partial t_j}P^{-1}-P\left(\int\frac{\partial\Lambda}{\partial t_j}\right)P^{-1}\right)\tag{9.17}$$

on Σ. Note that the right-hand side of the above is a matrix of formal power series in w without pole. So the left-hand side of (9.17) is bounded on any Σ. Since $-w^{mr-r-1}B_j$ is also a matrix of single valued functions on $(\Delta_w\setminus\{0\})\times\mathcal{L}$, it is holomorphic on $\Delta_w\times\mathcal{L}$. In other words, $B_j(w,t)$ is a matrix of meromorphic functions on $\Delta_w\times\mathcal{L}$, whose pole is of order at most $mr-r-1$.

Recall that the matrix of $\tilde{\nabla}'$ is given by

$$-\frac{\partial Y_\Sigma(w,t)}{\partial w}Y_\Sigma(w,t)^{-1}\mathrm{d}w=A'(w,t)\frac{\mathrm{d}w}{w^{mr-r}}$$

since Y_Σ is a fundamental solution of $\tilde{\nabla}'$. So we obtain a matrix of differential forms

$$A'(w,t)\frac{\mathrm{d}w}{w^{mr-r}}+\sum_{j=1}^N B_j(w,t)\mathrm{d}t_j$$

which determines a meromorphic connection

$$\left(\tilde{\nabla}'\right)^{\mathrm{flat}}\colon\ \tilde{E}'|_{\Delta_w\times\mathcal{L}}\longrightarrow\tilde{E}'|_{\Delta_w\times\mathcal{L}}\otimes\Omega^1_{\Delta_w\times\mathcal{L}}(\mathcal{D}_\mathcal{L}\cap(\Delta_w\times\mathcal{L})).$$

By the definition, $(\tilde{\nabla}')^{\mathrm{flat}}$ is an extension of the relative connection $\tilde{\nabla}'|_{\Delta_w\times\mathcal{L}}$.

The curvature form of $\left(\tilde{\nabla}'\right)^{\mathrm{flat}}$ is

$$\begin{aligned}
&\mathrm{d}\left(-\frac{\partial Y_\Sigma}{\partial w}Y_\Sigma^{-1}\mathrm{d}w-\sum_{j=1}^N\frac{\partial Y_\Sigma}{\partial t_j}Y_\Sigma^{-1}\mathrm{d}t_j\right)\\
&\qquad+\left(-\frac{\partial Y_\Sigma}{\partial w}Y_\Sigma^{-1}\mathrm{d}w-\sum_{j=1}^N\frac{\partial Y_\Sigma}{\partial t_j}Y_\Sigma^{-1}\mathrm{d}t_j\right)\wedge\left(-\frac{\partial Y_\Sigma}{\partial w}Y_\Sigma^{-1}\mathrm{d}w-\sum_{j=1}^N\frac{\partial Y_\Sigma}{\partial t_j}Y_\Sigma^{-1}\mathrm{d}t_j\right)\\
&\quad=-\sum_{j=1}^N\left(\frac{\partial^2Y_\Sigma}{\partial t_j\partial w}-\frac{\partial Y_\Sigma}{\partial w}Y_\Sigma^{-1}\frac{\partial Y_\Sigma}{\partial t_j}\right)Y_\Sigma^{-1}\mathrm{d}t_j\wedge\mathrm{d}w\\
&\qquad-\sum_{j=1}^N\left(\frac{\partial^2Y_\Sigma}{\partial w\partial t_j}-\frac{\partial Y_\Sigma}{\partial t_j}Y_\Sigma^{-1}\frac{\partial Y_\Sigma}{\partial w}\right)Y_\Sigma^{-1}\mathrm{d}w\wedge\mathrm{d}t_j\\
&\qquad-\sum_{j=1}^N\sum_{j'=1}^N\left(\frac{\partial^2Y_\Sigma}{\partial t_{j'}\partial t_j}\mathrm{d}t_{j'}-\frac{\partial Y_\Sigma}{\partial t_j}Y_\Sigma^{-1}\frac{\partial Y_\Sigma}{\partial t_{j'}}\mathrm{d}t_{j'}\right)\wedge Y_\Sigma^{-1}\mathrm{d}t_j\\
&\qquad+\frac{\partial Y_\Sigma}{\partial w}Y_\Sigma^{-1}\mathrm{d}w\wedge\sum_{j=1}^N\frac{\partial Y_\Sigma}{\partial t_j}Y_\Sigma^{-1}\mathrm{d}t_j
\end{aligned}$$

$$+\sum_{j=1}^{N}\frac{\partial Y_\Sigma}{\partial t_j}Y_\Sigma^{-1}\mathrm{d}t_j\wedge\frac{\partial Y_\Sigma}{\partial w}Y_\Sigma^{-1}\mathrm{d}w+\sum_{j=1}^{N}\sum_{j'=1}^{N}\frac{\partial Y_\Sigma}{\partial t_j}Y_\Sigma^{-1}\frac{\partial Y_\Sigma}{\partial t_{j'}}Y_\Sigma^{-1}\mathrm{d}t_j\wedge\mathrm{d}t_{j'}=0.$$

So $\big(\tilde{\nabla}'\big)^{\mathrm{flat}}$ is an integrable connection which is an extension of $\tilde{\nabla}'|_{\Delta_w\times\mathcal{L}}$.

The following proposition is in fact included in a more general framework by T. Mochizuki in [21, Section 20.3], which provides the existence of flat solution with asymptotic property in a general setting.

Proposition 9.6. *Let $\mathcal{L}\subset M^\circ$ be a submanifold and let $\big(\tilde{E}',\nabla'\big)|_{\Delta_w\times\mathcal{L}}$ be the restriction of the family of connections constructed in (9.7). Assume that for each point $t'\in\mathcal{L}$, there is an open neighborhood $\mathcal{L}'$ of t' in $\mathcal{L}$ and a meromorphic integrable connection*

$$\big(\tilde{\nabla}'\big)^{\mathrm{flat}}\colon\ \tilde{E}'|_{\Delta_w\times\mathcal{L}'}\longrightarrow\tilde{E}'|_{\Delta_w\times\mathcal{L}'}\otimes\Omega^1_{\Delta_w\times\mathcal{L}'}((mr-r)\tilde{x}'),$$

whose associated relative connection coincides with $\tilde{\nabla}'|_{\Delta_w\times\mathcal{L}'}$. Then $\big(\tilde{E}',\nabla'\big)|_{\Delta_w\times\mathcal{L}}$ is a local generalized isomonodromic deformation.

Proof. We have $\tilde{E}'|_{\Delta_w\times\mathcal{L}'}\cong\mathcal{O}^{\oplus r}_{\Delta_w\times\mathcal{L}'}$ and we can write

$$\big(\tilde{\nabla}'\big)^{\mathrm{flat}}=\mathrm{d}+A'(w,t)\frac{\mathrm{d}w}{w^{mr-r}}+\sum_{j=1}^{N}B_j(w,t)\mathrm{d}t_j.$$

After shrinking $\mathcal{L}'$ if necessary, we can take a covering $\{\Sigma=\Gamma_u\times\mathcal{L}'\}$ of $(\Delta_w\setminus\{0\})\times\mathcal{L}'$ with Γ_u a sector in $\Delta_w\setminus\{0\}$ and we can take a fundamental solution $Y_\Sigma(w,t)$ of $\tilde{\nabla}'|_\Sigma$ with the uniform asymptotic relation

$$Y_\Sigma(w,t)\exp\left(\int\Lambda(w)\right)\sim P(w,t),\qquad w\to 0,\ w\in\Sigma.\tag{9.18}$$

Since $(\nabla')^{\mathrm{flat}}$ is an integrable connection extending $\tilde{\nabla}'|_{\Delta_w\times\mathcal{L}'}$, we can take a fundamental solution $Y_\Sigma^{\mathrm{flat}}(w,t)$ of $(\nabla')^{\mathrm{flat}}$ on Σ satisfying $Y_\Sigma^{\mathrm{flat}}(w,t')=Y_\Sigma(w,t')$. We can write

$$Y_\Sigma^{\mathrm{flat}}(w,t)=Y_\Sigma(w,t)C(t),\qquad (w,t)\in\Sigma,\tag{9.19}$$

for a matrix $C(t)=(c_{ij}(t))$ of holomorphic functions in $t\in\mathcal{L}'$ such that $C(t')=I_r$ is the identity matrix. Differentiating (9.19) in t_j, we have

$$\frac{\partial Y_\Sigma^{\mathrm{flat}}}{\partial t_j}=\frac{\partial Y_\Sigma}{\partial t_j}C(t)+Y_\Sigma\frac{\partial C(t)}{\partial t_j},$$

from which we have

$$Y_\Sigma(w,t)\frac{\partial C(t)}{\partial t_j}C(t)^{-1}Y_\Sigma(w,t)^{-1}=\frac{\partial Y_\Sigma^{\mathrm{flat}}(w,t)}{\partial t_j}Y^{\mathrm{flat}}(w,t)^{-1}-\frac{\partial Y_\Sigma(w,t)}{\partial t_j}Y_\Sigma(w,t)^{-1}.\tag{9.20}$$

Since $Y^{\mathrm{flat}}(w,t)$ is a fundamental solution matrix of $(\nabla')^{\mathrm{flat}}$, we have

$$\frac{\partial Y_\Sigma^{\mathrm{flat}}(w,t)}{\partial t_j}\,Y_\Sigma^{\mathrm{flat}}(w,t)^{-1}=-B_j(w,t).\tag{9.21}$$

On the other hand, since the asymptotic relation (9.18) is uniform in $t\in\mathcal{L}$, we have the asymptotic relation

$$\frac{\partial Y_\Sigma}{\partial t_j}\exp\left(\textstyle\int\Lambda(w)\right)+Y_\Sigma\exp\left(\textstyle\int\Lambda(w)\right)\frac{\partial}{\partial t_j}\left(\textstyle\int\Lambda(w)\right)\ \sim\ \frac{\partial P}{\partial t_j}$$

on Σ. Multiplying $\left(Y_\Sigma \exp\left(\int \Lambda(w)\right)\right)^{-1} \sim P^{-1}$ from the right to the above, we have

$$\frac{\partial Y_\Sigma}{\partial t_j} Y_\Sigma^{-1} \sim \frac{\partial P}{\partial t_j} P^{-1} - P \frac{\partial}{\partial t_j}\left(\int \Lambda(w)\right) P^{-1} \qquad (w \to 0) \tag{9.22}$$

on Σ. Using the equality (9.20) and substituting (9.21) and (9.22), we have the asymptotic relation

$$\exp\left(\int \Lambda(w)\right)^{-1} \frac{\partial C(t)}{\partial t_j} C(t)^{-1} \exp\left(\int \Lambda(w)\right) \sim P^{-1} Y_\Sigma \frac{\partial C(t)}{\partial t_j} C(t)^{-1} Y_\Sigma^{-1} P$$
$$= P^{-1}\left(\frac{\partial Y_\Sigma^{\mathrm{flat}}}{\partial t_j}(Y_\Sigma^{\mathrm{flat}})^{-1} - \frac{\partial Y_\Sigma}{\partial t_j} Y_\Sigma^{-1}\right) P \sim -P^{-1} B_j P - P^{-1}\frac{\partial P}{\partial t_j} + \frac{\partial}{\partial t_j}\left(\int \Lambda(w)\right)$$

on Σ. So $w^N \exp\left(\int \Lambda(w)\right)^{-1} \frac{\partial C(t)}{\partial t_j} C(t)^{-1} \exp\left(\int \Lambda(w)\right)$ is bounded on Σ for a large N, because B_j is a matrix of meromorphic functions in w,.

Choose a point $(w_0, t') \in \Sigma$. After replacing a frame of E_0', we can write

$$\int \Lambda(w) = \begin{pmatrix} A_1(w) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_s(w) \end{pmatrix} \frac{1}{w^{mr-r-1}}, \qquad A_k(w) = \begin{pmatrix} a_1^{(k)}(w) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_{m_k}^{(k)}(w) \end{pmatrix},$$

such that $a_p^{(k)}(w) = a_p^{(k)}(0) + b_{p,1}^{(k)} w + \cdots + b_{p,mr-r-2}^{(k)} w^{mr-r-2} + b_{p,mr-r-1}^{(k)} w^{mr-r-1} \log w$ satisfies $a_p^{(k)}(0) \neq a_q^{(l)}(0)$ for $(k,p) \neq (l,q)$ and that $\rho_k = \mathrm{Re}\big(w_0^{-mr+r+1} a_p^{(k)}(0)\big)$ holds for $1 \le p \le m_k$ at t' with $\rho_1 > \rho_2 > \cdots > \rho_s$. Write

$$\frac{\partial C(t)}{\partial t_j} C(t)^{-1} =: \tilde{C}(t) = \begin{pmatrix} \tilde{C}_{11}(t) & \cdots & \tilde{C}_{1s}(t) \\ \vdots & \ddots & \vdots \\ \tilde{C}_{s1}(t) & \cdots & \tilde{C}_{ss}(t) \end{pmatrix}, \tag{9.23}$$

where $\tilde{C}_{kl}(t)$ is a matrix of size (m_k, m_l). Then we have

$$w^N \exp\left(\int \Lambda(w)\right)^{-1} \frac{\partial C(t)}{\partial t_j} C(t)^{-1} \exp\left(\int \Lambda(w)\right) \tag{9.24}$$
$$= w^N \begin{pmatrix} \exp\left(\frac{-A_1(w)}{w^{mr-r-1}}\right) \tilde{C}_{11}(t) \exp\left(\frac{A_1(w)}{w^{mr-r-1}}\right) & \cdots & \exp\left(\frac{-A_1(w)}{w^{mr-r-1}}\right) \tilde{C}_{1s}(t) \exp\left(\frac{A_s(w)}{w^{mr-r-1}}\right) \\ \vdots & \ddots & \vdots \\ \exp\left(\frac{-A_s(w)}{w^{mr-r-1}}\right) \tilde{C}_{1s}(t) \exp\left(\frac{A_1(w)}{w^{mr-r-1}}\right) & \cdots & \exp\left(\frac{-A_s(w)}{w^{mr-r-1}}\right) \tilde{C}_{ss}(t) \exp\left(\frac{A_s(w)}{w^{mr-r-1}}\right) \end{pmatrix},$$

which is bounded on Σ.

Suppose that $\tilde{C}_{kl}(t) \neq 0$ for $k > l$. Then the growth order of the (k,l) minor of (9.24) along the ray $\{\theta w_0 \mid 0 < \theta \le 1\}$ is the same as

$$(\theta w_0)^N \exp\left(\mathrm{Re}\big((\theta w_0)^{-mr+r+1}(A_k(0) - A_l(0))\big)\right) \tilde{C}_{k,l}(t) = \theta^N w_0^N \mathrm{e}^{\frac{\rho_k - \rho_l}{\theta^{mr-r-1}}} \tilde{C}_{kl}(t).$$

Since $\rho_k - \rho_l > 0$, it is divergent as $\theta \to 0$, which is a contradiction. If we write

$$\tilde{C}_{kk}(t) = \begin{pmatrix} \tilde{c}_{11}^{(k)}(t) & \cdots & \tilde{c}_{1m_k}^{(k)}(t) \\ \vdots & \ddots & \vdots \\ \tilde{c}_{m_k 1}^{(k)} & \cdots & \tilde{c}_{m_k m_k}^{(k)}(t) \end{pmatrix},$$

then we have

$$
\begin{aligned}
&w^N \exp\left(-w^{-mr+r+1}A_k(w)\right)\tilde{C}_{kk}(t)\exp\left(w^{-mr+r+1}A_k(w)\right)\\
&= w^N \begin{pmatrix} \tilde{c}^{(k)}_{11}(t) & \cdots & \mathrm{e}^{w^{-mr+r+1}(a^{(k)}_1(w)-a^{(k)}_{m_k}(w))}\tilde{c}^{(k)}_{1m_k}(t) \\ \vdots & \ddots & \vdots \\ \mathrm{e}^{w^{-mr+r+1}(a^{(k)}_{m_k}(w)-a^{(k)}_1(w))}\tilde{c}^{(k)}_{m_k1}(t) & \cdots & \tilde{c}_{m_km_k}(t) \end{pmatrix}.
\end{aligned} \tag{9.25}
$$

Suppose that $\tilde{c}^{(k)}_{pq}(t) \neq 0$ for $p \neq q$. Since $a^{(k)}_p(0) \neq a^{(k)}_q(0)$, we can find $\delta \neq 0$ with $|\delta|$ small such that $\left\{\theta \mathrm{e}^{\sqrt{-1}\delta}w_0 \mid 0 < \theta \leq 1\right\}$ is contained in Γ_u and that either $\mathrm{Re}\big(\frac{a^{(k)}_p(0)-a^{(k)}_q(0)}{(w_0\mathrm{e}^{\sqrt{-1}\delta})^{mr-r-1}}\big) > 0$ or $\mathrm{Re}\big(\frac{a^{(k)}_p(0)-a^{(k)}_q(0)}{(w_0\mathrm{e}^{\sqrt{-1}\delta})^{mr-r-1}}\big) < 0$ holds. After replacing δ with $\pm\delta$, we may assume the inequality $\mathrm{Re}\big(\frac{a^{(k)}_p(0)-a^{(k)}_q(0)}{(w_0\mathrm{e}^{\sqrt{-1}\delta})^{mr-r-1}}\big) > 0$. Then the growth order of the (p,q)-entry of (9.25) is the same as $(w_0\theta)^N \exp\big(\frac{a^{(k)}_p(0)-a^{(k)}_q(0)}{(w_0\mathrm{e}^{\sqrt{-1}\delta})^{mr-r-1}\theta^{mr-r-1}}\big)$, which is divergent along $\left\{\theta \mathrm{e}^{\sqrt{-1}\delta}w_0 \mid 0 < \theta \leq 1\right\}$ as $\theta \to 0$. Since (9.25) is bounded on $\Gamma_u \times \mathcal{L}$, it is a contradiction. So $\tilde{C}_{kk}(t)$ is a diagonal matrix for any k.

Thus we have proved that the matrix $\tilde{C}(t)$ given in (9.23) is a block upper triangular matrix in the sense that $\tilde{C}_{kl}(t) = 0$ for $k > l$ and that $\tilde{C}_{kk}(t)$ is are diagonal matrices for $1 \leq k \leq s$. We will show that $C(t)$ is also a block upper triangular matrix. Consider the Taylor expansion

$$
C(t) = \sum_{i_1,\dots,i_N} C_{i_1,\dots,i_n} t_1^{i_1}\cdots t_n^{i_n} \tag{9.26}
$$

around $t = t'$. Suppose that one of $C_{i_1,\dots,i_n}$ is not block upper triangular and put

$$
l = \min\left\{i_1 + \cdots + i_n \mid C_{i_1,\dots,i_n} \text{ is not a block upper triangular matrix}\right\}.
$$

By the minimality of l, $C(t) \pmod{(t_1,\dots,t_n)^{l-1}}$ is a block upper triangular matrix and so is $C(t)^{-1} \pmod{(t_1,\dots,t_n)^{l-1}}$. Differentiating (9.26), $\frac{\partial C(t)}{\partial t_j} \pmod{(t_1,\dots,t_n)^{l-1}}$ is not a block upper triangular matrix for some j. So we can see that $\frac{\partial C(t)}{\partial t_j}C(t)^{-1} \pmod{(t_1,\dots,t_n)^{l-1}}$ is not a block upper triangular matrix of the above form, which is a contradiction.

Thus $C(t)$ is also a block upper triangular matrix of the above form. Let $C_{\mathrm{diag}}(t)$ be the diagonal part of $C(t)$. Then we have

$$
\begin{aligned}
Y^{\mathrm{flat}}_\Sigma(w,t)\exp\left(\int\Lambda\right) &= Y_\Sigma(w,t)\exp\left(\int\Lambda\right)\exp\left(-\int\Lambda\right)C(t)\exp\left(\int\Lambda\right)\\
&\sim P(w,t)C_{\mathrm{diag}}(t)
\end{aligned} \tag{9.27}
$$

on Σ. If we take another sector $\Sigma' = \Gamma_{u'} \times \mathcal{L}$ and a fundamental solution $Y^{\mathrm{flat}}_{\Sigma'}$ of $\big(\tilde{\nabla}'\big)^{\mathrm{flat}}$ satisfying $Y^{\mathrm{flat}}_{\Sigma'} = Y_{\Sigma'}C'(t)$ with $C'(t') = I_r$, we have

$$
Y^{\mathrm{flat}}_{\Sigma'}(w,t)\exp\left(\int\Lambda\right) \sim P(w,t)C'_{\mathrm{diag}}(t) \tag{9.28}
$$

on Σ'. Since both of Y^{flat}_Σ and $Y^{\mathrm{flat}}_{\Sigma'}$ are fundamental solutions of the integrable connection $\big(\tilde{\nabla}'\big)^{\mathrm{flat}}$, we can write $Y^{\mathrm{flat}}_{\Sigma'} = Y^{\mathrm{flat}}_\Sigma K$ for a constant matrix K. Combining (9.27) and (9.28), we have

$$
C_{\mathrm{diag}}(t)^{-1}C'_{\mathrm{diag}}(t) \sim \exp\left(-\int\Lambda\right)(Y^{\mathrm{flat}}_\Sigma)^{-1}Y^{\mathrm{flat}}_{\Sigma'}\exp\left(\int\Lambda\right)
$$

$$= \exp\left(-\int\Lambda\right) K \exp\left(\int\Lambda\right)$$

on $\Sigma\cap\Sigma'$. Since the diagonal entries of the right-hand side of the above are those of K, which are constant in t, we can see that the left-hand side of the above is a constant matrix. Since $C'_{\mathrm{diag}}(t') = C'(t') = I_r = C(t') = C_{\mathrm{diag}}(t')$, we have $C'_{\mathrm{diag}}(t) = C_{\mathrm{diag}}(t)$.

Thus, the replacement of the formal transform $P(w,t)$ with $P(w,t)C_{\mathrm{diag}}(t)$ is independent of Σ. So the replacement of Y_Σ with Y_Σ^{flat} on each Σ satisfies the condition of Definition 9.4. ■

Summarizing the above arguments, we get the following theorem, which is the local version of a main consequence of the Jimbo–Miwa–Ueno theory. It is the significance of the formulation of generalized isomonodromic deformation introduced in Section 11 later.

Theorem 9.7 (Jimbo–Miwa–Ueno [16, Theorems 3.1 and 3.3])**.** *For a submanifold $\mathcal{L}$ of M°, the restriction $\big(\tilde{E}',\tilde{\nabla}'\big)|_{\Delta_w\times\mathcal{L}}$ of the family of connections to $\Delta_w\times\mathcal{L}$ is a local generalized isomonodromic deformation if and only if for each point t' of $\mathcal{L}$, there is a neighborhood $\mathcal{L}'$ of t' in $\mathcal{L}$ and a meromorphic integrable connection $\big(\tilde{\nabla}'\big)^{\mathrm{flat}}\colon \tilde{E}'|_{\Delta_w\times\mathcal{L}'} \longrightarrow \tilde{E}'|_{\Delta\times\mathcal{L}'}\otimes\Omega_{\Delta_w\times\mathcal{L}'}((mr-r)\tilde{x}')$ whose associated relative connection coincides with $\tilde{\nabla}'|_{\Delta_w\times\mathcal{L}'}$.*

Remark 9.8.

(i) In the precise setting of [16], each sector is taken sufficiently large so that the asymptotic solution Y_Σ is determined uniquely. Furthermore, the choice of formal transforms is also included in the system of differential equation in [16, Theorems 3.1 and 3.3].

(ii) In our setting of Theorem 9.7, there are ambiguities in the choice of asymptotic solutions $Y_\Sigma(w,t)$ and in the choice of the formal transforms $P(w,t)$. Our statement of Theorem 9.7 is a consequence of Proposition 9.6, which is essentially the result by T. Mochizuki in [21].

(iii) We introduce Definition 9.4 based on the naive meaning of Stokes data, but it will be better to explain the Stokes data by using the notion of local system with Stokes filtration as in [1, Section 4.6] or [21, Chapter 3].

(iv) Theorem 9.7 is also mentioned in the appendix of [5].

(v) We can see from (9.17) that the dt_j-coefficient of $\big(\tilde{\nabla}'\big)^{\mathrm{flat}}$ has a pole of order $mr-r-1$.

Proposition 9.9. *For the family of connections $\big(\tilde{E}',\tilde{\nabla}'\big)$ on $\Delta_w\times M^\circ$ which is constructed from $\big(\tilde{E},\tilde{\nabla}\big)|_{\Delta_z\times M^\circ}$ in (9.7) and for a submanifold $\mathcal{L}$ of M°, $\big(\tilde{E}',\tilde{\nabla}'\big)|_{\Delta_w\times\mathcal{L}}$ can be extended to an integrable connection if and only if $\big(\tilde{E},\tilde{\nabla}\big)|_{\Delta_z\times\mathcal{L}}$ can be extended to an integrable meromorphic connection on $\Delta_z\times\mathcal{L}$.*

Proof. Assume that $\tilde{\nabla}'|_{\Delta_w\times\mathcal{L}}$ can be extended to an integrable connection $\big(\tilde{\nabla}'\big)^{\mathrm{flat}}$. Note that there is a canonical inclusion $S(w)\colon p_{\mathcal{L}}^*\big(\tilde{E}|_{\Delta_z\times\mathcal{L}}\big)\hookrightarrow \tilde{E}'|_{\Delta_w\times\mathcal{L}}$ which is Galois equivariant and compatible with the connections. Consider the pullback $S(w)^*\big(\tilde{\nabla}'\big)^{\mathrm{flat}}$. If we write

$$\big(\tilde{\nabla}'\big)^{\mathrm{flat}} = \mathrm{d} + A'(w)\frac{\mathrm{d}w}{w^{mr-r}} + \sum_{j=1}^{N} B'_j(w)\mathrm{d}t_j,$$

then the connection $S(w)^*\big(\tilde{\nabla}'\big)^{\mathrm{flat}}$ on $p_{\mathcal{L}}^*\big(\tilde{E}|_{\Delta_z\times\mathcal{L}}\big)$ is given by

$$\mathrm{d} + S(w)^{-1}\left(\frac{\partial S(w)}{\partial w} + \frac{A'(w)S(w)}{w^{mr-r}}\right)\mathrm{d}w + \sum_{j=1}^{N} S(w)^{-1}\left(\frac{\partial S(w)}{\partial t_j} + B'_j(w)S(w)\right)\mathrm{d}t_j. \quad (9.29)$$

Note that there is a canonical action of $\mathrm{Gal}(\Delta_w/\Delta_z)$ on $p_{\mathcal{L}}^*\big(\tilde{E}|_{\Delta_z\times\mathcal{L}}\big) \cong p_{\mathcal{L}}^*\big(\mathcal{O}^{\mathrm{hol}}_{\Delta_z\times\mathcal{L}}\big)^{\oplus r}$, which induces a canonical Galois action on $\mathrm{End}\left(p_{\mathcal{L}}^*\big(\tilde{E}|_{\Delta_z\times\mathcal{L}}\big)\right)\otimes p_{\mathcal{L}}^*\Omega^1_{\Delta_z\times\mathcal{L}}(m\tilde{x})$. If we denote the matrix of $\tilde{\nabla}|_{\Delta_z\times\mathcal{L}}$ by $\frac{A(z)\mathrm{d}z}{z^m}$, then we have

$$\frac{A(z)\mathrm{d}z}{z^m} - \tilde{\nu}_0(z)I_r = S(w)^{-1}\left(\frac{\partial S(w)}{\partial w} + \frac{A'(w)S(w)}{w^{mr-r}}\right)\mathrm{d}w,$$

which is Galois invariant. On the other hand, the $\mathrm{d}t_j$-coefficient of (9.29) may not be Galois invariant. So we put

$$B_j := -\int \frac{\partial\tilde{\nu}_0(z)}{\partial t_j}I_r \;+\; \frac{1}{r}\sum_{\sigma\in\mathrm{Gal}(\Delta_w/\Delta_z)}\left[S(w)^{-1}\left(\frac{\partial S(w)}{\partial t_j} + B'_j(w)S(w)\right)\right]^{\sigma}.$$

Then B_j is $\mathrm{Gal}(\Delta_w/\Delta_z)$-invariant and becomes a matrix of meromorphic functions on $\Delta_z\times\mathcal{L}$. If we put

$$\tilde{\nabla}^{\mathrm{flat}} := \mathrm{d} + \frac{A(z)\mathrm{d}z}{z^m} + \sum_{j=1}^{N} B_j\mathrm{d}t_j,$$

then $\tilde{\nabla}^{\mathrm{flat}}$ defines a meromorphic integrable connection on $\tilde{E}|_{\Delta_z\times\mathcal{L}}$. The converse is immediate. ■

We can see by a calculation that

$$\Psi(z,t) := \begin{pmatrix} 1 & z^{\frac{1}{r}} & \cdots & z^{\frac{r-1}{r}} \\ 1 & \zeta_r z^{\frac{1}{r}} & \cdots & \zeta_r^{r-1} z^{\frac{r-1}{r}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_r^{r-1} z^{\frac{1}{r}} & \cdots & \zeta_r^{(r-1)^2} z^{\frac{r-1}{r}} \end{pmatrix}^{-1} \mathrm{e}^{\int\tilde{\nu}_0(z,t)} \exp\left(-\int \Lambda\big(z^{\frac{1}{r}},t\big)\right) \tag{9.30}$$

becomes a fundamental solution of

$$\mathrm{d} + \begin{pmatrix} \tilde{\nu}_0(z) & z\tilde{\nu}_{r-1}(z) & \cdots & z\tilde{\nu}_1(z) \\ \tilde{\nu}_1(z) & \tilde{\nu}_0(z) + \frac{\mathrm{d}z}{rz} & \cdots & z\tilde{\nu}_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\nu}_{r-1}(z) & \tilde{\nu}_{r-2}(z) & \cdots & \tilde{\nu}_0(z) + \frac{(r-1)\mathrm{d}z}{rz} \end{pmatrix}$$
$$+ \sum_{j=1}^{N} \begin{pmatrix} \frac{\partial}{\partial t_j}\int\tilde{\nu}_0(z) & zw^{-r+1}\frac{\partial}{\partial t_j}\int w^{r-1}\tilde{\nu}_{r-1}(z) & \cdots & zw^{-1}\frac{\partial}{\partial t_j}\int w\tilde{\nu}_1(z) \\ w^{-1}\frac{\partial}{\partial t_j}\int w\tilde{\nu}_1(z) & \frac{\partial}{\partial t_j}\int\tilde{\nu}_0(z) & \cdots & zw^{-2}\frac{\partial}{\partial t_j}\int w^2\tilde{\nu}_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ w^{-r+1}\frac{\partial}{\partial t_j}\int w^{r-1}\tilde{\nu}_{r-1}(z) & w^{-r+2}\frac{\partial}{\partial t_j}\int w^{r-2}\tilde{\nu}_{r-2}(z) & \cdots & \frac{\partial}{\partial t_j}\int\tilde{\nu}_0(z) \end{pmatrix}\mathrm{d}t_j$$

which is a matrix form of the integrable formal connection

$$\nabla_{\tilde{\nu}(w)+\sum\frac{\partial}{\partial t_j}(\int\tilde{\nu})\mathrm{d}t_j} \colon\ \mathcal{O}_{\mathcal{L}}[[w]] \longrightarrow \mathcal{O}_{\mathcal{L}}[[w]]\otimes\Omega_{\Delta_z\times\mathcal{L}}(m\tilde{x})$$
$$f(w) \mapsto \mathrm{d}f(w) + \left(\tilde{\nu}(w) + \sum_{j=1}^{N}\frac{\partial}{\partial t_j}\left(\int\tilde{\nu}(w)\right)\mathrm{d}t_j\right)f(w)$$

with respect to the basis $1, w, \dots, w^{r-1}$ of the free module $\mathcal{O}_{\mathcal{L}}[[w]]$ over $\mathcal{O}_{\mathcal{L}}[[z]]$. On the other hand, recall that the elementary transform $p_{\mathcal{L}}^*\big(\tilde{E},\tilde{\nabla}\big)|_{\Delta_w\times\mathcal{L}} \mapsto \big(\tilde{E}',\tilde{\nabla}'\big)$ is given by the rational gauge transform $S(w)\colon p_{\mathcal{L}}^*\big(\tilde{E}|_{\Delta_z\times\mathcal{L}}\big)\hookrightarrow \tilde{E}'$. If we put $\overline{\Sigma}:=p_{M^\circ}(\Sigma)$, then $p_{M^\circ}|_\Sigma\colon \Sigma\xrightarrow{\sim}\overline{\Sigma}$ is an isomorphism if Σ is sufficiently small. Substituting $z=w^{\frac{1}{r}}$ in the solution $Y_\Sigma(w)$, we can get a fundamental solution

$$Z_{\overline{\Sigma}}(z,t):=S\big(z^{\frac{1}{r}},t\big)^{-1}Y_\Sigma\big(z^{\frac{1}{r}},t\big)\mathrm{e}^{\int\tilde{\nu}_0(z,t)}$$

of $\tilde{\nabla}|_{\Delta_z\times\mathcal{L}}$. Using the asymptotic property $Y_\Sigma\exp\big(\int\Lambda(w)\big)\sim P(w)$ and the equality (9.12), we get the asymptotic relation

$$\begin{aligned}Z_{\overline{\Sigma}}(z)\Psi(z)^{-1}&=S\big(z^{\frac{1}{r}}\big)^{-1}Y_\Sigma\big(z^{\frac{1}{r}}\big)\\&\quad\times\exp\left(\int\Lambda\big(z^{\frac{1}{r}}\big)\right)\begin{pmatrix}1 & z^{\frac{1}{r}} & \cdots & z^{\frac{r-1}{r}}\\ 1 & \zeta_r z^{\frac{1}{r}} & \cdots & \zeta_r^{r-1}z^{\frac{r-1}{r}}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \zeta_r^{r-1}z^{\frac{1}{r}} & \cdots & \zeta_r^{(r-1)^2}z^{\frac{r-1}{r}}\end{pmatrix}\sim Q(z)\end{aligned}\tag{9.31}$$

on $(z,t)\in\overline{\Sigma}$. For another $\overline{\Sigma}'$, we have

$$Z_{\overline{\Sigma}'}(z,t)=Z_{\overline{\Sigma}}(z,t)C_{\overline{\Sigma},\overline{\Sigma}'}(t),$$

where $C_{\overline{\Sigma},\overline{\Sigma}'}(t)=C_{\Sigma,\Sigma'}(t)$. So we can in fact describe the Stokes data on Δ_z, without using a ramified cover, in the sense of patching data in [1, Theorem 4.5.1].

Definition 9.10. We say that a family of connections $\big(\tilde{E},\tilde{\nabla}\big)|_{\Delta_z\times\mathcal{L}}$ over a submanifold $\mathcal{L}\subset M^\circ$ is a local generalized isomonodromic deformation, if for each $t'\in\mathcal{L}$, we can take an open neighborhood $\mathcal{L}_{t'}$ of t' in $\mathcal{L}$, a replacement of the formal transform $Q(z,t)$ given in (9.11) and a replacement of the covering $\{\overline{\Sigma}\}$ of $(\Delta_z\setminus\{0\})\times\mathcal{L}_{t'}$ such that

(i) there is a fundamental solution $Z_{\overline{\Sigma}}(z,t)$ of $\tilde{\nabla}$ on each $\overline{\Sigma}$ with the asymptotic property (9.31) and

(ii) all the Stokes matrices $C_{\overline{\Sigma},\overline{\Sigma}'}(t)$ are constant in $t\in\mathcal{L}_{t'}$.

Corollary 9.11. *For a submanifold $\mathcal{L}$ of M°, the family $\big(\tilde{E},\tilde{\nabla}\big)|_{\Delta_z\times\mathcal{L}}$ is a local isomonodromic deformation in the sense of Definition 9.10 if and only if for each point t' of $\mathcal{L}$, there is a neighborhood $\mathcal{L}'$ of t' in $\mathcal{L}$ and an integrable meromorphic connection $\tilde{\nabla}\colon\tilde{E}|_{\Delta_z\times\mathcal{L}'}\longrightarrow \tilde{E}|_{\Delta_z\times\mathcal{L}'}\otimes\Omega^1_{\Delta_z\times\mathcal{L}'}(m\tilde{x})$ whose associated relative connection coincides with $\big(\tilde{E},\tilde{\nabla}\big)|_{\Delta_z\times\mathcal{L}'}$.*

Proof. Assume that there is an integrable connection $\tilde{\nabla}^{\text{flat}}$ on $\tilde{E}|_{\Delta_z\times\mathcal{L}'}$ which is an extension of $\tilde{\nabla}|_{\Delta_z\times\mathcal{L}'}$ as in Proposition 9.9. Then there is a canonically induced integrable connection $\big(\tilde{\nabla}'\big)^{\text{flat}}$ on $\tilde{E}'|_{\Delta_w\times\mathcal{L}'}$. If we take a fundamental solution $Y^{\text{flat}}(w,t)$ of $\big(\tilde{\nabla}'\big)^{\text{flat}}$ as in the proof of Proposition 9.6, then

$$Z^{\text{flat}}_{\overline{\Sigma}}(z):=S\big(z^{\frac{1}{r}}\big)^{-1}Y^{\text{flat}}\big(z^{\frac{1}{r}}\big)\mathrm{e}^{\int\tilde{\nu}_0(z)}$$

is a fundamental solution of $\tilde{\nabla}^{\text{flat}}$. Since $Y_\Sigma\exp\big(\int\Lambda(w)\big)\sim Y^{\text{flat}}\exp\big(\int\Lambda(w)\big)C_{\text{diag}}(t)^{-1}$ as in the proof of Proposition 9.6, we can see from (9.31) that the asymptotic relation

$$Z^{\text{flat}}_{\overline{\Sigma}}(z,t)C_{\text{diag}}(t)^{-1}\Psi(z,t)^{-1}\sim Q(z,t)$$

holds on $(z,t)\in\overline{\Sigma}$. Differentiating the above in t_j, we have

$$\frac{\partial Z^{\text{flat}}_{\overline{\Sigma}}}{\partial t_j}\big(Z^{\text{flat}}_{\overline{\Sigma}}\big)^{-1}Q(z)-Q(z)\Psi(z)\frac{\partial C_{\text{diag}}}{\partial t_j}C_{\text{diag}}^{-1}\Psi(z)^{-1}+Q(z)\frac{\partial\Psi(z)}{\partial t_j}\Psi(z)^{-1}\sim\frac{\partial Q(z)}{\partial t_j}\tag{9.32}$$

on $(z,t)\in\overline{\Sigma}$. Note that $-\frac{\partial Z^{\mathrm{flat}}_{\overline{\Sigma}}}{\partial t_j}\big(Z^{\mathrm{flat}}_{\overline{\Sigma}}\big)^{-1}$ is $\mathrm{Gal}(\Delta_w/\Delta_z)$-invariant because it is the $\mathrm{d}t_j$-coefficient of $\tilde{\nabla}^{\mathrm{flat}}$. We can see that $-\frac{\partial\Psi(z)}{\partial t_j}\Psi(z)^{-1}$ is also $\mathrm{Gal}(\Delta_w/\Delta_z)$-invariant because it is the $\mathrm{d}t_j$-coefficient of the formal connection $\nabla_{\nu(w)+\sum\int\frac{\partial\nu_0}{\partial t_j}\mathrm{d}t_j}$. The transform $Q(z)$ is also $\mathrm{Gal}(\Delta_w/\Delta_z)$-invariant as a matrix of formal power series. So, from the asymptotic relation (9.32), we can see that $\Psi(z)\frac{\partial C_{\mathrm{diag}}}{\partial t_j}C_{\mathrm{diag}}^{-1}\Psi(z)^{-1}$ is $\mathrm{Gal}(\Delta_w/\Delta_z)$-invariant. If the Galois transform $\sigma\in\mathrm{Gal}(\Delta_w/\Delta_z)$ is given by $\sigma(w)=\zeta_r^k w$, then the Galois transform by σ on $\Psi(z,t)$ in (9.30) is given by

$$
\begin{aligned}
\Psi(z,t)^{\sigma}&=\begin{pmatrix}1&\zeta^k z^{\frac{1}{r}}&\cdots&\zeta^{(r-1)k}z^{\frac{r-1}{r}}\\1&\zeta_r^{k+1}z^{\frac{1}{r}}&\cdots&\zeta_r^{(r-1)(k+1)}z^{\frac{r-1}{r}}\\\vdots&\vdots&\ddots&\vdots\\1&\zeta_r^{r-1+k}z^{\frac{1}{r}}&\cdots&\zeta_r^{(r-1)(r-1+k)}z^{\frac{r-1}{r}}\end{pmatrix}^{-1}\mathrm{e}^{\int\nu_0(z,t)}\exp\left(-\int\Lambda(\zeta_r^k z^{\frac{1}{r}},t)\right)\\
&=\begin{pmatrix}1&z^{\frac{1}{r}}&\cdots&z^{\frac{r-1}{r}}\\1&\zeta_r z^{\frac{1}{r}}&\cdots&\zeta_r^{r-1}z^{\frac{r-1}{r}}\\\vdots&\vdots&\ddots&\vdots\\1&\zeta_r^{r-1}z^{\frac{1}{r}}&\cdots&\zeta_r^{(r-1)^2}z^{\frac{r-1}{r}}\end{pmatrix}^{-1}P_\sigma\mathrm{e}^{\int\nu_0(z,t)}P_\sigma^{-1}\exp\left(-\int\Lambda(z^{\frac{1}{r}},t)\right)P_\sigma,
\end{aligned}
$$

where P_σ is the permutation matrix defined by $P_\sigma=(e_{k+1},e_{k+2},\dots,e_r,e_1,\dots,e_k)$ for the canonical basis $e_1,\dots,e_r$ of $\mathbb{C}^r$. So the equation of Galois invariance

$$
\Psi(z)^{\sigma}\frac{\partial C_{\mathrm{diag}}}{\partial t_j}C_{\mathrm{diag}}^{-1}\big(\Psi(z)^{\sigma}\big)^{-1}=\Psi(z)\frac{\partial C_{\mathrm{diag}}}{\partial t_j}C_{\mathrm{diag}}^{-1}\Psi(z)^{-1}
$$

deduces the equalities

$$
P_\sigma\,\frac{\partial C_{\mathrm{diag}}}{\partial t_j}C_{\mathrm{diag}}^{-1}\,P_\sigma^{-1}=\frac{\partial C_{\mathrm{diag}}}{\partial t_j}C_{\mathrm{diag}}^{-1}
$$

for cyclic permutation matrices P_σ corresponding to $\sigma\in\mathrm{Gal}(\Delta_w/\Delta_z)$. Thus all the diagonal entries of $\frac{\partial C_{\mathrm{diag}}}{\partial t_j}C_{\mathrm{diag}}^{-1}$ are the same, which implies that all the diagonal entries of $C_{\mathrm{diag}}(t)$ are the same. After replacing $Q(z)$ with $Q(z)C_{\mathrm{diag}}(t)$, we have the asymptotic relation

$$
Z^{\mathrm{flat}}_{\overline{\Sigma}}(z)\Psi(z)^{-1}\sim\ Q(z)\qquad\text{as } z\to 0\text{ on }\overline{\Sigma}
$$

for all $\overline{\Sigma}$. After replacing $Z_{\overline{\Sigma}}(z,t)$ with $Z^{\mathrm{flat}}_{\overline{\Sigma}}(z,t)$ and shrinking $\mathcal{L}$ if necessary, all the Stokes matrices $\big\{C_{\overline{\Sigma},\overline{\Sigma}'}\big\}$ become constant. So $\big(\tilde{E},\tilde{\nabla}\big)|_{\Delta_z\times\mathcal{L}}$ becomes a local generalized isomonodromic deformation.

Conversely, assume that $\big(\tilde{E},\tilde{\nabla}\big)|_{\Delta_z\times\mathcal{L}}$ is a local generalized isomonodromic deformation. For the fundamental solution $Z_{\overline{\Sigma}}(z,t)$ of $\tilde{\nabla}|_{\overline{\Sigma}}$ given in Definition 9.10,

$$
Y_{\Sigma}(z,t)=S(w,t)Z_{\overline{\Sigma}}(z,t)\mathrm{e}^{-\int\nu_0(z,t)}
$$

becomes a fundamental solution of $\tilde{\nabla}'|_{\Sigma}$. So we have $C_{\Sigma,\Sigma'}(t)=C_{\overline{\Sigma},\overline{\Sigma}'}$ which is constant in t. Thus $\tilde{\nabla}'|_{\Delta_w\times\mathcal{L}}$ is a local generalized isomonodromic deformation. By Theorem 9.7, we can extend $\tilde{\nabla}'|_{\Delta_w\times\mathcal{L}}$ to an integrable connection after shrinking $\mathcal{L}$ at each point. So $\big(\tilde{E},\tilde{\nabla}\big)|_{\Delta_z\times\mathcal{L}}$ can be extended to an integrable connection by Proposition 9.9. ■

Remark 9.12. The achievement of the construction of the generalized isomonodromic deformation by Bremer and Sage in [8] is based on the Jimbo–Miwa–Ueno theory, which becomes Corollary 9.11 in our setting.

10 Horizontal lift of a universal family of connections

We will extend the notion of local generalized isomonodromic deformation in Section 9 to a global setting on the moduli space of connections. Its differential equation is given as a subbundle of the tangent bundle of the moduli space, which satisfies the integrability condition. For its construction, we introduce the notion of horizontal lift of a universal family of connections.

Let the notations $\mathcal{T}$, $\mathcal{C}$, λ, $\tilde{\mu}$, $\tilde{\nu}$, $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$ be as in Section 5. There is an étale surjective morphism $\tilde{M} \longrightarrow M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$ such that there is a universal family of connections $\big(\tilde{E},\tilde{\nabla},\tilde{l},\tilde{\ell},\tilde{\mathcal{V}}\big)$ on $\mathcal{C}_{\tilde{M}}$. We may assume that the generic $\tilde{\nu}$-ramified structure $\tilde{\mathcal{V}}$ is given by a factorized $\tilde{\nu}$-ramified structure $\big(\tilde{V}_k,\tilde{\vartheta}_k,\tilde{\varkappa}_k\big)_{0\le k\le r-1}$.

For a Zariski open subset $\mathcal{T}'\subset\mathcal{T}$, we put $\tilde{M}' := \tilde{M}\times_{\mathcal{T}}\mathcal{T}'$. Take a vector field $v\in H^0(\mathcal{T}',T_{\mathcal{T}}|_{\mathcal{T}'})$. If we put $\mathcal{T}'[v] := \mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon]$ with $\epsilon^2=0$, then v is characterized by a morphism $I_v\colon \mathcal{T}'[v]\longrightarrow\mathcal{T}'$ whose restriction to $\mathcal{T}'$ is the identity. Put $\tilde{M}'[v] := \tilde{M}'\times\operatorname{Spec}\mathbb{C}[\epsilon]$ and consider the fiber product $\mathcal{C}_{\tilde{M}'[v]} := \mathcal{C}\times_{\mathcal{T}}(\tilde{M}'\times\operatorname{Spec}\mathbb{C}[\epsilon])$ with respect to the projection $\mathcal{C}\longrightarrow\mathcal{T}$ and the composition $\tilde{M}'\times\operatorname{Spec}\mathbb{C}[\epsilon]\longrightarrow\mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon]\xrightarrow{I_v}\mathcal{T}'\hookrightarrow\mathcal{T}$.

We define a divisor $\mathcal{D}'$ on $\mathcal{C}$ by setting

$$\mathcal{D}' := \sum_{i=1}^{n_{\rm un}}\big(m_i^{\rm un}-1\big)\tilde{x}_i^{\rm un} + \sum_{i=1}^{n_{\rm ram}}\big(m_i^{\rm ram}-1\big)\tilde{x}_i^{\rm ram}.$$

Consider the sheaf of differential forms $\Omega^1_{\mathcal{C}_{\tilde{M}'[v]/\tilde{M}'}}$ with respect to the composition of trivial projections

$$\mathcal{C}_{\tilde{M}'[v]} = \mathcal{C}\times_{\mathcal{T}}\big(\tilde{M}'\times\operatorname{Spec}\mathbb{C}[\epsilon]\big)\longrightarrow\tilde{M}'\times\operatorname{Spec}\mathbb{C}[\epsilon]\longrightarrow M'.$$

Take a local section $z_i^{\log}$ (resp. $z_i^{\rm un}$, $z_i^{\rm ram}$) of $\mathcal{O}_{\mathcal{C}_{\mathcal{T}'}}$ which is a local defining equation of $\tilde{x}_i^{\log}$ (resp. $\tilde{x}_i^{\rm un}$, $\tilde{x}_i^{\rm ram}$). We write the induced local section of $\mathcal{O}_{\mathcal{C}_{\tilde{M}'[v]}}$ by the same symbol $z_i^{\log}$ (resp. $z_i^{\rm un}$, $z_i^{\rm ram}$). Let $\tilde{\Omega}_v$ be the coherent subsheaf of $\Omega^1_{\mathcal{C}_{\tilde{M}'[v]}/\tilde{M}'}(\mathcal{D}_{\tilde{M}'[v]})$ which is locally defined by

$$\begin{aligned}
\tilde{\Omega}_v &= \mathcal{O}_{\mathcal{C}_{\tilde{M}'[v]}}\frac{dz_i^{\rm reg}}{z_i^{\log}} + \mathcal{O}_{\mathcal{C}_{\tilde{M}'}}d\epsilon \qquad \text{around } (\tilde{x}_i^{\log})_{\tilde{M}'[v]},\\
\tilde{\Omega}_v &= \mathcal{O}_{\mathcal{C}_{\tilde{M}'[v]}}\frac{dz_i^{\rm un}}{(z_i^{\rm un})^{m_i^{\rm un}}} + \mathcal{O}_{\mathcal{C}_{\tilde{M}'}}\frac{d\epsilon}{(z_i^{\rm un})^{m_i^{\rm un}-1}} \qquad \text{around } (\tilde{x}_i^{\rm un})_{\tilde{M}'[v]},\\
\tilde{\Omega}_v &= \mathcal{O}_{\mathcal{C}_{\tilde{M}'[v]}}\frac{dz_i^{\rm ram}}{(z_i^{\rm ram})^{m_i^{\rm ram}}} + \mathcal{O}_{\mathcal{C}_{\tilde{M}'}}\frac{d\epsilon}{(z_i^{\rm ram})^{m_i^{\rm ram}-1}} \qquad \text{around } (\tilde{x}_i^{\rm ram})_{\tilde{M}'[v]}.
\end{aligned}\tag{10.1}$$

Definition 10.1. We say that $\big(\mathcal{E}^v,\nabla^v,l^v,\ell^v,\mathcal{V}^v\big)$ is a global horizontal lift of $\big(\tilde{E},\tilde{\nabla},\tilde{l},\tilde{\ell},\tilde{\mathcal{V}}\big)_{\tilde{M}'}$ with respect to v, if

(i) $\mathcal{E}^v$ is a vector bundle on $\mathcal{C}_{\tilde{M}'[v]}$ of rank r,

(ii) $\nabla^v\colon\mathcal{E}^v\longrightarrow\mathcal{E}^v\otimes\tilde{\Omega}_v$ is a morphism such that $\nabla^v(fa)=a\otimes\mathrm{d}f+f\nabla^v(a)$ for $f\in\mathcal{O}_{\mathcal{C}_{\tilde{M}'[v]}}$, $a\in\mathcal{E}^v$ and that the matrix $\Gamma^v=\tilde{A}(z)\mathrm{d}z+B(z)\,\mathrm{d}\epsilon$ corresponding to ∇^v with respect to a local frame $e_0,\dots,e_{r-1}$ of $\mathcal{E}^v|_{U[v]}$ defined by $(\nabla^v(e_0),\dots,\nabla^v(e_{r-1}))=(e_0,\dots,e_{r-1})\Gamma^v$ satisfies $\tilde{A}(z)\in M_r(\mathcal{O}_{U[v]}(\mathcal{D}_{\tilde{M}'[v]}\cap U))$ and $B(z)\in M_r(\mathcal{O}_U(\mathcal{D}'_{\tilde{M}'}\cap U))$

(iii) ∇^v satisfies the integrability condition $\mathrm{d}\Gamma^v+\Gamma^v\wedge\Gamma^v=0$, which means that the equality $\frac{\partial\tilde{A}}{\partial\epsilon}\mathrm{d}z\wedge\mathrm{d}\epsilon=\mathrm{d}B(z)\wedge d\epsilon+[\tilde{A}(z),B(z)]\mathrm{d}z\wedge\mathrm{d}\epsilon$ holds,

(iv) for the relative connection $\overline{\nabla^v}\colon\mathcal{E}^v\longrightarrow\mathcal{E}^v\otimes\Omega^1_{\mathcal{C}_{\tilde{M}'[v]/\tilde{M}'[v]}}\big(\mathcal{D}_{\tilde{M}'[v]}\big)$ induced by ∇^v,

(a) $l^v = (l^v_k)_{0\le k\le r-1}$ is a logarithmic λ-parabolic structure on $\big(\mathcal{E}^v, \overline{\nabla^v}\big)$ such that the subsheaf $\ker\big(\mathcal{E}^v \to \mathcal{E}^v|_{(\mathcal{D}_{\log})_{\tilde{M}'[v]}}/l^v_k\big)$ of $\mathcal{E}^v$ is preserved by ∇^v for $0 \le k \le r-1$,

(b) $\ell^v = (\ell^v_k)_{0\le k\le r-1}$ is a generic unramified $I_v^*\tilde{\mu}$-parabolic structure on $\big(\mathcal{E}^v, \overline{\nabla^v}\big)$ such that the subsheaf $\ker\big(\mathcal{E}^v \to \mathcal{E}^v|_{(\mathcal{D}_{\mathrm{un}})_{\tilde{M}'[v]}}/\ell^v_k\big)$ of $\mathcal{E}^v$ is preserved by ∇^v for $0 \le k \le r-1$,

(c) $\mathcal{V}^v = \big(V^v_k, \vartheta^v_k, \varkappa^v_k\big)_{0\le k\le r-1}$ is a factorized $I_v^*\tilde{\nu}$-ramified structure on $\big(\mathcal{E}^v, \overline{\nabla^v}\big)$ such that the subsheaf $\ker\big(\mathcal{E}^v \to \mathcal{E}^v|_{(\mathcal{D}_{\mathrm{ram}})_{\tilde{M}'[v]}}/V^v_k\big)$ of $\mathcal{E}^v$ is preserved by ∇^v,

(v) $\big(\mathcal{E}^v, \overline{\nabla^v}, l^v, \ell^v, \mathcal{V}^v\big) \otimes \mathcal{O}_{\tilde{M}'[v]}/(\epsilon) \cong \big(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}}\big)_{\tilde{M}'}$ holds.

We will prove the existence and uniqueness of the horizontal lift in the above definition. For its proof, we will show the local existence and the uniqueness of the horizontal lift.

Definition 10.2. Let U be an open subset of $\mathcal{C}_{\tilde{M}'}$ such that $\tilde{E}|_U \cong \mathcal{O}_U^{\oplus r}$ and let $U[v]$ be the open subscheme of $\mathcal{C}_{\tilde{M}'[v]}$ whose underlying set is the same as U. We say that $\big(\mathcal{E}^v_U, \nabla^v_U, l^v_U, \ell^v_U, \mathcal{V}^v_U\big)$ is a local horizontal lift of $\big(\tilde{E}|_U, \tilde{\nabla}|_U, \tilde{l}|_U, \tilde{\ell}|_U, \tilde{\mathcal{V}}|_U\big)$ with respect to v, if

(i) $\mathcal{E}^v_U$ is a vector bundle on $U[v]$ of rank r,

(ii) $\nabla^v_U \colon \mathcal{E}^v_U \longrightarrow \mathcal{E}^v_U \otimes \tilde{\Omega}_v|_{U[v]}$ is an integrable connection in the sense of Definition 10.1 (ii) and (iii),

(iii) $\big(l^v_U, \ell^v_U, \mathcal{V}^v_U\big)$ satisfies the same condition as (a), (b), (c) of Definition 10.1 and for the induced relative connection $\overline{\nabla^v}$ on $\mathcal{E}^v$,

$$\big(\mathcal{E}^v_U, \overline{\nabla^v}_U, l^v_U, \ell^v_U, \mathcal{V}^v_U\big) \otimes \mathcal{O}_{\tilde{M}'[v]}/(\epsilon) \cong \big(\tilde{E}|_U, \tilde{\nabla}|_U, \tilde{l}|_U, \tilde{\ell}|_U, \tilde{\mathcal{V}}|_U\big)$$

holds.

Lemma 10.3 (logarithmic local horizontal lift)**.** *Let U be an affine open subset of $\mathcal{C}_{\tilde{M}'}$ such that $\tilde{E}|_U \cong \mathcal{O}_U^{\oplus r}$ and that $\mathcal{D}_{\tilde{M}'} \cap U = \big(\tilde{x}_i^{\log}\big)_{\tilde{M}'} \cap U$ for some i, which is defined by the equation $z_U = 0$ for a section z of $\mathcal{O}_{\mathcal{C}_{\mathcal{T}'}}$ on a Zariski open subset of $\mathcal{C}_{\mathcal{T}'}$. Then there exists a local horizontal lift $\big(\mathcal{E}^v_U, \nabla^v_U, l^v_U\big)$ of $\big(\tilde{E}|_U, \tilde{\nabla}|_U, \tilde{l}|_U\big)$ with respect to v, which is unique up to an isomorphism.*

Proof. Note that $\big(\tilde{\ell}|_U, \tilde{\mathcal{V}}|_U\big)$ is nothing in this case. Put $\tilde{x} := \big(\tilde{x}_i^{\log}\big)_{\tilde{M}'} \cap U$. For a suitable choice of a frame $e_0, \dots, e_{r-1}$ of $\tilde{E}|_U \cong \mathcal{O}_U^{\oplus r}$, we may assume that $\tilde{l}_k \cap U$ is given by $\langle e_k|_{\tilde{x}}, \dots, e_{r-1}|_{\tilde{x}}\rangle$. With respect to the frame $e_0, \dots, e_{r-1}$ of $\tilde{E}|_U$, we can write $\tilde{\nabla}|_U = \mathrm{d} + A(z)\mathrm{d}z/z$, where $A(z)$ is a matrix with values in $\mathcal{O}_U$ such that $A(0)$ is a lower triangular matrix with the diagonal entries $\lambda_0^{(i)}, \dots, \lambda_{r-1}^{(i)}$. Take a lift $\tilde{A}(z)$ of $A(z)$ as a matrix with values in $\mathcal{O}_{U[v]}$ such that $\tilde{A}(0)$ is a lower triangular matrix with the diagonal entries $\lambda_0^{(i)}, \dots, \lambda_{r-1}^{(i)}$. After replacing $\tilde{A}(z)$, we may assume that the $d\epsilon$-coefficient of each entry of $d\tilde{A}(z)$ in $\Omega^1_{U[v]/\tilde{M}'} = \mathcal{O}_{U[v]}\mathrm{d}z \oplus \mathcal{O}_U \mathrm{d}\epsilon$ vanishes. Then $\nabla^v_U := \mathrm{d} + \tilde{A}(z)\mathrm{d}z/z$ defines an integrable connection on $\mathcal{E}^v_U := \mathcal{O}_{U[v]}^{\oplus r}$, which preserves the parabolic structure l^v_U on $\mathcal{E}^v_U$ defined by $l^v_{U,k} = \langle e_k|_{\tilde{x}_{U[v]}}, \dots, e_{r-1}|_{\tilde{x}_{U[v]}}\rangle$.

Assume that $\big(\mathcal{E}'_U, \nabla'_U, l'_U\big)$ is another local horizontal lift of $\big(\tilde{E}|_U, \tilde{\nabla}|_U, \tilde{l}|_U\big)$. Then we have $\mathcal{E}'_U \cong \mathcal{O}_{U[v]}^{\oplus r}$ and we can write $\nabla'_U = \mathrm{d} + \tilde{A}'(z)\mathrm{d}z/z + B'(z)\mathrm{d}\epsilon$. After replacing the frame $e_0, \dots, e_{r-1}$ of $\mathcal{E}'_U \cong \mathcal{O}_{U[v]}^{\oplus r}$, we may assume that l'_U is given by $l'_{U,k} = \langle e_k|_{\tilde{x}_{U[v]}}, \dots, e_{r-1}|_{\tilde{x}_{U[v]}}\rangle$. Then $\tilde{A}'(0)$ is a lower triangular matrix and $B'(0)$ is also lower triangular by the condition (a) of Definition 10.1. Since ∇'_U is a lift of $\tilde{\nabla}|_U$, we can write $\tilde{A}'(z) = \tilde{A}(z) + \epsilon C'(z)$, with $C'(0)$ a lower triangular matrix whose diagonal entries are zero. The integrability condition of ∇' yields $C'(z)\mathrm{d}z/z = \mathrm{d}B'(z) + [\tilde{A}(z), B'(z)]\mathrm{d}z/z$. Applying the transform $I_r - \epsilon B'(z)$ to the connection ∇'_U, the matrix of connection becomes

$$(I_r + \epsilon B'(z))\mathrm{d}(I_r - \epsilon B'(z)) + (I_r + \epsilon B'(z))\big((\tilde{A}(z) + \epsilon C'(z))\mathrm{d}z/z + B'(z)\mathrm{d}\epsilon\big)(I_r - \epsilon B'(z))$$

$$= \tilde{A}\mathrm{d}z/z + \epsilon\big(C'(z)\mathrm{d}z/z - \mathrm{d}B'(z) - [\tilde{A}(z), B'(z)]\mathrm{d}z/z\big) - B'(z)\mathrm{d}\epsilon + B'(z)\mathrm{d}\epsilon = \tilde{A}(z)\mathrm{d}z/z.$$

So $I_r - \epsilon B'(z)$ transforms $\big(\mathcal{E}'_U, \nabla'_U, l'_U\big)$ to $\big(\mathcal{E}^v_U, \nabla^v_U, l^v_U\big)$. The transform $I_r - \epsilon B'(z)$ also preserves the parabolic structures on the both sides. Since the transform is uniquely determined by the $\mathrm{d}\epsilon$-coefficient, we can see the uniqueness of the transform. ■

The following lemma is essentially given in [15, Theorem 6.2].

Lemma 10.4 (unramified irregular singular local horizontal lift)**.** *Let U be an affine open subset of $\mathcal{C}_{\tilde{M}'}$ such that $\tilde{E}|_U \cong \mathcal{O}_U^{\oplus r}$, $\mathcal{D}_{\tilde{M}'} \cap U = m_i^{\mathrm{un}}\big(\tilde{x}_i^{\mathrm{un}}\big)_{\tilde{M}'} \cap U$ for some i and that $\big(\tilde{x}_i^{\mathrm{un}}\big)_{\tilde{M}'} \cap U$ is defined by the equation $z_U = 0$ for a section z of $\mathcal{O}_{\mathcal{C}_{\mathcal{T}'}}$ on a Zariski open subset of $\mathcal{C}_{\mathcal{T}'}$. Then there exists a local horizontal lift $\big(\mathcal{E}^v_U, \nabla^v_U, \ell^v_U\big)$ of $\big(\tilde{E}|_U, \tilde{\nabla}|_U, \tilde{\ell}|_U\big)$ with respect to v, which is unique up to an isomorphism.*

Proof. We put $\tilde{x} := \big(\tilde{x}_i^{\mathrm{un}}\big)_{\tilde{M}'} \cap U$ and $m := m_i^{\mathrm{un}}$. Write

$$\mu_k^{(i)}(z) = \sum_{j=0}^{m-2} a_{k,j}(z)\frac{\mathrm{d}z}{z^m} + c_k\frac{\mathrm{d}z}{z}.$$

We can write

$$I_v^*(a_{k,j}) = a_{k,j} + \epsilon b_{k,j} \in \mathcal{O}_{\mathcal{T}'[v]} = \mathcal{O}_{\mathcal{T}'\times \operatorname{Spec}\mathbb{C}[\epsilon]} = \mathcal{O}_{\mathcal{T}'} \oplus \epsilon\mathcal{O}_{\mathcal{T}'}.$$

We express the above equality by

$$I_v^*\mu_k^{(i)}(z) = \mu_k^{(i)}(z) + \epsilon\mu_{k,v}^{(i)}(z), \qquad \mu_{k,v}(z) = \sum_{j=0}^{m-2} b_{k,j} z^j \frac{\mathrm{d}z}{z^m}.$$

Take a local frame $e_0, \dots, e_{r-1}$ of $\tilde{E}|_U$ such that $\tilde{\ell}_k \cap U$ is given by $\langle e_k|_{\tilde{x}}, \dots, e_{r-1}|_{\tilde{x}}\rangle$. After a suitable replacement of the frame $e_0, \dots, e_{r-1}$, we can write $\tilde{\nabla}|_U = \mathrm{d} + A(z)\mathrm{d}z/z^m$ such that $A(z)\mathrm{d}z/z^m \pmod{z^{2m-1}\mathrm{d}z/z^m}$ is the diagonal matrix with the diagonal entries $\mu_0^{(i)}, \dots, \mu_{r-1}^{(i)}$. We can take a matrix $\tilde{A}(z)$ with entries in $\mathcal{O}_{U[v]}$ which is a lift of $A(z)$ such that $\partial\tilde{A}/\partial\epsilon = 0$ and that $\tilde{A}(z)\mathrm{d}z/z^m\big|_{z^{2m-1}=0}$ is a diagonal matrix with the diagonal entries $\mu_0^{(i)}, \dots, \mu_{r-1}^{(i)}$. Set

$$B(z) := \int \begin{pmatrix} \mu_{0,v}^{(i)}(z) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mu_{r-1,v}^{(i)}(z) \end{pmatrix}, \qquad C(z)\mathrm{d}z/z^m := \mathrm{d}B + [A, B]\frac{\mathrm{d}z}{z^m}.$$

Then $\nabla^v_U := \mathrm{d} + (\tilde{A}(z) + \epsilon C(z))\mathrm{d}z/z^m + B\mathrm{d}\epsilon$ defines an integrable connection on $\mathcal{E}^v_U = \mathcal{O}_{U[v]}^{\oplus r}$. By construction, the connection ∇^v_U preserves the parabolic structure ℓ^v_U on $\mathcal{E}^v_U$ induced by $e_0, \dots, e_{r-1}$. So we can see the existence of the local horizontal lift $\big(\mathcal{E}^v, \nabla^v, \ell^v\big)$.

Assume that $\big(\mathcal{E}'_U, \nabla'_U, \ell'_U\big)$ is another local horizontal lift of $\big(\tilde{E}|_U, \tilde{\nabla}|_U, \tilde{\ell}|_U\big)$. Note that $\mathcal{E}'_U \cong \mathcal{O}_{U[v]}^{\oplus r}$. So we may write $\nabla'_U = \mathrm{d} + (\tilde{A}(z) + \epsilon C'(z))\mathrm{d}z/z^m + B'(z)\mathrm{d}\epsilon$ with $C'(z) \equiv C(z) \pmod{z^m}$. The integrability condition

$$C'(z)\mathrm{d}z/z^m := \mathrm{d}B' + [A, B']\frac{\mathrm{d}z}{z^m} \tag{10.2}$$

yields $\big[A, z^{m-1}B'\big] \equiv 0 \pmod{z^{m-1}}$. Since $A(z)|_{z^{2m-1}=0}$ is a diagonal matrix whose constant term $A(0)$ has distinct eigenvalues, we can see that $z^{m-1}B'|_{z^{m-1}=0}$ is also a diagonal matrix. Looking at (10.2) again and using $C(z) \equiv C'(z) \pmod{z^m}$, we can see that $B'(z)|_{z^{2m-1}=0}$ is also a diagonal matrix with the diagonal entries $\mu^{(i)}_{0,v},\dots,\mu^{(i)}_{r-1,v}$. So $B(z)-B'(z)$ is a matrix of regular functions on U, whose constant term is diagonal. We can see by the same calculation as in the proof of Lemma 10.3 that the automorphism $I_r+\epsilon(B-B')$ transforms ∇'_U to ∇^v_U and it also preserves the parabolic structures on the both sides. We can see that such an automorphism is unique, because it is determined by the $d\epsilon$-coefficient of ∇'_U. ∎

Lemma 10.5 (existence of ramified irregular singular local horizontal lift)**.** *Let U be an affine open subset of $\mathcal{C}_{\tilde{M}'}$ such that $\tilde{E}|_U\cong\mathcal{O}_U^{\oplus r}$, $\mathcal{D}_{\tilde{M}'}\cap U=m_i^{\mathrm{ram}}\big(\tilde{x}_i^{\mathrm{ram}}\big)_{\tilde{M}'}\cap U$ for some i and that $\big(\tilde{x}_i^{\mathrm{ram}}\big)_{\tilde{M}'}\cap U$ is defined by the equation $z_U=0$ for a section z of $\mathcal{O}_{\mathcal{C}_{\mathcal{T}'}}$ on a Zariski open subset of $\mathcal{C}_{\mathcal{T}'}$. Then there exists a local horizontal lift $\big(\mathcal{E}^v_U,\nabla^v_U,\mathcal{V}^v_U\big)$ of $\big(\tilde{E}|_U,\tilde{\nabla}|_U,\mathcal{V}|_U\big)$ with respect to v.*

Proof. Write $\tilde{x}=\big(\tilde{x}_i^{\mathrm{ram}}\big)_{\tilde{M}'}\cap U$ and $m=m_i^{\mathrm{ram}}$. We denote the pullback of ν via the trivial first projection $\mathcal{T}'[v]\longrightarrow\mathcal{T}'\hookrightarrow\mathcal{T}$ by the same symbol ν. As in the proof of Lemma 10.4, we express

$$I_v^*\nu(w)=\nu(w)+\epsilon\nu_v(w),\qquad \nu(w)=\sum_{k=0}^{r-1}\sum_{j=0}^{m-1}a_{k,j}z^jw^k\frac{\mathrm{d}z}{z^m},\qquad \nu_v(w)=\sum_{k=0}^{r-1}\sum_{j=0}^{m-2}b_{k,j}z^jw^k\frac{\mathrm{d}z}{z^m},$$

where $a_{1,0}\in\mathcal{O}^\times_{\tilde{M}'}$ and $a_{k,m-1}=0$ for $1\le k\le r-1$.

We choose a local frame $e_0,\dots,e_{r-1}$ of $\tilde{E}|_U$ whose restriction to $(2m-1)\tilde{x}$ corresponds to $1,w,\dots,w^{r-1}$ via the isomorphism $\tilde{E}|_{(2m-1)\tilde{x}}\cong\mathcal{O}_{\tilde{M}'}[w]/\big(w^{(2m-1)r}\big)$ given by Proposition 4.1 in the case $q=2m-1$. Let

$$N\colon\ \tilde{E}|_U\longrightarrow\tilde{E}|_U$$

be the homomorphism defined by the representation matrix

$$\begin{pmatrix}0&0&\cdots&0&z\\1&0&\cdots&0&0\\\vdots&\ddots&\ddots&\vdots&\vdots\\0&\cdots&1&0&0\\0&\cdots&0&1&0\end{pmatrix}\tag{10.3}$$

with respect to the basis $e_0,\dots,e_{r-1}$ of $\tilde{E}|_U$. As in the proof of Theorem 7.5, we can construct homomorphisms $\theta\colon\tilde{E}_\alpha|^\vee_{m\tilde{x}}\longrightarrow\tilde{E}_\alpha|_{m\tilde{x}}$ and $\kappa\colon E_\alpha|_{m\tilde{x}}\longrightarrow E_\alpha|^\vee_{m\tilde{x}}$ which satisfy ${}^t\theta=\theta$, ${}^t\kappa=\kappa$ and $N|_{m\tilde{x}}=\theta\circ\kappa$. We may assume that $(\tilde{\theta}_k)$ and $(\tilde{\kappa}_k)$ are induced by θ and κ, respectively.

Write $\tilde{\nabla}|_{\bar{U}_\alpha}=\mathrm{d}+A(z)\frac{\mathrm{d}z}{z^m}$ with respect to the frame $e_0,\dots,e_{r-1}$ of $\tilde{E}|_U\cong\mathcal{O}_U^{\oplus r}$. Since $\big(\tilde{E}|_{(2m-1)\tilde{x}},\tilde{\nabla}|_{(2m-1)\tilde{x}}\big)\cong\big(\mathcal{O}_{\tilde{M}'}[w]/\big(w^{(2m-1)r}\big),\nabla_\nu\big)$ as in Proposition 4.1, we can write

$$A(z)=\sum_{k=0}^{r-1}\sum_{l=0}^{m-1}a_{k,l}z^lN^k+z^{m-1}R_r+z^{2m-1}A'(z)\tag{10.4}$$

for some matrix $A'(z)$ of regular functions, where we are putting

$$R_r:=\begin{pmatrix}0&0&\cdots&0\\0&\frac{1}{r}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\frac{r-1}{r}\end{pmatrix}.\tag{10.5}$$

Set $\mathcal{E}^v_U := \mathcal{O}^{\oplus r}_{U[v]}$ with the identification $\mathcal{E}^v_U \otimes \mathcal{O}_{U[v]}/(\epsilon) = \tilde{E}|_U$. Define the $\mathcal{O}_{U[v]}$-homomorphism

$$\tilde{N}\colon\ \mathcal{E}^v_U \longrightarrow \mathcal{E}^v_U$$

by the same matrix (10.3) as N. Then $\big(\mathcal{E}^v_U, \tilde{N}\big)$ becomes a lift of $\big(\tilde{E}|_U, N\big)$. Define matrices $\tilde{A}(z)$, $B(z)$, $C(z)$ by setting

$$\begin{aligned}
\tilde{A}(z) &:= \sum_{k=0}^{r-1}\sum_{l=0}^{m-1} a_{k,l} z^l \tilde{N}^k + z^{m-1}\tilde{R}_r + z^{2m-1}\tilde{A}'(z),\\
B(z) &:= \sum_{k=0}^{r-1}\sum_{l=0}^{m-2} \frac{r b_{k,l}}{(-mr+lr+r+k)z^{m-l-1}}\tilde{N}^k,\\
C(z)\frac{\mathrm{d}z}{z^m} &:= \mathrm{d}B(z) + [A(z),B(z)]\frac{\mathrm{d}z}{z^m},
\end{aligned}$$

where $\tilde{R}_r$ is the endomorphisms of $\mathcal{E}^v_U$ whose representation matrix with respect to the basis $e_0,\dots,e_{r-1}$ is the same as that of R_r in (10.5) and $\tilde{A}'(z)$ is a lift of $A'(z)$ such that $\frac{\partial \tilde{A}'(z)}{\partial\epsilon}=0$. Using the calculations

$$\tilde{N}^k = \begin{pmatrix}
0 & \cdots & 0 & z & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & 0 & \cdots & z\\
1 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 1 & 0 & \cdots & 0
\end{pmatrix}, \qquad
\big[\tilde{R}_r, \tilde{N}^k\big] = \begin{pmatrix}
0 & \cdots & 0 & -\frac{r-k}{r}z & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & 0 & \cdots & -\frac{r-k}{r}z\\
\frac{k}{r} & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & \frac{k}{r} & 0 & \cdots & 0
\end{pmatrix},$$

we can check the equality

$$\mathrm{d}\tilde{N}^k + \left[\tilde{R}_r\frac{\mathrm{d}z}{z}, \tilde{N}^k\right] = \frac{k}{r}\tilde{N}^k\frac{\mathrm{d}z}{z}. \tag{10.6}$$

Then we can see

$$\begin{aligned}
&\left(\mathrm{d}B(z) + [A(z),B(z)]\frac{\mathrm{d}z}{z^m}\right)\Big|_{(2m-1)\tilde{x}}\\
&= \sum_{k=0}^{r-1}\sum_{l=0}^{m-2}\frac{r b_{k,l}}{-mr+lr+r+k}\left(\mathrm{d}\left(\frac{1}{z^{m-l-1}}\tilde{N}^k\right) + \left[\tilde{R}_r\frac{\mathrm{d}z}{z}, \frac{1}{z^{m-l-1}}\tilde{N}^k\right]\right)\Big|_{(2m-1)\tilde{x}}\\
&\qquad= \sum_{k=0}^{r-1}\sum_{l=0}^{m_i-2}\frac{r b_{k,l}}{-mr+lr+r+k}\left(\frac{-m+l+1}{z^{m-l}}\tilde{N}^k\mathrm{d}z + \frac{1}{z^{m-l-1}}\frac{k}{r}\tilde{N}^k\frac{\mathrm{d}z}{z}\right)\Big|_{(2m-1)\tilde{x}}\\
&\qquad= \sum_{k=0}^{r-1}\sum_{l=0}^{m-2}\frac{r(-m+l+1)+k}{-mr+lr+r+k}\frac{b_{k,l}}{z^{m-l}}\tilde{N}^k\mathrm{d}z\Big|_{(2m-1)\tilde{x}} = \nu_v(\tilde{N})\big|_{(2m-1)\tilde{x}}.
\end{aligned}$$

So the matrix

$$\big(\tilde{A}(z)+\epsilon C(z)\big)\frac{\mathrm{d}z}{z^m} + B(z)\mathrm{d}\epsilon$$

determines an integrable connection $\nabla^{\mathrm{flat}}_{U[v]}\colon \mathcal{O}^{\oplus r}_{U[v]} \longrightarrow \mathcal{O}^{\oplus r}_{U[v]} \otimes_{\mathcal{O}_{U[v]}} \tilde{\Omega}_v$ such that the induced relative connection $\overline{\nabla^{\mathrm{flat}}_{U[v]}}\colon \mathcal{O}^{\oplus r}_{U[v]} \longrightarrow \mathcal{O}^{\oplus r}_{U[v]} \otimes \Omega^1_{U[v]/\mathcal{T}'[v]}(\mathcal{D}_{\mathcal{T}'}\cap U)$ satisfies

$$\overline{\nabla^{\mathrm{flat}}_{U[v]}}\big|_{(2m-1)\tilde{x}} = I_v^*\nu\big(\tilde{N}\big)\big|_{(2m-1)\tilde{x}}.$$

We can give a filtration $\mathcal{O}_{U[v]}^{\oplus r}\big|_{\mathcal{D}_{\tilde{M}'[v]}} = V^v_{U,0} \supset V^v_{U,1} \supset \cdots \supset V^v_{U,r-1} \supset V^v_{U,r} = zV^v_{U,0}$ by setting $V^v_{U,k} := \operatorname{Im}\big(\tilde{N}^k\big|_{\mathcal{D}_{\tilde{M}'[v]}}\big)$ for $k=0,1,\dots,r$. So we can see that $\{V^v_{U,k}\}$ induces $\overline{V}^v_{U,k}$, $\overline{W}^v_{U,k}$ and that the homomorphism $\overline{\tilde{N}|_{V^v_{U,k}}}\colon \overline{V}^v_{U,k} \longrightarrow \overline{V}^v_{U,k}$ induced by the restriction $\tilde{N}|_{V^v_{U,k}}$ has a factorization

$$\overline{\tilde{N}|_{V^v_{U,k}}}\colon\ \overline{V}^v_{U,k} \xrightarrow{\kappa^v_{U,k}} \overline{W}^v_{U,k} \xrightarrow[\sim]{\theta^v_{U,k}} \overline{V}^v_{U,k}.$$

Then $\big(V^v_{U,k},\theta^v_{U,k},\kappa^v_{U,k}\big)$ induces a factorized ramified structure $\big(V^v_{U,k},\vartheta^v_{U,k},\varkappa^v_{U,k}\big)$ on $\big(\mathcal{E}^v_U,\overline{\nabla^{\mathrm{flat}}_{U[v]}}\big)$, where $\overline{\nabla^{\mathrm{flat}}_{U[v]}}$ is the relative connection induced by $\nabla^{\mathrm{flat}}_{U[v]}$. Thus $\big(\mathcal{E}^v_U,,\nabla^{\mathrm{flat}}_{U[v]},\{V^v_{U,k},\vartheta^v_{U,k},\varkappa^v_{U,k}\}\big)$ becomes a local horizontal lift of $\big(\tilde{E},\tilde{\nabla},\{\tilde{V}_k,\tilde{\vartheta}_k,\tilde{\varkappa}_k\}\big)\big|_U$. ■

Lemma 10.6 (uniqueness of ramified irregular singular local horizontal lift)**.** *Under the same assumption as Lemma 10.5, a local horizontal lift $\big(\mathcal{E}^v_U,\nabla^v_U,\mathcal{V}^v_U\big)$ of $\big(\tilde{E}|_U,\tilde{\nabla}|_U,\mathcal{V}|_U\big)$ with respect to v is unique up to an isomorphism.*

Proof. Let $\big(\mathcal{E}^v_U,\nabla^{\mathrm{flat}}_{U[v]},\{V^v_{U,k},\vartheta^v_{U,k},\varkappa^v_{U,k}\}\big)$ be the local horizontal lift constructed in Lemma 10.5. Take another local horizontal lift $\big(\mathcal{O}^{\oplus r}_{U_\alpha[v]},\nabla',\{V'_k,\vartheta'_k,\varkappa'_k\}\big)$ of $\big(\tilde{E},\tilde{\nabla},\{\tilde{V}_k,\tilde{\vartheta}_k,\tilde{\varkappa}_k\}\big)\big|_U$. The connection $\nabla'\colon \mathcal{O}^{\oplus r}_{U[v]} \longrightarrow \mathcal{O}^{\oplus r}_{U[v]}\otimes\Omega^1_{U[v]/M'}(m(\tilde{x})_{M'})$ can be given by

$$\nabla'\begin{pmatrix} f_1\\ \vdots\\ f_r\end{pmatrix} = \begin{pmatrix} \mathrm{d}f_1\\ \vdots\\ \mathrm{d}f_r\end{pmatrix} + \left(\big(\tilde{A}(z)+\epsilon C'(z)\big)\frac{\mathrm{d}z}{z^m} + B'(z)\mathrm{d}\epsilon\right)\begin{pmatrix} f_1\\ \vdots\\ f_r\end{pmatrix}$$

with $B'(z)$ a rational function on U admitting a pole at $z=0$ of order at most $m-1$. Note that ∇' satisfies the integrability condition

$$C'(z)\frac{\mathrm{d}z}{z^m} = \mathrm{d}B'(z) + [A(z),B'(z)]\frac{\mathrm{d}z}{z^m}. \tag{10.7}$$

Now we apply Proposition 4.1 in the case $q=2m-1$ to the relative connection $\overline{\nabla'}$ on $\mathcal{O}^{\oplus r}_{U[v]}$ induced by ∇'. Then, after applying an automorphism of $\mathcal{O}^{\oplus r}_{U[v]}$ of the form $I_r+\epsilon h$, we may assume that

$$C'(z)\frac{\mathrm{d}z}{z^m}\Big|_{(2m-1)\tilde{x}} = \nu_v\big(\tilde{N}\big)\big|_{(2m-1)\tilde{x}}, \tag{10.8}$$

$V'_k = \operatorname{Im}\big(\tilde{N}^k|_{m\tilde{x}}\big)$ and that $\theta'_k\circ\kappa'_k$ is induced by the restriction $\tilde{N}|_{V'_k}$ for $0\le k\le r-1$.

By the equality (10.7), we have $\big[A(z),z^{m-1}B'(z)\big]\equiv 0 \pmod{z^{m-1}}$. Note that $A(z)$ satisfies the equality (10.4) with $a_{1,0}\in\mathcal{O}_{\mathcal{T}}^{\times}$ and $B'(z)\in M_r(\mathcal{O}_U(\mathcal{D}'_{\tilde{M}'}\cap U))$ by the condition (ii) of Definition 10.1. So we can find $c_0(z),\dots,c_{r-1}(z)\in\mathcal{O}_{U[v]}$ satisfying

$$z^{m-1}B'(z) \equiv \sum_{k=0}^{r-1} c_k(z)\tilde{N}^k \pmod{z^{m-1}\operatorname{End}\big(\mathcal{O}^{\oplus r}_{U_\alpha[v]}\big)}$$

since the equality $\ker\big(\operatorname{ad}\big(\tilde{N}|_{z=0}\big)\big) = \mathcal{O}_{U[v]}\big[\tilde{N}|_{z=0}\big]$ holds. Then we can write

$$B'(z) = \sum_{k=0}^{r-1}\frac{c_k(z)}{z^{m-1}}\tilde{N}^k + B_m(z).$$

with $B_m(z)$ a matrix of regular functions. Furthermore, we can see that $B_m(0)$ is a lower triangular matrix, since $\nabla^v = \mathrm{d} + \big(\tilde{A}(z) + \epsilon C(z)\big)\frac{\mathrm{d}z}{z^m} + B'(z)\mathrm{d}\epsilon$ preserves the filtration (V^v_k). Looking at the equality (10.7) again, we can see that

$$\begin{aligned}
&C'(z)\mathrm{d}z - z^m\left(\mathrm{d}B'(z) + [R_r, B'(z)]\frac{\mathrm{d}z}{z}\right) \\
&\quad = \big[\tilde{A}(z) - z^{m-1}R_r, B'(z)\big]\mathrm{d}z \\
&\quad = \left[\sum_{k=0}^{r-1}\sum_{l=0}^{m-1} a_{k,l}z^l\tilde{N}^k + z^{2m-1}\tilde{A}', \sum_{k=0}^{r-1}\frac{c_k(z)}{z^{m-1}}\tilde{N}^k + B_m(z)\right]\mathrm{d}z \\
&\quad \equiv \sum_{k=0}^{r-1}\left[z^m c_k(z)\tilde{A}'(z) - \sum_{l=0}^{m-1} a_{k,l}z^l B_m(z), \tilde{N}^k\right]\mathrm{d}z \in \mathrm{ad}\big(\tilde{N}\big)\mathrm{d}z \quad \big(\mathrm{mod}\ z^{2m-1}\mathrm{d}z\big).
\end{aligned}$$

In particular, we have

$$\mathrm{Tr}\left(\tilde{N}^l\left(C'(z)\mathrm{d}z - z^m\left(\mathrm{d}B'(z) + [R_r, B'(z)]\frac{\mathrm{d}z}{z}\right)\right)\right) \equiv 0 \quad \big(\mathrm{mod}\ z^{2m-1}\mathrm{d}z\big) \tag{10.9}$$

for $0 \le l \le r-1$. Since

$$\begin{aligned}
&z^m\left(\mathrm{d}B'(z) + [R_r, B'(z)]\frac{\mathrm{d}z}{z}\right) \\
&\quad \equiv \sum_{k=0}^{r-1} z^m\left(\mathrm{d}\left(\frac{c_k(z)}{z^{m-1}}\right) + \frac{kc_k(z)}{rz^m}\mathrm{d}z\right)\tilde{N}^k + z^{m-1}[R_r, B_m(z)]\mathrm{d}z \quad \big(\mathrm{mod}\ z^m\mathrm{d}z\big)
\end{aligned}$$

and since $[R_r, B_m]|_{z=0}$ is lower triangular nilpotent matrix, we can see that the condition (10.9) implies

$$\begin{aligned}
&z^m\left(\mathrm{d}\left(\frac{c_0(z)}{z^{m-1}}\right) - \nu_{0,v}(z)\right) \equiv 0 \quad (\mathrm{mod}\ z^m), \\
&z^{m+1}\left(\mathrm{d}\left(\frac{c_{r-l}(z)}{z^{m-1}}\right) + \frac{(r-l)c_{r-l}(z)}{rz^m}\mathrm{d}z - \nu_{r-l,v}(z)\right) \equiv 0 \quad (\mathrm{mod}\ z^m) \quad (1 \le l \le r-1).
\end{aligned}$$

In other words, we have

$$\mathrm{d}\left(\frac{c_0(z)}{z^{m-1}}\right)\Big|_{m\tilde{x}} = \nu_{0,v}(z)\big|_{m\tilde{x}}, \qquad \left(\mathrm{d}\left(\frac{c_k(z)}{z^{m-1}}\right) + \frac{kc_k(z)}{rz^m}\mathrm{d}z\right)\Big|_{(m-1)\tilde{x}} = \nu_{k,v}(z)\big|_{(m-1)\tilde{x}}$$

for $1 \le k \le r-1$, which implies that

$$c_k(z) \equiv \sum_{l=0}^{m-2}\frac{rb_{k,l}}{-mr + lr + r + k}z^l \quad \big(\mathrm{mod}\ z^{m-1}\big).$$

Thus

$$Q(z) := B(z) - B'(z)$$

becomes a matrix of regular functions. Furthermore, (10.7) and (10.8) implies the equality

$$\mathrm{d}Q(z) + [A(z), Q(z)]\frac{\mathrm{d}z}{z^m} \equiv 0 \quad \big(\mathrm{mod}\ z^{2m-1}\mathrm{d}z/z^m\big),$$

from which we can see $Q(z)|_{m\tilde{x}} \in \mathcal{O}_{m\tilde{x}}[\tilde{N}]$. If we apply the transform $I_r + \epsilon Q(z)$ to the connection ∇', then the consequent connection has the matrix form

$$
\begin{aligned}
&(I_r + \epsilon Q(z))^{-1}\mathrm{d}(I_r + \epsilon Q(z)) + (I_r + \epsilon Q(z))^{-1}\left(\big(\tilde{A}(z) + \epsilon C'(z)\big)\frac{\mathrm{d}z}{z^m} + B'(z)\mathrm{d}\epsilon\right)(I_r + \epsilon Q(z)) \\
&\quad = (I_r - \epsilon Q(z))(\epsilon \mathrm{d}Q(z) + Q(z)\mathrm{d}\epsilon) + \tilde{A}(z)\frac{\mathrm{d}z}{z^m} + \epsilon([A(z), Q(z)] + C'(z))\frac{\mathrm{d}z}{z^m} + B'(z)\mathrm{d}\epsilon \\
&\quad = \tilde{A}(z)\frac{\mathrm{d}z}{z^m} + \epsilon\left(\mathrm{d}B(z) - \mathrm{d}B'(z) + \big([A(z), B(z) - B'(z)] + C'(z)\big)\frac{\mathrm{d}z}{z^m}\right) \\
&\qquad + (Q(z) + B'(z))\mathrm{d}\epsilon \\
&\quad = \tilde{A}(z)\frac{\mathrm{d}z}{z^m} + \epsilon\left(\mathrm{d}B(z) + [A(z), B(z)]\frac{\mathrm{d}z}{z^m} - \mathrm{d}B'(z) - [A(z), B'(z)]\frac{\mathrm{d}z}{z^m} + C'(z)\frac{\mathrm{d}z}{z^m}\right) \\
&\qquad + B(z))\mathrm{d}\epsilon \\
&\quad = \big(\tilde{A}(z) + \epsilon C(z)\big)\frac{\mathrm{d}z}{z^m} + B(z)\mathrm{d}\epsilon,
\end{aligned}
$$

which means that $\big(\mathcal{O}^{\oplus r}_{U_\alpha[v]}, \nabla'\big)$ is isomorphic to $\big(\mathcal{O}^{\oplus r}_{U_\alpha[v]}, \nabla^{\mathrm{flat}}_{U_\alpha[v]}\big)$ via $I_r + \epsilon Q(z)$. Since $Q(z)|_{m\tilde{x}}$ belongs to $\mathcal{O}_{m\tilde{x}}[\tilde{N}|_{m\tilde{x}}]$, we can see that $I_r + \epsilon Q(z)$ induces an isomorphism which transforms $\big(\mathcal{O}^{\oplus r}_{U[v]}, \nabla', \{V'_k, \vartheta'_k, \varkappa'_k\}\big)$ to $\big(\mathcal{O}^{\oplus r}_{U[v]}, \nabla^{\mathrm{flat}}_{U[v]}, \{V^v_k, \vartheta^v_k, \varkappa^v_k\}\big)$. We can see that such an isomorphism is unique, because it is determined by the coefficient of $d\epsilon$. ■

Proposition 10.7. *For any vector field $v \in H^0(\mathcal{T}', T_{\mathcal{T}'})$, there is a unique global horizontal lift $\big(\mathcal{E}^v, \nabla^v, l^v, \ell^v, \mathcal{V}^v\big)$ of $\big(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}}\big)_{\tilde{M}'}$.*

Proof. We take an affine open covering $\mathcal{C}_{\tilde{M}'} = \bigcup_\alpha U_\alpha$ such that $\tilde{E}|_{U_\alpha} \cong \mathcal{O}^{\oplus r}_{U_\alpha}$ for each α. We may assume that $\#\{\alpha \mid U_\alpha \supset \tilde{x}\} = 1$ for each $\tilde{x} = \big(\tilde{x}^{\log}_i\big)_{\tilde{M}'}$, $\tilde{x} = \big(\tilde{x}^{\mathrm{un}}_i\big)_{\tilde{M}'}$ and $\tilde{x} = \big(\tilde{x}^{\mathrm{ram}}_i\big)_{\tilde{M}'}$. We may further assume that, for each α, $U_\alpha \cap \mathcal{D}_{\tilde{M}'} = \varnothing$ holds or $U_\alpha \cap \mathcal{D}_{\tilde{M}'} = \tilde{x}$ holds for some $\tilde{x} = \big(\tilde{x}^{\log}_i\big)_{\tilde{M}'}$, $\tilde{x} = \big(\tilde{x}^{\mathrm{un}}_i\big)_{\tilde{M}'}$ or $\tilde{x} = \big(\tilde{x}^{\mathrm{ram}}_i\big)_{\tilde{M}'}$.

Let $U_\alpha[v]$ be the open subscheme of $\mathcal{C}_{\tilde{M}'[v]}$ whose underlying set is U_α. If $U_\alpha \cap \mathcal{D}_{\tilde{M}'} = \varnothing$, then we can write $\tilde{\nabla}|_{U_\alpha} = \mathrm{d} + A_\alpha(z)\mathrm{d}z$ for a matrix A_α with values in $\mathcal{O}_{U_\alpha}$. We can take a matrix $\tilde{A}_\alpha$ with values in $\mathcal{O}_{U_\alpha}[v]$ which is a lift of A_α. After adding an element of $\epsilon M_r(\mathcal{O}_{U_\alpha})$, we can assume that $\partial\tilde{A}_\alpha/\partial\epsilon = 0$. Then $\nabla_\alpha = \mathrm{d} + \tilde{A}_\alpha \mathrm{d}z$ is an integrable connection and $\big(\mathcal{O}^{\oplus r}_{U_\alpha[v]}, \nabla_\alpha\big)$ is a local horizontal lift of $\big(\tilde{E}|_{U_\alpha}, \tilde{\nabla}|_{U_\alpha}\big)$. Furthermore, we can prove the uniqueness of the local horizontal lift by the same proof as Lemma 10.3.

If α satisfies $U_\alpha \cap \mathcal{D}_{\tilde{M}'} = \tilde{x}$ for some $\tilde{x} = \big(\tilde{x}^{\log}_i\big)_{\tilde{M}'}$, $\tilde{x} = \big(\tilde{x}^{\mathrm{un}}_i\big)_{\tilde{M}'}$ or $\tilde{x} = \big(\tilde{x}^{\mathrm{ram}}_i\big)_{\tilde{M}'}$, we can take a local horizontal lift $\big(\mathcal{E}^v_{U_\alpha}, \nabla^v_{U_\alpha}, l^v_{U_\alpha}, \ell^v_{U_\alpha}, \mathcal{V}^v_{U_\alpha}\big)$ of $\big(\tilde{E}|_{U_\alpha}, \tilde{\nabla}|_{U_\alpha}, \tilde{l}|_{U_\alpha}, \tilde{\ell}|_{U_\alpha}, \tilde{\mathcal{V}}|_{U_\alpha}\big)$ by Lemmas 10.3, 10.4 and 10.5. Since the local horizontal lifts are unique up to unique isomorphisms, we can patch them and get a global horizontal lift $\big(\mathcal{E}^v, \nabla^v, l^v, \ell^v, \mathcal{V}^v\big)$ of $\big(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}}\big)_{\tilde{M}'}$, which is unique up to an isomorphism. ■

For a Zariski open subset $\mathcal{T}' \subset \mathcal{T}$, consider a morphism

$$u\colon\ \operatorname{Spec}\mathcal{O}_{\mathcal{T}'}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) \longrightarrow \mathcal{T}'$$

such that $u|_{\mathcal{T}'} = \mathrm{id}_{\mathcal{T}'}$. Let

$$\bar{u}\colon\ \operatorname{Spec}\mathcal{O}_{\mathcal{T}'}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2\big) \longrightarrow \mathcal{T}'$$

be the induced morphism which corresponds to a pair (u_1, u_2) of vector fields. We write

$$\mathcal{T}'[\bar{u}] := \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2\big), \qquad \mathcal{T}'[u] := \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big)$$

with the structure morphisms $\mathcal{T}'[\bar{u}] \xrightarrow{\bar{u}} \mathcal{T}'$ and $\mathcal{T}'[u] \xrightarrow{u} \mathcal{T}'$, respectively. We further set

$$\begin{aligned} \tilde{M}'[\bar{u}] &:= \tilde{M}' \times_{\mathcal{T}'} \mathcal{T}'[\bar{u}], & \mathcal{C}_{\mathcal{M}'[\bar{u}]} &:= \mathcal{C} \times_{\mathcal{T}'} \mathcal{M}'[\bar{u}], \\ \tilde{M}'[u] &:= \tilde{M}' \times_{\mathcal{T}'} \mathcal{T}'[u], & \mathcal{C}_{\mathcal{M}'[u]} &:= \mathcal{C} \times_{\mathcal{T}'} \mathcal{M}'[u]. \end{aligned}$$

We define a coherent subsheaf $\tilde{\Omega}_u$ of $\Omega^1_{\mathcal{C}_{\tilde{M}'[u]}/\tilde{M}'}(\mathcal{D}_{\tilde{M}'[u]})$ in the same way as in (10.1) and define a coherent subsheaf $\tilde{\Omega}_{\bar{u}}$ of $\Omega^1_{\mathcal{C}_{\tilde{M}'[\bar{u}]}/\tilde{M}'}(\mathcal{D}_{\tilde{M}'[\bar{u}]})$ similarly.

Definition 10.8. We say that $(\mathcal{E}^u, \nabla^u, l^u, \ell^u, \mathcal{V}^u)$ (resp. $(\mathcal{E}^{\bar{u}}, \nabla^{\bar{u}}, l^{\bar{u}}, \ell^{\bar{u}}, \mathcal{V}^{\bar{u}})$) is a horizontal lift of $(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}})_{\tilde{M}'}$ with respect to u (resp. $\bar{u}$) if the conditions (i), (ii), (iii), (iv) and (v) of Definition 10.1 are satisfied after replacing $\tilde{M}'[v]$ with $\tilde{M}'[u]$ (resp. $\tilde{M}'[\bar{u}]$), replacing $\tilde{\Omega}_v$ with $\tilde{\Omega}_u$ (resp. $\tilde{\Omega}_{\bar{u}}$), replacing $(\lambda, I_v^*\tilde{\mu}, I_v^*\tilde{\nu})$-structure in (vi) with $(\lambda, I_u^*\tilde{\mu}, I_u^*\tilde{\nu})$-structure (resp. $(\lambda, I_{\bar{u}}^*\tilde{\mu}, I_{\bar{u}}^*\tilde{\nu})$-structure) and replacing the equality of integrability condition in (iii) with

$$\begin{aligned} &\frac{\partial A}{\partial \epsilon_1} \mathrm{d}z \wedge \mathrm{d}\epsilon_1 + \frac{\partial A}{\partial \epsilon_2} \mathrm{d}z \wedge \mathrm{d}\epsilon_2 + \frac{\partial B_1}{\partial \epsilon_2} \mathrm{d}\epsilon_2 \wedge \mathrm{d}\epsilon_1 + \frac{\partial B_2}{\partial \epsilon_1} \mathrm{d}\epsilon_1 \wedge \mathrm{d}\epsilon_2 \\ &\qquad = \mathrm{d}B_1 \wedge \mathrm{d}\epsilon_1 + [A, B_1] \mathrm{d}z \wedge \mathrm{d}\epsilon_1 + \mathrm{d}B_2 \wedge \mathrm{d}\epsilon_1 + [A, B_2] \mathrm{d}z \wedge d\epsilon_2 + [B_1, B_2] \mathrm{d}\epsilon_1 \wedge \mathrm{d}\epsilon_2 \end{aligned}$$

for $\Gamma^u = A\mathrm{d}z + B_1\mathrm{d}\epsilon_1 + B_2\mathrm{d}\epsilon_2$ (resp. replacing with

$$\frac{\partial A}{\partial \epsilon_1} \mathrm{d}z \wedge \mathrm{d}\epsilon_1 + \frac{\partial A}{\partial \epsilon_2} \mathrm{d}z \wedge \mathrm{d}\epsilon_2 = \mathrm{d}B_1 \wedge d\epsilon_1 + [A, B_1] \mathrm{d}z \wedge \mathrm{d}\epsilon_1 + \mathrm{d}B_2 \wedge \mathrm{d}\epsilon_1 + [A, B_2] \mathrm{d}z \wedge \mathrm{d}\epsilon_2$$

for $\Gamma^{\bar{u}} = Adz + B_1\mathrm{d}\epsilon_1 + B_2\mathrm{d}\epsilon_2$).

The following proposition can be proved in the same way as Proposition 10.7. So we omit its proof.

Proposition 10.9. *There exists a unique horizontal lift* $(\mathcal{E}^{\bar{u}}, \nabla^{\bar{u}}, l^{\bar{u}}, \ell^{\bar{u}}, \mathcal{V}^{\bar{u}})$ *of* $(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}})_{\tilde{M}'}$ *with respect to* $\bar{u}\colon \mathcal{T}'[\bar{u}] = \operatorname{Spec} \mathcal{O}_{\mathcal{T}'}[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2) \longrightarrow \mathcal{T}'$.

If a horizontal lift $(\mathcal{E}^u, \nabla^u, l^u, \ell^u, \mathcal{V}^u)$ of $(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}})_{\tilde{M}'}$ with respect to u exists, it can be obtained as a lift of $(\mathcal{E}^{\bar{u}}, \nabla^{\bar{u}}, l^{\bar{u}}, \ell^{\bar{u}}, \mathcal{V}^{\bar{u}})$ whose existence is ensured by Proposition 10.9.

Proposition 10.10. *There exists a unique horizontal lift* $(\mathcal{E}^u, \nabla^u, l^u, \ell^u, \mathcal{V}^u)$ *of* $(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}})_{\tilde{M}'}$ *with respect to* $u\colon \mathcal{T}'[u] = \operatorname{Spec} \mathcal{O}_{\mathcal{T}'}[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2) \longrightarrow \mathcal{T}'$.

Proof. By Proposition 10.9, there is a unique horizontal lift $(\mathcal{E}^{\bar{u}}, \nabla^{\bar{u}}, l^{\bar{u}}, \ell^{\bar{u}}, \mathcal{V}^{\bar{u}})$ of $(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}})_{\tilde{M}'}$ with respect to $\bar{u}\colon \operatorname{Spec} \mathcal{O}_{\mathcal{T}'}[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2) \to \mathcal{T}'$. So we only have to show the existence and the uniqueness of a lift of $(\mathcal{E}^{\bar{u}}, \nabla^{\bar{u}}, l^{\bar{u}}, \ell^{\bar{u}}, \mathcal{V}^{\bar{u}})$, which is a horizontal lift of $(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}})_{\tilde{M}'}$ with respect to the morphism $u\colon \mathcal{T}'[u] = \operatorname{Spec} \mathcal{O}_{\mathcal{T}'}[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2) \longrightarrow \mathcal{T}'$. The method of the proof is similar to that of Proposition 10.7.

We take an affine open covering $\mathcal{C} \times_{\mathcal{T}} \tilde{M}' = \bigcup U_\alpha$ as in the proof of Proposition 10.7. If U_α is an open neighborhood of $(\tilde{x}_i^{\mathrm{un}})_{\tilde{M}'}$, then the existence and the uniqueness of the local horizontal lift with respect to u is given in the proof of [12, Lemma 5.5]. If U_α is an open neighborhood of $(\tilde{x}_i^{\log})_{\tilde{M}'}$, then it is much easier to prove the existence and the uniqueness of a logarithmic local horizontal lift.

So assume that $\tilde{x} := (\tilde{x}_i^{\mathrm{ram}})_{\tilde{M}'}$ is contained in U_α. If u is given by

$$\begin{aligned} u^*(\nu(w)) &= \nu(w) + \epsilon_1\nu_{u_1}(w) + \epsilon_2\nu_{u_2}(w) + \epsilon_1\epsilon_2\nu_{u_{12}}(w) \\ &= \sum_{k=0}^{r-1} \left(a_{k,m-1} z^{m-1} + \sum_{l=0}^{m-2} (a_{k,l} + \epsilon_1 b_{1,k,l} + \epsilon_2 b_{2,k,l} + \epsilon_1\epsilon_2 b_{1,2,k,l}) z^l \right) w^k, \end{aligned}$$

then, by the proof of Proposition 10.7, the restriction of $\nabla^{\bar{u}}$ to $U_\alpha[\bar{u}] = U_\alpha[u] \otimes \mathcal{O}_{\mathcal{T}'[u]/(\epsilon_1\epsilon_2)}$ can be given by

$$A(z)\frac{\mathrm{d}z}{z^m} + \epsilon_1 C_1(z)\frac{\mathrm{d}z}{z^m} + \epsilon_2 C_2(z)\frac{\mathrm{d}z}{z^m} + B_1(z)\mathrm{d}\epsilon_1 + B_2(z)\mathrm{d}\epsilon_2$$

where $\frac{\partial A(z)}{\partial \epsilon_1} = \frac{\partial A(z)}{\partial \epsilon_2} = 0$ and

$$\begin{aligned}
A(z) &= \sum_{k=0}^{r-1}\sum_{l=0}^{m-1} a_{k,l} z^l \tilde{N}^k + z^{m-1} R_r + z^{3m-1} A'(z), \qquad (10.10)\\
B_1(z) &= \sum_{k=0}^{r-1}\sum_{l=0}^{m-2} \frac{r b_{1,k,l}}{(-mr+lr+r+k) z^{m-l-1}} \tilde{N}^k,\\
B_2(z) &= \sum_{k=0}^{r-1}\sum_{l=0}^{m-2} \frac{r b^{(i)}_{2,k,l}}{(-mr+lr+r+k) z^{m-l-1}} \tilde{N}^k,\\
C_1(z)\frac{\mathrm{d}z}{z^m} &= \mathrm{d}B_1(z) + [A(z), B_1(z)]\frac{\mathrm{d}z}{z^m}, \qquad C_2(z)\frac{\mathrm{d}z}{z^m} = \mathrm{d}B_2(z) + [A(z), B_2(z)]\frac{\mathrm{d}z}{z^m}.
\end{aligned}$$

Then we can see by the above equality that

$$C_j(z)|_{2m\tilde{x}} = \sum_{k=0}^{r-1}\sum_{l=0}^{m-2} b_{j,k,l} z^l \tilde{N}^k|_{2m\tilde{x}}$$

for $j = 1, 2$. So we have $[C_1(z), B_2(z)], [C_2(z), B_1(z)] \in z^{m+1}\operatorname{End}(\tilde{E}|_{U_\alpha})$.

Claim 10.11. $[C_1(z), B_2(z)] = [C_2(z), B_1(z)]$.

Proof. First notice that we can check the equality

$$\mathrm{d}B_1(z) = \sum_{k=0}^{r-1}\sum_{l=0}^{m-2} \frac{r(-m+l+1)+k}{-mr+lr+r+k}\frac{b_{1,k,l}}{z^{m-l}}\tilde{N}^k \mathrm{d}z + [B_1(z), R_r]\,\frac{\mathrm{d}z}{z}$$

using (10.6). So we have

$$\begin{aligned}
[\mathrm{d}B_1(z), B_2(z)] &= [[B_1(z), R_r], B_2(z)]\,\frac{\mathrm{d}z}{z}\\
&= [[B_2(z), R_r], B_1(z)]\,\frac{\mathrm{d}z}{z} + [[B_1(z), B_2(z)], R_r]\,\frac{\mathrm{d}z}{z}\\
&= [\mathrm{d}B_2(z), B_1(z)],
\end{aligned}$$

because $[B_1(z), B_2(z)] = 0$. Thus we have

$$\begin{aligned}
\left[C_1(z)\frac{\mathrm{d}z}{z^m}, B_2(z)\right] &= \left[\mathrm{d}B_1(z) + [A(z), B_1(z)]\frac{\mathrm{d}z}{z^m}, B_2(z)\right]\\
&= [\mathrm{d}B_1(z), B_2(z)] + [[A(z), B_1(z)], B_2(z)]\frac{\mathrm{d}z}{z^m}\\
&= [\mathrm{d}B_2(z), B_1(z)] + [[A(z), B_2(z)], B_1(z)]\frac{\mathrm{d}z}{z^m} = \left[C_2(z)\frac{\mathrm{d}z}{z^m}, B_1(z)\right]. \qquad \blacksquare
\end{aligned}$$

We put

$$\tilde{A}(z) := \sum_{k=0}^{r-1}\sum_{l=0}^{m-1} a_{k,l} z^l \tilde{N}^k + z^{m-1} R_r + z^{3m-1} \tilde{A}'(z),$$

$$B_{1,2}(z) := \sum_{k=0}^{r-1}\sum_{l=0}^{m-2} \frac{rb_{1,2,k,l}}{(-mr+lr+r+k)z^{m-l-1}}\tilde{N}^k,$$
$$C_{1,2}(z)\frac{\mathrm{d}z}{z^m} := \mathrm{d}B_{1,2}(z) + ([A(z),B_{1,2}(z)]+[C_1(z),B_2(z)])\,\frac{\mathrm{d}z}{z^m},$$

where $\tilde{A}'(z)$ is a lift of $A'(z)$ as a matrix with coefficients in $\mathcal{O}_{U_\alpha[u]}$ such that $\frac{\partial\tilde{A}(z)}{\partial\epsilon_1} = \frac{\partial\tilde{A}(z)}{\partial\epsilon_2} = 0$. Define a connection $\nabla^u_\alpha\colon \mathcal{O}^{\oplus r}_{U_\alpha[u]} \longrightarrow \mathcal{O}^{\oplus r}_{U_\alpha[u]} \otimes \Omega^1_{\mathcal{C}_{\tilde{M}'[u]/\tilde{M}'}}(\mathcal{D}_{\tilde{M}'[u]})$ by setting

$$\nabla^u_\alpha = \mathrm{d} + \big(\tilde{A}+\epsilon_1C_1+\epsilon_2C_2+\epsilon_1\epsilon_2C_{1,2}\big)\frac{\mathrm{d}z}{z^m} + B_1\mathrm{d}\epsilon_1 + B_2\mathrm{d}\epsilon_2 + B_{1,2}(\epsilon_1\mathrm{d}\epsilon_2+\epsilon_2\mathrm{d}\epsilon_1).$$

Then ∇^u_α is an integrable connection because its curvature form becomes

$$\begin{aligned}
&(C_1+\epsilon_2C_{1,2})\mathrm{d}\epsilon_1\wedge\frac{\mathrm{d}z}{z^m} + (C_2+\epsilon_1C_{1,2})\mathrm{d}\epsilon_2\wedge\frac{\mathrm{d}z}{z^m} + (\mathrm{d}B_1+\epsilon_2\mathrm{d}B_{1,2})\wedge\mathrm{d}\epsilon_1\\
&\quad + (\mathrm{d}B_2+\epsilon_1\mathrm{d}B_{1,2})\wedge\mathrm{d}\epsilon_2 + B_{1,2}\mathrm{d}\epsilon_1\wedge\mathrm{d}\epsilon_2 + B_{1,2}\mathrm{d}\epsilon_2\wedge\mathrm{d}\epsilon_1\\
&\quad + [A,B_1+\epsilon_2B_{1,2}]\frac{\mathrm{d}z}{z^{m_i}}\wedge\mathrm{d}\epsilon_1 + [A,B_2+\epsilon_1B_{1,2}]\frac{\mathrm{d}z}{z^{m_i}}\wedge\mathrm{d}\epsilon_2 + \epsilon_2[C_2,B_1]\frac{\mathrm{d}z}{z^m}\wedge\mathrm{d}\epsilon_1\\
&\quad + \epsilon_1[C_1,B_2]\frac{\mathrm{d}z}{z^m}\wedge\mathrm{d}\epsilon_2 + [B_1,B_2]\mathrm{d}\epsilon_1\wedge\mathrm{d}\epsilon_2\\
&= \left(\mathrm{d}B_1 + (-C_1+[A,B_1])\frac{\mathrm{d}z}{z^m}\right)\wedge\mathrm{d}\epsilon_1 + \left(\mathrm{d}B_2+(-C_2+[A,B_2])\frac{\mathrm{d}z}{z^m}\right)\wedge\mathrm{d}\epsilon_2\\
&\quad + \epsilon_2\left(\mathrm{d}B_{1,2} + (-C_{1,2}+[A,B_{1,2}]+[C_2,B_1])\frac{\mathrm{d}z}{z^m}\right)\wedge\mathrm{d}\epsilon_1\\
&\quad + \epsilon_1\left(\mathrm{d}B_{1,2} + (-C_{1,2}+[A,B_{1,2}]+[C_1,B_2])\frac{\mathrm{d}z}{z^m}\right)\wedge\mathrm{d}\epsilon_2 = 0.
\end{aligned}$$

We can define $V^u_{k,\alpha}$, $\vartheta^u_{k,\alpha}$, $\varkappa^u_{k,\alpha}$ on $\mathcal{O}^{\oplus r}_{\mathcal{D}[u]}$ in the same way as in the proof of Lemma 10.5. So we can get a local horizontal lift $\big(\mathcal{O}^{\oplus r}_{U_\alpha[u]}, \nabla^u_\alpha, (V^u_{k,\alpha}, \vartheta^u_{k,\alpha}, \varkappa^u_{k,\alpha})_{0\le k\le r-1}\big)$, which is a lift of the restriction $\big(\mathcal{E}^{\bar{u}}, \nabla^{\bar{u}}, l^{\bar{u}}, \ell^{\bar{u}}, \mathcal{V}^{\bar{u}}\big)|_{U_\alpha[\bar{u}]}$.

Let $\big(\mathcal{O}^{\oplus r}_{U_\alpha[u]}, \nabla'_\alpha, (V'_{k,\alpha}, \vartheta'_{k,\alpha}, \varkappa'_{k,\alpha})_{0\le k\le r-1}\big)$ be another local horizontal lift with respect to u, which is a lift of $\big(\mathcal{E}^{\bar{u}}, \nabla^{\bar{u}}, l^{\bar{u}}, \ell^{\bar{u}}, \mathcal{V}^{\bar{u}}\big)|_{U_\alpha[\bar{u}]}$. Then we can write

$$\nabla'_\alpha = \mathrm{d} + \big(\tilde{A}+\epsilon_1C_1+\epsilon_2C_2+\epsilon_1\epsilon_2C'_{1,2}\big)\frac{\mathrm{d}z}{z^m} + B_1\mathrm{d}\epsilon_1 + B_2\mathrm{d}\epsilon_2 + B'_{1,2}\epsilon_1\mathrm{d}\epsilon_2 + B'_{2,1}\epsilon_2\mathrm{d}\epsilon_1.$$

The integrability condition of ∇' implies the equalities

$$\begin{aligned}
C'_{1,2}(z)\frac{\mathrm{d}z}{z^m} &= \mathrm{d}B'_{1,2}(z) + \big([A(z),B'_{1,2}(z)]+[C_1(z),B_2(z)]\big)\frac{\mathrm{d}z}{z^m}\\
&= \mathrm{d}B'_{2,1}(z) + \big([A(z),B'_{2,1}(z)]+[C_1(z),B_2(z)]\big)\frac{\mathrm{d}z}{z^m}
\end{aligned} \tag{10.11}$$

and $B'_{1,2} = B'_{2,1}$. Since ∇' has the property of local horizontal lift, we have

$$C'_{1,2}(z)\big|_{m\tilde{x}} = \sum_{k=0}^{r-1}\sum_{l=0}^{m-2} b_{1,2,k,l}z^l\big(\tilde{N}\big)^k\big|_{m\tilde{x}}.$$

We can see that $[A(z),B'_{1,2}(z)]$ is regular from the equality (10.11). Since $A(z)$ satisfies (10.10), we can first verify $z^{m-1}B'_{1,2}(z) \in \mathcal{O}_{(m-1)\tilde{x}}\big[\tilde{N}|_{(m-1)\tilde{x}}\big]$. Combining with the condition (c) of Definition 10.1 (iv), we can take $\beta_{k,l}$ such that

$$B'_{1,2}(z) - \sum_{k=0}^{r-1}\sum_{l=0}^{m-1}\beta_{k,l}z^{l-m-1}\tilde{N}^k \in \operatorname{End}\big(\tilde{E}|_{U_\alpha}\big).$$

is a matrix of regular functions whose constant term is a lower triangular matrix. Using the same argument as in the proof of Lemma 10.6, we can see

$$\beta_{k,l} = \frac{rb_{1,2,k,l}}{-mr+l+r+k}.$$

So $B_{12}(z) - B'_{12}(z)$ becomes a matrix of regular functions and $I_r + \epsilon_1\epsilon_2(B_{12}(z) - B'_{12}(z))$ gives an automorphism of $\mathcal{O}^{\oplus r}_{U_\alpha[u]}$ which transform ∇'_α to ∇^u_α and which sends $V'_{k,\alpha}$ to $V^u_{k,\alpha}$. Furthermore, we can see that such a transform is uniquely determined by the coefficient of $\epsilon_2\, d\epsilon_1$. Thus the existence and the uniqueness of a ramified local horizontal lift with respect to u is proved.

Patching the local horizontal lifts together, we get a unique horizontal lift $\left(\mathcal{E}^{\bar{u}}, \nabla^{\bar{u}}, l^{\bar{u}}, \ell^{\bar{u}}, \mathcal{V}^{\bar{u}}\right)$ of $\left(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}}\right)_{\tilde{M}'}$ on $\mathcal{C} \times_{\mathcal{T}} \operatorname{Spec} \mathcal{O}_{\tilde{M}'}[\epsilon_1, \epsilon_2]/\left(\epsilon_1^2, \epsilon_2^2\right)$ with respect to u. ■

11 Global generalized isomonodromic deformation

Definition 11.1. For each vector field $v \in T_{\mathcal{T}'}$, the relative connection $\left(\mathcal{E}^v, \overline{\nabla^v}, l^v, \ell^v, \mathcal{V}^v\right)$ induced by the global horizontal lift $\left(\mathcal{E}^v, \nabla^v, l^v, \ell^v, \mathcal{V}^v\right)$ (which exists by Proposition 10.7) defines a morphism

$$I_{\Phi(v)} \colon\ \tilde{M}' \times \operatorname{Spec} \mathbb{C}[\epsilon]/\left(\epsilon^2\right) \longrightarrow \tilde{M}'$$

which makes the diagram

$$\begin{array}{ccc} \tilde{M}' \times \operatorname{Spec} \mathbb{C}[\epsilon]/\left(\epsilon^2\right) & \xrightarrow{I_{\Phi(v)}} & \tilde{M}' \\ {\scriptstyle \pi_{\mathcal{T}'} \times \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathcal{T}'}} \\ \mathcal{T}' \times \operatorname{Spec} \mathbb{C}[\epsilon] & \xrightarrow{I_v} & \mathcal{T}' \end{array} \tag{11.1}$$

commutative. We can see by the uniqueness of the horizontal lift that the morphism $I_{\Phi(v)}$ descends to a morphism $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})_{\mathcal{T}'} \times \mathbb{C}[\epsilon] \longrightarrow M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})_{\mathcal{T}'}$ which corresponds to a vector field

$$\Phi(v) \in H^0\left(M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})_{\mathcal{T}'}, T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})_{\mathcal{T}'}}\right).$$

We call this vector field $\Phi(v)$ a generalized isomonodromic vector field.

Proposition 11.2. *The map*

$$\Phi\colon\ H^0(\mathcal{T}', T_{\mathcal{T}'}) \ni v \mapsto \Phi(v) \in H^0\left(M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})_{\mathcal{T}'}, T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda, \tilde{\mu}, \tilde{\nu})_{\mathcal{T}'}}\right)$$

is a homomorphism of $H^0(\mathcal{T}', \mathcal{O}_{\mathcal{T}'})$-modules.

Proof. Take vector fields $v_1, v_2 \in H^0(\mathcal{T}', T_{\mathcal{T}'})$. Then (v_1, v_2) corresponds to a morphism

$$\bar{u}\colon\ \mathcal{T}' \times \operatorname{Spec} \mathbb{C}[\epsilon_1, \epsilon_2]/\left(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2\right) \longrightarrow \mathcal{T}'$$

such that the composition $\mathcal{T}' \times \operatorname{Spec} \mathbb{C}[\epsilon_i]/\left(\epsilon_i^2\right) \hookrightarrow \mathcal{T}' \times \operatorname{Spec} \mathbb{C}[\epsilon_1, \epsilon_2]/\left(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2\right) \xrightarrow{\bar{u}} \mathcal{T}'$ coincides with the morphism I_{v_i} for $i = 1, 2$. Let

$$\Delta_{\mathcal{T}'}\colon\ \mathcal{T}' \times \operatorname{Spec} \mathbb{C}[\epsilon]/\left(\epsilon^2\right) \longrightarrow \mathcal{T}' \times \operatorname{Spec} \mathbb{C}[\epsilon_1, \epsilon_2]/\left(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2\right)$$

be the morphism corresponding to the ring homomorphism

$$\mathcal{O}_{\mathcal{T}'}[\epsilon_1, \epsilon_2]/\left(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2\right) \ni a + b_1\epsilon_1 + b_2\epsilon_2 \mapsto a + b_1\epsilon + b_2\epsilon \in \mathcal{O}_{\mathcal{T}'}[\epsilon]/\left(\epsilon^2\right).$$

Then the composition

$$\bar{u}\circ\Delta_{\mathcal{T}'}\colon\ \mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)\xrightarrow{\Delta_{\mathcal{T}'}}\mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon_1,\epsilon_2]/(\epsilon_1^2,\epsilon_1\epsilon_2,\epsilon_2^2)\xrightarrow{\bar{u}}\mathcal{T}'$$

coincides with the morphism $I_{v_1+v_2}$ corresponding to the vector field v_1+v_2. By virtue of Proposition 10.9, there exists a horizontal lift $\big(\mathcal{E}^{\bar{u}},\nabla^{\bar{u}},l^{\bar{u}},\ell^{\bar{u}},\mathcal{V}^{\bar{u}}\big)$ of $\big(\tilde{E},\tilde{\nabla},\tilde{l},\tilde{\ell},\tilde{\mathcal{V}}\big)_{\tilde{M}'}$ with respect to $\bar{u}$. By the same procedure as Definition 11.1, the flat family of connections induced by the horizontal lift $\big(\mathcal{E}^{\bar{u}},\nabla^{\bar{u}},l^{\bar{u}},\ell^{\bar{u}},\mathcal{V}^{\bar{u}}\big)$ provides a morphism $I_{\Phi(\bar{u})}\colon\tilde{M}'\times\mathbb{C}[\epsilon_1,\epsilon_2]/(\epsilon_1^2,\epsilon_1\epsilon_2,\epsilon_2^2)\longrightarrow\tilde{M}'$ such that the right square of the diagram

$$\begin{tikzcd}
\tilde{M}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \arrow[r, "\Delta_{\tilde{M}'}"] \arrow[d] & \tilde{M}'\times\operatorname{Spec}\mathbb{C}[\epsilon_1,\epsilon_2]/(\epsilon_1^2,\epsilon_1\epsilon_2,\epsilon_2^2) \arrow[r, "I_{\Phi(\bar{u})}"] \arrow[d] & \tilde{M}' \arrow[d] \\
\mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \arrow[r, "\Delta_{\mathcal{T}'}"] & \mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon_1,\epsilon_2]/(\epsilon_1^2,\epsilon_1\epsilon_2,\epsilon_2^2) \arrow[r, "\bar{u}"] & \mathcal{T}'
\end{tikzcd}$$

is commutative. The left square of the above diagram is defined as a Cartesian diagram. By the definition of horizontal lift, the pullback $\Delta_{\tilde{M}'}^*\big(\mathcal{E}^{\bar{u}},\nabla^{\bar{u}},l^{\bar{u}},\ell^{\bar{u}},\mathcal{V}^{\bar{u}}\big)$ is a horizontal lift of $\big(\tilde{E},\tilde{\nabla},\tilde{l},\tilde{\ell},\tilde{\mathcal{V}}\big)_{\tilde{M}'}$ with respect to $I_{v_1+v_2}$. So the composition $I_{\Phi(\bar{u})}\circ\Delta_{\tilde{M}'}$ coincides with the morphism $I_{\Phi(v_1+v_2)}$ determined by the vector field $\Phi(v_1+v_2)$. On the other hand, the morphism $I_{\Phi(\bar{u})}$ corresponds to the pair $(\Phi(v_1),\Phi(v_2))$ of vector fields and the composition $I_{\Phi(\bar{u})}\circ\Delta_{\tilde{M}'}$ corresponds to the vector field $\Phi(v_1)+\Phi(v_2)$. So we have the equality

$$I_{\Phi(v_1+v_2)}=I_{\Phi(v_1)+\Phi(v_2)}$$

which means the equality $\Phi(v_1+v_2)=\Phi(v_1)+\Phi(v_2)$.

Take a vector field $v\in H^0(\mathcal{T}',T_{\mathcal{T}'})$ and a regular function $f\in H^0(\mathcal{T}',\mathcal{O}_{\mathcal{T}'})$. Consider the morphism

$$\alpha_f\colon\ \mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)\longrightarrow\mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$$

corresponding to the ring homomorphism

$$\mathcal{O}_{\mathcal{T}'}[\epsilon]/(\epsilon^2)\ni a+\epsilon b\mapsto a+\epsilon fb\in\mathcal{O}_{\mathcal{T}'}[\epsilon]/(\epsilon^2).$$

Then the composition

$$\mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)\xrightarrow{\alpha_f}\mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)\xrightarrow{I_v}\mathcal{T}'$$

coincides with the morphism I_{fv} corresponding to the vector field fv. As in Definition 11.1, the horizontal lift $\big(\mathcal{E}^v,\nabla^v,l^v,\ell^v,\mathcal{V}^v\big)$ induces a morphism $I_{\Phi(v)}\colon\tilde{M}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)\longrightarrow\tilde{M}'$ which makes the diagram

$$\begin{tikzcd}
\tilde{M}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \arrow[r, "(\alpha_f)_{\tilde{M}'}"] \arrow[d] & \tilde{M}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \arrow[r, "I_{\Phi(v)}"] \arrow[d] & \tilde{M}' \arrow[d] \\
\mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \arrow[r, "\alpha_f"] & \mathcal{T}'\times\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \arrow[r, "I_v"] & \mathcal{T}'
\end{tikzcd}$$

commutative, where the right square is Cartesian. By the definition of horizontal lift, the pullback $(\alpha_f)_{\tilde{M}'}^*\big(\mathcal{E}^v,\nabla^v,l^v,\ell^v,\mathcal{V}^v\big)$ is a horizontal lift of $\big(\tilde{E},\tilde{\nabla},\tilde{l},\tilde{\ell},\tilde{\mathcal{V}}\big)_{\tilde{M}'}$ with respect to fv. So the composition $I_{\Phi(v)}\circ(\alpha_f)_{\tilde{M}'}$ coincides with the morphism $I_{\Phi(fv)}$ corresponding to $\Phi(fv)$. On the other hand, the composition $I_{\Phi(v)}\circ(\alpha_f)_{\tilde{M}'}$ coincides with the morphism $I_{f\Phi(v)}$ corresponding to the vector field $f\Phi(v)$. So we have $\Phi(fv)=f\Phi(v)$. ■

By Proposition 11.2, Φ defines a homomorphism

$$\Phi\colon\ T_{\mathcal{T}} \longrightarrow (\pi_{\mathcal{T}})_* T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$$

of sheaves of $\mathcal{O}_{\mathcal{T}}$-modules. By the adjoint property, Φ corresponds to a homomorphism

$$\Psi\colon\ (\pi_{\mathcal{T}})^* T_{\mathcal{T}} \longrightarrow T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})} \tag{11.2}$$

Since the diagram (11.1) in Definition 11.1 is commutative, we can see $\mathrm{d}\pi_{\mathcal{T}} \circ \Psi = \mathrm{id}_{T_{\mathcal{T}}}$ for the canonical surjection $\mathrm{d}\pi_{\mathcal{T}}\colon T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})} \longrightarrow (\pi_{\mathcal{T}})^* T_{\mathcal{T}}$. In particular, the image $\operatorname{Im}\Psi$ is a subbundle of $T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$.

Definition 11.3. We call $\operatorname{Im}\Psi$ the generalized isomonodromic subbundle of $T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$.

By using the generalized isomonodromic subbundle $\operatorname{Im}\Psi$, we can extend the relative symplectic form $\omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ constructed in Theorem 8.1 to a total 2-form on the moduli space $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$ in the following.

Definition 11.4. We define a 2-form $\omega^{\mathrm{GIM}}_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ on $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$ by setting

$$\omega^{\mathrm{GIM}}_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}(v_1,v_2) = \omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}\big(v_1 - \Psi(\mathrm{d}\pi_{\mathcal{T}}(v_1)), v_2 - \Psi(\mathrm{d}\pi_{\mathcal{T}}(v_2))\big)$$

for $v_1, v_2 \in T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ and call it the generalized isomonodromic 2-form.

Remark 11.5. In the logarithmic case, the above formulation of isomonodromic 2-form is given by A. Komyo in [18]. For a vector field $v \in T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ we can immediately see the equivalence

$$v \in \operatorname{Im}\Psi \Leftrightarrow \omega^{\mathrm{GIM}}_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}(v,w) = 0 \qquad \text{for any } w \in T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$$

from the definition of the generalized isomonodromic 2-form. So the generalized isomonodromic 2-form recovers the generalized isomonodromic subbundle.

Theorem 11.6. *For any vector fields $v_1, v_2 \in T_{\mathcal{T}}$, the equality*

$$\Phi([v_1,v_2]) = [\Phi(v_1), \Phi(v_2)]$$

holds, where $[v_1,v_2] = v_1v_2 - v_2v_1$ is the commutator of the vector fields v_1, v_2. In particular, the generalized isomonodromic subbundle $\operatorname{Im}\Psi$ of $T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ satisfies the integrability condition

$$[\operatorname{Im}\Psi, \operatorname{Im}\Psi] \subset \operatorname{Im}\Psi.$$

Proof. Take vector fields $v_1, v_2 \in H^0(\mathcal{T}', T_{\mathcal{T}}|_{\mathcal{T}'})$ over a Zariski open subset $\mathcal{T}'$ of $\mathcal{T}$. Let

$$\tilde{I}_{v_1}\colon\ \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1,\epsilon_2]/\big(\epsilon_1^2,\epsilon_2^2\big) \longrightarrow \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1,\epsilon_2]/\big(\epsilon_1^2,\epsilon_2^2\big)$$

be the automorphism corresponding to the ring automorphism $\tilde{I}^*_{v_1}$ of $\mathcal{O}_{\mathcal{T}'}[\epsilon_1,\epsilon_2]/\big(\epsilon_1^2,\epsilon_2^2\big)$ defined by

$$\tilde{I}^*_{v_1}\,(a + b_1\epsilon_1 + b_2\epsilon_2 + c\,\epsilon_1\epsilon_2) = a + (v_1(a) + b_1)\epsilon_1 + b_2\epsilon_2 + (v_1(b_2) + c)\epsilon_1\epsilon_2.$$

Similarly, we can define an automorphism $\tilde{I}_{v_2}$ of $\mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1,\epsilon_2]/\big(\epsilon_1^2,\epsilon_2^2\big)$ corresponding to v_2. By construction, we can see that $\tilde{I}_{-v_1} = \tilde{I}^{-1}_{v_1}$ and $\tilde{I}_{-v_2} = \tilde{I}^{-1}_{v_2}$. The composition $\tilde{I}_{v_2} \circ \tilde{I}_{v_1} \circ \tilde{I}_{-v_2} \circ \tilde{I}_{-v_1}$ corresponds to the ring automorphism of $\mathcal{O}_{\mathcal{T}'}[\epsilon_1,\epsilon_2]/\big(\epsilon_1^2,\epsilon_2^2\big)$ determined by

$$\tilde{I}^*_{-v_1} \circ \tilde{I}^*_{-v_2} \circ \tilde{I}^*_{v_1} \circ \tilde{I}^*_{v_2}(a + b_1\epsilon_1 + b_2\epsilon_2 + c\,\epsilon_1\epsilon_2)$$

$$
\begin{aligned}
&= \tilde{I}^*_{-v_1} \circ \tilde{I}^*_{-v_2} \circ \tilde{I}^*_{v_1}(a + b_1\epsilon_1 + (v_2(a) + b_2)\epsilon_2 + (c + v_2(b_1))\epsilon_1\epsilon_2) \\
&= \tilde{I}^*_{-v_1} \circ \tilde{I}^*_{-v_2}(a + (v_1(a) + b_1)\epsilon_1 + (v_2(a) + b_2)\epsilon_2 + (v_1v_2(a) + v_1(b_2) + c + v_2(b_1))\epsilon_1\epsilon_2) \\
&= \tilde{I}^*_{-v_1}(a + (v_1(a) + b_1)\epsilon_1 + b_2\epsilon_2 + (-v_2v_1(a) + v_1v_2(a) + v_1(b_2) + c)\epsilon_1\epsilon_2) \\
&= a + b_1\epsilon_1 + b_2\epsilon_2 + ((v_1v_2 - v_2v_1)(a) + c)\epsilon_1\epsilon_2.
\end{aligned}
$$

Let

$$
\rho\colon\ \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) \longrightarrow \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon]/\big(\epsilon^2\big)
$$

be the morphism corresponding to the ring homomorphism $\rho^*\colon \mathcal{O}_{\mathcal{T}'}[\epsilon]/\big(\epsilon^2\big) \to \mathcal{O}_{\mathcal{T}'}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big)$ determined by $\rho^*(a + c\epsilon) = a + c\epsilon_1\epsilon_2$. Then the composition

$$
\mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) \xrightarrow{\rho} \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon]/\big(\epsilon^2\big) \xrightarrow{I_{v_1v_2 - v_2v_1}} \mathcal{T}' \tag{11.3}
$$

coincides with the composition

$$
\begin{aligned}
\mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) &\xrightarrow{\tilde{I}_{v_2}\circ\tilde{I}_{v_1}\circ\tilde{I}^{-1}_{v_2}\circ\tilde{I}^{-1}_{v_1}} \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) \\
&\xrightarrow{\text{trivial projection}} \mathcal{T}'.
\end{aligned} \tag{11.4}
$$

By Proposition 10.9, there exists a horizontal lift $\big(\mathcal{E}^{\tilde{v}_i}, \nabla^{\tilde{v}_i}, l^{\tilde{v}_i}, \ell^{\tilde{v}_i}, \mathcal{V}^{\tilde{v}_i}\big)$ of $\big(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}}\big)_{\tilde{M}'}$ with respect to the morphism

$$
\mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) \xrightarrow{\tilde{I}_{v_i}} \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) \xrightarrow{\text{trivial projection}} \mathcal{T}'.
$$

For the relative connection $\overline{\nabla^{\tilde{v}_i}}$ induced by $\nabla^{\tilde{v}_i}$, the flat family $\big(\mathcal{E}^{\tilde{v}_i}, \overline{\nabla^{\tilde{v}_i}}, l^{\tilde{v}_i}, \ell^{\tilde{v}_i}, \mathcal{V}^{\tilde{v}_i}\big)$ determines a morphism $I_{\Phi(\tilde{v}_i)}\colon \tilde{M}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) \longrightarrow \tilde{M}'$ which is canonically extended to a morphism

$$
\tilde{I}_{\Phi(\tilde{v}_i)}\colon\ \tilde{M}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) \longrightarrow \tilde{M}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big)
$$

over $\operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big)$. Furthermore, the diagram

$$
\begin{array}{ccc}
\tilde{M}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) & \xrightarrow{\tilde{I}_{\Phi(\tilde{v}_2)}\circ\tilde{I}_{\Phi(\tilde{v}_1)}\circ\tilde{I}^{-1}_{\Phi(\tilde{v}_2)}\circ\tilde{I}^{-1}_{\Phi(\tilde{v}_1)}} & \tilde{M}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) \\
\downarrow & & \downarrow \\
\mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) & \xrightarrow{\tilde{I}_{v_2}\circ\tilde{I}_{v_1}\circ\tilde{I}^{-1}_{v_2}\circ\tilde{I}^{-1}_{v_1}} & \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big)
\end{array}
$$

is commutative.

By the definition of horizontal lift, we can see that the pullback

$$
\big(\tilde{I}^{-1}_{\Phi(\tilde{v}_1)}\big)^*\big(\tilde{I}^{-1}_{\Phi(\tilde{v}_2)}\big)^*\tilde{I}^*_{\Phi(\tilde{v}_1)}\big(\mathcal{E}^{\tilde{v}_2}, \nabla^{\tilde{v}_2}, l^{\tilde{v}_2}, \ell^{\tilde{v}_2}, \mathcal{V}^{\tilde{v}_2}\big)
$$

becomes a horizontal lift of $\big(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}}\big)_{\tilde{M}'}$ with respect to the morphism (11.4). On the other hand, there is a canonical commutative diagram

$$
\begin{array}{ccccc}
\tilde{M}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) & \xrightarrow{\rho_{\tilde{M}'}} & \tilde{M}' \times \operatorname{Spec}\mathbb{C}[\epsilon]/\big(\epsilon^2\big) & \xrightarrow{I_{\Phi(v_1v_2 - v_2v_1)}} & \tilde{M}' \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\big(\epsilon_1^2, \epsilon_2^2\big) & \xrightarrow{\rho} & \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon]/\big(\epsilon^2\big) & \xrightarrow{I_{v_1v_2 - v_2v_1}} & \mathcal{T}',
\end{array}
$$

whose left square is Cartesian. So we can see that the pullback

$$\rho^*_{\tilde{M}'}\big(\mathcal{E}^{v_1v_2-v_2v_1}, \nabla^{v_1v_2-v_2v_1}, l^{v_1v_2-v_2v_1}, \ell^{v_1v_2-v_2v_1}, \mathcal{V}^{v_1v_2-v_2v_1}\big)$$

becomes a horizontal lift of $\big(\tilde{E}, \tilde{\nabla}, \tilde{l}, \tilde{\ell}, \tilde{\mathcal{V}}\big)_{\tilde{M}'}$ with respect to the morphism (11.3). Since the morphism (11.4) coincides with the morphism (11.3), we can deduce an isomorphism

$$\begin{aligned}&\big(\tilde{I}^{-1}_{\Phi(\tilde{v}_1)}\big)^*\big(\tilde{I}^{-1}_{\Phi(\tilde{v}_2)}\big)^*\tilde{I}^*_{\Phi(\tilde{v}_1)}\big(\mathcal{E}^{\tilde{v}_2}, \nabla^{\tilde{v}_2}, l^{\tilde{v}_2}, \ell^{\tilde{v}_2}, \mathcal{V}^{\tilde{v}_2}\big)\\ &\qquad\cong \rho^*_{\tilde{M}'}\big(\mathcal{E}^{v_1v_2-v_2v_1}, \nabla^{v_1v_2-v_2v_1}, l^{v_1v_2-v_2v_1}, \ell^{v_1v_2-v_2v_1}, \mathcal{V}^{v_1v_2-v_2v_1}\big)\end{aligned}$$

by the uniqueness of horizontal lift proved in Proposition 10.9. Considering the induced morphism, we have

$$(\text{trivial projection}) \circ \tilde{I}_{\Phi(\tilde{v}_2)\circ\tilde{I}_{\Phi(\tilde{v}_1)}\circ\tilde{I}^{-1}_{\Phi(\tilde{v}_2)}\circ\tilde{I}^{-1}_{\Phi(\tilde{v}_1)}} = I_{\Phi(v_1v_2-v_2v_1)} \circ \rho_{\tilde{M}'},$$

from which we get $\Phi(v_1v_2 - v_2v_1) = \Phi(v_1)\Phi(v_2) - \Phi(v_2)\Phi(v_1)$. ∎

Definition 11.7. Since the subbundle $\operatorname{Im}\Psi \subset T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ satisfies the integrability condition by Theorem 11.6, it determines a foliation $\mathcal{F}^{\mathrm{GIM}}_{T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}}$ on the moduli space $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$. We call it the generalized isomonodromic foliation.

Take a point $t_0 \in \mathcal{T}$ and a point y of the fiber $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_{t_0}$ over t_0. Then we can take an analytic open neighborhood $\mathcal{M}'$ of y in $M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})$ and an analytic open neighborhood $\mathcal{T}'$ of t_0 in $\mathcal{T}$ together with an analytic isomorphism

$$\mathcal{M}' \cong \mathcal{M}'_{t_0} \times \mathcal{T}' \tag{11.5}$$

such that the restriction $\pi_{\mathcal{T}}|_{\mathcal{M}'}$ of $\pi_{\mathcal{T}}\colon M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu}) \longrightarrow \mathcal{T}$ coincides with the second projection and that the fibers $\{\{y'\}\times\mathcal{T}'\}_{y'\in\mathcal{M}'_{t_0}}$ over $\mathcal{M}'_{t_0}$ are leaves in $\mathcal{F}^{\mathrm{GIM}}_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$.

Take a holomorphic system of coordinates $\theta = (\theta_1,\dots,\theta_N)$ of $\mathcal{T}'$. If we set

$$\mathcal{T}'[\partial\theta] := \mathcal{T}' \times \operatorname{Spec}\mathbb{C}[\epsilon_1,\dots,\epsilon_N]/\big(\epsilon_i\epsilon_j \mid 1 \le i,j \le N\big),$$

then the tuple $\partial\theta = (\partial/\partial\theta_1,\dots,\partial/\partial\theta_N)$ of vector fields on $\mathcal{T}'$ corresponds to a morphism

$$I_{\partial\theta}\colon\ \mathcal{T}'[\partial\theta] \longrightarrow \mathcal{T}',$$

whose restriction to $\mathcal{T}' \subset \mathcal{T}'[\partial\theta]$ is the identity morphism.

By the same proof as Proposition 10.9, we can construct a horizontal lift $\big(\mathcal{E}^{\partial\theta}, \nabla^{\partial\theta}, l^{\partial\theta}, \ell^{\partial\theta}, \mathcal{V}^{\partial\theta}\big)$ of the universal family $(E,\nabla,l,\ell,\mathcal{V})$ on $\mathcal{C}\times_{\mathcal{T}}\mathcal{M}'$ with respect to the morphism $I_{\partial\theta}$. On a small open subset $U \subset \mathcal{C}\times_{\mathcal{T}}\mathcal{M}'$, we may assume $E|_U \cong \mathcal{O}_U^{\oplus r}$. Then we can write $\nabla|_U = \mathrm{d} + A\mathrm{d}z$ where z is a holomorphic coordinate on $\mathcal{C}\times_{\mathcal{T}}\mathcal{M}'$ over $\mathcal{M}'$ and A is a matrix of meromorphic functions in z. Let $U[\partial\theta] \subset \mathcal{C}\times_{\mathcal{T}}\mathcal{M}'[\partial\theta]$ be the open subscheme whose underlying set is U. Then we have $\mathcal{E}|_{U[\partial\theta]} \cong \mathcal{O}^{\oplus r}_{U[\partial\theta]}$ and we can write $\nabla^{\partial\theta} = \mathrm{d} + A(\epsilon)\mathrm{d}z + \sum_{j=1}^N B_j\mathrm{d}\epsilon_j$, where $A(\epsilon)$ is a lift of A. By the integrability condition of $\nabla^{\partial\theta}$, we have the equality

$$\sum_{j=1}^N \frac{\partial A(\epsilon)}{\partial\epsilon_j}\mathrm{d}\epsilon_j\wedge\mathrm{d}z + \sum_{j=1}^N \frac{\partial B_j}{\partial z}\mathrm{d}z\wedge\mathrm{d}\epsilon_j + \sum_{j=1}^N [A,B_j]\mathrm{d}z\wedge\mathrm{d}\epsilon_j = 0.$$

Take a holomorphic coordinate system $x_1,\dots,x_\delta$ of $\mathcal{M}'_{t_0}$. With respect to the coordinate system $z, x_1,\dots,x_\delta,\theta_1,\dots,\theta_N$, the partial derivative $\partial/\partial\theta_j$ coincides with the vector field $\Phi(\partial/\partial\theta_j)$ and the partial derivative $\partial A/\partial\theta_j$ coincides with $\partial A(\epsilon)/\partial\epsilon_j$. So the above integrability condition of $\nabla^{\partial\theta}$ is the same as the integrability condition

$$\sum_{j=1}^{N}\frac{\partial A}{\partial\theta_j}\mathrm{d}\theta_j\wedge\mathrm{d}z+\sum_{j=1}^{N}\frac{\partial B_j}{\partial z}\mathrm{d}z\wedge\mathrm{d}\theta_j+\sum_{j=1}^{N}[A,B_j]\mathrm{d}z\wedge\mathrm{d}\theta_j=0$$

of the connection

$$\nabla_U^{\mathrm{flat}}=\mathrm{d}+A\mathrm{d}z+\sum_{j=1}^{N}B_j\mathrm{d}\theta_j$$

on $E|_U$ relative to the composition

$$U\hookrightarrow\mathcal{C}\times_{\mathcal{T}}\mathcal{M}'\longrightarrow\mathcal{M}'\cong\mathcal{M}'_{t_0}\times\mathcal{T}'\xrightarrow{\pi_{\mathcal{M}'_{t_0}}}\mathcal{M}'_{t_0},$$

where $\pi_{\mathcal{M}'_{t_0}}\colon\mathcal{M}'\cong\mathcal{M}'_{t_0}\times\mathcal{T}'\longrightarrow\mathcal{M}'_{t_0}$ is the first projection. So we can see from Theorem 9.7 and Corollary 9.11 that $\nabla|_U$ is a local generalized isomonodromic deformation in the sense of Definition 9.4 or Definition 9.10.

We can patch ∇_U^{flat} together to get a global connection on E. Indeed, take another open subset $U'\subset\mathcal{C}\times_{\mathcal{T}}\mathcal{M}'$ and write $\nabla|_{U'}=\mathrm{d}+A'\mathrm{d}z$. Then we have $P^{-1}\mathrm{d}P+P^{-1}A\mathrm{d}zP=A'\mathrm{d}z$ for a transition matrix P. There is a local horizontal lift $\mathrm{d}+A'(\epsilon)\mathrm{d}z+\sum_{j=1}^{N}B'_j\mathrm{d}\epsilon_j$ of $\nabla|_{U'}$ and by the uniqueness of the local horizontal lift, we have a uniquely lift $P(\epsilon)$ of P satisfying

$$\begin{aligned}&P(\epsilon)^{-1}\left(\frac{\partial P(\epsilon)}{\partial z}\mathrm{d}z+\sum_{j=1}^{N}\frac{\partial P(\epsilon)}{\partial\epsilon_j}\mathrm{d}\epsilon_j\right)+P(\epsilon)^{-1}\left(A(\epsilon)\mathrm{d}z+\sum_{j=1}^{N}B_j\mathrm{d}\epsilon_j\right)P(\epsilon)\\&\quad=A'(\epsilon)\mathrm{d}z+\sum_{j=1}^{N}B'_j\mathrm{d}\epsilon_j.\end{aligned}$$

Since $\partial P(\epsilon)/\partial\epsilon_j=\partial P/\partial\theta_j$, the above equality yields the equality

$$P^{-1}\left(\frac{\partial P}{\partial z}\mathrm{d}z+\sum_{j=1}^{N}\frac{\partial P}{\partial\theta_j}\mathrm{d}\theta_j\right)+P^{-1}\left(A\mathrm{d}z+\sum_{j=1}^{N}B_j\mathrm{d}\theta_j\right)P=A'\mathrm{d}z+\sum_{j=1}^{N}B'_j\mathrm{d}\theta_j.$$

So we can patch the local connections ∇_U^{flat} together to get an integrable connection

$$\nabla^{\mathrm{flat}}\colon\ E\longrightarrow E\otimes\Omega_{\mathcal{C}\times_{\mathcal{T}}\mathcal{M}'/\mathcal{M}'_{t_0}}(D_{\mathcal{M}'})\tag{11.6}$$

relative to the composition $\mathcal{C}\times_{\mathcal{T}}\mathcal{M}'\longrightarrow\mathcal{M}'\cong\mathcal{M}'_{t_0}\times\mathcal{T}'\longrightarrow\mathcal{M}'_{t_0}$.

Corollary 11.8. *The generalized isomonodromic* 2*-form* $\omega^{\mathrm{GIM}}_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$ *constructed in Definition 11.4 is* d*-closed.*

Proof. Under the above notations, we will prove the equality

$$\omega^{\mathrm{GIM}}_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}\big|_{\mathcal{M}'}=\pi^*_{\mathcal{M}'_{t_0}}\left(\omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}\big|_{\mathcal{M}'_{t_0}}\right)\tag{11.7}$$

where $\pi_{\mathcal{M}'_{t_0}}\colon\mathcal{M}'\cong\mathcal{M}'_{t_0}\times\mathcal{T}'\longrightarrow\mathcal{M}'_{t_0}$ corresponds to the first projection with respect to the isomorphism (11.5). The corollary follows from this equality, since $\omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}|_{\mathcal{M}'_{t_0}}$ is d-closed by Theorem 8.1.

Take two tangent vectors $v, v' \in T_{\mathcal{M}'}(y, t_0)$ at $(y, t) \in \mathcal{M}'_{t_0} \times \mathcal{T}'$. We have the equalities

$$(\pi_{\mathcal{M}'_{t_0}})_*(v) = (\pi_{\mathcal{M}'_{t_0}})_*(v - \Phi(\pi_{\mathcal{T}'})_*(v)), \qquad (\pi_{\mathcal{M}'_{t_0}})_*(v') = (\pi_{\mathcal{M}'_{t_0}})_*(v - \Phi(\pi_{\mathcal{T}'})_*(v'))$$

because $\{y \times \mathcal{T}'\}_{y \in \mathcal{M}'_{t_0}}$ are leaves of the foliation $\mathcal{F}^{\mathrm{GIM}}_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$, which is determined by the subbundle $\operatorname{Im}\Phi$ of $T_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}$. The tangent vector $(\pi_{\mathcal{M}'_{t_0}})_*(v - \Phi(\pi_{\mathcal{T}'})_*(v))$ corresponds to a morphism $\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \longrightarrow \mathcal{M}'_{t_0}$. Let

$$\tilde{I}_v\colon \ \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \times \mathcal{T}' \longrightarrow \mathcal{M}'_{t_0} \times \mathcal{T}' \cong \mathcal{M}'$$

be its base change.

We can construct a complex $\mathcal{F}^\bullet$ of sheaves on $\mathcal{C} \times_{\mathcal{T}} \mathcal{M}'$ from the universal family $(E, \nabla, l, \ell, \mathcal{V})$ in the same way as (6.9) in Section 6. Since $(\mathrm{id} \times \tilde{I}_v)^*(E, \nabla, l, \ell, \mathcal{V})$ is a lift of $(E, \nabla, l, \ell, \mathcal{V})|_{\mathcal{C}_{y \times \mathcal{T}'}}$, it induces a gluing data $\{u_{\alpha\beta}, v_\alpha, \eta_\alpha\}$ with respect to an open covering $\{U_\alpha\}$ of $\mathcal{C}_{y\times\mathcal{T}'} := \mathcal{C} \times_{\mathcal{T}} (y \times \mathcal{T}')$. as in Proposition 6.3. Set

$$\tilde{v} := [\{u_{\alpha\beta}, v_\alpha, \eta_\alpha\}] \in \mathbb{R}^1(p_{y\times\mathcal{T}'})_*\big(\mathcal{F}^\bullet|_{\mathcal{C}_{y\times\mathcal{T}'}}\big).$$

Then we can see from the construction of $\tilde{v}$ that the equalities

$$\begin{aligned}
&\tilde{v}|_{(y,t_0)} = (\pi_{\mathcal{M}'t_0})_*(v) \in \mathbb{H}^1\big(\mathcal{F}^\bullet|_{\mathcal{C}_{(y,t_0)}}\big),\\
&\tilde{v}|_{(y,t)} = v - \Phi(\pi_{\mathcal{T}'})_*(v) \in \mathbb{H}^1\big(\mathcal{F}^\bullet|_{\mathcal{C}_{(y,t)}}\big).
\end{aligned}$$

hold. We can similarly construct an element $\tilde{v'} = [\{u'_{\alpha\beta}, v'_\alpha, \eta'_\alpha\}]$ of $\mathbb{R}^1(p_{y\times\mathcal{T}'})_*\big(\mathcal{F}^\bullet|_{\mathcal{C}_{y\times\mathcal{T}'}}\big)$ from the tangent vector v'. Recall the construction of the complex $\mathcal{F}^\bullet$ in (6.9). Since the map $\mathcal{G}^1 \longrightarrow G^1$ is surjective and the map $\mathcal{G}^0 \longrightarrow \mathcal{S}^1_{\mathrm{ram}}$ is a surjection to the kernel of the surjection $\mathcal{S}^1_{\mathrm{ram}} \longrightarrow A^1$, we can replace $u_{\alpha\beta}$, v_α so that $\eta_\alpha = 0$ holds. Similarly we may assume $\eta'_\alpha = 0$.

Consider the pairing

$$\omega\big(\tilde{v}, \tilde{v'}\big) := \big[\big\{\operatorname{Tr}(u_{\alpha\beta}u'_{\beta\gamma}), -\operatorname{Tr}(u_{\alpha\beta}v_\beta - v_\alpha u'_{\alpha\beta})\big\}\big] \in \mathbb{R}^2(p_{y\times\mathcal{T}'})_*\big(\Omega^\bullet_{\mathcal{C}_{y\times\mathcal{T}'}/\mathcal{T}'}\big) \cong \mathcal{O}_{\mathcal{T}'}$$

of $\tilde{v}$ and $\tilde{v'}$. Then we have the equalities

$$\begin{aligned}
&\pi^*_{\mathcal{M}'t_0}\big(\omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}|_{\mathcal{M}'t_0}\big)(v, v') = \omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_{t_0}}\big((\pi_{\mathcal{M}'t_0})_*(v), (\pi_{\mathcal{M}'t_0})_*(v')\big) = \omega\big(\tilde{v}, \tilde{v'}\big)|_{(y,t_0)},\\
&\omega^{\mathrm{GIM}}_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})}(v, v') = \omega_{M^{\boldsymbol{\alpha}}_{\mathcal{C},\mathcal{D}}(\lambda,\tilde{\mu},\tilde{\nu})_t}\big(v - \Phi(\pi_{\mathcal{T}'})_*(v), v' - \Phi(\pi_{\mathcal{T}'})_*(v')\big) = \omega\big(\tilde{v}, \tilde{v'}\big)|_{(y,t)}.
\end{aligned}$$

So, in order to prove (11.7), we only have to prove that $\omega\big(\tilde{v}, \tilde{v'}\big) \in \mathcal{O}_{\mathcal{T}'}$ is constant on $\mathcal{T}'$. We may assume that $\mathcal{T}'$ is isomorphic to a polydisk. Then it is sufficient to show that $\omega\big(\tilde{v}, \tilde{v'}\big)$ belongs to the image of the canonical map

$$\begin{aligned}
&\mathbb{C} \cong \mathbb{H}^2\big(\mathcal{O}_{\mathcal{C}_{y\times\mathcal{T}'}} \xrightarrow{\mathrm{d}} \Omega^1_{\mathcal{C}_{y\times\mathcal{T}'}} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{N+1}_{\mathcal{C}_{y\times\mathcal{T}'}}\big)\\
&\quad \hookrightarrow \mathbb{R}^2(p_{y\times\mathcal{T}'})_*\big(\mathcal{O}_{\mathcal{C}_{y\times\mathcal{T}'}} \xrightarrow{\mathrm{d}} \Omega^1_{\mathcal{C}_{y\times\mathcal{T}'}/\mathcal{T}'}\big) \cong \mathcal{O}_{\mathcal{T}'}.
\end{aligned} \tag{11.8}$$

Recall that (E, ∇) can be extended to the family of integrable connections $\big(E, \nabla^{\mathrm{flat}}\big)$ in (11.6). Then the pullback $\big(\mathrm{id} \times \tilde{I}_v\big)^*\big(E, \nabla^{\mathrm{flat}}\big)$ is a family of integrable connections relative to $\operatorname{Spec}\mathbb{C}[\epsilon]$ whose induced relative connection is $\big(\mathrm{id}\times\tilde{I}_v\big)^*(E, \nabla)$. So we can extend the relative meromorphic differential $v_\alpha = B_\alpha \mathrm{d}z$ to a total differential $v^{\mathrm{flat}}_\alpha = B_\alpha \mathrm{d}z + \sum_{i=1}^N C^i_\alpha \mathrm{d}\theta_i$ which satisfies the patching condition

$$(\mathrm{id} + \epsilon u_{\alpha\beta}) \circ \big(\nabla^{\mathrm{flat}} + \epsilon v^{\mathrm{flat}}_\beta\big) = \big(\nabla^{\mathrm{flat}} + \epsilon v^{\mathrm{flat}}_\alpha\big) \circ (\mathrm{id} + \epsilon u_{\alpha\beta})$$

on $E|_{U_{\alpha\beta}} \otimes \mathbb{C}[\epsilon]$ and the integrability condition

$$\big(\nabla^{\rm flat} + \epsilon v_\alpha^{\rm flat}\big) \circ \big(\nabla^{\rm flat} + \epsilon v_\alpha^{\rm flat}\big) = 0.$$

Let

$$\nabla^{\rm flat}_\dagger \colon\ \mathcal{E}{\rm nd}(E_{y\times\mathcal{T}'}) \ni u \mapsto \nabla^{\rm flat} \circ u - (u \otimes {\rm id}) \circ \nabla^{\rm flat} \in \mathcal{E}{\rm nd}(E_{y\times\mathcal{T}'}) \otimes \Omega^1_{\mathcal{C}\times_{\mathcal{T}}(y\times\mathcal{T}')}(D_{y\times\mathcal{T}'})$$

be the induced connection on $\mathcal{E}{\rm nd}(E|_{\mathcal{C}_{y\times\mathcal{T}'}})$. Then the above two equalities become

$$\nabla^{\rm flat}_\dagger(u_{\alpha\beta}) = v_\beta^{\rm flat} - v_\alpha^{\rm flat}, \qquad \nabla^{\rm flat}_\dagger\big(v_\alpha^{\rm flat}\big) = 0.$$

We can check the equalities

$$\begin{aligned}
{\rm d}\,{\rm Tr}\,\big(u_{\alpha\beta}u'_{\beta\gamma}\big) &= {\rm Tr}\,\big(\nabla^{\rm flat}_\dagger(u_{\alpha\beta}u'_{\beta\gamma})\big) = {\rm Tr}\,\big(\nabla^{\rm flat}_\dagger(u_{\alpha\beta})u'_{\beta\gamma} + u_{\alpha\beta}\nabla^{\rm flat}_\dagger(u'_{\beta\gamma})\big) \\
&= {\rm Tr}\,\big((v_\beta^{\rm flat} - v_\alpha^{\rm flat})u'_{\beta\gamma} + u_{\alpha\beta}(v_\gamma'^{\rm flat} - v_\beta'^{\rm flat})\big) \\
&= {\rm Tr}\,\big(v_\beta^{\rm flat}u'_{\beta\gamma} - v_\alpha^{\rm flat}(u'_{\alpha\gamma} - u'_{\alpha\beta}) + (u_{\alpha\gamma} - u_{\beta\gamma})v_\gamma'^{\rm flat} - u_{\alpha\beta}v_\beta'^{\rm flat}\big) \\
&= {\rm Tr}\,\big(-u_{\beta\gamma}v_\gamma'^{\rm flat} + v_\beta^{\rm flat}u'_{\beta\gamma}\big) - {\rm Tr}\,\big(-u_{\alpha\gamma}v_\gamma'^{\rm flat} + v_\alpha^{\rm flat}u'_{\alpha\gamma}\big) \\
&\quad + {\rm Tr}\,\big(-u_{\alpha\beta}v_\beta'^{\rm flat} + v_\alpha^{\rm flat}u'_{\alpha\beta}\big), \\
{\rm d}\,{\rm Tr}\,\big(-u_{\alpha\beta}v_\beta'^{\rm flat} + v_\alpha^{\rm flat}u'_{\alpha\beta}\big) &= {\rm Tr}\,\big(\nabla^{\rm flat}_\dagger\big(-u_{\alpha\beta}v_\beta'^{\rm flat} + v_\alpha^{\rm flat}u'_{\alpha\beta}\big)\big) \\
&= {\rm Tr}\,\big(-\nabla^{\rm flat}_\dagger(u_{\alpha\beta}) \wedge v_\beta'^{\rm flat} - v_\alpha^{\rm flat} \wedge \nabla^{\rm flat}_\dagger(u'_{\alpha\beta})\big) \\
&= {\rm Tr}\,\big(\big(v_\alpha^{\rm flat} - v_\beta^{\rm flat}\big) \wedge v_\beta'^{\rm flat}\big) - {\rm Tr}\,\big(v_\alpha^{\rm flat} \wedge \big(v_\beta'^{\rm flat} - v_\alpha'^{\rm flat}\big)\big) \\
&= -\,{\rm Tr}\,\big(v_\beta^{\rm flat} \wedge v_\beta'^{\rm flat}\big) + {\rm Tr}\,\big(v_\alpha^{\rm flat} \wedge v_\alpha'^{\rm flat}\big)
\end{aligned}$$

and

$${\rm d}\,{\rm Tr}\,\big(v_\alpha^{\rm flat} \wedge v_\alpha'^{\rm flat}\big) = {\rm Tr}\,\big(\nabla^{\rm flat}_\dagger\big(v_\alpha^{\rm flat} \wedge v_\alpha'^{\rm flat}\big)\big) = 0.$$

Therefore, $\big[\big\{\,{\rm Tr}(u_{\alpha\beta}u'_{\beta\gamma}), -{\rm Tr}\,\big(u_{\alpha\beta}v_\beta'^{\rm flat} + v_\alpha^{\rm flat}u'_{\alpha\beta}\big), {\rm Tr}\,\big(v_\alpha^{\rm flat} \wedge v_\alpha'^{\rm flat}\big)\big\}\big]$ defines an element of $\mathbb{H}^2\big(\Omega^\bullet_{\mathcal{C}_{y\times\mathcal{T}'}}\big) \cong \mathbb{C}$ and its image by the map (11.8) coincides with

$$\omega\big(\tilde{v}, \tilde{v'}\big) = \big[\big\{\,{\rm Tr}(u_{\alpha\beta}u'_{\beta\gamma}), -{\rm Tr}(u_{\alpha\beta}v'_\beta + v_\alpha u'_{\alpha\beta})\big\}\big].$$

Thus we have proved the corollary. ■

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