Seven Concepts Attributed to Siméon-Denis Poisson

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Abstract. Siméon-Denis Poisson was 25 years old when he was appointed Professor of Mathematics at the École Polytechnique in 1806. Elected to the Paris Académie des Sciences six years later, he soon became one of its most influential members. The origin and later developments of the many concepts in mathematics and physics that bear his name make interesting stories, a few of which we shall attempt to sketch in this paper.

Key words: Poisson; École Polytechnique; Académie des Sciences; Poisson's equation; Poisson's ratio; Poisson distribution; Poisson kernel; Poisson brackets

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Pour Jean-Pierre Bourguignon
à l'occasion de son 75e anniversaire

1 The mathematician and physicist Poisson (1781–1840)

Pithiviers is a small town in France, 50 miles south of Paris, renowned for a special, tasty kind of pastry, called a "pithiviers", and for the high-quality honey from the neighboring countryside. But it has another claim to fame. It was the birthplace in 1781 of Siméon-Denis Poisson, who would become the mathematician and mathematical physicist whose name is attached to the Poisson distribution, the Poisson brackets, Poisson geometry, Poisson algebras and many other concepts, formulas, equations and theorems.

He was just eight years old when the French Revolution broke out. His father was a retired soldier who held a modest administrative position. The Revolution allowed boys from families such as his to obtain a decent education. In 1794, the École Polytechnique, first called École centrale des travaux publics, was created to prepare engineers with a solid scientific background. Then, as now, admission was by a competitive examination. Encouraged by one of his teachers in high school, and equipped with a certificate attesting to his deep love of liberty, equality and all the fundamental beliefs of the Republic, including a "hate for tyrants", Poisson sat for the entrance examination in 1798 and took first place. This was the beginning of a very brilliant career, through many regime changes, first the revolutionary Republic, then Napoleon's Empire, the Restauration of the royalty in 1815, under Louis XVIII until his death in 1824, followed by the more autocratic Charles X, then the revolution of 1830 and the constitutional monarchy of Louis-Philippe. Poisson did not live to see France's later regime, the short-lived republic of 1848, since he died in 1840 at the age of 58.

Ten years after his death, a lifesize statue of Poisson was installed in his hometown. To this day, there is a square in the center of Pithiviers that bears the name "Place Denis Poisson", but the brass statue has disappeared, like so many others in France and elsewhere, having been melted down during the German occupation of Pithiviers in the Second World War.

When Poisson entered the École Polytechnique, the professors were the most distinguished scientists of the time, Joseph Louis Lagrange (1736–1813) and Gaspard Monge (1746–1818) taught mathematics, Pierre-Simon Laplace (1749–1827) was an examiner for mathematics, Jean Baptiste Biot (1774–1862) was an examiner for physics, and Antoine-François de Fourcroy (1755–1809) was professor of chemistry.

Volume 4 of the *Journal de l'École polytechnique*, dated 1801–1802, contains three papers written by Poisson in 1799 or early 1800, when he was still a student. One deals with the classification of quadrics and is an "Addition" to an article by Monge and Jean Nicolas Pierre Hachette (1769–1834) who were both professors at the École. This 3-page note [7] must have been a remark that he made on a publication of his teachers, and it is signed by one of them, Hachette, together with Poisson. It is both the first and last paper ever co-authored by Poisson.

In the same volume, we find the paper "Essay on elimination in algebraic equations",[1] an essay on the elimination of variables in systems of algebraic equations, which contains a new, simplified proof of the theorem of Étienne Bézout (1730–1783) on the degree of the resultant attached to a pair of algebraic curves, and the "Essay on the plurality of integrals in the calculus of differences",[2] that he had read before the Institut National, which had replaced the Académie des Sciences in 1795, on the 16th of Frimaire in the year 9 of the French revolutionary calendar, i.e., December 8, 1800. In that paper, he generalized a remark of Laplace on the solutions of first-order difference equations. The report by two members of the Academy, Adrien-Marie Legendre (1752–1833) and Sylvestre François Lacroix (1765–1843), is preserved in the Archives de l'Académie des Sciences. In the conclusion of their four-page long "Report on a paper by Citizen Poisson on the number of complete integrals of which equations of finite differences are susceptible",[3] they wrote: "It follows at a minimum that the theory established by this young geometer is correct, and even though it may not be susceptible to useful applications in the problems that lead to this type of equations, one must always regard the clarification of a problem of analysis which, until the present, remained in great obscurity as contributing to the progress of science",[4] and they recommended the publication of Poisson's paper. Legendre and Lacroix were not enthusiastic but they certainly did not discourage the promising young mathematician.

Upon finishing his studies at the École, Poisson was immediately appointed as an assistant, and in 1802 he was called upon to "take over temporarily the duties of Citizen Fourier", Joseph Fourier (1768–1830) who would soon develop the Fourier series and integrals. Four years later, at the age of 25, he was appointed "Instituteur d'Analyse", i.e., full professor of mathematics, to replace Fourier for whom he had already been substituting for four years. The archives of the École Polytechnique contain the confirmation of Poisson's nomination to replace Fourier, dated 11 Brumaire, year 11 (November 2, 1802), as well as the covering letter, dated March 17, 1806, of the Emperor's official decree naming Poisson "Instituteur d'Analyse" at the École Polytechnique.

In 1804, Poisson appears as a handsome young professor in a portrait by the painter E. Marcellot, that is now in the collection of École Polytechnique.

In 1809, Napoleon decreed the opening of a re-organized "Université Impériale", and Poisson was named its first professor of mechanics. A poster announcing the opening of classes in April 1811 may still be seen in the collection of the Bibliothèque Nationale de France and it specifies that Poisson's lectures at the Sorbonne, to use the old name that has survived all reforms, student

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[1]Mémoire sur l'élimination dans les équations algébriques [11].

[2]Mémoire sur la pluralité des intégrales dans le calcul des différences [10].

[3]Rapport sur un Mémoire du Citoyen Poisson relatif au nombre d'intégrales complettes [sic] dont les équations aux différences finies sont susceptibles.

[4]Il résulte au moins que la théorie établie par ce jeune géomètre est exacte, et quand même elle ne serait pas susceptible d'applications utiles dans les problèmes qui conduisent à ce genre d'équations, on doit toujours regarder commme contribuant au progrès de la science, l'éclairicssement d'un problème d'analyse qui jusqu'à présent était resté dans une grande obscurité.

revolts and re-organizations to this day, would be delivered on Mondays and Fridays. (The other professors were Lacroix, for differential and integral calculus, Louis-Benjamin Francœur (1773–1849) for advanced algebra, Hachette for descriptive geometry, and Biot for astronomy.)

When there was finally a vacancy in the Academy of Sciences it was in the physics section in 1812 and Poisson was elected to fill it. His role in the Academy was soon preeminent, particularly in the election of new members. He was charged with writing numerous reports. One, written in 1816, I find particularly interesting because it shows Poisson, already a respected member of the Academy, in a position to judge an unknown, young scientist, just as he had been judged by Legendre and Lacroix, back in 1800. In it, together with André-Marie Ampère (1775–1836), he reports on a paper by Claude Pouillet (1790–1868) on the phenomenon of colored rings. Their conclusion on the work submitted by this "young physicist" is very reminiscent of that of Legendre and Lacroix on the "young geometer" Poisson, sixteen years earlier, and they too recommend the publication of the essay. Their judgment was fair, since Pouillet went on to teach at Polytechnique and at the Sorbonne, and to be elected to the Academy. This report in the Archives de l'Académie des Sciences is an autograph manuscript of Poisson.

Another aspect of Poisson's role in the Académie was important even if it was sometimes controversial. He often had to read papers submitted for publication to the *Mémoires de l'Académie*. In several instances he rendered a great service to the scientific community by summarizing in his own terms the main points of these papers, before, sometimes many years before, they were revised and published. A controversy arose when Poisson made use of such material, with insufficient acknowledgement of his source. The best known case is his bitter dispute with Fourier. Having had access to the manuscript on the theory of heat that Fourier had submitted to the Academy in 1807, he published a detailed account of the subject in 1808 in the *Bulletin de la Société Philomatique* (*Bulletin des Sciences*), under the title "On the propagation of heat in solid bodies, by M. Fourier"[5] and signed P, his initial. While, from 1814 to 1825, Poisson published many essays on trigonometric series and the theory of heat, Fourier's text, revised, would be published in 1822 [5], much later than Poisson's first papers. A virulent debate over precedence broke out in 1815 between the two scientists, during which Fourier wrote a letter to Laplace, blaming both Poisson and Biot: "[They] recognize that they could not obtain up to now any results different from mine [. . .] but they say that they have another method for formulating them, and that this method is excellent and the true one. [. . .] But it does not extend the limits of science to present results that one has not found in a form that one says is different".[6]

When Gaston Darboux (1842–1917) edited the works of Fourier in 1890, he included Poisson's 1808 account, explaining, "This article [. . .] is not by Fourier. Signed with the initial "P", it was written by Poisson who was an editor of the mathematical portion of the *Bulletin des Sciences*. Because of the historic interest that it presents as the first publication that made Fourier's theory known, we believed that we should reproduce it in its entirety".[7] In 1808, Fourier had derived the heat equation and solved it in a particular case, by expanding the sought-after solution into a trigonometric series. Poisson established the heat equation for the case of variable conductivity in his "Essay on the distribution of heat in solid bodies",[8] published in the *Journal de l'École polytechnique* in 1823, and included it in his 1835 book, "Mathematical Theory of Heat"[9] [20].

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[5]Mémoire sur la propagation de la chaleur dans les corps solides, par M. Fourier.

[6][Ils] reconnaissent qu'ils n'ont pu donner jusqu'ici aucun résultat différent des miens [. . .] mais ils disent qu'ils ont une autre manière de les exposer et que cette manière est excellente et la véritable. [. . .] Mais ce n'est pas reculer les limites des sciences que de présenter sous une forme que l'on dit être différente des résultats que l'on n'a pas trouvés soi-même.

[7]Cet Article [. . .] n'est pas de Fourier. Signé de l'initiale P, il a été rédigé par Poisson qui était un rédacteur du *Bulletin des Sciences* pour la partie mathématique. À raison de l'intérêt historique qu'il présente comme étant le premier écrit où l'on ait fait connaître la théorie de Fourier, nous avons cru devoir le reproduire intégralement.

[8]Mémoire sur la distribution de la chaleur dans les corps solides.

[9]Théorie mathématique de la chaleur.

From 1820 until his death, Poisson played an important role in the organization of education in France, as a member and then, after 1822, as treasurer of the Royal Council of Public Education.[10] As head of mathematics in France, he wielded considerable influence and used his authority to do battle to insure the teaching of mathematics to all students, including those primarily studying humanities. As president of the jury of the "agrégation" for the selection of teachers, he strove to maintain a single examination for both mathematics and physics. He championed mathematics but he also understood that the two fields had to be developed simultaneously at the middle and high school levels and at the university, just as they were in his research.

2 Some stories

Poisson's name appears in so many contexts in mathematics and in physics that discovering the earliest formulation of each of his concepts or theorems in his more that 200 publications would be a very time-consuming enterprise. Here, I shall mention only a few of them.

2.1 Poisson's equation in electromagnetism

Poisson's first important contributions to the theory of electrostatics date from 1811 to 1813, when he took up the problem of determining the distribution of electrical charges in charged bodies by applying analytical techniques, such as series expansions, while he began treating magnetism in an article published later. In his "Essay on the theory of magnetism in motion"[11] [16], which appeared in the *Mémoires de l'Académie des Sciences* of 1823, one finds, on p. 463, "$\Delta V = 0$, $= -2k\pi$, $= -4k\pi$, depending on whether point M is located outside, on the surface of, or inside the volume in question".[12] Here k denotes the constant charge density of the charged body. (It was Friedrich Gauss (1777–1855) who would later treat the case of variable density.) The case of the equation $\Delta V = 0$ was already well known, being the famous Laplace equation. Poisson had derived an equation satisfied by the potential at a point interior to the charged body, but the novelty in the 1823 paper was to treat the case of a point on the surface of the body. This case became known as "Poisson's equation" in electromagnetism. It was indeed discovered by Poisson.

The importance of these articles was immediately recognized by George Green (1793–1841), after whom are named the Green function and the Green–Riemann theorem. In 1828, in the preface to his *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism,* Green cited Poisson's work prominently and, speaking of the papers of 1811 and 1812, he wrote, "Little appears to have been effected in the mathematical theory of electricity [. . .] when M. Poisson presented to the French Institut two memoirs of singular elegance, relative to the distribution of electricity on the surfaces of conducting spheres, previously electrified and put in presence of each other".

The fame of Poisson's mathematical theory of electrostatics is reflected in the judgment of E.T. Whittaker (1873–1956) in his *History of the Theories of Æther and Electricity* (1910). Regarding Poisson's 1812 essay he wrote, "Electrostatical theory was, however, suddenly advanced to quite a mature state of development by Siméon-Denis Poisson, in a memoir which was read to the French academy in 1812 [. . .]. The rapidity with which in a single memoir Poisson passed from the barest elements of the subject to such recondite problems as those just mentioned may well excite admiration". He concluded: "His success is, no doubt, partly explained by the high

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[10]Conseil royal de l'instruction publique.

[11]Mémoire sur la théorie du magnétisme en mouvement.

[12]$\Delta V = 0$, $= -2k\pi$, $= -4k\pi$, selon que le point M sera situé en dehors, à la surface ou en dedans du volume que l'on considère".

state of development to which analysis had been advanced by the great mathematicians of the eighteenth century; but [...] Poisson's investigation must be accounted a splendid memorial of his genius". Later, he examined "Poisson's theory of magnetic induction", rejecting his interpretation of the physics of the situation but noting that the formulas derived by Poisson are valid.

And later, in 1939, the historian of science, Mario Gliozzi, in a paper analyzing "Poisson's contribution to electricity theory",[13] concluded that Poisson's 1813 publication was a most remarkable paper.

2.2 Poisson's ratio

The story of Poisson's ratio is that of a concept whose present, everyday applications are as surprising as they are numerous. I heard a beautiful talk by Tadashi Tokieda in 2012 in Paris, that started with Poisson, continued with origami, and went on to an astonishing variety of contemporary questions in materials science, and I later read the article by the physicist George Neville Greaves (1945–2019), "Poisson's ratio over two centuries: challenging hypotheses" [6], which gave both a historical account and a detailed description of the current theory and practice of this concept, which I shall briefly summarize here. It started with a "shape versus volume concept", a hint already given by Poisson in his early *Traité de Mécanique* [13], first published in 1811, where he wrote on page 176 of volume 2:

> For each of the elements into which we have divided the amount of fluid matter, its shape will be altered during the time dt, and also its volume will change if the fluid is compressible; but, since its mass must remain unaltered, it follows that, if we seek to determine its volume and its density at the end of time $t + dt$, their product will necessarily be the same as after time t.[14]

In a Note in the *Annales de chimie et de physique* in 1827, "On the extension of elastic threads and plates"[15] [18], Poisson introduced the dimensionless ratio that bears his name, and, by means of a computation based on Laplace's theory of molecular interaction, announced that its value is $\frac{1}{2}$, in accordance with recent experiments on brass [in French, *laiton*] by baron Charles Cagniard de La Tour (1777–1859) and Félix Savart (1791–1841), on the vibrations of plates, whose results had been recently presented to the Academy. Poisson further developped the theory of elasticity in several papers that he read before the Académie des Sciences between 1823 and 1828 and published from 1828 to 1830, introducing the ratio of the strain in the transverse direction to the strain in the primary direction.

About ten years later, as the precision of the measurements in experiments increased, the constancy of Poisson's ratio for all materials was proved to be wrong, but the conflict between the hypothesis of interacting molecules and the continuum theory of Sophie Germain (1776–1831) and Augustin Cauchy (1789–1857) was not resolved until much later. In the 1860's, James Clerk Maxwell (1831–1879) defended Poisson's viewpoint, but William Thomson (Lord Kelvin, 1824–1907) declared that it had already been proved false by George Stokes (1819–1903) in 1845. Until the 1970's the variability of Poisson's ratio with every kind of material had only been established experimentally by engineers "for whom macroscopic properties were sovereign" [6]. Still, it was shown that its variability, unlike that of the other elastic moduli, is restricted within the range $[-1, \frac{1}{2}]$. The number of publications concerning the Poisson ratio increased exponentially after

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[13]Il contributo del Poisson all'electrologia.

[14]Chacun des éléments dans lesquels nous avons partagé la masse fluide, changera de forme pendant l'instant dt, et il changera même de volume, si le fluide est compressible; mais comme sa masse devra toujours rester la même, il s'ensuit que, si nous cherchons ce que deviennent son volume et sa densité à la fin du temps $t + dt$, leur produit devra être le même qu'à la fin du temps t.

[15]Sur l'extension des fils et des plaques élastiques.

1970, when it was discovered that this concept helped to understand "the narrowing of arteries during hypertension, the resilience of bones and medical implants, the rheology of liquid crystals, the shaping of ocean floors, the oblateness of the Earth, and planetary seismology after meteor impact" [6]. This is when materials with negative Poisson's ratio began to appear and found countless, important applications that were aptly evoked in Tokieda's entertaining lecture. This was the result of the work of many physicists, beginning with Roderic Lakes in 1987 and including Greaves, the author of the article that we have attempted to summarize here, who observed that "Siméon-Denis Poisson is particularly remembered for a ratio, a dimensionless quantity, which today has surprisingly acquired a ubiquitous physical significance".[16]

2.3   Poisson's spot

Poisson's contribution to optics was not a successful treatment of the general phenomena of light but a prediction based on his ability to compute. Following Laplace, he held that all phenomena could be explained by molecular interaction and was opposed to the theory of Augustin Fresnel (1788–1827), based on a wave theory. When, in 1817, submissions for a grand prize for a study of the diffraction of light were being examined by the Academy of Sciences, Poisson was a member of the committee in charge of examining Fresnel's submission. Convinced that Fresnel was wrong, Poisson suggested an experiment that would prove that a mathematical consequence of Fresnel's formulas was contrary to intuition and would disprove his theory. When the consequence of Fresnel's theory that Poisson had derived and considered absurd was tested experimentally, what he had judged to be absurd was actually observed: a bright spot appeared in the centre of the shadow of a disk lit by a source situated on its axis. This phenomenon was then called derisively "Poisson's spot". The experiment suggested by Poisson yielded a result in Fresnel's favor who was awarded the prize.

2.4   The Poisson distribution

This is probably the best known occurrence of Poisson's name in the scientific literature. In French it is referred to as "la loi de Poisson" (Poisson's law).

Poisson was not the first to deal with probabilities. Blaise Pascal (1623–1662), Christiaan Huygens (1629–1695), John Arbuthnot (1667–1735), Giovanni Battista Vico (1668–1744), Georges-Louis Leclerc de Buffon (1707–1788) in his *Essay on Moral Arithmetic*,[17] of 1777, had all written on this subject, and even Voltaire had written a pamphlet, *Essay on Probabilities Applied to Justice*,[18] in 1772, but it was not mathematical, and the revised and augmented fifth edition of Laplace's *Philosophical Essay on Probabilities*[19] of 1814 was published in volume 7 of his works in 1825. In 1981, Bernard Bru, in his essay on Poisson and probability theory [1], wrote that a precursor of the probability law that bears the name of Poisson can be found as early as 1718 in the work of the huguenot mathematician working in England, Abraham de Moivre (1667–1754), *The Doctrine of Chances*. But in its present form, the Poisson distribution appeared for the first time on page 262 of Poisson's essay of 1829, "Essay on the proportion of new-born females and males",[20] which was published in the *Mémoires de l'Académie des Sciences* [19] of 1830, and reappeared in his subsequent book, *Recherches sur la probabilité des*

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[16]See also the comprehensive article by G.N. Greaves et al., "Poisson's ratio and modern materials", *Nature Materials*, vol. 10, November 2011.

[17]Essai d'arithmétique morale.

[18]Essai sur les probabilités en fait de justice.

[19]Essai philosophique sur les probabilités.

[20]Mémoire sur la proportion des naissances des filles et des garçons.

jugements [21], published in 1837, where one reads on p. 206,

$$P = \left(1 + \omega + \frac{\omega^2}{1 \cdot 2} + \frac{\omega^3}{1 \cdot 2 \cdot 3} + \cdots + \frac{\omega^n}{1 \cdot 2 \cdot 3 \cdots n}\right) \mathrm{e}^{-\omega}.$$

Related to the Poisson distribution are the so-called Poisson–Charlier polynomials, whose sequence first appeared in the work of the Swedish astronomer and statistician, Carl Vilhelm Ludwig Charlier (1862–1934). It was the German mathematician Gustav Doetsch (1892–1977) who called them Charlier polynomials in the title of an article published in *Mathematische Annalen* in 1933 where he discussed the differential-difference equation they satisfied. The reviewer of Doetsch's article for *Zentralblatt* wrote the definition of these polynomials in terms of a sequence of functions defined by recursion from the Poisson distribution (Poissonsche Verteilung) and noted the orthogonality property which they satisfy with respect to the Poisson density, whence the present terminology. In a note in the *Annals of Mathematical Statistics* in 1947, Clifford Truesdell (1919–2000) derived their properties from those of the F-functions which he introduced, and he entitled his article, "A note on the Poisson–Charlier functions" [23]. Thus Poisson's name became attached to concepts invented a hundred years after his death.

2.5   The Poisson summation formula

The Poisson summation formula is so well known that it is often called, simply, "the Poisson formula". Why is the formula

$$\sum_{n=-\infty}^{n=\infty} f(n) = \sum_{n=-\infty}^{n=\infty} \mathcal{F}f(n),$$

where $\mathcal{F}f$ is the Fourier transform of f, defined by $\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} x \xi} f(x)\, \mathrm{d}x$, attributed to Poisson? In the hope of finding references that would lead us to his original papers, we opened a modern textbook, *Sphere Packings, Lattices and Groups* (1999), by John Conway and N.J.A. Sloane. Introducing the Jacobi theta functions, they write on p. 103, "These functions are related by a labyrinth of identities [...] One may regard [them] as consequences of the general version of the Poisson summation formula".

How did the reference to Poisson reach Conway and Sloane at the end of the 20th century, and what do we find in Poisson? Fortunately, they refer to p. 475 of the classical treatise, *A Course of Modern Analysis* (1927), by Whittaker and Watson who in turn refer to Poisson's article of 1823 in the *Journal de l'École polytechnique*, "Continuation of the essay on definite integrals and the summation of series"[21] [17]. There, on p. 420, we read the formula

$$\pi + 2\pi \sum_{n=1}^{\infty} \mathrm{e}^{-4k\pi^2 n^2} = \frac{\sqrt{\pi}}{2\sqrt{k}} + \frac{\sqrt{\pi}}{\sqrt{k}} \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{n^2}{4k}},$$

which Poisson derived when working on a precise evaluation of the remainder in the summation formula that Leonhard Euler had obtained in 1736 and which Colin Maclaurin stated in his 1742 "Treatise of Fluxions". Since the Fourier transform of $f(x) = \mathrm{e}^{-\alpha x^2}$ is $\mathcal{F}f(\xi) = \sqrt{\frac{\pi}{\alpha}} \mathrm{e}^{-\frac{\pi^2 \xi^2}{\alpha}}$, when the preceding formula is rewritten as

$$\sum_{n=-\infty}^{n=\infty} \mathrm{e}^{-4k\pi^2 n^2} = \frac{1}{\sqrt{4\pi k}} \sum_{n=-\infty}^{n=\infty} \mathrm{e}^{-\frac{n^2}{4k}},$$

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[21] Suite du mémoire sur les intégrales définies et sur la sommation des séries.

we recognize instances of the summation formula,

$$\sum_{n=-\infty}^{n=\infty} f(n) = \sum_{n=-\infty}^{n=\infty} \mathcal{F}f(n),$$

for each function f_k defined by $f_k(x) = e^{-4k\pi^2 x^2}$.

What Whittaker and Watson observed was that, setting $4k\pi = -i\tau$, Poisson's formula can be rewritten as

$$\theta_3(0|\tau) = \frac{1}{\sqrt{-i\tau}}\theta_3\left(0\left|-\frac{1}{\tau}\right.\right),$$

the particular case for $z = 0$ of the general transformation formula for the third theta function,

$$\theta_3(z|\tau) = (-i\tau)^{-\frac{1}{2}}e^{\frac{z^2}{\pi i \tau}}\theta_3\left(\frac{z}{\tau}\left|-\frac{1}{\tau}\right.\right).$$

They also stated that a more general case is to be found in Poisson's "Essay on the numerical calculation of definite integrals",[22] published in 1827 in the *Mémoires de l'Académie des Sciences*, a year before Jacobi published "Continuation of the notices on elliptical functions"[23] in *Crelle's Journal*, which was followed by his comprehensive treatment of the identities satisfied by the theta functions in his *Fundamenta nova theorias functionum ellipticarum*, published in Kœnigsberg in 1829.

Today, the summation formula, generalized in the theory of group representations, has applications to network theory and error-correcting codes, that Poisson could not have anticipated.

2.6   The Poisson kernel and the Poisson integral formula

If one opens any modern book on potential theory, one will no doubt find a definition of "the Poisson kernel" and a proof of "the Poisson integral formula", often simply called "the Poisson formula", for the case of a half-plane and for a disk in the plane, often also for the sphere in 3-space or in higher-dimensional spaces. How did these formulas reach the modern authors and where did they appear in Poisson's vast mathematical production?

What became known as "the Dirichlet problem" for a domain in n-space, which consists of determining the value of a harmonic function in the interior of the domain, given the value of the function on the boundary of the domain, was formulated for a disk in the plane by Peter Gustav Lejeune Dirichlet (1805–1859) in an article in *Crelle's Journal* in 1828.

In a paper on the later history of Fourier series, Jean-Pierre Kahane (1926–2017) listed, among the advances made in the 19th century on the subject of the trigonometric series, the solution by Hermann Amandus Schwarz (1843–1921) of the Dirichlet problem for the circle by means of the Poisson formula, in 1872. In fact, Schwarz's article [22] in the *Journal für die reine und angewandte Mathematik* of 1872, contains the formula that expresses the value of a harmonic function in the interior of a disk as an integral involving only the values of that function on the boundary circle,

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(1,\phi)\frac{1-r^2}{1-2r\cos(\psi-\phi)+r^2}\,d\psi,$$

but he writes: "It is easy to recognize the fundamental idea of Poisson's proof in the proof that is to be found in Section 5.b".[24] Schwarz attributed this formula to Carl Neumann (1832–1925)

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[22]Mémoire sur le calcul numérique des intégrales définies.

[23]Suite des notices sur les fonctions elliptiques.

[24]Man wird auch in dem Beweise, der in Section 5 unter b enthalten ist, die Grundgedanken des Poissonschen Beweises leicht wiedererkennen.

in his article [9] in volume 59 of the same journal. One does find this formula on p. 365 of Neumann's article. While there is no mention of Poisson in Neumann's article, Schwarz on the other hand gave numerous references to Poisson's articles of 1815 and 1823, and to his book on the theory of heat, *Théorie mathématique de la chaleur*, of 1835, as well as to three other papers published in 1827, 1829 and 1831. He thus gave us a useful map to enter the maze of sometimes very long essays that Poisson wrote, many of them with a "Suite" and an "Addition".

Poisson's search started as early as 1813 in his "Essay on definite integrals"[25] [14] published in the *Journal de l'École polytechnique*. This paper was followed by another essay, sixty pages long, in 1820, "Essay on the manner of expressing functions by series of periodic quantities and on the use of this transformation in the solution of various problems"[26] [15], published in the same journal, and a "Continuation of the essay on definite integrals and the summation of series"[27] [17] in 1823.

Poisson was trying to establish interpolation formulas à la Lagrange. In 1820, in his essay on series of periodic functions, he proved that, given a finite sequence of $m-1$ quantities, $y_1, \ldots, y_{m-1}$, setting $z_j = \frac{2}{m} \sum_{k=1}^{m-1} y_k \sin \frac{kj\pi}{m}$, it follows that $y_n = \sum_{j=1}^{m-1} z_j \sin \frac{nj\pi}{m}$. He wrote that this formula is a particular case of Lagrange's formula in his "Researches on the nature of sound and its propagation"[28] which had appeared in the first volume of the *Mémoires de l'Académie de Turin* in 1759. Then, using a limiting procedure and exchanging summation and integration, he derived the following formula

$$ fx = \frac{2}{l} \int_0^l \sum_{k=1}^\infty \sin \frac{k\pi x}{m} \sin \frac{k\pi \alpha}{m} f\alpha \, d\alpha, $$

which he also attributed to Lagrange. In this long essay, he expressed his aim of replacing the summation of a series by the computation of an integral or conversely, and he treated the question of the summation of series of sines and cosines. He wrote: "It will be advantageous to bring all of them together under the same point of view and to deduce these values by a uniform method".[29] On p. 422, he introduced the evaluation of a function with the help of integration by way of what would later be called a Poisson kernel, and he gave numerous applications for it, including to the motion of a vibrating string composed of two parts of different material and the motion of a heavy body suspended from an elastic wire.

What is clear from reading Poisson is that he was not trying to solve the so-called Dirichlet problem, except maybe in the case of some applied problem related to his more general researches, but that he returned many times to the theory of trigonometric series and that he was in fact trying to prove the so-called Fourier theorem, that is, he was working towards a proof of the convergence of the "Fourier series" of a given function to the function itself. His attempts at a rigorous treatment of this question, as well as the later treatment by Cauchy in 1823, were not successful, but it is in the course of such a research that Poisson introduced the function that became known as the Poisson kernel and the integral that became known as the Poisson integral. Both terms are justified, but their appearance in the theory of harmonic functions and potential theory came later, with Dirichlet and Schwarz. In conclusion, we can affirm that, on the one hand, the Poisson kernel was indeed introduced by Poisson in his attempts to prove the convergence of the Fourier series of general types of functions, and that, on the other hand, Poisson did not use the corresponding integral formula in the search for a general solution of the Laplace equation, but only in particular cases that arose from physical problems.

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[25]Mémoire sur les intégrales définies.

[26]Mémoire sur la manière d'exprimer les fonctions par des séries de quantités périodiques, et sur l'usage de cette transformation dans la résolution de différents problèmes.

[27]Suite du mémoire sur les intégrales définies et sur la sommation des séries.

[28]Recherches sur la nature, et la propagation du son.

[29]Il ne sera pas inutile de les réunir toutes sous un même point de vue et de déduire ces valeurs d'une méthode uniforme.

2.7   Poisson brackets

It all began in celestial mechanics. Following Lagrange's essays of 1808 and 1809 on the variation of the principal axes of the orbits of planets and on a general theory of the variation of arbitrary constants in mechanics, but it was in Poisson's famous essay of 1809, "Essay on the variation of arbitrary constants in questions of mechanics"[30] [12] that the Poisson brackets appeared in their own right. Lagrange had derived their expression by an involved procedure which amounted to – in modern terms – inverting the matrix of components of the canonical symplectic 2-form. Poisson denoted the coordinates of the position of the body by ϕ, ψ, θ and the components of its velocity by s, u, v, and he wrote:

> It is clear that the left hand side of this equation is a complete differential with respect to t; integrating, we thus obtain the very simple equation
>
> $$\frac{\mathrm{d}b}{\mathrm{d}s} \cdot \frac{\mathrm{d}a}{\mathrm{d}\phi} - \frac{\mathrm{d}a}{\mathrm{d}s} \cdot \frac{\mathrm{d}b}{\mathrm{d}\phi} + \frac{\mathrm{d}b}{\mathrm{d}u} \cdot \frac{\mathrm{d}a}{\mathrm{d}\psi} - \frac{\mathrm{d}a}{\mathrm{d}u} \cdot \frac{\mathrm{d}b}{\mathrm{d}\psi} + \frac{\mathrm{d}b}{\mathrm{d}v} \cdot \frac{\mathrm{d}a}{\mathrm{d}\theta} - \frac{\mathrm{d}a}{\mathrm{d}v} \cdot \frac{\mathrm{d}b}{\mathrm{d}\theta} = \mathrm{const.}$$
>
> One sees that the constant on the right hand side of this equation will, in general, be a function of a and b, and of the arbitrary constants appearing in the other integrals of the motion; [...] but in order to recall the origin of this quantity, which represents a certain combination of the partial differentials of the values of a and b, we shall make use of the notation (b, a) to denote it; so that we shall have generally[31]
>
> $$\frac{\mathrm{d}b}{\mathrm{d}s} \cdot \frac{\mathrm{d}a}{\mathrm{d}\phi} - \frac{\mathrm{d}a}{\mathrm{d}s} \cdot \frac{\mathrm{d}b}{\mathrm{d}\phi} + \frac{\mathrm{d}b}{\mathrm{d}u} \cdot \frac{\mathrm{d}a}{\mathrm{d}\psi} - \frac{\mathrm{d}a}{\mathrm{d}u} \cdot \frac{\mathrm{d}b}{\mathrm{d}\psi} + \frac{\mathrm{d}b}{\mathrm{d}v} \cdot \frac{\mathrm{d}a}{\mathrm{d}\theta} - \frac{\mathrm{d}a}{\mathrm{d}v} \cdot \frac{\mathrm{d}b}{\mathrm{d}\theta} = (b, a).$$

(There is a short but detailed discussion of Poisson's paper in René Dugas's *Histoire de la Mécanique* [4], in a somewhat modernized notation, which renders Poisson's argument easy to understand and facilitates the reading of his original text.)

It is in this early article that Poisson introduced the change of variables from $(q_i, \dot{q}_i)$ to (q_i, p_i), where the p_i's are the conjugate quantities, or momenta, of the q_i's, defined by $\frac{\partial L}{\partial \dot{q}_i} = p_i$, when L is the Lagrangian function, paving the way for the Hamiltonian form of the equations of motion, already implicit in Lagrange, eventually derived by Cauchy in a lithographed memoir in 1831, which was only later printed, in 1834 in Italian and in 1835 in French, and eventually published by William Rowan Hamilton (1805–1865) in his "Second essay on a general method in dynamics" in 1835. Poisson returned to the subject in 1816.

Jacobi, in 1850, read a biography of Poisson (maybe Arago's?), and he re-discovered what became known as Poisson's theorem, that the Poisson bracket of two integrals of the motion is an integral of the motion. He exclaimed that this theorem was a "truly prodigious theorem" (ce théorème vraiment prodigieux), and he endeavored to explain what he claimed its author and later authors had not perceived.

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[30]Mémoire sur la variation des constantes arbitraires dans les questions de mécanique.

[31]Il est visible que le premier membre de cette équation est une différentielle complète par rapport à t; en intégrant, nous aurons donc cette équation fort simple

$$\frac{\mathrm{d}b}{\mathrm{d}s} \cdot \frac{\mathrm{d}a}{\mathrm{d}\phi} - \frac{\mathrm{d}a}{\mathrm{d}s} \cdot \frac{\mathrm{d}b}{\mathrm{d}\phi} + \frac{\mathrm{d}b}{\mathrm{d}u} \cdot \frac{\mathrm{d}a}{\mathrm{d}\psi} - \frac{\mathrm{d}a}{\mathrm{d}u} \cdot \frac{\mathrm{d}b}{\mathrm{d}\psi} + \frac{\mathrm{d}b}{\mathrm{d}v} \cdot \frac{\mathrm{d}a}{\mathrm{d}\theta} - \frac{\mathrm{d}a}{\mathrm{d}v} \cdot \frac{\mathrm{d}b}{\mathrm{d}\theta} = \mathrm{const.}$$

On conçoit que la constante qui fait le second membre de cette équation, sera en général une fonction de a et b, et des constantes arbitraires contenues dans les autres intégrales des équations du mouvement; [...] mais, afin de rappeler l'origine de cette quantité, qui représente une certaine combinaison des différences partielles des valeurs de a et b, nous ferons usage de cette notation (b, a), pour la désigner; de manière que nous aurons généralement

$$\frac{\mathrm{d}b}{\mathrm{d}s} \cdot \frac{\mathrm{d}a}{\mathrm{d}\phi} - \frac{\mathrm{d}a}{\mathrm{d}s} \cdot \frac{\mathrm{d}b}{\mathrm{d}\phi} + \frac{\mathrm{d}b}{\mathrm{d}u} \cdot \frac{\mathrm{d}a}{\mathrm{d}\psi} - \frac{\mathrm{d}a}{\mathrm{d}u} \cdot \frac{\mathrm{d}b}{\mathrm{d}\psi} + \frac{\mathrm{d}b}{\mathrm{d}v} \cdot \frac{\mathrm{d}a}{\mathrm{d}\theta} - \frac{\mathrm{d}a}{\mathrm{d}v} \cdot \frac{\mathrm{d}b}{\mathrm{d}\theta} = (b, a).$$

The ubiquity of the concepts of Poisson brackets, Poisson algebras and Poisson manifolds in mechanics, theoretical physics and an impressive number of fields of mathematics suggests the question: how did this happen? The story of Poisson brackets involves, in addition to Lagrange, Cauchy and Hamilton, mainly Jacobi, Liouville and Sophus Lie (1842–1899), and culminates in the explanation of the role they played in the development of quantum mechanics. It is of course too long a story to outline here. Even a short history of Poisson geometry would imply an excursus into the history of symplectic geometry. I shall only comment here on the fact that the name "Poisson brackets" does not seem to have been adopted until Whittaker used it in his *History of the Theories of Æther and Electricity* in 1910. They had previously been called "expressions" generally, while one author, Joseph Graindorge (1843–1889), wrote in 1872 that he was using "Poisson's notation" (la notation de Poisson).

Already in 1857 Arthur Cayley (1821–1895), in his *Report of the British Association for the Advancement of Science*, had predicted the importance that the Poisson brackets would assume later, as opposed to the Lagrange parentheses, "The theory of Poisson gives rise to developments which seem to have nothing corresponding to them in the theory of Lagrange". In fact, while the Lagrange parentheses are the components of a closed 2-form, a concept that appeared only with Élie Cartan (1869–1951) just before 1900, the Poisson brackets satisfy the identity that Jacobi proved ca. 1840 (published posthumously in 1862), and the Jacobi identity has become the foundation of the theory of Lie algebras.

3 Conclusion

Most, but not all, of the judgments that were passed on Poisson's œuvre in the 19th century were extremely laudatory. The physicist and astronomer François Arago (1786–1853), who was then Secrétaire perpétuel (president) de l'Académie des Sciences, declared at Poisson's funeral in 1840: "Genius does not die thus; it survives in its works",[32] and ten years later he wrote in another eulogy in the form of a scientific biography that Poisson had "three qualities: genius, devotion to work and mathematical erudition".[33] But the author of a history of the mathematical and physical sciences in 12 volumes published in the 1880's, Maximilien Marie, devoted several pages to a derogatory summary of Poisson's activity, "Poisson was very far from accomplishing the promise of his youth",[34] claiming that his position in any debate would always have been the wrong one, and Pierre Costabel in [2], although less negative, wrote a harsh appreciation.

When the first new buildings of the Sorbonne were inaugurated in 1890, fifty years after Poisson's death, it was Charles Hermite (1822–1901) – after whom the "Hermite polynomials" and "Hermitian matrices" are named – who delivered a speech [8] at the ceremony in the presence of the then French President, Sadi Carnot. He chose to review the work and the legacy of the professors who held the first chairs at the Faculté des Sciences at its creation in 1809. He declared, "Poisson is one of the great geometers of this century",[35] before reviewing Poisson's achievements in mathematical physics: "For Laplace and Poisson, pure analysis is not the object, but the tool",[36] and he pursues: "But, having a different object, Poisson and Fourier contributed to the development of analysis which they enriched with methods, new results, fundamental concepts".[37] He underlined the fact that Poisson was a disciple of Laplace, but he also announced

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[32] Le génie ne meurt pas ainsi; il se survit dans ses œuvres.

[33] On se demandera sans doute comment, durant une vie si courte et consacrée en grande partie au professorat, notre confrère était parvenu à attaquer et à résoudre tant de problèmes. Je répondrai que c'est par la réunion de trois qualités: le génie, l'amour du travail et l'érudition mathématique.

[34] Poisson n'a pas tenu, à beaucoup près, les promesses de sa jeunesse.

[35] Poisson est l'un des grands géomètres de ce siècle.

[36] Pour Laplace et Poisson, l'Analyse pure n'est point le but, mais l'instrument.

[37] Mais en ayant un autre but, Poisson et Fourier contribuent au développement de l'Analyse qu'ils enrichissent de méthodes, de résultats nouveaux, de notions fondamentales.

the following, referring to Poisson's work on what was already known as "the Poisson theorem", "But he is also related to contemporary analysis regarding a question of the utmost importance and of particular interest from the point of view of mathematical invention".[38] And he goes on to quote a sentence in Latin written by Jacobi which I understand to mean, "We here have an example that clearly shows that, if problems are not already formulated in our minds, it may well be that we would not see most important discoveries that are set in front of our eyes".

Was Poisson a mathematician or a physicist? If he was called "a geometer" by Arago, Hermite and others, it is because "geometer" was the generic term for mathematician until much later in the 19th century. His ambition was to write a comprehensive treatise of mathematical physics. Lagrange, Laplace, Gauss, Cauchy, all contributed both to "pure mathematics" and to the solution of problems in physics, sometimes very practical problems, such as geodesy, working towards the latter with the mathematial tools that they forged. Poisson's work on the theory of magnetism had important applications to the navigation instruments for ships. His molecular theory, following Laplace, did not win against the wave theory advocated by Fresnel, but in the history of elasticity some of his insights which had been discredited have regained their importance in the latest development of composite materials.

Poisson never traveled. After he came to Paris at the age of 17 to enter the École Polytechnique, he left the capital only to visit Laplace at Arcueil, a few miles from the southern edge of Paris, where he joined the other members of the Société d'Arcueil, a small circle of renowned scientists, named after their meeting place. But his publications abroad were numerous and his influence in Germany, in England, and in Russia was considerable. Several of his books were translated and extracts or summaries of his articles appeared in the *Zeitschrift für Physik*, in the *Annalen der Physik*, and the *Philosophical Transactions*.[39]

But Lie algebra theory had to wait for other geniuses to be developed, and Poisson geometry had to wait for more than a century and a half to be developed in various forms in the work of, among others, Wolfgang Pauli, George W. Mackey, Wlodzimierz Tulczyjew, Vladimir Maslov, Robert Hermann, Alexander Kirillov, Moshé Flato, André Lichnerowicz and Alan Weinstein.

Poisson's role in French science was dominant while he lived. His explanations of physical phenomena were mostly proved wrong by his contemporaries or by later physicists, but his achievements in mathematical physics remain. Often, he found the right equations using the wrong physical arguments. He was a formidable "computer" and his legacy in mathematics is essential. He advanced mathematics by trying to solve physical problems, sometimes successfully, sometimes not, and he is to be remembered for concepts, equations, formulas and theorems. He demonstrated how physical problems can suggest entirely abstract mathematics, what we now call mathematical physics.

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[38]Mais il se rattache aussi à l'analyse de notre temps dans une question de la plus grande importance et qui présente un intérêt singulier au point de vue de l'invention mathématique.

[39]These references are not in the autograph list of his works – now in the Bibliothèque de l'Institut – that Poisson himself drew up not long before he died. They had escaped the meticulous, invaluable work of Pierre Dugac in [3].

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