

# Affine Kac–Moody Algebras and Tau-Functions for the Drinfeld–Sokolov Hierarchies: the Matrix-Resolvent Method

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**Abstract.** For each affine Kac–Moody algebra  $X_n^{(r)}$  of rank  $\ell$ ,  $r = 1, 2$ , or  $3$ , and for every choice of a vertex  $c_m$ ,  $m = 0, \dots, \ell$ , of the corresponding Dynkin diagram, by using the matrix-resolvent method we define a gauge-invariant tau-structure for the associated Drinfeld–Sokolov hierarchy and give explicit formulas for generating series of logarithmic derivatives of the tau-function in terms of matrix resolvents, extending the results of [*Mosc. Math. J.* **21** (2021), 233–270, arXiv:1610.07534] with  $r = 1$  and  $m = 0$ . For the case  $r = 1$  and  $m = 0$ , we verify that the above-defined tau-structure agrees with the axioms of Hamiltonian tau-symmetry in the sense of [*Adv. Math.* **293** (2016), 382–435, arXiv:1409.4616] and [arXiv:math.DG/0108160].

*Key words:* Kac–Moody algebra; tau-function; Drinfeld–Sokolov hierarchy; matrix resolvent

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## 1 Introduction

Let  $X_n^{(r)}$  be an affine Kac–Moody algebra of rank  $\ell$ , with  $r = 1, 2, 3$  (here  $n = n(\ell)$ , for example,  $n(\ell) = \ell$  when  $r = 1$ ), and let  $C = (C_{ij})_{i,j=0}^{\ell}$  be its Cartan matrix [23]. In  $X_n^{(r)}$  there is a set of Chevalley generators  $\{e_i, h_i, f_i \mid i = 0, \dots, \ell\}$  satisfying the following relations:

$$[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \quad [h_i, e_j] = C_{ij} e_j, \quad [h_i, f_j] = -C_{ij} f_j, \quad \forall 0 \leq i, j \leq \ell, \quad (1.1)$$

and for  $i \neq j$  we have

$$(\operatorname{ad} e_i)^{1-C_{ij}} e_j = (\operatorname{ad} f_i)^{1-C_{ij}} f_j = 0. \quad (1.2)$$

Let  $a_i$  (respectively  $a_i^\vee$ ) be the positive integers satisfying  $\sum_{j=0}^{\ell} C_{ij} a_j = 0$  (respectively  $\sum_{j=0}^{\ell} C_{ji} a_j^\vee = 0$ ), for all  $i = 0, \dots, \ell$ , such that their greatest common divisor is 1. The number

$$h = \sum_{i=0}^{\ell} a_i \quad \left( \text{respectively } h^\vee = \sum_{i=0}^{\ell} a_i^\vee \right)$$

is called *Coxeter number* (respectively *dual Coxeter number*) of  $X_n^{(r)}$  [23].

Let  $\tilde{\mathfrak{g}}$  be the quotient of  $X_n^{(r)}$  by the one-dimensional space generated by the central element  $K = \sum_{i=0}^{\ell} a_i^\vee h_i$ . The *principal gradation* on  $\tilde{\mathfrak{g}}$  is defined by assigning  $\deg^{\text{pr}} e_i = -\deg^{\text{pr}} f_i = 1$ ,  $i = 0, \dots, \ell$ . Clearly,  $\tilde{\mathfrak{g}}$  decomposes into the direct sum of homogeneous subspaces

$$\tilde{\mathfrak{g}} = \bigoplus_{k \in \mathbb{Z}} \tilde{\mathfrak{g}}^k,$$

where elements in  $\tilde{\mathfrak{g}}^k$  have principal degree  $k$ . In this paper we are interested with a completion of  $\tilde{\mathfrak{g}}$  rather than with  $\tilde{\mathfrak{g}}$  itself. By an abuse of notation we denote these two objects with the same symbol and let

$$\widehat{\tilde{\mathfrak{g}}} = \widehat{\bigoplus_{k \in \mathbb{Z}} \tilde{\mathfrak{g}}^k}, \quad (1.3)$$

where the direct sum is completed by allowing infinite series in negative degree. Given an element  $a \in \tilde{\mathfrak{g}}$  we denote by  $a^+$  its projection on  $\tilde{\mathfrak{g}}^{\geq 0} = \bigoplus_{k \geq 0} \tilde{\mathfrak{g}}^k$  and by  $a^-$  its projection on  $\tilde{\mathfrak{g}}^{< 0} = \bigoplus_{k < 0} \tilde{\mathfrak{g}}^k$ .

Introduce the *cyclic element*

$$\Lambda = \sum_{i=0}^{\ell} e_i \in \tilde{\mathfrak{g}}. \quad (1.4)$$

Note that  $\Lambda$  is homogeneous of principal degree 1. Let  $\mathcal{H} = \text{Ker ad } \Lambda$  be the so-called *principal (centerless) Heisenberg subalgebra* of  $\tilde{\mathfrak{g}}$ . According to [22] (cf. [26]),  $\mathcal{H}$  is abelian and we have the direct sum decomposition

$$\tilde{\mathfrak{g}} = \mathcal{H} \oplus \text{Im ad } \Lambda. \quad (1.5)$$

Given  $A \in \tilde{\mathfrak{g}}$  we denote by  $\pi_{\mathcal{H}}(A) \in \mathcal{H}$  its projection with respect to the direct sum decomposition (1.5) (namely  $\text{Ker } \pi_{\mathcal{H}} = \text{Im ad } \Lambda$ ).

It is known that  $\mathcal{H}$  and  $\text{Im ad } \Lambda$  admit the following decomposition:

$$\mathcal{H} = \widehat{\bigoplus_{i \in E} \mathbb{C} \Lambda_i}, \quad \text{Im ad } \Lambda = \widehat{\bigoplus_{i \in \mathbb{Z}} (\text{Im ad } \Lambda)^i}, \quad (1.6)$$

where  $E \subset \mathbb{Z}$  is the set of exponents of  $X_n^{(r)}$  (see [23] for the definition of exponents),  $\deg^{\text{pr}} \Lambda_i = i$ , and  $(\text{Im ad } \Lambda)^i = \text{Im ad } \Lambda \cap \tilde{\mathfrak{g}}^i$ ,  $i \in \mathbb{Z}$ . A convenient normalization of the basis elements  $\Lambda_i$  can be found in Section 2.2. Recall that the set  $E$  has the following form

$$E = \bigsqcup_{a=1}^n (m_a + r h \mathbb{Z}), \quad (1.7)$$

where

$$1 = m_1 < m_2 \leq \dots \leq m_{n-1} < m_n = r h - 1,$$

satisfy the following relation:

$$m_a + m_{n+1-a} = r h, \quad a = 1, \dots, n. \quad (1.8)$$

Let  $m$  be an integer from 0 to  $\ell$ . Take  $c_m$  to be the  $m$ th vertex of the Dynkin diagram of  $X_n^{(r)}$ . Here we label the Dynkin diagram according to [23]. Recall that the vertex  $c_0$  is the so-called *special vertex* [23]. The *standard gradation* corresponding to  $c_m$  is defined by assigning

$\deg_{\text{st}} e_m = -\deg_{\text{st}} f_m = 1$ , and degree 0 to all the remaining Chevalley generators. Then, we also have the direct sum decomposition

$$\tilde{\mathfrak{g}} = \widehat{\bigoplus_{k \in \mathbb{Z}} \tilde{\mathfrak{g}}_k}, \quad (1.9)$$

where  $\tilde{\mathfrak{g}}_k$  denotes the homogeneous subspace of elements with standard degree  $k$ . Given an element  $a \in \tilde{\mathfrak{g}}$  we denote by  $a_+$  its projection on  $\tilde{\mathfrak{g}}_{\geq 0} = \bigoplus_{k \geq 0} \tilde{\mathfrak{g}}_k$  and by  $a_-$  its projection on  $\tilde{\mathfrak{g}}_{< 0} = \bigoplus_{k < 0} \tilde{\mathfrak{g}}_k$ .

Denote  $\mathfrak{a} = \tilde{\mathfrak{g}}_0$ . Note that  $\mathfrak{a}$  is a semisimple Lie algebra with Cartan matrix  $C_{\mathfrak{a}} = (C_{ij})_{i,j \neq m}$  and Chevalley generators  $e_i, h_i$  and  $f_i, i \in \{0, \dots, \ell\} \setminus \{m\}$ . With respect to the principal gradation we can write

$$\mathfrak{a} = \bigoplus_{i=-h_{\mathfrak{a}}+1}^{h_{\mathfrak{a}}-1} \mathfrak{a}^i, \quad (1.10)$$

where  $\mathfrak{a}^i = \mathfrak{a} \cap \tilde{\mathfrak{g}}^i$  and  $h_{\mathfrak{a}}$  is the Coxeter number of  $\mathfrak{a}$ . In particular,  $\mathfrak{a}^0$ , that is generated by  $h_i, i \in \{0, \dots, \ell\} \setminus \{m\}$ , is equal to  $\tilde{\mathfrak{g}}^0$ , and it is a Cartan subalgebra of  $\mathfrak{a}$ . Let us further denote by  $\mathfrak{n}$  the nilpotent subalgebra  $\mathfrak{n} = \mathfrak{a}^{< 0}$  of  $\mathfrak{a}$  and by  $\mathfrak{b}$  the Borel subalgebra  $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{a}^0$  of  $\mathfrak{a}$ . Clearly,  $\mathfrak{n}$  is generated by  $f_i, i \in \{0, \dots, \ell\} \setminus \{m\}$ , and  $\mathfrak{b}$  is generated by  $f_i, h_i, i \in \{0, \dots, \ell\} \setminus \{m\}$ . From the defining relations (1.1) of the Kac–Moody algebra  $\tilde{\mathfrak{g}}$  we have

$$[\mathfrak{n}, \mathfrak{b}] \subset \mathfrak{n}, \quad [\mathfrak{n}, e_m] = 0, \quad [\mathfrak{n}, e_i] \subset \mathfrak{b}, \quad i \in \{0, \dots, \ell\} \setminus \{m\}. \quad (1.11)$$

Let  $e = \Lambda - e_m \in \mathfrak{a}$ , where  $\Lambda \in \tilde{\mathfrak{g}}$  is the cyclic element in (1.4). The element  $e$  is called a *principal nilpotent element*.

Define a linear operator associated to the pair  $(\tilde{\mathfrak{g}}, c_m)$ , called a Lax operator, by

$$\mathcal{L} = \partial + \Lambda + q,$$

where  $\partial := \partial_x$  and  $q := q(x) \in C^\infty(S^1, \mathfrak{b})$  is a smooth function from the circle to the Borel subalgebra  $\mathfrak{b}$ . We denote by  $\mathcal{A}^q$  the algebra of differential polynomials in  $q$ , namely, an element of  $\mathcal{A}^q$  is a polynomial in the entries of the smooth function  $q$  and their  $x$ -derivatives. Recall from [14] that there exist a unique pair of elements  $U \in \mathcal{A}^q \otimes (\text{Im ad } \Lambda)^{< 0}$  and  $H \in \mathcal{A}^q \otimes \mathcal{H}^{< 0}$  such that

$$e^{\text{ad } U}(\partial + \Lambda + q) = \partial + \Lambda + H. \quad (1.12)$$

(Observe that the element  $U$  used in this paper is  $-U$  in [2].)

**Definition 1.1.** An element  $R \in \mathcal{A}^q \otimes \tilde{\mathfrak{g}}$  such that  $[\mathcal{L}, R] = 0$  is called a *resolvent* for  $\mathcal{L}$ .

For every  $i \in E$ , we denote

$$R_i = e^{-\text{ad } U}(\Lambda_i) \in \mathcal{A}^q \otimes \tilde{\mathfrak{g}}. \quad (1.13)$$

Clearly,  $R_i, i \in E$ , are resolvents for  $\mathcal{L}$ . Indeed, using (1.12) and the fact that  $\mathcal{H}$  is abelian, we have

$$[\mathcal{L}, R_i] = e^{-\text{ad } U}[\partial + \Lambda + H, \Lambda_i] = 0. \quad (1.14)$$

We call  $R_{m_a}, a = 1, \dots, n$ , the *basic resolvents*.

Recall that the following system of evolutionary partial differential equations (PDEs)

$$\frac{\partial \mathcal{L}}{\partial t_i} = [(R_i)_+, \mathcal{L}], \quad i \in E \cap \mathbb{Z}_{>0}, \quad (1.15)$$

is called the *pre-Drinfeld–Sokolov (pre-DS) hierarchy* associated to the pair  $(\tilde{\mathfrak{g}}, c_m)$ . The proof of the fact that (1.15) indeed defines evolutionary PDEs can be found in [14]. Also according to [14], the flows in this system of PDEs all commute. Recall also that, for all  $j \in E$  and  $i \in E \cap \mathbb{Z}_{>0}$ , we have

$$\frac{\partial R_j}{\partial t_i} = [(R_i)_+, R_j], \quad (1.16)$$

A *gauge transformation* is a change of variables  $q \mapsto \tilde{q} \in \mathcal{A}^q \otimes \mathfrak{b}$  of the form

$$\tilde{\mathcal{L}} = e^{\text{ad } N} \mathcal{L} = \partial + \Lambda + \tilde{q}, \quad N \in \mathcal{A}^q \otimes \mathfrak{n}. \quad (1.17)$$

Explicitly, we have

$$\tilde{q} = q - \sum_{k \geq 1} \frac{(\text{ad } N)^{k-1}}{k!} (\partial N) + \sum_{k \geq 1} \frac{(\text{ad } N)^k}{k!} (q + \Lambda). \quad (1.18)$$

Due to the commutation relations in (1.11), the expression for  $\tilde{q}$  in (1.18) is a well-defined element of  $\mathcal{A}^q \otimes \mathfrak{b}$ . Via the gauge transformation (1.18), a resolvent  $R$  transforms as follows:

$$R \mapsto e^{\text{ad } N} R = \tilde{R}. \quad (1.19)$$

By a *gauge invariant*, we mean a differential polynomial  $g(q, q_x, q_{xx}, \dots)$  in  $\mathcal{A}^q$ , such that  $g(\tilde{q}, \tilde{q}_x, \tilde{q}_{xx}, \dots) = g(q, q_x, q_{xx}, \dots)$  for all gauge transformations (1.17). The space of all gauge invariants, denoted by  $\mathcal{R}$ , is a differential algebra [14] that can be identified with the classical  $\mathcal{W}$ -algebra  $\mathcal{W}(\mathfrak{a}, e)$  associated to the Lie algebra  $\mathfrak{a}$  and its principal nilpotent element  $e$  [9].

Since,  $\text{ad } e: \mathfrak{n} \rightarrow \mathfrak{b}$  is injective [14] (where we recall that  $e$  is the principal nilpotent element), we may choose a space  $V \subset \mathfrak{b}$  complementary to  $[e, \mathfrak{n}]$  and compatible with the direct sum decomposition (1.10), i.e.,

$$\mathfrak{b} = V \oplus [e, \mathfrak{n}]. \quad (1.20)$$

Note that  $\dim V = \dim \mathfrak{b} - \dim \mathfrak{n} = \dim \mathfrak{a}^0 = \ell$ . The vector space  $V \subset \mathfrak{b}$  is called a *Drinfeld–Sokolov (DS) gauge*. It is proved by Drinfeld and Sokolov that there exists a unique  $N^{\text{can}} \in \mathcal{A}^q \otimes \mathfrak{n}$  such that

$$\mathcal{L}^{\text{can}} = e^{\text{ad } N^{\text{can}}} \mathcal{L} = \partial + \Lambda + q^{\text{can}}, \quad q^{\text{can}} \in \mathcal{A}^q \otimes V. \quad (1.21)$$

If  $g(q, q_x, q_{xx}, \dots)$  is an element in  $\mathcal{R}$ , then  $g(q, q_x, q_{xx}, \dots) = g(q^{\text{can}}, q_x^{\text{can}}, q_{xx}^{\text{can}}, \dots)$ . Hence  $\mathcal{R}$  can be realized as an algebra of polynomials in the entries  $u_1, \dots, u_\ell$  of  $q^{\text{can}}$  and their  $x$ -derivatives.

By the results of [14], the differential algebra  $\mathcal{R}$  is preserved by the flows of the pre-DS hierarchy (1.15), namely, for every  $i \in E \cap \mathbb{Z}_{>0}$  we have that  $\frac{\partial}{\partial t_i}(\mathcal{R}) \subset \mathcal{R}$ .

**Definition 1.2.** The *DS hierarchy* associated to the affine Kac–Moody algebra  $\tilde{\mathfrak{g}}$  and a vertex  $c_m$  of its Dynkin diagram is the set of equations

$$\frac{\partial u_s}{\partial t_i} = P_{s,i} \in \mathcal{R}, \quad s = 1, \dots, \ell, \quad i \in E \cap \mathbb{Z}_{>0}, \quad (1.22)$$

where the RHS of (1.22) can be computed by applying the flow in (1.15) to the gauge invariants  $u_s$ .

It is known from [14] that

$$\frac{\partial u_s}{\partial t_1} = -\frac{\partial u_s}{\partial x}.$$

Therefore, for the DS hierarchy, we identify  $t_1$  with  $-x$ , and a solution  $u_s(x, \mathbf{t})$  to the DS hierarchy (1.22) will be simply denoted by  $u_s(\mathbf{t})$ . We also remark that, as it is shown in [14], if a vertex of the Dynkin diagram of  $\tilde{\mathfrak{g}}$  is the image of the vertex  $c_m$  under an automorphism of the diagram, then the corresponding DS hierarchies coincide.

The main theme of this paper is on computing logarithmic derivatives of tau-functions for the DS hierarchy using the matrix-resolvent method [2, 3, 33]. To proceed let us realize  $\tilde{\mathfrak{g}}$  as a subalgebra  $L(\mathfrak{g}, \sigma_m)$  of  $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}((\lambda^{-1})) = \mathfrak{g}((\lambda^{-1}))$ , where  $\mathfrak{g}$  is a certain simple Lie algebra and  $\sigma_m: \mathfrak{g} \rightarrow \mathfrak{g}$  is a finite-order automorphism both depending on the pair  $(\tilde{\mathfrak{g}}, c_m)$  (see Section 2.1 for the details). Denote  $N_m = ra_m$ , and let  $\pi_\lambda: \mathbb{C}((\lambda^{-1})) \rightarrow \lambda^{-1}\mathbb{C}[[\lambda^{-N_m}]]$  be defined by (2.21) and  $\pi_{\lambda, \mu} := \pi_\mu \circ \pi_\lambda$ .

**Definition 1.3.** Define the series  $F_{a,b}(\lambda, \mu) \in \mathcal{A}^q \otimes \lambda^{-1}\mu^{-1}\mathbb{C}[[\lambda^{-N_m}, \mu^{-N_m}]]$ ,  $a, b = 1, \dots, n$ , by

$$F_{a,b}(\lambda, \mu) = \pi_{\lambda, \mu} \left( \frac{(R_{m_a}(\lambda) | R_{m_b}(\mu)) - \frac{\delta_{a+b, n+1}}{r} (m_a \lambda^{N_m} + m_b \mu^{N_m})}{(\lambda - \mu)^2} \right). \quad (1.23)$$

Here  $(\cdot | \cdot)$  denotes the normalized Cartan–Killing form of  $\mathfrak{g}$  with the natural extension to  $L(\mathfrak{g})$  (cf. (2.7)–(2.8) for its precise definition).

The fact that the right-hand side of (1.23) belongs to  $\mathcal{A}^q \otimes \lambda^{-1}\mu^{-1}\mathbb{C}[[\lambda^{-N_m}, \mu^{-N_m}]]$  will be proved in the beginning of Section 2.3.

Write

$$F_{a,b}(\lambda, \mu) =: \sum_{l, k \in \mathbb{Z}_{\geq 0}} \Omega_{a,l;b,k} \lambda^{-N_m l - 1} \mu^{-N_m k - 1}. \quad (1.24)$$

It follows from (1.19) and the invariance property of  $(\cdot | \cdot)$  that the differential polynomials  $\Omega_{a,l;b,k}$ ,  $a, b = 1, \dots, n$ ,  $l, k \in \mathbb{Z}_{\geq 0}$ , defined via (1.24) belong to  $\mathcal{R}$ . In particular,  $F_{a,b}(\lambda, \mu)$  does not change if we replace  $R_{m_c}$  in the right-hand side of (1.23) with

$$R_{m_c}^{\text{can}} := e^{\text{ad } N^{\text{can}}} R_{m_c} \quad (1.25)$$

(cf. (1.19)). We will prove in Section 2.3 that the differential polynomials  $\Omega_{a,l;b,k} \in \mathcal{R}$  also have the following properties:

$$\Omega_{a,l;b,k} = \Omega_{b,k;a,l}, \quad (1.26)$$

$$\partial_{t_{m_c + m r h}} \Omega_{a,l;b,k} = \partial_{t_{m_a + l r h}} \Omega_{b,k;c,m}. \quad (1.27)$$

We call  $\{\Omega_{a,l;b,k} \mid a, b = 1, \dots, n, l, k \in \mathbb{Z}_{\geq 0}\}$  the *tau-structure* for the DS hierarchy. For  $N \geq 3$ ,  $c_1, \dots, c_N \in \{1, \dots, n\}$ , and  $k_1, \dots, k_N \geq 0$ , we also define

$$\Omega_{c_1, k_1; \dots; c_N, k_N} := \partial_{t_{m_{c_N} + k_N r h}} \cdots \partial_{t_{m_{c_2} + k_2 r h}} (\Omega_{c_1, k_1; c_2, k_2}). \quad (1.28)$$

Clearly, these elements all belong to  $\mathcal{R}$ . It follows from (1.26)–(1.27) that  $\Omega_{c_1, k_1; \dots; c_N, k_N}$  are totally symmetric with respect to permuting its index-pairs.

For every  $N \geq 2$ , we define a cyclic symmetric invariant  $N$ -linear form  $B: \mathfrak{g} \times \cdots \times \mathfrak{g} \rightarrow \mathbb{C}$  by

$$B(x_1, \dots, x_N) = \text{tr}(\text{ad } x_1 \circ \cdots \circ \text{ad } x_N), \quad x_1, \dots, x_N \in \mathfrak{g}.$$

We extend  $B$  to a cyclic symmetric invariant linear  $N$ -form on  $L(\mathfrak{g}, \sigma_m) \times \cdots \times L(\mathfrak{g}, \sigma_m)$  with values in  $\mathbb{C}((\lambda_1^{-1}, \dots, \lambda_N^{-1}))$  in the obvious way. The main result of the paper is given by the following theorem.

**Theorem 1.4.** *For each  $N \geq 2$ , let  $c_1, \dots, c_N$  be arbitrarily given integers in  $\{1, \dots, n\}$ . We have*

$$\sum_{k_1, \dots, k_N \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^N \lambda_j^{-N_m k_j - 1} \Omega_{c_1, k_1; \dots; c_N, k_N} = -\frac{\pi_{\lambda_1, \dots, \lambda_N}}{2h_{\mathfrak{g}}^N} \quad (1.29)$$

$$\times \left( \sum_{s \in S_N / C_N} \frac{B(R_{c_{s_1}}^{\text{can}}(\lambda_{s_1}), \dots, R_{c_{s_N}}^{\text{can}}(\lambda_{s_N}))}{\prod_{j=1}^N (\lambda_{s_j} - \lambda_{s_{j+1}})} - \frac{\delta_{N,2} \delta_{c_1 + c_2, n+1} m_{c_1} \lambda_1^{N_m} + m_{c_2} \lambda_2^{N_m}}{2r (\lambda_1 - \lambda_2)^2} \right),$$

where  $S_N$  denotes the symmetric group and  $C_N$  the cyclic group, and  $s_{N+1} = s_1$ .

It also follows from (1.26)–(1.27) and the fact that  $\Omega_{a,l;b,k} \in \mathcal{R}$  that for an arbitrary solution  $u_s(\mathbf{t})$ ,  $s = 1, \dots, n$ , to the DS hierarchy (1.22), there exists a power series  $\tau(\mathbf{t})$ , such that

$$\frac{\partial^2 \log \tau(\mathbf{t})}{\partial t_{m_a + lrh} \partial t_{m_b + krh}} = \Omega_{a,l;b,k}(\mathbf{t}).$$

Here  $\Omega_{a,l;b,k}(\mathbf{t})$  are understood as the substitution of the solution in  $\Omega_{a,l;b,k}$ . We call  $\tau(\mathbf{t})$  the *tau-function of the solution  $u_s(\mathbf{t})$*  to the DS hierarchy. Note that the tau-function  $\tau(\mathbf{t})$  is uniquely determined by the solution  $u_s(\mathbf{t})$  to the DS hierarchy (1.22) up to only a factor of the form

$$\exp\left(d_0 + \sum_{i \in E \cap \mathbb{Z}_{>0}} d_i t_i\right), \quad d_0, d_i \text{ are arbitrary constants.}$$

Clearly, for any  $N \geq 3$ ,

$$\frac{\partial^N \log \tau(\mathbf{t})}{\partial t_{m_{c_1} + k_1 rh} \cdots \partial t_{m_{c_N} + k_N rh}} = \Omega_{c_1, k_1; \dots; c_N, k_N}(\mathbf{t}).$$

It immediately follows from Theorem 1.4 the next corollary.

**Corollary 1.5.** *For each  $N \geq 2$ , let  $c_1, \dots, c_N$  be arbitrarily given integers in  $\{1, \dots, n\}$ . For an arbitrary solution  $u_s(\mathbf{t})$  to the DS hierarchy (1.22), let  $\tau(\mathbf{t})$  be the tau-function of the solution. Then the generating series of logarithmic derivatives of  $\tau(\mathbf{t})$  (replacing  $\Omega_{c_1, k_1; \dots; c_N, k_N}$  in the left-hand side of (1.29) by  $\frac{\partial^N \log \tau(\mathbf{t})}{\partial t_{m_{c_1} + k_1 rh} \cdots \partial t_{m_{c_N} + k_N rh}}$ ) is equal to the right-hand side of (1.29) with  $R_{m_a}(\lambda)$  being replaced by  $R_{m_a}(\lambda; \mathbf{t})$ .*

A further investigation of the interplay between the Hamiltonian structure [9, 14] of the DS hierarchy and the tau-structure  $\Omega_{a,l;b,k}$  will be given in Section 4. In particular, for an untwisted affine Kac–Moody algebra and the choice of the special vertex  $c_0$  of its Dynkin diagram, we verify in Theorem 4.7 that  $\Omega_{a,l;b,k}$  agree with the axioms of Hamiltonian tau-symmetry in the sense of [16, 19].

The paper is organized as follows. In Section 2 we apply the matrix-resolvent method to the study of tau-functions for the DS hierarchies. In Section 3 we apply the matrix-resolvent method to the DS hierarchies for the affine Kac–Moody algebra  $A_2^{(2)}$ . In Section 4 we investigate relationship between the Hamiltonian structure of the DS hierarchy and the tau-structure.

## 2 The matrix-resolvent method to tau-functions for the DS hierarchy

In this section, we apply the matrix-resolvent (MR) method to the study of tau-functions for the DS hierarchies. In particular, we will prove Theorem 1.4.

## 2.1 The standard realization of affine Kac–Moody algebras

Let  $\mathfrak{g}$  be a simple finite-dimensional Lie algebra of rank  $n$ , and let  $\sigma$  be an automorphism of  $\mathfrak{g}$  satisfying  $\sigma^N = 1$  for a positive integer  $N$ . Since  $\sigma$  is diagonalizable, we have the direct sum decomposition

$$\mathfrak{g} = \bigoplus_{\bar{k} \in \mathbb{Z}/N\mathbb{Z}} \mathfrak{g}_{\bar{k}}, \quad (2.1)$$

where  $\mathfrak{g}_{\bar{k}}$  is the eigenspace of  $\sigma$  with eigenvalue  $e^{\frac{2\pi i k}{N}}$ .

As in Section 1, denote by  $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}((\lambda^{-1})) = \mathfrak{g}((\lambda^{-1}))$  the space of Laurent series in the variable  $\lambda^{-1}$  with coefficients in  $\mathfrak{g}$ . The Lie algebra structure of  $\mathfrak{g}$  extends to a Lie algebra structure on  $L(\mathfrak{g})$  in the obvious way. We extend  $\sigma$  to a Lie algebra homomorphism (which we still denote by  $\sigma$ )  $\sigma: L(\mathfrak{g}) \rightarrow L(\mathfrak{g})$  given by

$$\sigma(a \otimes f(\lambda)) = \sigma(a) \otimes f(e^{-\frac{2\pi i}{N}} \lambda), \quad (2.2)$$

for  $a \in \mathfrak{g}$ ,  $f \in \mathbb{C}((\lambda^{-1}))$ . The subalgebra of invariant elements with respect to  $\sigma$  is the twisted algebra of Laurent series in the variable  $\lambda^{-1}$  with coefficients in  $\mathfrak{g}$ , and we denote it by

$$L(\mathfrak{g}, \sigma) = L(\mathfrak{g})^\sigma = \{a \in L(\mathfrak{g}) \mid \sigma(a) = a\}.$$

On  $L(\mathfrak{g}, \sigma)$  we have the following gradation induced by the gradation (2.1) of  $\mathfrak{g}$  and the action of  $\sigma$  given by (2.2):

$$L(\mathfrak{g}, \sigma) = \widehat{\bigoplus_{k \in \mathbb{Z}} L(\mathfrak{g}, \sigma)_k}, \quad (2.3)$$

where  $L(\mathfrak{g}, \sigma)_k = \mathfrak{g}_{\bar{k}} \otimes \lambda^k$ .

Let  $r$  be the least positive integer such that  $\sigma^r$  is an inner automorphism of  $\mathfrak{g}$ . Then  $r = 1, 2$  or  $3$ , and we have that  $\tilde{\mathfrak{g}}$ , the quotient of the affine Kac–Moody algebra  $X_n^{(r)}$  by the central element  $K$  (cf. Section 1), can be realized as

$$\tilde{\mathfrak{g}} \cong L(\mathfrak{g}, \sigma).$$

For every vertex  $c_m$  of the Dynkin diagram of  $X_n^{(r)}$  there exists an automorphism  $\sigma_m$  of  $\mathfrak{g}$  of order  $N_m = r a_m$ , such that  $\tilde{\mathfrak{g}} \cong L(\mathfrak{g}, \sigma_m)$ , and the standard gradation (1.9) of  $\tilde{\mathfrak{g}}$  becomes the gradation of  $L(\mathfrak{g}, \sigma_m)$  in powers of  $\lambda$  given by (2.3), see [23]. We call  $L(\mathfrak{g}, \sigma_m)$  the *standard realization* of  $\tilde{\mathfrak{g}}$  corresponding to the vertex  $c_m$ .

For the remaining of the section we will work with the standard realization of  $\tilde{\mathfrak{g}}$  corresponding to the vertex  $c_m$ .

Recall from Section 1 that  $\mathfrak{a} := L(\mathfrak{g}, \sigma_m)_0 = \mathfrak{g}_0$  is a semisimple Lie algebra, and  $e = \Lambda - e_m$  is a principal nilpotent element. By the Jacobson–Morozov theorem [6] there exist  $\rho^\vee$  and  $f$  in  $\mathfrak{a}$  such that

$$[\rho^\vee, e] = e, \quad [\rho^\vee, f] = -f, \quad [e, f] = \rho^\vee.$$

The decomposition (1.10) of  $\mathfrak{a}$  is the decomposition in  $\text{ad } \rho^\vee$ -eigenspaces. Note that  $\rho^\vee \in \mathfrak{a}^0$ , which is a Cartan subalgebra of  $\mathfrak{a}$ . The centralizer of  $\mathfrak{a}^0$  is a Cartan subalgebra of  $\mathfrak{g}$  (see [23]). Hence,  $\rho^\vee \in \mathfrak{g}$  is a semisimple element and by the representation theory of  $\mathfrak{sl}_2$  we have the  $\text{ad } \rho^\vee$ -eigenspace decomposition of  $\mathfrak{g}$ :

$$\mathfrak{g} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}^i, \quad \mathfrak{g}^i = \{a \in \mathfrak{g} \mid [\rho^\vee, a] = ia\}.$$

For an eigenvector  $a \in \mathfrak{g}$  with respect to the adjoint action of  $\rho^\vee$ , we denote by  $\delta(a)$  the corresponding eigenvalue, namely  $[\rho^\vee, a] = \delta(a)a$ . Note that the maximal eigenvalue for the adjoint action of  $\rho^\vee$  is  $\frac{r(h-1)}{N_m}$ .

The principal gradation (1.3) on  $\tilde{\mathfrak{g}} \cong L(\mathfrak{g}, \sigma_m)$  is then defined as follows: if  $a \otimes \lambda^k \in \tilde{\mathfrak{g}}$ ,  $k \in \mathbb{Z}$ , and  $a$  is an eigenvector for  $\text{ad } \rho^\vee$ , then

$$\text{deg}^{\text{pr}}(a \otimes \lambda^k) = \delta(a) + k \frac{rh}{N_m}. \quad (2.4)$$

From equation (2.4) we have that the principal gradation (1.3) on  $\tilde{\mathfrak{g}}$  is defined by the following linear map:

$$\text{ad } \rho^\vee \otimes 1 + 1 \otimes \frac{rh}{N_m} \lambda \frac{d}{d\lambda}: \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}. \quad (2.5)$$

(In the sequel we will often omit the tensor product sign.)

As in Section 1 we write

$$L(\mathfrak{g}, \sigma_m) = \bigoplus_{k \in \mathbb{Z}} L(\mathfrak{g}, \sigma_m)^k, \quad (2.6)$$

where elements in  $\tilde{\mathfrak{g}}^k \cong L(\mathfrak{g}, \sigma)^k$  have principal degree  $k$ .

Denote by  $(\cdot | \cdot)$  the normalized invariant bilinear form on  $\mathfrak{g}$ :

$$(a|b) = \frac{1}{2h_{\mathfrak{g}}^\vee} \text{tr}(\text{ad } a \circ \text{ad } b), \quad a, b \in \mathfrak{g}, \quad (2.7)$$

where  $h_{\mathfrak{g}}^\vee$  is the dual Coxeter number of  $\mathfrak{g}$ . We extend it to a bilinear form on  $L(\mathfrak{g})$  with values in  $\mathbb{C}((\lambda^{-1}))$  by

$$(a \otimes f(\lambda) | b \otimes g(\lambda)) = (a|b) f(\lambda) g(\lambda), \quad a, b \in \mathfrak{g}, \quad f(\lambda), g(\lambda) \in \mathbb{C}((\lambda^{-1})). \quad (2.8)$$

Throughout the paper we will consider the restriction of this  $\mathbb{C}((\lambda^{-1}))$ -valued bilinear form to  $\tilde{\mathfrak{g}}$ .

**Remark 2.1.** If  $a(\lambda) \in \tilde{\mathfrak{g}} \subset L(\mathfrak{g})$  and  $b(\lambda) \in L(\mathfrak{g})$ , then we can compute  $(a(\lambda)|b(\lambda))$ . Note that  $\tilde{\mathfrak{g}}$  is not preserved by  $\partial_\lambda$  if  $r > 1$ . Nevertheless, an expression of the form  $(\partial_\lambda a(\lambda)|b(\lambda))$ ,  $a(\lambda), b(\lambda) \in \tilde{\mathfrak{g}}$  still makes sense. We note that, however, the operator  $\lambda \partial_\lambda$  does preserve  $\tilde{\mathfrak{g}}$ , and one could think of  $(\partial_\lambda a(\lambda)|b(\lambda))$  as defined by  $(\partial_\lambda a(\lambda)|b(\lambda)) = (\lambda \partial_\lambda a(\lambda)|b(\lambda)) \lambda^{-1}$ .

## 2.2 Basis of the principal Heisenberg subalgebra and basic resolvents

Under the standard realization, we often write  $\Lambda = \Lambda(\lambda)$ , and let us fix a basis  $\{\Lambda_i(\lambda) \mid i \in E\}$  of  $\mathcal{H}$  (cf. (1.6)), with  $\text{deg}^{\text{pr}} \Lambda_i(\lambda) = i$ , as follows. We let  $\Lambda_1(\lambda) = \Lambda(\lambda)$  and

$$\Lambda_{m_a + rhk}(\lambda) = \Lambda_{m_a}(\lambda) \lambda^{kN_m}, \quad k \in \mathbb{Z}, \quad 1 \leq a, b \leq n, \quad (2.9)$$

where  $\Lambda_{m_a}(\lambda)$  are normalized by the condition

$$(\Lambda_{m_a}(\lambda) | \Lambda_{m_b}(\lambda)) = \delta_{a+b, n+1} h \lambda^{N_m}, \quad 1 \leq a, b \leq n. \quad (2.10)$$

Recall that  $\Lambda(\lambda) = e + e_m(\lambda)$ , with  $e_m(\lambda) \in \tilde{\mathfrak{g}}^1$ . By (2.3) we have that  $e_m(\lambda) = \tilde{e}_m \lambda$ , for some  $\tilde{e}_m \in \mathfrak{g}_{\bar{1}}$ .

We note that the invariance of the bilinear form (2.8) and the fact that  $\mathcal{H}$  is abelian imply that the decomposition (1.5) is orthogonal with respect to  $(\cdot | \cdot)$ .



**Lemma 2.2.** *For  $a, b = 1, \dots, n$ , we have*

$$(\partial_\lambda \Lambda_{m_a}(\lambda) | \Lambda_{m_b}(\lambda)) = \delta_{a+b, n+1} \frac{m_a N_m}{r} \lambda^{N_m-1}. \quad (2.11)$$

**Proof.** Since  $\deg^{\text{pr}} \Lambda_{m_a}(\lambda) = m_a$ , using the grading operator defined in (2.5), we get the identity

$$m_a \Lambda_{m_a}(\lambda) = [\rho^\vee, \Lambda_{m_a}(\lambda)] + \frac{r\hbar}{N_m} \lambda \partial_\lambda \Lambda_{m_a}(\lambda). \quad (2.12)$$

By pairing both sides of (2.12) with  $\Lambda_{m_b}(\lambda)$  we get

$$\begin{aligned} m_a (\Lambda_{m_a}(\lambda) | \Lambda_{m_b}(\lambda)) &= ([\rho^\vee, \Lambda_{m_a}(\lambda)] | \Lambda_{m_b}(\lambda)) + \frac{r\hbar}{N_m} \lambda (\partial_\lambda \Lambda_{m_a}(\lambda) | \Lambda_{m_b}(\lambda)) \\ &= \frac{r\hbar}{N_m} \lambda (\partial_\lambda \Lambda_{m_a}(\lambda) | \Lambda_{m_b}(\lambda)), \end{aligned}$$

where in the last identity we used the invariance of the bilinear form and the fact that  $\mathcal{H}$  is abelian. Equation (2.19) follows by using the normalization condition given in (2.10) in the LHS of the above identity. ■

The following result will be used in Section 4.

**Lemma 2.3.** *For  $a = 1, \dots, n$ , we have that*

$$\pi_{\mathcal{H}}(\lambda \partial_\lambda \Lambda_{m_a}(\lambda)) = \frac{m_a N_m}{r\hbar} \Lambda_{m_a}(\lambda), \quad (2.13)$$

where we recall that  $\pi_{\mathcal{H}}$  denotes the projection onto  $\mathcal{H}$  with respect to (1.5).

**Proof.** From (1.6) we have that  $\pi_{\mathcal{H}}(\lambda \partial_\lambda \Lambda_{m_a}(\lambda)) = \sum_{j \in E} c_j \Lambda_j(\lambda)$ . Using (1.7) and equations (2.9)–(2.10) we get

$$\begin{aligned} (\pi_{\mathcal{H}}(\lambda \partial_\lambda \Lambda_{m_a}(\lambda)) | \Lambda_{m_b+r\hbar k}(\lambda)) &= \sum_{i=1, \dots, n, l \in \mathbb{Z}} \delta_{i+b, n+1} c_{i,l} \hbar \lambda^{(l+k+1)N_m} \\ &= \sum_{l \in \mathbb{Z}} c_{n+1-b, l} \hbar \lambda^{(l+k+1)N_m}. \end{aligned} \quad (2.14)$$

Hence, since the decomposition (1.5) is orthogonal with respect to  $(\cdot | \cdot)$ , from equations (2.9)–(2.10) and (2.11) we have that

$$\begin{aligned} (\pi_{\mathcal{H}}(\lambda \partial_\lambda \Lambda_{m_a}(\lambda)) | \Lambda_{m_b+r\hbar k}(\lambda)) &= (\lambda \partial_\lambda \Lambda_{m_a}(\lambda) | \Lambda_{m_b+r\hbar k}(\lambda)) \\ &= \delta_{a+b, n+1} \frac{m_a N_m}{r} \lambda^{(k+1)N_m}. \end{aligned} \quad (2.15)$$

Combining equations (2.14) and (2.15) it follows that  $c_{i,l} = \delta_{i,a} \delta_{l,0} \frac{m_a N_m}{r\hbar}$ , for  $i = 1, \dots, n$  and  $l \in \mathbb{Z}$  thus proving equation (2.13). ■

Recall from Section 1 the definition (cf. Definition 1.1) of the resolvents  $R_i$ ,  $i \in E$ . Under the standard realization of  $\tilde{\mathfrak{g}}$ , we will often write  $R_i = R_i(\lambda)$ . Using the normalization (2.10) and the invariance of the  $\mathbb{C}((\lambda^{-1}))$ -valued bilinear form on  $\tilde{\mathfrak{g}}$  we get

$$(R_{m_a}(\lambda) | R_{m_b}(\lambda)) = (\Lambda_{m_a}(\lambda) | \Lambda_{m_b}(\lambda)) = \delta_{a+b, n+1} \hbar \lambda^{N_m}, \quad 1 \leq a, b \leq n. \quad (2.16)$$

For every  $a = 1, \dots, n$ , we decompose  $R_a(\lambda)$  according to (2.3) as follows:

$$R_a(\lambda) = \sum_{k \in \mathbb{Z}} R_{a;k}(\lambda), \quad (2.17)$$

where  $R_{a;k}(\lambda) \in \mathfrak{g}_{\bar{k}} \otimes \lambda^k$ . On the other hand, by (1.13) we have that

$$R_a(\lambda) = \Lambda_{m_a}(\lambda) + \text{lower order terms}, \quad (2.18)$$

where lower order terms are considered with respect to the principal gradation (2.6).

**Lemma 2.4.** *For  $a, b = 1, \dots, n$ , we have*

$$(\partial_\lambda R_a(\lambda)|R_b(\lambda)) = \delta_{a+b, n+1} \frac{m_a N_m}{r} \lambda^{N_m-1}. \quad (2.19)$$

**Proof.** It is immediate to check, using equation (1.14), the invariance of the bilinear form  $(\cdot|\cdot)$  and the fact that  $[R_{m_a}(\lambda), R_{m_b}(\lambda)] = 0$ , that  $\frac{\partial}{\partial x}(\partial_\lambda R_{m_a}(\lambda)|R_{m_b}(\lambda)) = 0$ . Hence,

$$(\partial_\lambda R_{m_a}(\lambda)|R_{m_b}(\lambda)) = (\partial_\lambda \Lambda_{m_a}(\lambda)|\Lambda_{m_b}(\lambda)).$$

The claim follows from Lemma 2.2. ■

### 2.3 From basic resolvents to tau-function

Using the basis of  $\mathcal{H}$  given by (2.9) and the fact that the standard gradation of  $\tilde{\mathfrak{g}}$  corresponds to the gradation of  $L(\mathfrak{g}, \sigma_m)$  in powers of  $\lambda$ , we write the pre-DS hierarchy (1.15) as

$$\frac{\partial \mathcal{L}}{\partial t_{m_a + rhk}} = [(\lambda^{k N_m} R_{m_a}(\lambda))_+, \mathcal{L}], \quad a = 1, \dots, n, \quad k \in \mathbb{Z}_+, \quad (2.20)$$

where the subscript  $+$  stands for the polynomial part in  $\lambda$  (we are choosing  $i = m_a + rhk \in E \cap \mathbb{Z}_{>0}$ ). Let  $\pi_\lambda: \mathbb{C}((\lambda^{-1})) \rightarrow \lambda^{-1} \mathbb{C}[[\lambda^{-N_m}]]$  be the linear map defined via

$$\lambda^k \mapsto \begin{cases} \lambda^k & \text{if } k \equiv -1 \pmod{N_m}, \quad k < 0, \\ 0 & \text{otherwise,} \end{cases}$$

where  $k \in \mathbb{Z}$ . Let  $\epsilon$  be a primitive  $N_m$ -root of unity. Recall that, for  $h \in \mathbb{Z}$ , we have

$$\sum_{k=0}^{N_m-1} \epsilon^{kh} = \begin{cases} N_m & \text{if } h \equiv 0 \pmod{N_m}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\pi_\lambda$  can be equivalently defined as follows:

$$\pi_\lambda(f(\lambda)) = \frac{1}{N_m} \sum_{k=0}^{N_m-1} \epsilon^k f(\epsilon^k \lambda)_-, \quad f(\lambda) \in \mathbb{C}((\lambda^{-1})), \quad (2.21)$$

where  $f(\lambda)_- \in \lambda^{-1} \mathbb{C}[[\lambda^{-1}]]$  denotes the singular part of  $f(\lambda)$ . Similarly, we will denote

$$\pi_{\lambda, \mu} = \pi_\lambda \circ \pi_\mu: \mathbb{C}((\lambda^{-1}, \mu^{-1})) \rightarrow \lambda^{-1} \mu^{-1} \mathbb{C}[[\lambda^{-N_m}, \mu^{-N_m}]]$$

and

$$\pi_{\lambda, \mu, \eta} = \pi_\lambda \circ \pi_\mu \circ \pi_\eta: \mathbb{C}((\lambda^{-1}, \mu^{-1}, \eta^{-1})) \rightarrow \lambda^{-1} \mu^{-1} \eta^{-1} \mathbb{C}[[\lambda^{-N_m}, \mu^{-N_m}, \eta^{-N_m}]].$$

Clearly, the maps  $\pi_\lambda$ ,  $\pi_\mu$  and  $\pi_\eta$  commute. We extend  $\pi_\lambda$  to a map  $\mathcal{A}^q \otimes \mathbb{C}((\lambda^{-1})) \rightarrow \mathcal{A}^q \otimes \lambda^{-1} \mathbb{C}((\lambda^{-N_m}))$  in the obvious way. By using equations (2.16), (2.19) and (1.8) we find

$$(\lambda - \mu)^{-2} \left( (R_{m_a}(\lambda)|R_{m_b}(\mu)) - \frac{\delta_{a+b, n+1}}{r} (m_a \lambda^{N_m} + m_b \mu^{N_m}) \right) \in \mathcal{A}^q((\lambda^{-1}, \mu^{-1})).$$

Hence, the LHS of (1.23) is well defined; in other words,  $\Omega_{a,k;b,l} \in \mathcal{A}^q$  are well defined (see (1.24)). Recall also from Section 1 that  $\Omega_{a,k;b,l}$  are gauge invariant, hence they actually belong to  $\mathcal{R} \subset \mathcal{A}^q$ .

For a Laurent series  $a(\lambda) = \sum_{i \leq M} a_i \lambda^i$ , we denote  $\text{Res}_\lambda a(\lambda) = a_{-1}$  (which is equal to  $-\text{Res}_{\lambda=\infty} a(\lambda)$ ). The following result extends Proposition 2.3.2 in [3] to our current more general setting.

**Lemma 2.5.** *For  $a, b = 1, \dots, n$ , we have*

$$\operatorname{Res}_\mu F_{a,b}(\lambda, \mu) = \pi_\lambda(R_{m_a}(\lambda)|\partial_\lambda R_{m_b}(\lambda)_+). \quad (2.22)$$

*In particular, for every  $a = 1, \dots, n$ , we have*

$$\operatorname{Res}_\mu F_{a,1}(\lambda, \mu) = \pi_\lambda(R_{m_a}(\lambda)|\tilde{e}_m). \quad (2.23)$$

**Proof.** Equation (2.22) follows by taking the residue in  $\mu$  in both sides of equation (1.23) and using the identity (which holds for an arbitrary Laurent series  $a(\mu)$ )

$$\operatorname{Res}_\mu a(z)\iota_\mu(\mu - \lambda)^{-2} = \partial_\lambda a(\lambda)_+,$$

where  $\iota_\mu$  denotes the expansion in the region  $|\mu| > |\lambda|$ . Equation (2.23) is obtained from (2.22) by recalling that  $R_1(\lambda)_+ = \Lambda(\lambda) = e + \tilde{e}_m \lambda$ .  $\blacksquare$

For simplicity of notation, let us denote  $G_a(\lambda) = \pi_\lambda(R_{m_a}(\lambda)|\tilde{e}_m)$ ,  $a = 1, \dots, n$ . Using (1.24) and (2.23), we have

$$G_a(\lambda) = \sum_{k \in \mathbb{Z}_{\geq 0}} \Omega_{a,k;1,0} \lambda^{-N_m k - 1} \in \mathcal{R}[[\lambda^{-N_m}]] \lambda^{-1}. \quad (2.24)$$

It follows from the relations (1.27) with  $c = 1$ ,  $m = 0$  and the identification  $x = -t_1$  that  $\Omega_{a,k;1,0}$  are densities of conservation laws for the DS hierarchy (1.22). The relationship between the series  $G_a(\lambda)$  and the Hamiltonian structure of the DS hierarchies will be studied in Section 4.

Let us introduce the *loop operator* for the pre-DS hierarchy (2.20) as follows:

$$\nabla_a(\lambda) = \sum_{k \in \mathbb{Z}_{\geq 0}} \lambda^{-N_m k - 1} \frac{\partial}{\partial t_{m_a + r h k}}, \quad a = 1, \dots, n.$$

**Lemma 2.6.** *For every  $a = 1, \dots, n$ , we have*

$$\nabla_a(\lambda) R_{m_b}(\mu) = \pi_\lambda \frac{[R_{m_a}(\lambda), R_{m_b}(\mu)]}{\lambda - \mu}. \quad (2.25)$$

**Proof.** We have

$$\begin{aligned} \nabla_a(\lambda) R_{m_b}(\mu) &= \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{\partial R_{m_b}(\mu)}{\partial t_{m_a + r h k}} \lambda^{-N_m k - 1} = \sum_{k \in \mathbb{Z}_{\geq 0}} [(\mu^{N_m k} R_{m_a}(\mu))_+, R_{m_b}(\mu)] \lambda^{-N_m k - 1} \\ &= \sum_{k \in \mathbb{Z}_{\geq 0}} \operatorname{Res}_\rho [\rho^{N_m k} R_{m_a}(\rho), R_{m_b}(\mu)] \lambda^{-N_m k - 1} \iota_\rho(\rho - \mu)^{-1}. \end{aligned} \quad (2.26)$$

In the second identity we used equation (1.16) and in the third identity we used the fact that

$$a(w)_+ = \operatorname{Res}_z a(z) \iota_z(z - w)^{-1},$$

which holds for any formal series  $a(z)$  (here  $\operatorname{Res}_z$  is the coefficient of  $z^{-1}$  and  $\iota_z$  is the expansion in the region  $|z| > |w|$ ). Note that

$$\sum_{k \in \mathbb{Z}_{\geq 0}} \rho^{N_m k} \lambda^{-N_m k - 1} = \lambda^{N_m - 1} \iota_\rho(\lambda^{N_m} - \rho^{N_m})^{-1}. \quad (2.27)$$

Using equation (2.27), we can rewrite (2.26) as

$$\begin{aligned}
\nabla_a(\lambda)R_{m_b}(\mu) &= \frac{\lambda^{N_m-1}}{2\pi i} \oint_{|\mu| < |\rho| < |\lambda|} \frac{[R_{m_a}(\rho), R_{m_b}(\mu)]}{(\lambda^{N_m} - \rho^{N_m})(\rho - \mu)} \\
&= \lambda^{N_m-1} \left( \sum_{k=0}^{N_m-1} \frac{[R_{m_a}(\epsilon^k \lambda), R_{m_b}(\mu)]}{N_m(\epsilon^k \lambda)^{N_m-1}(\epsilon^k \lambda - \mu)} - \frac{[R_{m_a;N_m}(\lambda), R_{m_b}(\mu)]}{\lambda^{N_m}} \right) \\
&= \frac{1}{N_m} \sum_{k=0}^{N_m-1} \epsilon^k \frac{[R_{m_a}(\epsilon^k \lambda), R_{m_b}(\mu)]}{(\epsilon^k \lambda - \mu)} - \frac{[R_{m_a;N_m}(\lambda), R_{m_b}(\mu)]}{\lambda}, \tag{2.28}
\end{aligned}$$

where the second identity follows by the residue theorem. Equation (2.25) is obtained by combining equations (2.28) and (2.21) and the fact that (cf. (2.17))

$$\frac{1}{N} \sum_{k=0}^{N_m-1} \epsilon^k \left( \frac{[R_{m_a}(\epsilon^k \lambda), R_{m_b}(\mu)]}{(\epsilon^k \lambda - \mu)} \right)_+ = \frac{[R_{m_a;N_m}(\lambda), R_{m_b}(\mu)]}{\lambda},$$

where  $(\cdot)_+$  denotes the non-negative part in powers of  $\lambda$ . This concludes the proof.  $\blacksquare$

Recall the definitions of the differential polynomials  $\Omega_{c_1, k_1; \dots; c_N, k_N}$  from (1.24) and (1.28). We have the following proposition.

**Proposition 2.7.** *For each  $N \geq 2$ , let  $c_1, \dots, c_N$  be arbitrarily given integers in  $\{1, \dots, n\}$ . We have*

$$\begin{aligned}
\sum_{k_1, \dots, k_N \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^N \lambda_j^{-N_m k_j - 1} \Omega_{c_1, k_1; \dots; c_N, k_N} &= -\frac{\pi_{\lambda_1, \dots, \lambda_N}}{2h_g^\vee} \\
&\times \left( \sum_{s \in S_N / C_N} \frac{B(R_{c_{s_1}}(\lambda_{s_1}; \mathbf{t}), \dots, R_{c_{s_N}}(\lambda_{s_N}; \mathbf{t}))}{\prod_{j=1}^N (\lambda_{s_j} - \lambda_{s_{j+1}})} - \frac{\delta_{N, 2} \delta_{c_1 + c_2, n+1}}{2r} \frac{m_{c_1} \lambda_1^{N_m} + m_{c_2} \lambda_2^{N_m}}{(\lambda_1 - \lambda_2)^2} \right), \tag{2.29}
\end{aligned}$$

where  $S_N$  denotes the symmetric group and  $C_N$  the cyclic group, and  $s_{N+1} = s_1$ .

**Proof.** With the above Lemma 2.6, the proof could now follow the same lines as the proof of the KdV case [2] (cf. also [3, 18]); so we omit the details here.  $\blacksquare$

The properties (1.26), (1.27) follow from the symmetric nature of identities with  $N = 2$  and  $N = 3$  of Proposition 2.7, respectively.

**Proof of Theorem 1.4.** The theorem follows from (2.29), equation (1.25) and the invariance of the multilinear form  $B$ .  $\blacksquare$

**Remark 2.8.** DS hierarchies associated to untwisted affine Kac–Moody algebras and their tau functions were intensively studied via various methods [1, 3, 5, 17, 20, 21, 24, 32]. In particular, using the matrix-resolvent method, gauge invariant differential polynomials  $\Omega_{a, l; b, k}$  satisfying the tau-structure properties (1.26) and (1.27) (hence leading to the construction of tau-function following the scheme of [19]) were defined in [3] (cf. also references therein for earlier results). In the twisted cases, these were also constructed in [28, 29] with a different method. It would be interesting to extend the matrix-resolvent method to the *generalized Drinfeld–Sokolov hierarchies* [7, 8, 9, 12, 13, 21].

### 3 Examples

In this section, we apply the matrix-resolvent construction to the Drinfeld–Sokolov hierarchies associated to the affine Kac–Moody algebra  $A_2^{(2)}$ .

The Dynkin diagram of the affine Kac–Moody algebra  $A_2^{(2)}$  is

$$\begin{array}{c} \circ \quad \Leftarrow \quad \circ \\ c_0 \quad c_1 \end{array} \quad (3.1)$$

and its Cartan matrix is

$$C = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}. \quad (3.2)$$

As discussed in Section 1 we have a set of Chevalley generators  $e_0, e_1, h_0, h_1, f_0, f_1$  satisfying the relations (1.1) and (1.2). It follows immediately from the definition (3.2) of  $C$  that

$$a_0 = a_1^\vee = 2, \quad a_1 = a_0^\vee = 1,$$

hence the Coxeter number and dual Coxeter number of  $A_2^{(2)}$  are  $h = h^\vee = 3$ . The set of exponents  $E$  has the form (cf. (1.7))

$$E = (1 + 6\mathbb{Z}) \cup (5 + 6\mathbb{Z}),$$

that is we have  $m_1 = 1$  and  $m_2 = 5$  in (1.7).

Let  $K = h_0 + 2h_1$  be the central element. Recall that we are interested in the quotient of  $A_2^{(2)}$  by the one-dimensional space generated by  $K$ , which we denote by  $\tilde{\mathfrak{g}}$ . We describe this quotient as a subspace of  $L(\mathfrak{sl}_3)$  following the discussion in Section 2.1. The normalized invariant bilinear form (2.7) on  $\mathfrak{sl}_3$  is the trace form which we extend to  $L(\mathfrak{sl}_3)$  in the natural way.

#### 3.1 The Sawada–Kotera hierarchy

The Drinfeld–Sokolov hierarchy associated to  $A_2^{(2)}$  and the choice of the vertex  $c_0$  of its Dynkin diagram (3.1) is known to be the Sawada–Kotera hierarchy [31]. Following Section 2 we compute the basic matrix resolvents for this hierarchy.

##### 3.1.1 Principal and standard gradations for $A_2^{(2)}$ and the $c_0$ vertex

In this case there exists an automorphism  $\sigma_0$  of  $\mathfrak{sl}_3$  of order  $N_0 = 4$  such that  $\tilde{\mathfrak{g}} = L(\mathfrak{sl}_3, \sigma_0) \subset L(\mathfrak{sl}_3)$  (cf. Section 2.1). Explicitly,

$$\tilde{\mathfrak{g}} = \left\{ \left( \begin{array}{ccc} a_1(\lambda) + a_2(\lambda) & b_1(\lambda) + b_2(\lambda) & p(\lambda) \\ c_1(\lambda) + c_2(\lambda) & -2a_2(\lambda) & b_1(\lambda) - b_2(\lambda) \\ r(\lambda) & c_1(\lambda) - c_2(\lambda) & a_2(\lambda) - a_1(\lambda) \end{array} \right) \middle| \begin{array}{l} a_1(\lambda), p(\lambda), r(\lambda) \in \mathbb{C}((\lambda^{-4})) \\ b_2(\lambda), c_1(\lambda) \in \mathbb{C}((\lambda^{-4}))\lambda \\ a_2(\lambda) \in \mathbb{C}((\lambda^{-4}))\lambda^2 \\ b_1(\lambda), c_2(\lambda) \in \mathbb{C}((\lambda^{-4}))\lambda^3 \end{array} \right\}.$$

Let us consider the following Chevalley generators for  $\tilde{\mathfrak{g}}$  ( $E_{ij}$  denotes the elementary matrix):

$$\begin{aligned} e_0(\lambda) &= (E_{21} + E_{32})\lambda, & h_0 &= -2(E_{11} - E_{33}), & f_0(\lambda) &= 2(E_{12} + E_{23})\lambda^{-1}, \\ e_1 &= E_{13}, & h_1 &= E_{11} - E_{33}, & f_1 &= E_{31}. \end{aligned} \quad (3.3)$$

The *principal gradation* is defined by the linear map (2.5), where  $\rho^\vee = h_1/2$ . Explicitly, we have

$$\tilde{\mathfrak{g}}^{6k} = \mathbb{C}h_1\lambda^{4k}, \quad \tilde{\mathfrak{g}}^{6k+1} = \mathbb{C}e_0(\lambda)\lambda^{4k} \oplus \mathbb{C}e_1\lambda^{4k}, \quad \tilde{\mathfrak{g}}^{6k+2} = \mathbb{C}(E_{12} - E_{23})\lambda^{4k+1},$$

$$\begin{aligned}\widetilde{\mathfrak{g}}^{6k+3} &= \mathbb{C}(E_{11} - 2E_{22} + E_{33})\lambda^{4k+2}, & \widetilde{\mathfrak{g}}^{6k+4} &= \mathbb{C}(E_{21} - E_{32})\lambda^{4k+3}, \\ \widetilde{\mathfrak{g}}^{6k+5} &= \mathbb{C}f_0(\lambda)\lambda^{4k+4} \oplus \mathbb{C}f_1\lambda^{4k+4},\end{aligned}$$

where  $k \in \mathbb{Z}$ . The *standard gradation* corresponding to  $c_0$  is the gradation in powers of  $\lambda$ , i.e.,

$$\begin{aligned}\widetilde{\mathfrak{g}}_{4k} &= \mathbb{C}f_1\lambda^{4k} \oplus \mathbb{C}h_1\lambda^{4k} \oplus \mathbb{C}e_1\lambda^{4k}, & \widetilde{\mathfrak{g}}_{4k+1} &= \mathbb{C}e_0(\lambda)\lambda^{4k} \oplus \mathbb{C}(E_{12} - E_{23})\lambda^{4k+1}, \\ \widetilde{\mathfrak{g}}_{4k+2} &= \mathbb{C}(E_{11} - 2E_{22} + E_{33})\lambda^{4k+2}, & \widetilde{\mathfrak{g}}_{4k+3} &= \mathbb{C}f_0(\lambda)\lambda^{4k+4} \oplus \mathbb{C}(E_{21} - E_{32})\lambda^{4k+3},\end{aligned}$$

where  $k \in \mathbb{Z}$ . Note that  $\mathfrak{a} = \widetilde{\mathfrak{g}}_0 \cong \mathfrak{sl}_2$ , moreover,  $\mathfrak{n} = \mathbb{C}f_1 \subset \mathfrak{b} = \mathbb{C}f_1 \oplus \mathbb{C}h_1$ .

Recall that the element  $\Lambda(\lambda) = e_0(\lambda) + e_1 \in \widetilde{\mathfrak{g}}$  is semisimple and we have the direct sum decomposition (1.5). Let, as in Section 1,  $\mathcal{H} = \text{Ker ad } \Lambda(\lambda)$ . It is immediate to check that

$$\mathcal{H} = \left( \widehat{\bigoplus_{k \in \mathbb{Z}} \mathbb{C}\Lambda(\lambda)\lambda^{4k}} \right) \oplus \left( \widehat{\bigoplus_{k \in \mathbb{Z}} \mathbb{C}(f_0(\lambda) + 2f_1)\lambda^{4k}} \right). \quad (3.4)$$

We rewrite (3.4) as in (1.6) using the following basis  $\{\Lambda_i(\lambda) \mid i \in E\}$ :

$$\begin{aligned}\Lambda_1(\lambda) &= \Lambda(\lambda), & \Lambda_5(\lambda) &= \left( \frac{1}{2}f_0(\lambda) + f_1 \right) \lambda^4, \\ \Lambda_{1+6k}(\lambda) &= \Lambda_1(\lambda)\lambda^{4k}, & \Lambda_{5+6k}(\lambda) &= \Lambda_5(\lambda)\lambda^{4k}.\end{aligned}$$

Here  $k \in \mathbb{Z}$ . This basis satisfies the normalization conditions (2.9) and (2.10).

### 3.1.2 The matrix resolvent

Take the DS gauge  $V = \mathbb{C}f_1$  (cf. (1.20)). The element  $\mathcal{L}^{\text{can}}$  in (1.21) has the form

$$\mathcal{L}^{\text{can}} = \partial + \Lambda(\lambda) + uf_1 = \begin{pmatrix} \partial & 0 & 1 \\ \lambda & \partial & 0 \\ u & \lambda & \partial \end{pmatrix}.$$

We have that  $\mathcal{R} = \mathbb{C}[u, u_x, u_{2x}, \dots]$  is the algebra of differential polynomials in  $u$ . (Here and in what follows, for a smooth function  $y = y(x)$  of  $x$ , we denote  $y_{nx} := \partial_x^n(y)$ ,  $n \geq 0$ .) Let

$$R(\lambda) = \begin{pmatrix} a_1(\lambda) + a_2(\lambda) & b_1(\lambda) + b_2(\lambda) & p(\lambda) \\ c_1(\lambda) + c_2(\lambda) & -2a_2(\lambda) & b_1(\lambda) - b_2(\lambda) \\ r(\lambda) & c_1(\lambda) - c_2(\lambda) & a_2(\lambda) - a_1(\lambda) \end{pmatrix} \in \mathcal{R} \otimes \widetilde{\mathfrak{g}} \quad (3.5)$$

be a resolvent of  $\mathcal{L}^{\text{can}}$ . Then  $[\mathcal{L}^{\text{can}}, R(\lambda)] = 0$ ; cf. (1.14). It follows that

$$\begin{aligned}a_1(\lambda) &= \frac{1}{2}p_x(\lambda), & a_2(\lambda) &= \frac{1}{3}(ub_1(\lambda) - b_{1,2x}(\lambda))\lambda^{-1}, \\ b_2(\lambda) &= \frac{1}{3}(ub_{1,x}(\lambda) + u_x b_1(\lambda) - b_{1,3x}(\lambda))\lambda^{-2}, \\ c_1(\lambda) &= p(\lambda)\lambda - \frac{1}{3}(2b_{1,x}(\lambda)u_x + ub_{1,2x}(\lambda) + b_1(\lambda)u_{2x} - b_{1,4x}(\lambda))\lambda^{-2}, \\ c_2(\lambda) &= b_{1,x}(\lambda), & r(\lambda) &= b_1(\lambda)\lambda + up(\lambda) - \frac{1}{2}p_{2x}(\lambda),\end{aligned}$$

where  $b_1 = b_1(\lambda) \in \mathcal{R}((\lambda^{-4}))\lambda^3$  and  $p = p(\lambda) \in \mathcal{R}((\lambda^{-4}))$  satisfy the following system of ODEs:

$$\begin{cases} 9p_x\lambda^3 + 2(uu_x - u_{3x})b_1 + 2(u^2 - 3u_{2x})b_{1,x} - 6u_x b_{1,2x} - 4ub_{1,3x} + 2b_{1,5x} = 0, \\ 6\lambda b_{1,x} + 2u_x p + 4up_x - p_{3x} = 0. \end{cases} \quad (3.6)$$

To find the basic resolvent  $R_1(\lambda)$ , we write it as in (2.18) as follows:

$$R_1(\lambda) = \Lambda(\lambda) + \text{terms of lower degree.}$$

Then one can solve (3.6) for  $b_1(\lambda) = \sum_{k \geq 0} b_{1,k} \lambda^{-4k-1}$  and  $p(\lambda) = 1 + \sum_{k \geq 1} p_k \lambda^{-4k}$  recursively. The first few terms are given by

$$\begin{aligned} b_1(\lambda) &= -\frac{u}{3} \lambda^{-1} + \frac{1}{243} (-7u^4 + 42u^2 u_{2x} + 21uu_x^2 - 21uu_{4x} - 21u_{2x}^2 - 21u_x u_{3x} + 3u_{6x}) \lambda^{-5} \\ &\quad + \dots, \\ p(\lambda) &= 1 + \frac{1}{81} (4u^3 - 9u_x^2 - 18uu_{2x} + 6u_{4x}) \lambda^{-4} + \frac{1}{6561} (-18u_{10x} + 162uu_{8x} \\ &\quad - 522u^2 u_{6x} + 798u^3 u_{4x} + 1395u_{4x}^2 - 3456uu_{3x}^2 - 630u_{2x} u^4 + 3591u^2 u_{2x}^2 \\ &\quad - 3252u_{2x}^3 - 1260u^3 u_x^2 + 1134u_x^4 + 648u_x u_{7x} + 1548u_{2x} u_{6x} + 2376u_{3x} u_{5x} \\ &\quad - 3132uu_x u_{5x} - 6066uu_{2x} u_{4x} - 4428u_x^2 u_{4x} + 4788u_x^2 u_x u_{3x} + 9324uu_x^2 u_{2x} \\ &\quad - 14184u_x u_{2x} u_{3x} + 35u^6) \lambda^{-8} + \dots. \end{aligned} \quad (3.7)$$

Similarly, to find the basic resolvent  $R_5(\lambda)$ , we write it as in (2.18) as follows:

$$R_5(\lambda) = \Lambda_5(\lambda) + \text{terms of lower degree.}$$

Then one can solve (3.6) for  $b_1(\lambda) = \lambda^3 + \sum_{k \geq 0} b_{1,k} \lambda^{-4k-1}$  and  $p(\lambda) = \sum_{k \geq 0} p_k \lambda^{-4k}$  recursively. The first few terms are

$$\begin{aligned} b_1(\lambda) &= \lambda^3 + \frac{1}{81} (5u^3 - 15uu_{2x} + 3u_{4x}) \lambda^{-1} + \frac{1}{6561} (-9u_{10x} + 99uu_{8x} - 396u^2 u_{6x} \\ &\quad + 726u^3 u_{4x} + 693u_{4x}^2 - 2079uu_{3x}^2 - 660u^4 u_{2x} + 2772u^2 u_{2x}^2 - 1716u_{2x}^3 \\ &\quad - 990u^3 u_x^2 + 297u_x^4 + 297u_x u_{7x} + 792u_{2x} u_{6x} + 1188u_{3x} u_{5x} - 1782uu_x u_{5x} \\ &\quad - 3762uu_{2x} u_{4x} - 1782u_x^2 u_{4x} + 3366u^2 u_x u_{3x} + 4950uu_x^2 u_{2x} - 6732u_x u_{2x} u_{3x} \\ &\quad + 44u^6) \lambda^{-5} + \dots, \\ p(\lambda) &= \frac{1}{9} (2u_{2x} - u^2) + \frac{1}{729} (-6u_{8x} + 42uu_{6x} - 96u^2 u_{4x} + 117u_{3x}^2 + 100u^3 u_{2x} \\ &\quad - 288uu_{2x}^2 + 150u^2 u_x^2 + 126u_x u_{5x} + 222u_{2x} u_{4x} - 384uu_x u_{3x} \\ &\quad - 396u_x^2 u_{2x} - 8u^5) \lambda^{-4} + \dots. \end{aligned} \quad (3.8)$$

Recall from Section 2.2 that  $\Lambda(\lambda) = e + e_0(\lambda)$ , where  $e = e_1$  and  $e_0(\lambda) = \tilde{e}_0 \lambda$ . From (3.3) we have that  $\tilde{e}_0 = E_{21} + E_{32}$ . Let  $R(\lambda)$  be as in (3.5), then  $(R(\lambda)|\tilde{e}_0) = 2b_1(\lambda)$ . Hence, recalling the definition of the series  $G_a(\lambda)$ ,  $a = 1, 2$ , given in (2.24) and using (3.7)–(3.8) we have the following expression for the first few terms of the tau-structure of the DS hierarchy

$$\begin{aligned} \Omega_{1,0;1,0} &= -\frac{2}{3}u, \\ \Omega_{1,1;1,0} &= \frac{2}{243} (-7u^4 + 42u^2 u_{2x} + 21uu_x^2 - 21uu_{4x} - 21u_{2x}^2 - 21u_x u_{3x} + 3u_{6x}), \\ \Omega_{2,0;1,0} &= \frac{2}{81} (5u^3 - 15uu_{2x} + 3u_{4x}), \\ \Omega_{2,1;1,0} &= \frac{2}{6561} (-9u_{10x} + 99uu_{8x} - 396u^2 u_{6x} + 726u^3 u_{4x} + 693u_{4x}^2 - 2079uu_{3x}^2 \\ &\quad - 660u^4 u_{2x} + 2772u^2 u_{2x}^2 - 1716u_{2x}^3 - 990u^3 u_x^2 + 297u_x^4 + 297u_x u_{7x} \\ &\quad + 792u_{2x} u_{6x} + 1188u_{3x} u_{5x} - 1782uu_x u_{5x} - 3762uu_{2x} u_{4x} - 1782u_x^2 u_{4x} \\ &\quad + 3366u^2 u_x u_{3x} + 4950uu_x^2 u_{2x} - 6732u_x u_x u_{3x} + 44u^6). \end{aligned} \quad (3.9)$$

Taking  $b = c = 1$  and  $k = m = 0$  in (1.27), and using the first equation in (3.9) and the fact that  $\partial_x = -\partial_{t_1}$ , the DS hierarchy for  $A_2^{(2)}$  and  $c_0$  can be written as

$$\frac{\partial u}{\partial t_{m_a+6l}} = \frac{3}{2} \partial_x \Omega_{a,l;1,0}, \quad a = 1, 2, \quad l \geq 0. \quad (3.10)$$

From (3.9) and (3.10) we get the following first few equations of the hierarchy:

$$\begin{aligned} \frac{\partial u}{\partial t_1} &= -u_x, \\ \frac{\partial u}{\partial t_7} &= \frac{1}{27} u_{7x} - \frac{7}{27} u u_{5x} + \frac{14}{27} u^2 u_{3x} - \frac{28}{81} u^3 u_x + \frac{7}{27} u_x^3 - \frac{14}{27} u_x u_{4x} - \frac{7}{9} u_{2x} u_{3x} + \frac{14}{9} u u_x u_{2x}, \\ \frac{\partial u}{\partial t_5} &= \frac{1}{9} u_{5x} - \frac{5}{9} u_x u_{2x} - \frac{5}{9} u u_{3x} + \frac{5}{9} u^2 u_x, \\ \frac{\partial u}{\partial t_{11}} &= -\frac{1}{243} u_{11x} + \frac{11}{243} u u_{9x} - \frac{44}{243} u^2 u_{7x} + \frac{242}{729} u^3 u_{5x} - \frac{220}{729} u^4 u_{3x} + \frac{88}{729} u^5 u_x \\ &\quad - \frac{110}{81} u^2 u_x^3 + \frac{44}{243} u_x u_{8x} + \frac{121}{243} u_{2x} u_{7x} + \frac{220}{243} u_{3x} u_{6x} - \frac{286}{243} u u_x u_{6x} + \frac{286}{243} u_{4x} u_{5x} \\ &\quad - \frac{616}{243} u u_{2x} u_{5x} - \frac{44}{27} u_x^2 u_{5x} - \frac{880}{243} u u_{3x} u_{4x} + \frac{616}{243} u^2 u_x u_{4x} + \frac{110}{27} u^2 u_{2x} u_{3x} \\ &\quad - \frac{440}{81} u_{2x}^2 u_{3x} + \frac{1298}{243} u u_x^2 u_{3x} - \frac{979}{243} u_x u_{3x}^2 - \frac{1540}{729} u^3 u_x u_{2x} + \frac{572}{81} u u_x u_{2x}^2 \\ &\quad + \frac{682}{243} u_x^3 u_{2x} - \frac{1562}{243} u_x u_{2x} u_{4x}. \end{aligned}$$

The equation corresponding to the flow  $\frac{\partial}{\partial t_5}$  is the Sawada–Kotera equation [31].

### 3.1.3 Matrix resolvent and residues of fractional powers of Lax operators

Let us consider the space  $\mathcal{R}((\lambda^{-1}))^3$  and let us denote  $\psi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  and  $\psi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ . We have the following decomposition

$$\mathcal{R}((\lambda^{-1}))^3 = W_1 \oplus W_2,$$

where

$$W_1 = \widehat{\bigoplus}_{k \in \mathbb{Z}} (\mathcal{R} \otimes \Lambda(\lambda)^k \psi_1) \quad \text{and} \quad W_2 = \widehat{\bigoplus}_{k \in \mathbb{Z}} (\mathcal{R} \otimes \Lambda(\lambda)^k \psi_2).$$

This decomposition can be checked directly using the formulas

$$\Lambda(\lambda)^{3k} = \lambda^{2k} \mathbb{1}_3, \quad \Lambda(\lambda)^{3k+1} = \lambda^{2k} \Lambda(\lambda), \quad \Lambda(\lambda)^{3k+2} = \lambda^{2k} \Lambda(\lambda)^2, \quad k \in \mathbb{Z}.$$

Following [14] we introduce a  $\mathcal{R}((\partial^{-1}))$ -module structure on  $\mathcal{R}((\lambda^{-1}))^3$  by setting

$$\partial^n \cdot \eta(\lambda) := (\partial + q^{\text{can}} + \Lambda(\lambda))^n (\eta(\lambda)), \quad \eta(\lambda) \in \mathcal{R}((\lambda^{-1}))^3, \quad n \in \mathbb{Z}.$$

Note that  $\Lambda(\lambda)$  is invertible, hence the action of  $\partial^{-1}$  is well defined using the geometric series expansion

$$\begin{aligned} (\partial + q^{\text{can}} + \Lambda(\lambda))^{-1} &= (\Lambda(\lambda)(\mathbb{1}_3 + \Lambda(\lambda)^{-1}(\partial + q^{\text{can}})))^{-1} \\ &= \sum_{k \in \mathbb{Z}_{\geq 0}} (-1)^k (\Lambda(\lambda)^{-1}(\partial + q^{\text{can}}))^k \Lambda(\lambda)^{-1}, \end{aligned} \quad (3.11)$$



which gives a well-defined operator on  $\mathcal{R}((\lambda^{-1}))^3$ . Indeed, we note that multiplication by

$$\Lambda(\lambda)^{-1} = \begin{pmatrix} 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \\ 1 & 0 & 0 \end{pmatrix}$$

does not increase the orders of powers of  $\lambda$  of elements in  $\mathcal{R}((\lambda^{-1}))^3$ .

Since  $q^{\text{can}}(W_i) \subset W_i$  we have that  $(\partial + q^{\text{can}} + \Lambda(\lambda))(W_i) \subset W_i$ ,  $i = 1, 2$ . Using the arguments in [14] we can show that any vector in  $W_i$ ,  $i = 1, 2$ , which contains only non-negative powers of  $\lambda$  can be uniquely expressed as  $A(\partial) \cdot \psi_i$ , where  $A(\partial) \in \mathcal{R}[\partial]$  is a differential operator.

Note that  $\lambda\psi_1 \in W_2$  and  $\lambda\psi_2 \in W_1$ . Hence, there exist unique  $L_1(\partial), L_2(\partial) \in \mathcal{R}[\partial]$  such that

$$L_1(\partial) \cdot \psi_1 = \lambda\psi_2, \quad L_2(\partial) \cdot \psi_2 = \lambda\psi_1.$$

It is immediate to check that  $L_1(\partial) = \partial^2 - u$  and  $L_2(\partial) = \partial$ . Moreover, we have that  $L_1(\partial)L_2(\partial)\psi_2 = \lambda^2\psi_2$ , from which it follows (denoting  $L(\partial) = L_1(\partial)L_2(\partial)$  and using the standard arguments in [14]) that  $(\lambda^{4k}R_1(\lambda))\psi_2 = L(\partial)^{\frac{6k+1}{3}} \cdot \psi_2$  and  $(\lambda^{4k}R_5(\lambda))\psi_2 = L(\partial)^{\frac{6k+5}{3}} \cdot \psi_2$ ,  $k \in \mathbb{Z}_{\geq 0}$ .

Recall from [14] that the DS hierarchy for  $A_2^{(2)}$  and the 0-th vertex of its Dynkin diagram can be rewritten as

$$\frac{\partial L(\partial)}{\partial t_i} = [L(\partial), (L(\partial)^{\frac{i}{3}})_+],$$

where  $i \in E \cap \mathbb{Z}_{>0}$  (recall that  $i = 1 + 6k$  or  $i = 5 + 6k$ ,  $k \geq 0$ ).

**Proposition 3.1.** *For  $a = 1, 2$  and  $k \in \mathbb{Z}_{\geq 0}$ , we have*

$$\Omega_{a,k;1,0} = 2 \text{Res}_{\partial} L(\partial)^{\frac{m_a+6k}{3}},$$

where  $\text{Res}_{\partial} L(\partial)^{\frac{m_a+6k}{3}}$  denotes the coefficient of  $\partial^{-1}$  of the pseudodifferential operator  $L(\partial)^{\frac{m_a+6k}{3}}$ .

**Proof.** Note that, if  $A(\partial) \in \mathcal{R}[\partial]$ , then  $A(\partial) \cdot \psi_2 \in \mathcal{R}[\lambda]^3$ . Hence, using (3.11) we have that

$$L(\partial)^{\frac{6k+m_a}{3}} \cdot \psi_2 = \text{a polynomial in } \lambda + \begin{pmatrix} \text{Res}_{\partial} L(\partial)^{\frac{6k+m_a}{3}} \\ * \\ * \end{pmatrix} \lambda^{-1} + O(\lambda^{-2}). \quad (3.12)$$

On the other hand, let  $R_{m_a}(\lambda)$  be written as in (3.16). Then we have that

$$(\lambda^{4k}R_{m_a}(\lambda))\psi_2 = \begin{pmatrix} b_1(\lambda) + b_2(\lambda) \\ -2a_2(\lambda) \\ c_1(\lambda) - c_2(\lambda) \end{pmatrix} \lambda^{4k}. \quad (3.13)$$

Recall that  $b_2(\lambda) \in \mathcal{R}((\lambda^{-4}))\lambda$ , hence  $\text{Res}_{\lambda} b_2(\lambda) = 0$ . Since  $(\lambda^{4k}R_{m_a}(\lambda))\psi_2 = L(\partial)^{\frac{6k+m_a}{3}} \cdot \psi_2$ , from equations (3.12) and (3.13) we then have

$$2 \text{Res}_{\partial} L(\partial)^{\frac{6k+m_a}{3}} = \text{Res}_{\lambda} 2b_1(\lambda)\lambda^{4k} = \text{Res}_{\lambda} (R_{m_a}(\lambda)|_{\tilde{e}_0})\lambda^{4k} = \text{Res}_{\lambda} G_a(\lambda)\lambda^{4k} = \Omega_{a,k;1,0}.$$

In the last identity we used (2.24). This concludes the proof.  $\blacksquare$

**Remark 3.2.** It is claimed in [14] that the differential polynomials  $\text{Res}_{\partial} L^{\frac{m_a+6k}{3}}(\partial)$ ,  $k \geq 0$ ,  $a = 1, 2$ , up to constant multiples can be served as Hamiltonian densities for the Hamiltonian structure of the SK hierarchy (for a review of the Hamiltonian formalism of DS hierarchies [14] see Section 4).

### 3.2 The Kaup–Kupershmidt hierarchy

The Drinfeld–Sokolov hierarchy associated to  $A_2^{(2)}$  and the vertex  $c_1$  of its Dynkin diagram (3.1) is known as the Kaup–Kupershmidt hierarchy [25]. Following Section 2 we compute the basic matrix resolvents for this hierarchy.

#### 3.2.1 Principal and standard gradations for $A_2^{(2)}$ and the $c_1$ vertex

In this case there exists an automorphism  $\sigma_1$  of  $\mathfrak{sl}_3$  of order  $N_1 = 2$  (cf. Section 2.1) such that  $\tilde{\mathfrak{g}} = L(\mathfrak{sl}_3, \sigma_1) \subset L(\mathfrak{sl}_3)$ :

$$\tilde{\mathfrak{g}} = \left\{ \left( \begin{array}{ccc} a_1(\lambda) + a_2(\lambda) & b_1(\lambda) + b_2(\lambda) & p(\lambda) \\ c_1(\lambda) + c_2(\lambda) & -2a_2(\lambda) & b_1(\lambda) - b_2(\lambda) \\ r(\lambda) & c_1(\lambda) - c_2(\lambda) & a_2(\lambda) - a_1(\lambda) \end{array} \right) \middle| \begin{array}{l} a_1(\lambda), b_1(\lambda), c_1(\lambda) \in \mathbb{C}((\lambda^{-2})) \\ a_2(\lambda), b_2(\lambda), c_2(\lambda), \\ p(\lambda), r(\lambda) \in \mathbb{C}((\lambda^{-2}))\lambda \end{array} \right\}.$$

Let us consider the following Chevalley generators for  $\tilde{\mathfrak{g}}$ :

$$\begin{aligned} e_0 &= E_{12} + E_{23}, & h_0 &= 2(E_{11} - E_{33}), & f_0 &= 2(E_{21} + E_{32}), \\ e_1(\lambda) &= E_{31}\lambda, & h_1 &= E_{33} - E_{11}, & f_1(\lambda) &= E_{13}\lambda^{-1}. \end{aligned} \quad (3.14)$$

The principal gradation is defined by the linear map (2.5), where  $\rho^\vee = h_0/2$ . Explicitly, we have

$$\begin{aligned} \tilde{\mathfrak{g}}^{6k} &= \mathbb{C}h_0\lambda^{2k}, & \tilde{\mathfrak{g}}^{6k+1} &= \mathbb{C}e_0\lambda^{2k} \oplus \mathbb{C}e_1(\lambda)\lambda^{2k}, & \tilde{\mathfrak{g}}^{6k+2} &= \mathbb{C}(E_{21} - E_{32})\lambda^{2k+1}, \\ \tilde{\mathfrak{g}}^{6k+3} &= \mathbb{C}(E_{11} - 2E_{22} + E_{33})\lambda^{2k+1}, & \tilde{\mathfrak{g}}^{6k+4} &= \mathbb{C}(E_{12} - E_{23})\lambda^{2k+1}, \\ \tilde{\mathfrak{g}}^{6k+5} &= \mathbb{C}f_0\lambda^{2k+2} \oplus \mathbb{C}f_1(\lambda)\lambda^{2k+2}, \end{aligned}$$

where  $k \in \mathbb{Z}$ . The standard gradation corresponding to the vertex  $c_1$  is the gradation in powers of  $\lambda$ . Hence we have

$$\begin{aligned} \tilde{\mathfrak{g}}_{2k} &= \mathbb{C}f_0\lambda^{2k} \oplus \mathbb{C}h_0\lambda^{2k} \oplus \mathbb{C}e_0\lambda^{4k}, \\ \tilde{\mathfrak{g}}_{2k+1} &= \mathbb{C}f_1\lambda^{2k} \oplus \mathbb{C}(E_{21} - E_{32})\lambda^{2k+1} \oplus \mathbb{C}(E_{11} - 2E_{22} + E_{33})\lambda^{2k+1} \\ &\quad \oplus \mathbb{C}(E_{12} - E_{23})\lambda^{2k+1} \oplus \mathbb{C}e_1\lambda^{2k}. \end{aligned}$$

Note that  $\mathfrak{a} = \tilde{\mathfrak{g}}_0 \cong \mathfrak{sl}_2$ , moreover,  $\mathfrak{n} = \mathbb{C}f_0 \subset \mathfrak{b} = \mathbb{C}f_0 \oplus \mathbb{C}h_0$ .

Recall that the element  $\Lambda(\lambda) = e_0 + e_1(\lambda) \in \tilde{\mathfrak{g}}$  is semisimple and we have the direct sum decomposition (1.5). Let, as in Section 1,  $\mathcal{H} = \text{Ker ad } \Lambda(\lambda)$ . It is immediate to check that

$$\mathcal{H} = \left( \widehat{\bigoplus_{k \in \mathbb{Z}} \mathbb{C}\Lambda(\lambda)\lambda^{2k}} \right) \oplus \left( \widehat{\bigoplus_{k \in \mathbb{Z}} \mathbb{C}(f_0 + 2f_1(\lambda))\lambda^{2k}} \right). \quad (3.15)$$

We rewrite (3.15) as in (1.6) using the following basis  $\{\Lambda_i(\lambda) \mid i \in E\}$ :

$$\Lambda_1(\lambda) = \Lambda(\lambda), \quad \Lambda_5(\lambda) = \left( \frac{1}{2}f_0 + f_1(\lambda) \right) \lambda^2, \quad \Lambda_{1+6k}(\lambda) = \Lambda_1(\lambda)\lambda^{2k}, \quad \Lambda_{5+6k}(\lambda) = \Lambda_5(\lambda)\lambda^{2k},$$

where  $k \in \mathbb{Z}$ . This basis satisfies the normalization conditions (2.9) and (2.10).

#### 3.2.2 The matrix resolvent

Take the DS gauge  $V = \mathbb{C}f_0$  (cf. (1.20)). The element  $\mathcal{L}^{\text{can}}$  in (1.21) has the form

$$\mathcal{L}^{\text{can}} = \partial + \Lambda(\lambda) + u \frac{f_0}{2} = \begin{pmatrix} \partial & 1 & 0 \\ u & \partial & 1 \\ \lambda & u & \partial \end{pmatrix}.$$

We have that  $\mathcal{R} = \mathbb{C}[u, u_x, u_{2x}, \dots]$  is the algebra of differential polynomials in  $u$ . Let

$$R(\lambda) = \begin{pmatrix} a_1(\lambda) + a_2(\lambda) & b_1(\lambda) + b_2(\lambda) & p(\lambda) \\ c_1(\lambda) + c_2(\lambda) & -2a_2(\lambda) & b_1(\lambda) - b_2(\lambda) \\ r(\lambda) & c_1(\lambda) - c_2(\lambda) & a_2(\lambda) - a_1(\lambda) \end{pmatrix} \in \mathcal{R} \otimes \tilde{\mathfrak{g}} \quad (3.16)$$

be a resolvent of  $\mathcal{L}^{\text{can}}$ . Then  $[\mathcal{L}^{\text{can}}, R(\lambda)] = 0$ ; cf. (1.14). Solving this linear system we find that

$$\begin{aligned} a_1(\lambda) &= b_{1,x}(\lambda), & a_2(\lambda) &= -\frac{1}{3}up(\lambda) + \frac{1}{6}p_{2x}(\lambda), & b_2(\lambda) &= \frac{1}{2}p_x(\lambda), \\ c_1(\lambda) &= ub_1(\lambda) + p(\lambda)\lambda - b_{1,2x}(\lambda), & c_2(\lambda) &= \frac{1}{3}u_xp(\lambda) + \frac{5}{6}up_x(\lambda) - \frac{1}{6}p_{3x}(\lambda), \\ r(\lambda) &= b_1(\lambda)\lambda + \frac{1}{3}(3u^2 - u_{2x})p(\lambda) - \frac{7}{6}u_xp_x(\lambda) - \frac{4}{3}up_{2x}(\lambda) + \frac{1}{6}p_{4x}(\lambda), \end{aligned}$$

where  $b_1 = b_1(\lambda) \in \mathcal{R}((\lambda^{-2}))\lambda$  and  $p = p(\lambda) \in \mathcal{R}((\lambda^{-2}))$  satisfy the following system of ODEs:

$$\begin{cases} 18b_{1,x}\lambda + 2(8uu_x - u_{3x})p + (16u^2 - 9u_{2x})p_x - 15u_xp_{xx} - 10up_{3x} + p_{5x} = 0, \\ 3p_x\lambda + 2u_xb_1 + 4ub_{1,x} - 2b_{1,3x} = 0. \end{cases} \quad (3.17)$$

To find the basic resolvent  $R_1(\lambda)$ , we write it as in (2.18) as follows:

$$R_1(\lambda) = \Lambda(\lambda) + \text{terms of lower degree.}$$

Then one can solve (3.17) for  $p(\lambda) = \sum_{k \geq 0} p_k \lambda^{-2k-1}$  and  $b_1(\lambda) = \sum_{k \geq 0} b_{1,k} \lambda^{-2k}$  recursively. The first few terms are as follows:

$$\begin{aligned} p(\lambda) &= -\frac{2}{3}u\lambda^{-1} + \frac{1}{243}(6u_{6x} - 84uu_{4x} + 336u^2u_{2x} - 147u_{2x}^2 + 420uu_x^2 \\ &\quad - 210u_xu_{3x} - 112u^4)\lambda^{-3} + \dots, \\ b_1(\lambda) &= 1 + \frac{1}{81}(32u^3 - 18u_x^2 - 36uu_{2x} + 3u_{4x})\lambda^{-2} + \frac{1}{6561}(-9u_{10x} + 216uu_{8x} \\ &\quad - 1908u^2u_{6x} + 7728u^3u_{4x} + 2718u_{4x}^2 - 15174uu_{3x}^2 - 15120u^4u_{2x} + 34776u^2u_{2x}^2 \\ &\quad - 11463u_{2x}^3 - 30240u^3u_x^2 + 7749u_x^4 + 864u_xu_{7x} + 2493u_{2x}u_{6x} + 4455u_{3x}u_{5x} \\ &\quad - 11448uu_xu_{5x} - 24714uu_{2x}u_{4x} - 14067u_x^2u_{4x} + 46368u^2u_xu_{3x} \\ &\quad + 77364uu_x^2u_{2x} - 48456u_xu_{2x}u_{3x} + 2240u^6)\lambda^{-4} + \dots. \end{aligned} \quad (3.18)$$

Similarly, we write

$$R_5(\lambda) = \Lambda_5(\lambda) + \text{terms of lower degree,}$$

and we solve (3.17) for  $p(\lambda) = \lambda + \sum_{k \geq 0} p_k \lambda^{-2k-1}$  and  $b_1(\lambda) = \sum_{k \geq 0} b_{1,k} \lambda^{-2k}$  recursively. The first few terms are as follows

$$\begin{aligned} p(\lambda) &= \lambda + \frac{1}{81}(40u^3 - 45u_x^2 - 60uu_{2x} + 6u_{4x})\lambda^{-1} + \frac{1}{6561}(150480uu_x^2u_{2x} \\ &\quad - 100188u_xu_{2x}u_{3x} - 47520u^3u_x^2 + 77616u^2u_xu_{3x} - 21384uu_xu_{5x} + 19602u_x^4 \\ &\quad - 30888u_x^2u_{4x} + 1782u_xu_{7x} - 21120u^4u_{2x} + 56232u^2u_{2x}^2 - 44352uu_{2x}u_{4x} \\ &\quad - 22044u_{2x}^3 + 4950u_{2x}u_{6x} + 11616u^3u_{4x} - 3168u^2u_{6x} - 27324uu_{3x}^2 + 396uu_{8x} \\ &\quad + 5445u_{4x}^2 + 8910u_{3x}u_{5x} - 18u_{10x} + 2816u^6)\lambda^{-3} + \dots, \end{aligned}$$

$$\begin{aligned}
b_1(\lambda) &= \frac{1}{9}(u_{2x} - 4u^2) + \frac{1}{729}(-3u_{8x} + 60uu_{6x} - 408u^2u_{4x} + 252u_{3x}^2 + 1120u^3u_{2x} \\
&\quad - 1224uu_{2x}^2 + 1680u^2u_x^2 + 180u_xu_{5x} + 402u_{2x}u_{4x} - 1632uu_xu_{3x} \\
&\quad - 1188u_x^2u_{2x} - 256u^5)\lambda^{-2} + \dots.
\end{aligned} \tag{3.19}$$

Recall from Section 2.2 that  $\Lambda(\lambda) = e + e_1(\lambda)$ , where  $e = e_0$  and  $e_1(\lambda) = \tilde{e}_1\lambda$ . From (3.14) we have that  $\tilde{e}_1 = E_{31}$ . Let  $R(\lambda)$  be as in (3.16), then  $(R(\lambda)|\tilde{e}_1) = p(\lambda)$ . Hence, recalling the definition of the series  $G_a(\lambda)$ ,  $a = 1, 2$ , given in (2.24) and (3.18)–(3.19) we have the following expression for the first few terms of the tau-structure of the DS hierarchy

$$\begin{aligned}
\Omega_{1,0;1,0} &= -\frac{2}{3}u, \\
\Omega_{1,1;1,0} &= \frac{1}{243}(6u_{6x} - 84uu_{4x} + 336u^2u_{2x} - 147u_{2x}^2 + 420uu_x^2 - 210u_xu_{3x} - 112u^4), \\
\Omega_{2,0;1,0} &= \frac{1}{81}(40u^3 - 45u_x^2 - 60uu_{2x} + 6u_{4x}), \\
\Omega_{2,1;1,0} &= \frac{1}{6561}(150480uu_x^2u_{2x} - 100188u_xu_{2x}u_{3x} - 47520u^3u_x^2 + 77616u^2u_xu_{3x} \\
&\quad - 21384uu_xu_{5x} + 19602u_x^4 - 30888u_x^2u_{4x} + 1782u_xu_{7x} - 21120u^4u_{2x} \\
&\quad + 56232u^2u_{2x}^2 - 44352uu_{2x}u_{4x} - 22044u_{2x}^3 + 4950u_{2x}u_{6x} + 11616u^3u_{4x} \\
&\quad - 3168u^2u_{6x} - 27324u_{3x}^2u + 396uu_{8x} + 5445u_{4x}^2 + 8910u_{3x}u_{5x} \\
&\quad - 18u_{10x} + 2816u^6).
\end{aligned} \tag{3.20}$$

Similarly to what done in Section 3.1, one can show that the DS hierarchy for  $A_2^{(2)}$  and  $c_1$  can be written again using equation (3.10). Hence, from (3.10) and (3.20) we get the first few equations of the hierarchy:

$$\begin{aligned}
\frac{\partial u}{\partial t_1} &= -u_x, \\
\frac{\partial u}{\partial t_6} &= \frac{1}{27}u_{7x} - \frac{14}{27}uu_{5x} + \frac{56}{27}u^2u_{3x} - \frac{224}{81}u^3u_x + \frac{70}{27}u_x^3 - \frac{49}{27}u_xu_{4x} - \frac{28}{9}u_{2x}u_{3x} + \frac{28}{3}uu_xu_{2x}, \\
\frac{\partial u}{\partial t_5} &= \frac{1}{9}u_{5x} - \frac{25}{9}u_xu_{2x} - \frac{10}{9}uu_{3x} + \frac{20}{9}u^2u_x, \\
\frac{\partial u}{\partial t_{11}} &= -\frac{1}{243}u_{11x} + \frac{22}{243}uu_{9x} - \frac{176}{243}u^2u_{7x} + \frac{1936}{729}u^3u_{5x} - \frac{3520}{729}u^4u_{3x} + \frac{2816}{729}u^5u_x \\
&\quad - \frac{880}{27}u^2u_x^3 + \frac{121}{243}u_xu_{8x} + \frac{374}{243}u_{2x}u_{7x} + \frac{770}{243}u_{3x}u_{6x} - \frac{1540}{243}uu_xu_{6x} + \frac{1100}{243}u_{4x}u_{5x} \\
&\quad - \frac{3652}{243}uu_{2x}u_{5x} - \frac{968}{81}u_x^2u_{5x} - \frac{5500}{243}uu_{3x}u_{4x} + \frac{6248}{243}u^2u_xu_{4x} + \frac{3520}{81}u^2u_{2x}u_{3x} \\
&\quad - \frac{3080}{81}u_{2x}^2u_{3x} + \frac{16984}{243}uu_x^2u_{3x} - \frac{7084}{243}u_xu_{3x}^2 - \frac{29920}{729}u^3u_xu_{2x} + \frac{2552}{27}uu_xu_{2x}^2 \\
&\quad + \frac{12716}{243}u_x^3u_{2x} - \frac{11462}{243}u_xu_{2x}u_{4x}.
\end{aligned}$$

The equation corresponding to the flow  $\frac{\partial}{\partial t_5}$  is the Kaup–Kupershmidt equation [25].

### 3.2.3 Matrix resolvent and residues of fractional powers of Lax operators

Let us consider the space  $\mathcal{R}((\lambda^{-1}))^3$ . Following [14] we introduce a  $\mathcal{R}((\partial^{-1}))$ -module structure on  $\mathcal{R}((\lambda^{-1}))^3$  by setting

$$\partial^n \cdot \eta(\lambda) = (\partial + q^{\text{can}} + \Lambda(\lambda))^n(\eta(\lambda)), \quad \eta(\lambda) \in \mathcal{R}((z^{-1}))^3, \quad n \in \mathbb{Z}.$$

Note that  $\Lambda$  is invertible, hence the action of  $\partial^{-1}$  is well defined using the geometric series expansion as in Section 3.1.3 (cf. (3.11)).

Let  $\psi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . Using the arguments in [14] we can show that any vector in  $\mathcal{R}[\lambda]^3$  can be uniquely expressed as  $A(\partial).\psi$ , where  $A(\partial) \in \mathcal{R}[\partial]$  is a differential operator. Hence, there exists unique  $L(\partial) \in \mathcal{R}[\partial]$  such that

$$L(\partial).\psi = \lambda\psi.$$

It is straightforward to check that  $L(\partial) = \partial^3 - u\partial - \partial \circ u$  from which it follows, using the standard arguments in [14], that  $(\lambda^{2k}R_1(\lambda))\psi = L(\partial)^{\frac{6k+1}{3}}.\psi$  and  $(\lambda^{2k}R_5(\lambda))\psi = L(\partial)^{\frac{6k+5}{3}}.\psi$ ,  $k \in \mathbb{Z}_{\geq 0}$ .

Recall from [14] that the DS hierarchy for  $A_2^{(2)}$  and vertex  $c_1$  of its Dynkin diagram can be rewritten as

$$\frac{\partial L(\partial)}{\partial t_i} = [L(\partial), (L(\partial)^{\frac{i}{3}})_+],$$

where  $i \in E \cap \mathbb{Z}_{\geq 1}$ .

**Proposition 3.3.** *For  $a = 1, 2$  and  $k \in \mathbb{Z}_+$ , we have*

$$\Omega_{a,k;1,0} = \text{Res}_{\partial} L(\partial)^{\frac{m_a+6k}{3}}.$$

**Proof.** The argument is the same as in the proof of Proposition 3.1. Recall that, if  $A(\partial) \in \mathcal{R}[\partial]$ , then  $A(\partial).\psi \in \mathcal{R}[\lambda]^3$ . Hence, using (3.11) we have that

$$L(\partial)^{\frac{6k+m_a}{3}}.\psi = \text{a polynomial in } \lambda + \begin{pmatrix} \text{Res}_{\partial} L(\partial)^{\frac{6k+m_a}{3}} \\ 0 \\ * \end{pmatrix} \lambda^{-1} + O(\lambda^{-2}). \quad (3.21)$$

On the other hand, let  $R_{m_a}(\lambda)$  be written as in (3.16). Then we have that

$$(\lambda^{2k}R_{m_a}(\lambda))\psi = \begin{pmatrix} p(\lambda) \\ b_1(\lambda) - b_2(\lambda) \\ a_2(\lambda) - a_1(\lambda) \end{pmatrix} \lambda^{2k}. \quad (3.22)$$

Since  $(\lambda^{2k}R_{m_a}(\lambda))\psi = L(\partial)^{\frac{6k+m_a}{3}}.\psi$ , from equations (3.21) and (3.22) we then have

$$\text{Res}_{\partial} L^{\frac{6k+m_a}{3}} = \text{Res}_{\lambda} p(\lambda)\lambda^{2k} = \text{Res}_{\lambda}(R_{m_a}(\lambda)|\tilde{e}_1)\lambda^{2k} = \text{Res}_{\lambda} G_a(\lambda)\lambda^{4k} = \Omega_{a,k;1,0}.$$

In the last identity we used (2.24). This concludes the proof.  $\blacksquare$

**Remark 3.4.** Similarly to Remark 3.2 we have that the differential polynomials  $\text{Res}_{\partial} L^{\frac{m_a+6k}{3}}(\partial)$ ,  $k \geq 0$ ,  $a = 1, 2$ , up to constant multiples can be served as Hamiltonian densities for the Hamiltonian structure of the KK hierarchy.

## 4 Hamiltonian structure and tau-structure<sup>1</sup>

### 4.1 Review of known results on Hamiltonian structures of the DS hierarchies

#### 4.1.1 Poisson structures

Recall from [14] that (up to a constant factor) there exists a unique non-degenerate symmetric invariant  $\mathbb{C}$ -valued bilinear form  $\kappa: \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \rightarrow \mathbb{C}$  which is coordinated with the principal and

<sup>1</sup>Section 4 is written by D.V. and D.Y. after Boris Dubrovin passed away. This section contains a generalization of Propositions 3.1 and 3.3 for the explicit  $A_2^{(2)}$  examples, and a proof of Theorem 4.7 that generalizes the  $A_1^{(1)}$  case.

standard gradations, that is,  $\kappa(a, b) = 0$  if  $a \in \tilde{\mathfrak{g}}^k$ ,  $b \in \tilde{\mathfrak{g}}^l$  and  $k+l \neq 0$  (similarly for the standard gradation). We note that the direct sum decomposition (1.5) is orthogonal with respect to  $\kappa$ . The restriction  $(\cdot | \cdot) = \kappa|_{\mathfrak{a} \times \mathfrak{a}}$  of  $\kappa$  to the semisimple subalgebra  $\mathfrak{a}$  is a non-degenerate symmetric invariant bilinear form [14]. Let us extend the bilinear form  $(\cdot | \cdot)$  on  $\mathfrak{a}$  to a bilinear form (which we still denote with the same symbol with an abuse of notation) on smooth functions  $u = u(x)$ ,  $v = v(x) \in C^\infty(S^1, \mathfrak{a})$ , from the circle  $S^1$  in  $\mathfrak{a}$ , in the natural way by letting

$$(u|v) = \int (u(x)|v(x)) dx. \quad (4.1)$$

Here and below, for a smooth function  $g(x)$ , we denote

$$\int g(x) dx := \frac{1}{2\pi} \int_{S^1} g(x) dx.$$

Recall from Section 1 that  $\mathcal{R} \subset \mathcal{A}^q$  denotes the space of gauge invariants. Let

$$\mathcal{F} = \{ \bar{f}[q] := \int f(q, q_x, q_{2x}, \dots) dx \mid f \in \mathcal{R} \}$$

be the space of local functionals whose densities are gauge invariant differential polynomials. The space  $\mathcal{F}$  can be canonically identified with the quotient space  $\mathcal{R}/\partial\mathcal{R}$ .

Let  $\bar{f} := \int f(q, q_x, q_{2x}, \dots) dx \in \mathcal{F}$ . Then,  $f(q, q_x, q_{2x}, \dots) \in \mathcal{R} \subset \mathcal{A}^q$  is a differential polynomial in the entries of  $q$ , and their  $x$ -derivatives.

Let  $V$  be a Drinfeld–Sokolov gauge (cf. (1.20)). Let  $v_1, \dots, v_{\dim \mathfrak{b}}$  be a basis of  $\mathfrak{b}$  homogeneous with respect to (1.10), such that  $v_1, \dots, v_\ell$  is a basis for  $V$ , and  $v_{\ell+1}, \dots, v_{\dim \mathfrak{b}}$  is a basis for  $[e, \mathfrak{n}]$ . Write  $q = \sum_{i=1}^{\dim \mathfrak{b}} q_i v_i$ . We define the variational derivative of  $\bar{f}$  with respect to  $q_i$  as

$$\frac{\delta \bar{f}}{\delta q_i} = \sum_{k \geq 0} (-\partial)^k \frac{\partial f}{\partial (\partial^k q_i)}, \quad i = 1, \dots, \dim \mathfrak{b}. \quad (4.2)$$

Let  $v^1, \dots, v^{\dim \mathfrak{b}}$  be the basis of  $\mathfrak{a}^{\geq 0}$ , which is dual, with respect to the non-degenerate bilinear form  $(\cdot | \cdot)$  on  $\mathfrak{a}$ , to the basis  $v_1, \dots, v_{\dim \mathfrak{b}}$  of  $\mathfrak{b}$ . Recall from [9, Lemma 3.13] that  $(v^i)_{i=1, \dots, \ell}$  is a basis of  $\mathfrak{a}^e = \{a \in \mathfrak{a} \mid [a, e] = 0\}$ , the centralizer of  $e$  in  $\mathfrak{a}$ . We identify the collection  $\frac{\delta \bar{f}}{\delta q} = \left( \frac{\delta \bar{f}}{\delta q_i} \right)_{i=1, \dots, \dim \mathfrak{b}}$  of all variational derivatives (4.2) of  $\bar{f}$  with respect to the variables  $q_i$  with a smooth functions with values in  $\mathfrak{a}^{\geq 0}$  by

$$\frac{\delta \bar{f}}{\delta q} = \sum_{i=1}^{\dim \mathfrak{b}} \frac{\delta \bar{f}}{\delta q_i} v^i. \quad (4.3)$$

In [14] it is shown that the following formula:

$$\{ \bar{f}, \bar{g} \}_2[q] = \int \left( \frac{\delta \bar{f}}{\delta q} \left[ \frac{\delta \bar{g}}{\delta q}, \partial + f + q \right] \right) dx, \quad \forall \bar{f}, \bar{g} \in \mathcal{F}, \quad (4.4)$$

defines a local Poisson bracket on  $\mathcal{F}$  (namely a Lie algebra structure on  $\mathcal{F}$ ). In particular, for any gauge transformation (1.18) given by  $N \in \mathcal{A}^q \otimes \mathfrak{n}$ ,  $\{ \bar{f}, \bar{g} \}_2[q^N] = \{ \bar{f}, \bar{g} \}_2[q]$ , so it is a well-defined element of  $\mathcal{F}$ .

Recall from the Section 1 that,  $f(q, q_x, q_{xx}, \dots) \in \mathcal{R}$  is a differential polynomial in the variables  $u_i$ ,  $i = 1, \dots, \ell$  ( $q^{\text{can}} = \sum_{i=1}^{\ell} u_i v_i$ , cf. (1.21)). Hence, we can also consider the vector of variational derivatives

$$\frac{\delta \bar{f}}{\delta q^{\text{can}}} = \left( \frac{\delta \bar{f}}{\delta u_i} \right)_{i=1}^{\ell},$$

where  $\frac{\delta \bar{f}}{\delta u_i}$  is defined as in (4.2), where  $q_i$  is replaced by  $u_i$ ,  $i = 1, \dots, \ell$ . As before, we will also use the following notation

$$\frac{\delta \bar{f}}{\delta q^{\text{can}}} = \sum_{i=1}^{\ell} \frac{\delta \bar{f}}{\delta u_i} v^i \quad (4.5)$$

to identify the variational derivative with a function with values in  $\mathfrak{a}^e$ . The Lie bracket (4.4) can be rewritten as

$$\{\bar{f}, \bar{g}\}_2(q^{\text{can}}) = \int \frac{\delta \bar{f}}{\delta q^{\text{can}}} P(\partial) \frac{\delta \bar{g}}{\delta q^{\text{can}}} dx, \quad \bar{f}, \bar{g} \in \mathcal{F}, \quad (4.6)$$

where  $P(\partial) = (P_{ij}(\partial))_{i,j=1}^{\ell}$  is a local Hamiltonian operator [9, 10, 14, 19].

Let us extend the bilinear form  $\kappa$  on  $\tilde{\mathfrak{g}}$  to a bilinear form on smooth functions with values in  $\tilde{\mathfrak{g}}$  in the natural way as in (4.1). For any  $\Theta \in \mathcal{H}$ , define  $\bar{h}_{\Theta} = \int \kappa(\Theta, H) dx \in \mathcal{F}$  (we review the proof of the fact  $\bar{h}_{\Theta}$  belongs to  $\mathcal{F}$  in Section 4.1.2). Here  $H$  is defined in (1.12). Then, the DS hierarchy (1.22) can be written in Hamiltonian form (using (4.4) or (4.6)) as

$$\frac{\partial u_s}{\partial t_{\Theta}} = \{\bar{h}_{\Theta}, u_s(x)\}_2 = \sum_{j=1}^{\ell} P_{sj}(\partial) \frac{\delta \bar{h}_{\Theta}}{\delta u_j}, \quad s = 1, \dots, \ell, \quad \Theta \in \mathcal{H}. \quad (4.7)$$

It is proved in [14] that  $\{\bar{h}_{\Theta_1}, \bar{h}_{\Theta_2}\}_2 = 0$ , for every  $\Theta_1, \Theta_2 \in \mathcal{H}$ .

**Example 4.1.** For both the SK hierarchy (see Section 3.1) and the KK hierarchy (see Section 3.2), we have that  $\mathcal{R} = \mathbb{C}[u, u_x, u_{xx}, \dots]$ . The Poisson structure (4.6) for the SK hierarchy is given by the Hamiltonian operator  $P(\partial) = -u_x - 2u\partial + \frac{1}{2}\partial^3$ , while the Poisson structure (4.6) for the KK hierarchy is given by the Hamiltonian operator  $P(\partial) = -\frac{u_x}{2} - u\partial + \frac{1}{2}\partial^3$  (see [9, 14]).

Recall also from [14], that for untwisted affine Kac–Moody algebras (this is the case when  $r = 1$ ) and the choice of the special vertex of the Dynkin diagram  $c_0$ , we have that  $\tilde{\mathfrak{g}} = \mathfrak{a}((\lambda^{-1}))$  and  $e_0 = e_{-\theta}\lambda$ ,  $e_{-\theta}$  being the lowest root vector of  $\mathfrak{a}$ . (With this realization, the standard gradation defined in Section 1 coincides with the gradation in powers of  $\lambda$ .) In this case it is possible to endow  $\mathcal{F}$  with another local Poisson bracket compatible with (4.4). It is defined as

$$\{\bar{f}, \bar{g}\}_1[q] = \int \left( \frac{\delta \bar{f}}{\delta q} \left[ \left[ e_{-\theta}, \frac{\delta \bar{g}}{\delta q} \right] \right] \right) dx, \quad \forall \bar{f}, \bar{g} \in \mathcal{F}.$$

As in the previous discussion, this can be rewritten as

$$\{\bar{f}, \bar{g}\}_1[q^{\text{can}}] = \int \frac{\delta \bar{f}}{\delta q^{\text{can}}} Q(\partial) \frac{\delta \bar{g}}{\delta q^{\text{can}}} dx, \quad \bar{f}, \bar{g} \in \mathcal{F}, \quad (4.8)$$

where  $Q(\partial) = (Q_{ij}(\partial))_{i,j=1}^{\ell}$  is another local Hamiltonian operator compatible to  $P(\partial)$  [9, 10, 14, 19].

Then, the DS hierarchy (1.22) can be written in another Hamiltonian form as

$$\frac{\partial u_s}{\partial t_{\Theta}} = \{\bar{h}_{\lambda\Theta}, u_s(x)\}_1 = \sum_{j=1}^{\ell} Q_{sj}(\partial) \frac{\delta \bar{h}_{\lambda\Theta}}{\delta u_j}, \quad s = 1, \dots, \ell, \quad \Theta \in \mathcal{H}, \quad (4.9)$$

and we have that  $\{\bar{h}_{\Theta_1}, \bar{h}_{\Theta_2}\}_1 = 0$ , for every  $\Theta_1, \Theta_2 \in \mathcal{H}$ . Furthermore, the following Lenard–Magri recursion relation [30] holds:

$$\{\bar{h}_{\Theta}, u(x)\}_2 = \{\bar{h}_{\lambda\Theta}, u(x)\}_1,$$

for every  $u(x) \in \mathcal{R}$  and  $\Theta \in \mathcal{H}$ . Thus, in this case, the DS hierarchy (1.22) is bi-Hamiltonian.

### 4.1.2 Hamiltonian densities

As in Section 1, let  $U \in \mathcal{A}^q \otimes (\text{Im ad } \Lambda)^{<0}$  and  $H \in \mathcal{A}^q \otimes \mathcal{H}^{<0}$  be such that (1.12) holds, namely  $e^{\text{ad} U} \mathcal{L} = \partial + \Lambda + H$ . Recall from Section 4.1.1 that the DS hierarchy (1.22) (cf. (4.7)) can be written in Hamiltonian form with respect to the Poisson structure (4.4) and the Hamiltonian

$$\bar{h}_{a,k} := \int \kappa(\Lambda_{m_a+rhk}, H) dx, \quad a = 1, \dots, n, \quad k \geq 0. \quad (4.10)$$

Let us recall the following result from [14].

**Proposition 4.2** ([14]). *The elements  $\bar{h}_{a,k}$  all belong to  $\mathcal{F}$ , that is, there exist differential polynomials  $g_{a,k} \in \mathcal{A}^q$  such that  $h_{a,k} - \partial_x(g_{a,k}) \in \mathcal{R}$ .*

**Proof.** Let  $N \in \mathcal{A}^q \otimes \mathfrak{n}$  and  $\tilde{\mathcal{L}} = e^{\text{ad} N} \mathcal{L} = \partial + \Lambda + \tilde{q}$  be as in (1.17). Let  $U_N \in \mathcal{A}^q \otimes (\text{Im ad } \Lambda)^{<0}$  and  $\tilde{H} \in \mathcal{A}^q \otimes \mathcal{H}^{<0}$  be such that  $e^{\text{ad} U_N} \tilde{\mathcal{L}} = \partial + \Lambda + \tilde{H}$ . Then,  $e^{\text{ad} U_N} e^{\text{ad} N} \mathcal{L} = \partial + \Lambda + \tilde{H}$ . Since  $\mathfrak{n} \subset \mathfrak{g}^{<0}$ , by the Baker–Campbell–Hausdorff formula  $e^{\text{ad} U_N} e^{\text{ad} N} = e^{\text{ad} \tilde{U}}$  for some  $\tilde{U} \in \mathcal{A}^q \otimes \mathfrak{g}^{<0}$ . Hence,  $\tilde{U}$  and  $\tilde{H}$  is another solution to equation (1.12), namely,  $e^{\text{ad} \tilde{U}} \mathcal{L} = \partial + \Lambda + \tilde{H}$ . Recall from [14] that this implies that there exists  $S \in \mathcal{A}^q \otimes \mathcal{H}^{<0}$  such that  $e^{\text{ad} \tilde{U}} = e^{\text{ad} S} e^{\text{ad} U_N}$ . Hence,

$$\begin{aligned} H - \tilde{H} &= (1 - e^{\text{ad} S}) e^{\text{ad} U} (\partial + q + \Lambda) = (1 - e^{\text{ad} S}) (\partial + \Lambda + H) = - \sum_{k \geq 1} \frac{(\text{ad} S)^k}{k!} (\partial + \Lambda + H) \\ &= \partial S - \sum_{k \geq 2} \frac{(\text{ad} S)^{k-1}}{k!} (\partial S) - \sum_{k \geq 1} \frac{(\text{ad} S)^k}{k!} (\Lambda + H). \end{aligned} \quad (4.11)$$

Let  $\Lambda_i \in \mathcal{H}$ ,  $i = m_a + rhk$ . By pairing both sides of (4.11) with  $\Lambda_i$ , using the invariance of the bilinear form and the fact that  $\mathcal{H}$  is abelian we get

$$\kappa(\Lambda_i, H) = \kappa(\Lambda_i, \tilde{H}) + \partial_x \kappa(\Lambda_i, S). \quad (4.12)$$

Equation (4.12) implies that the densities  $h_{a,k} \in \mathcal{A}^q$ ,  $a = 1, \dots, n$ ,  $k \geq 0$ , are gauge invariant up to total  $x$ -derivatives, hence  $\bar{h}_{a,k} \in \mathcal{F}$ .  $\blacksquare$

As in Section 2, let us take the standard realization of  $\tilde{\mathfrak{g}}$  corresponding to the vertex  $c_m$ . The  $\mathbb{C}$ -valued bilinear form  $\kappa$  on  $\tilde{\mathfrak{g}}$ , coordinated with the principal and standard gradations, can be realized as follows:

$$\kappa(a \otimes f(\lambda), b \otimes g(\lambda)) = \text{Res}_\lambda (a \otimes f(\lambda) | b \otimes g(\lambda)) \lambda^{-1} = (a|b) \text{Res}_\lambda f(\lambda) g(\lambda) \lambda^{-1}. \quad (4.13)$$

Noting that  $\mathcal{H}^{<0}$  is spanned by  $\Lambda_{m_a-rhk}(\lambda)$ ,  $a = 1, \dots, n$ ,  $k \geq 0$ , and using (2.9), we rewrite  $H = H(\lambda) \in \mathcal{A}^q \otimes \mathcal{H}^{<0}$  more explicitly as

$$H(\lambda) = \sum_{b=1}^n \sum_{l \in \mathbb{Z}_{\geq 0}} H_{b,l} \Lambda_{m_b-rh(l+1)}(\lambda) = \sum_{b=1}^n \sum_{l \in \mathbb{Z}_{\geq 0}} H_{b,l} \Lambda_{m_b}(\lambda) \lambda^{-(l+1)N_m}, \quad (4.14)$$

where  $H_{b,l} \in \mathcal{A}^q$ . Recalling the definition of the bilinear form  $\kappa$  in (4.13) and  $\bar{h}_{a,k} \in \mathcal{F}$  in (4.10), we have, for every  $a = 1, \dots, n$  and  $k \geq 0$ :

$$\begin{aligned} \bar{h}_{a,k} &= \int \kappa(\Lambda_{m_a+rhk}(\lambda), H(\lambda)) = \int \text{Res}_\lambda \sum_{b=1}^n \sum_{l \in \mathbb{Z}_{\geq 0}} (\Lambda_{m_a}(\lambda) | \Lambda_{m_b}(\lambda)) H_{b,l} \lambda^{(k-l-1)N_m-1} \\ &= \int \text{Res}_\lambda \sum_{l \in \mathbb{Z}_{\geq 0}} h H_{n+1-a,l} \lambda^{(k-l)N_m-1} = \int h H_{n+1-a,k}, \end{aligned} \quad (4.15)$$

where in the third equality we used equation (2.10).



Let us collect the densities  $h_{a,k}$ ,  $a = 1, \dots, n$ ,  $k \geq 0$ , into  $n$  generating series using the  $\mathbb{C}((\lambda^{-1}))$ -valued bilinear form on  $\tilde{\mathfrak{g}}$  (see [2, 3] and [9] for more details on the analogous construction for the untwisted case) by letting

$$g_a(\lambda) = (\Lambda_{m_a}(\lambda)|H(\lambda)) \in \mathcal{A}^q((\lambda^{-1})), \quad a = 1, \dots, n. \quad (4.16)$$

Indeed, using equations (4.14), (2.10) and (4.15) we get

$$g_a(\lambda) = \sum_{k \in \mathbb{Z}_{\geq 0}} h_{n+1-a,k} \lambda^{-kN_m} = \sum_{k \in \mathbb{Z}_{\geq 0}} h_{a,k} \lambda^{-kN_m} \in \mathcal{A}^q[[\lambda^{-N_m}]]. \quad (4.17)$$

## 4.2 The series $G_a(\lambda)$

Recall from equation (2.24) the gauge-invariant differential polynomials  $\Omega_{a,k;1,0}$  and its generating series  $G_a(\lambda)$ . In this subsection we establish a relation between  $\Omega_{a,k;1,0}$  and  $h_{a,k}$  by deriving an identity between the series  $G_a(\lambda)$  and  $g_a(\lambda)$ . To proceed, we need the following results.

**Lemma 4.3** ([9]).

(a) *Let  $D$  be a derivation of  $\mathcal{A}^q \otimes \tilde{\mathfrak{g}}$ . For every  $\alpha \in \mathbb{C}$ ,  $A, U_1, \dots, U_k \in \mathcal{A}^q \otimes \tilde{\mathfrak{g}}$ , with  $k \geq 1$ , we have*

$$\begin{aligned} D(\operatorname{ad} U_1 \cdots \operatorname{ad} U_k (\alpha \partial_x + A)) &= \sum_{h=1}^k \operatorname{ad} U_1 \cdots \operatorname{ad} D(U_h) \cdots \operatorname{ad} U_k (\alpha \partial_x + A) \\ &\quad + \operatorname{ad} U_1 \cdots \operatorname{ad} U_k (D(A)) \\ &\quad - \alpha \operatorname{ad} U_1 \cdots \operatorname{ad} U_{k-1} ([D, \partial_x](U_k)). \end{aligned} \quad (4.18)$$

(b) *For any  $\alpha \in \mathbb{C}$  and  $A, U, V \in \mathcal{A}^q \otimes \tilde{\mathfrak{g}}$  we have*

$$\begin{aligned} &\left[ \sum_{h \in \mathbb{Z}_{\geq 0}} \frac{1}{(h+1)!} (\operatorname{ad} U)^h (V), e^{\operatorname{ad} U} (\alpha \partial_x + A) \right] \\ &= \sum_{h, k \in \mathbb{Z}_{\geq 0}} \frac{1}{(h+k+1)!} (\operatorname{ad} U)^h \operatorname{ad} V (\operatorname{ad} U)^k (\alpha \partial_x + A). \end{aligned}$$

Now we can prove the following proposition.

**Proposition 4.4.** *We have that*

$$G_a(\lambda) = \left( \partial_\lambda - \frac{m_a N_m}{r h \lambda} \right) g_a(\lambda) + \partial_x X(\lambda),$$

for some  $X(\lambda) \in \mathcal{A}^q[[\lambda^{-N_m}]]\lambda^{-1}$ .

**Proof.** Let us start by computing  $\partial_\lambda H(\lambda)$ , where  $H(\lambda)$  is the series appearing in (1.12). We have

$$\begin{aligned} \partial_\lambda H(\lambda) &= \partial_\lambda (e^{-\operatorname{ad} U(\lambda)}(\mathcal{L}) - \partial_x - \Lambda(\lambda)) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \partial_\lambda ((\operatorname{ad} U(\lambda))^k(\mathcal{L})) \\ &= \sum_{k=1}^{\infty} \sum_{h=0}^{k-1} \frac{(-1)^k}{k!} (\operatorname{ad} U(\lambda))^h (\operatorname{ad} \partial_\lambda U(\lambda)) (\operatorname{ad} U(\lambda))^{k-1-h}(\mathcal{L}) \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} (\text{ad } U(\lambda))^k (\tilde{e}_m) \\
& = \sum_{h,k \in \mathbb{Z}_{\geq 0}} \frac{(-1)^{h+k+1}}{(h+k+1)!} (\text{ad } U(\lambda))^h (\text{ad } \partial_\lambda U(\lambda)) (\text{ad } U(\lambda))^k (\mathcal{L}) + e^{-\text{ad } U(\lambda)} (\tilde{e}_m) - \tilde{e}_m \\
& = [e^{-\text{ad } U(\lambda)} (\mathcal{L}), Y(\lambda)] + e^{-\text{ad } U(\lambda)} (\tilde{e}_m) - \tilde{e}_m \\
& = \partial_x Y(\lambda) + [\Lambda(\lambda) + H(\lambda), Y(\lambda)] + e^{-\text{ad } U(\lambda)} (\tilde{e}_m) - \tilde{e}_m,
\end{aligned} \tag{4.19}$$

where  $Y(\lambda) = \sum_{h \in \mathbb{Z}_{\geq 0}} \frac{(-1)^h}{(h+1)!} (\text{ad } U(\lambda))^h (\partial_\lambda U(\lambda))$ . In the first equality we used equation (1.12), in the second equality we used the definition of exponential function, in the third equality equation (4.18) and the facts that  $\partial_\lambda \mathcal{L} = \tilde{e}_m$  and  $[\partial_\lambda, \partial_x] = 0$ , the fourth equality is trivial, in the fifth equality we used Lemma 4.3(b), and in the last equality we used again equation (1.12).

From the definition of  $g_a(\lambda)$  we then have

$$\lambda \partial_\lambda g_a(\lambda) = (\lambda \partial_\lambda \Lambda_{m_a}(\lambda) | H(\lambda)) + (\Lambda_{m_a}(\lambda) | \lambda \partial_\lambda H(\lambda)). \tag{4.20}$$

Since  $H(\lambda) \in \mathcal{A}^q \otimes \text{Ker ad } \Lambda(\lambda)$  and the direct sum decomposition (1.5) is orthogonal with respect to  $(\cdot | \cdot)$  we have, using equation (2.13),

$$\begin{aligned}
(\lambda \partial_\lambda \Lambda_{m_a}(\lambda) | H(\lambda)) & = (\pi_{\mathcal{H}}(\lambda \partial_\lambda \Lambda_{m_a}(\lambda)) | H(\lambda)) = \frac{m_a N_m}{rh} (\Lambda_{m_a}(\lambda) | H(\lambda)) \\
& = \frac{m_a N_m}{rh} g_a(\lambda).
\end{aligned} \tag{4.21}$$

Furthermore, using equation (4.19) we obtain

$$\begin{aligned}
(\Lambda_{m_a}(\lambda) | \partial_\lambda H) & = (\Lambda_{m_a}(\lambda) | \partial_x Y(\lambda)) + (\Lambda_{m_a}(\lambda) | [\Lambda(\lambda) + H, Y(\lambda)]) + (\Lambda_{m_a}(\lambda) | e^{-\text{ad } U}(\tilde{e}_m)) \\
& \quad - (\Lambda_{m_a}(\lambda) | \tilde{e}_m) = (\Lambda_{m_a}(\lambda) | \partial_x Y(\lambda)) + ([\Lambda_{m_a}(\lambda), \Lambda(\lambda) + H(\lambda)] | Y(\lambda)) \\
& \quad + (R_{m_a}(\lambda) | \tilde{e}_m) - \delta_{a,n} \frac{N_m}{r} \lambda^{N_m-1} \\
& = \partial_x (\Lambda_{m_a}(\lambda) | Y(\lambda)) + (R_{m_a}(\lambda) | \tilde{e}_m) - \delta_{a,n} \frac{N_m}{r} \lambda^{N_m-1}.
\end{aligned} \tag{4.22}$$

Here, in the second equality we used the invariance of the Cartan–Killing form, the definition of  $R_{m_a}$  and equation (2.11); in the third equality we used the facts that  $\partial_x \Lambda_{m_a}(\lambda) = 0$  and  $\mathcal{H}$  is abelian. Combining equations (4.20), (4.21) and (4.22) we get the identity

$$(R_{m_a}(\lambda) | \tilde{e}_m) = \left( \partial_\lambda - \frac{m_a N_m}{rh\lambda} \right) g_a(\lambda) - \partial_x (\Lambda_{m_a}(\lambda) | Y(\lambda)) + \delta_{a,n} N_m \lambda^{N_m-1}. \tag{4.23}$$

Recall from equation (4.17) that  $g_a(\lambda) \in \mathcal{A}^q[[\lambda^{-N_m}]]$ . Hence, the claim follows by applying  $\pi_\lambda$  to both sides of (4.23).  $\blacksquare$

**Corollary 4.5.** *The elements  $\Omega_{a,k;1,0}$  are related to  $h_{a,k}$  by*

$$\Omega_{a,k;1,0} = -(m_a + rhk) \frac{N_m}{rh} h_{a,k} + \partial_x X_{a,k}$$

for some  $X_{a,k} \in \mathcal{A}^q$ . In other words, the DS hierarchy (1.22) can be written in terms of the gauge invariant densities  $\Omega_{a,k;1,0}$  as follows:

$$\frac{\partial u_s}{\partial t_{m_a + krh}} = \frac{-rh}{(m_a + rhk) N_m} \{ \bar{\Omega}_{a,k;1,0}, u_s(x) \}_2. \tag{4.24}$$

**Proof.** Follows straightforwardly from Proposition 4.4. ■

**Example 4.6.** Let  $L(\partial)$  be the Lax operator of the SK hierarchy given in Section 3.1.3 (respectively of the KK hierarchy given in Section 3.2.3). We have (see [11])

$$h_{a,k} = -\frac{m_a + 6k}{3} \operatorname{Res}_{\partial} L^{\frac{m_a+6k}{3}}(\partial) + \partial_x X_{a,k}, \quad a = 1, 5, \quad k \geq 0, \quad (4.25)$$

for some  $X_{a,k} \in \mathcal{A}^q$ . The identity (4.25) agrees with Corollary 4.5 and Proposition 3.1 (respectively Proposition 3.3).

In the last part of this section we will prove the following theorem, which extends the result in [2] (cf. also [18]) for the  $A_1^{(1)}$  case.

**Theorem 4.7.** *For an untwisted affine Kac–Moody algebra and the choice of the special vertex  $c_0$  of its Dynkin diagram, under a suitable Miura-type transformation, the gauge invariant differential polynomials  $\Omega_{a,k;1,0}$  satisfy the axioms of tau-symmetric bi-Hamiltonian structure given in [19].*

**Proof.** It is assumed here that  $r = 1$  (untwisted case) and that we choose the special vertex  $c_0$ . In particular, we have that  $n = \ell$  and  $N_0 = 1$  (see Section 2.1). Moreover, recall from Section 4.1.1 that in this case the DS hierarchy is bi-Hamiltonian and we can rewrite equation (4.24) in Corollary 4.5 using the first Hamiltonian structure as follows (cf. (2.10) and (4.9))

$$\frac{\partial u_s}{\partial t_{m_a+kh}} = \frac{-h}{m_a + h(k+1)} \left\{ \bar{\Omega}_{a,k+1;1,0}, u_s(x) \right\}_1. \quad (4.26)$$

This means that  $\Omega_{a,k+1,0} \in \mathcal{R}$  are (up to a scalar factor) Hamiltonian densities associated to the first Hamiltonian structure (4.8) for the DS hierarchy.

Next, from Corollary 4.5 and from the fact that  $\bar{h}_{a,0}$  are Casimirs of the first Hamiltonian structure proved in [9] we know that  $r_a = \Omega_{a,0;1,0}$  are densities of these Casimirs. Moreover, as it was shown in [3, Lemma 4.1.3], the map  $(u_1, \dots, u_n) \mapsto (r_1, \dots, r_n)$  is a Miura-type transformation meaning that the dispersionless limit of this map has non-degenerate Jacobian.

On the other hand, it is shown in [17] that the genus zero part of the DS hierarchy is equivalent to the principal Hierarchy of the Frobenius manifold of type  $\mathfrak{a}$  [15]. Finally, using (1.26), (1.27) and the normalization of times given in [3] (see  $t^{a,k}$  therein), it follows that under the Miura transformation  $(u_1, \dots, u_n) \mapsto (r_1, \dots, r_n)$  (see, e.g., [16]) the DS hierarchy written in the coordinate  $r_a$  (cf. (4.26))

$$\frac{\partial r_a}{\partial t_{m_b+krh}} = \left\{ \bar{h}_{b,k+1}, r_a(x) \right\}_1 = -\partial_x (\Omega_{b,k;a,0}) \quad (4.27)$$

is a tau-symmetric bi-Hamiltonian deformation of the principal Hierarchy in the sense of [19] (cf. also [16]), and the differential polynomials  $\Omega_{a,k;1,0}$  are tau-symmetric Hamiltonian densities in the sense of [19]. ■

According to Theorem 4.7, the tau-structure  $(\Omega_{a,k;b,l})_{k,l \geq 0, a,b=1, \dots, n}$  coincides with the axiomatic tau-structure in [19] for the DS hierarchies under the assumption of Theorem 4.7. The coordinates  $r_a$  introduced in the above proof are called *normal coordinates* [16, 19].

**Remark 4.8.** Under the condition of Theorem 4.7, one easily sees from (4.27) (choosing  $a = 1$ ) and a homogeneity argument that the tau-symmetric hamiltonian densities for the DS hierarchy if exist must be unique. The construction of tau-symmetric hamiltonian densities for DS hierarchies was previously given in [32] (cf. also [27, 29]) by using the central extension of  $\tilde{\mathfrak{g}}$ . Our construction, however, uses only the geometry of the resolvent manifold  $\mathcal{M}_{\mathcal{L}} = \{R \in \mathcal{A}^q \otimes \tilde{\mathfrak{g}} \mid [R, \mathcal{L}] = 0\}$  [14] and is therefore simpler from the computational point of view.

### 4.3 Variational derivative of the series $G_a(\lambda)$

Recall from Section 4.1.1 that  $(v_i)_{i=1,\dots,\dim \mathfrak{b}}$  is a basis of  $\mathfrak{b}$  and  $(v^i)_{i=1,\dots,\dim \mathfrak{b}}$  is the dual basis of  $\mathfrak{a}^{\geq 0}$ . Moreover, the basis is chosen so that  $(v_i)_{i=1,\dots,\ell}$  is a basis of the Drinfeld–Sokolov gauge  $V \subset \mathfrak{b}$  and  $(v_i)_{i=\ell+1,\dots,\dim \mathfrak{b}}$  is a basis of  $[e, \mathfrak{n}]$ . Recall from Section 4.1.1 that  $(v^i)_{i=1,\dots,\ell}$  is a basis for  $\mathfrak{a}^e$ .

For  $A(\lambda) \in \tilde{\mathfrak{g}}$  we define its projection on  $\tilde{\mathfrak{a}}^{\geq 0} := \mathfrak{a}^{\geq 0}((\lambda^{-N_m}))$  (respectively  $\tilde{\mathfrak{a}}^e := \mathfrak{a}^e((\lambda^{-N_m}))$ ) by

$$\pi_{\tilde{\mathfrak{a}}^{\geq 0}}(A(\lambda)) = \sum_{i=1}^{\dim \mathfrak{b}} (A(\lambda)|v_i)v^i \quad \left( \text{respectively } \pi_{\tilde{\mathfrak{a}}^e}(A(\lambda)) = \sum_{i=1}^{\ell} (A(\lambda)|v_i)v^i \right). \quad (4.28)$$

Recall the definition of variational derivatives given in (4.3) and (4.5). With a similar method used in [9] we can prove that

$$\frac{\delta \bar{g}_a(\lambda)}{\delta q} = \pi_{\tilde{\mathfrak{a}}^{\geq 0}} R_{m_a}(\lambda), \quad a = 1, \dots, n.$$

Moreover, in a similar way for which we are going to provide the details for completeness, we can prove the following result.

**Proposition 4.9.** *For every  $a = 1, \dots, n$  we have*

$$\frac{\delta \bar{g}_a(\lambda)}{\delta q^{\text{can}}} = \pi_{\tilde{\mathfrak{a}}^e} (e^{\text{ad } N^{\text{can}}} R_{m_a}(\lambda)) \in \mathcal{R} \otimes \mathfrak{a}^e [[\lambda^{-N_m}]],$$

where  $N^{\text{can}} \in \mathcal{A}^q \otimes \mathfrak{n}$  is defined by (1.21).

**Proof.** Let  $U^{\text{can}}(\lambda) \in \mathcal{R} \otimes (\text{Im ad } \Lambda(\lambda))^{\leq 0}$  and  $H^{\text{can}}(\lambda) \in \mathcal{R} \otimes \mathcal{H}_{<0}$  be the unique elements such that

$$e^{\text{ad } U^{\text{can}}(\lambda)} (\partial + \Lambda(\lambda) + q^{\text{can}}) = \partial + \Lambda(\lambda) + H^{\text{can}}(\lambda). \quad (4.29)$$

By the definition (4.5) of the variational derivative, the definition (4.16) of  $\bar{g}_a(\lambda)$  and equation (4.12), we have

$$\begin{aligned} \frac{\delta \bar{g}_a(\lambda)}{\delta q^{\text{can}}} &= \sum_{i=1}^{\ell} \sum_{m \geq 0} (-\partial)^m \frac{\partial g_a(\lambda)}{\partial u_{i,mx}} v^i = \sum_{i=1}^{\ell} \sum_{m \geq 0} (-\partial)^m \left( \Lambda_{m_a}(\lambda) \left| \frac{\partial H^{\text{can}}(\lambda)}{\partial u_{i,mx}} \right. \right) v^i \\ &= \sum_{i=1}^{\ell} \sum_{m \geq 0} (-\partial)^m \left( \Lambda_{m_a}(\lambda) \left| \frac{\partial}{\partial u_{i,mx}} (e^{\text{ad } U^{\text{can}}(\lambda)} (\mathcal{L}^{\text{can}}) - \partial - \Lambda(\lambda)) \right. \right) v^i. \end{aligned} \quad (4.30)$$

In the last identity we used equation (4.29). We next expand  $e^{\text{ad } U^{\text{can}}(\lambda)}$  in power series. Since  $q^{\text{can}} = \sum_{i=1}^{\ell} u_i v_i$  and using the definition (4.28) of  $\pi_{\tilde{\mathfrak{a}}^e}$  we find that the first term of the expansion is

$$\sum_{i=1}^{\ell} \sum_{m \geq 0} (-\partial)^m \left( \Lambda_{m_a}(\lambda) \left| \frac{\partial q^{\text{can}}}{\partial u_{i,mx}} \right. \right) = \sum_{i=1}^{\ell} (\Lambda(\lambda)|v_i)v^i = \pi_{\tilde{\mathfrak{a}}^e} \Lambda_{m_a}(\lambda). \quad (4.31)$$

By Lemma 4.3, all the other terms in the power series expansion of the RHS of (4.30) are

$$\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i=1}^{\ell} \sum_{m \geq 0} (-\partial)^m \left( \Lambda_{m_a}(\lambda) \left| \frac{\partial}{\partial u_{i,mx}} (\text{ad } U^{\text{can}}(\lambda))^k \mathcal{L}^{\text{can}} \right. \right)$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i=1}^{\ell} \sum_{m \geq 0} (-\partial)^m \left( \Lambda_{m_a}(\lambda) \left| \sum_{h=0}^{k-1} (\text{ad } U^{\text{can}}(\lambda))^h \left( \text{ad } \frac{\partial U^{\text{can}}(\lambda)}{\partial u_{i,mx}} \right) \right. \right) \\
&\quad \times (\text{ad } U^{\text{can}}(\lambda))^{k-h-1} \mathcal{L}^{\text{can}} \\
&\quad + (\text{ad } U^{\text{can}}(\lambda))^k \frac{\partial}{\partial u_{i,mx}} (q^{\text{can}} + \Lambda(\lambda)) - (\text{ad } U^{\text{can}}(\lambda))^{k-1} \frac{\partial U^{\text{can}}(\lambda)}{\partial u_i^{(m-1)}} \\
&= \sum_{h,k \geq 0} \frac{1}{(h+k+1)!} \sum_{i=1}^{\ell} \sum_{m \geq 0} (-\partial)^m \left( \Lambda_{m_a}(\lambda) \left| (\text{ad } U^{\text{can}}(\lambda))^h \left( \text{ad } \frac{\partial U^{\text{can}}(\lambda)}{\partial u_{i,mx}} \right) \right. \right) \\
&\quad \times (\text{ad } U^{\text{can}}(\lambda))^k \mathcal{L}^{\text{can}} \Big) + \sum_{k=1}^{\infty} \frac{1}{k!} \pi_{\bar{a}e} \left( (-\text{ad } U^{\text{can}}(\lambda))^k (\Lambda_{m_a}(\lambda)) \right) \\
&\quad - \sum_{k \geq 0} \frac{1}{(k+1)!} \sum_{i=1}^{\ell} \sum_{m \geq 0} (-\partial)^m \left( \Lambda_{m_a}(\lambda) \left| (\text{ad } U^{\text{can}}(\lambda))^k \frac{\partial U^{\text{can}}(\lambda)}{\partial u_i^{(m-1)}} \right. \right). \tag{4.32}
\end{aligned}$$

For the first and last terms in the RHS we just changed the summation indices, while for the second term we used the definition (4.28) of the map  $\pi_{\bar{a}e}$  and the invariance of the bilinear form. Combining (4.31) and the second term in the RHS of (4.32), we get  $\pi_{\bar{a}e}(e^{-\text{ad } U^{\text{can}}}(\Lambda_{m_a}(\lambda))) = \pi_{\bar{a}e}(R_{m_a}^{\text{can}}(\lambda))$ , where we recall that  $R_{m_a}^{\text{can}}$  is defined in (1.25). Hence, in order to complete the proof of the proposition, we are left to show that the first and last term in the RHS of (4.32) cancel out. The last term of the RHS of (4.32) can be rewritten as

$$-\sum_{i=1}^{\ell} \sum_{m \geq 1} (-\partial)^m (\Lambda_{m_a}(\lambda) | X_{i,m-1}(\lambda)), \tag{4.33}$$

where  $X_{i,m}(\lambda) = \sum_{k \geq 0} \frac{1}{(k+1)!} (\text{ad } U^{\text{can}}(\lambda))^k \frac{\partial U^{\text{can}}(\lambda)}{\partial u_{i,mx}}$ . On the other hand, by Lemma 4.3b), the first term of the RHS of (4.32) is equal to

$$\sum_{i=1}^{\ell} \sum_{m \geq 0} (-\partial)^m (\Lambda_{m_a}(\lambda) | [X_{i,m}(\lambda), e^{\text{ad } U^{\text{can}}(\lambda)} \mathcal{L}^{\text{can}}]).$$

By equation (4.29), the invariance of the bilinear form and the fact that  $\mathcal{H}$  is abelian the above expression is equal to

$$\sum_{i=1}^{\ell} \sum_{m \geq 0} (-\partial)^{m+1} (\Lambda_{m_a}(\lambda) | X_{i,m}(\lambda)),$$

which, combined with (4.33), gives zero. The fact that  $\pi_{\bar{a}e}(e^{\text{ad } N^{\text{can}}} R_{m_a}(\lambda)) \in \mathcal{R} \otimes \mathfrak{a}^e[[\lambda^{-N_m}]]$  follows by simple degree considerations.  $\blacksquare$

**Corollary 4.10.** *For every  $a = 1, \dots, n$  we have*

$$\frac{\delta \bar{G}_a(\lambda)}{\delta q^{\text{can}}} = \left( \partial_{\lambda} - \frac{m_a N_m}{r h \lambda} \right) (\pi_{\bar{a}e}(e^{\text{ad } N^{\text{can}}} R_{m_a}(\lambda))) \in \mathcal{R} \otimes \mathfrak{a}^e[[\lambda^{-N_m}]] \lambda^{-1}, \tag{4.34}$$

where  $N^{\text{can}} \in \mathcal{A}^q \otimes \mathfrak{n}$  is defined by (1.21).

**Proof.** Immediate from Propositions 4.4 and 4.9 and the fact that total derivatives are in the kernel of the operator of variational derivative.  $\blacksquare$

Let us now assume that  $r = 1$  and let us choose the special vertex  $c_0$  of the Kac–Moody algebra, so that in this case  $\tilde{\mathfrak{g}} = \mathfrak{g}((\lambda^{-1}))$  for a simple Lie algebra  $\mathfrak{g}$  of rank  $n$  (we also have  $\ell = n$ ). It is possible to choose a basis  $\{v_i\}_{i=1}^n$  of  $V$  such that  $v_n = e_{-\theta}$  (up to a constant multiple), where  $\theta$  is the highest root of  $\mathfrak{g}$ . Moreover,  $\tilde{e}_m = e_{-\theta}$ , namely  $\Lambda(\lambda) = e + e_{-\theta}\lambda$ . In this setting, we have the following corollary.

**Corollary 4.11.** *For an untwisted affine Kac–Moody algebra with the choice of the special vertex  $c_0$ , the following identity holds:*

$$\frac{\delta \overline{G}_a(\lambda)}{\delta u_n} = \left( \partial_\lambda - \frac{m_a}{h\lambda} \right) G_a(\lambda) \in \mathcal{R}[[\lambda^{-N_m}]] \lambda^{-1}.$$

**Proof.** From the definition of the variational derivative (4.5) and the fact that  $v_n = e_{-\theta}$ , we have

$$\frac{\delta \overline{G}_a(\lambda)}{\delta u_n} = \left( \frac{\delta \overline{G}_a(\lambda)}{\delta q^{\text{can}}} \Big|_{v_n} \right) = \left( \frac{\delta \overline{G}_a(\lambda)}{\delta q^{\text{can}}} \Big|_{e_{-\theta}} \right).$$

Hence, by equation (4.34) we have

$$\begin{aligned} \frac{\delta \overline{G}_a(\lambda)}{\delta u_n} &= \left( \partial_\lambda - \frac{m_a N_m}{rh\lambda} \right) (\pi_{\tilde{a}e} (e^{\text{ad } N^{\text{can}}} R_{m_a}(\lambda)) \Big|_{e_{-\theta}}) \\ &= \left( \partial_\lambda - \frac{m_a N_m}{rh\lambda} \right) \pi_\lambda (R_{m_a}(\lambda) \Big|_{e^{-\text{ad } N^{\text{can}}} (e_{-\theta})}) \\ &= \left( \partial_\lambda - \frac{m_a N_m}{rh\lambda} \right) \pi_\lambda (R_{m_a}(\lambda) \Big|_{e_{-\theta}}) = \left( \partial_\lambda - \frac{m_a N_m}{rh\lambda} \right) G_a(\lambda). \end{aligned}$$

In the second equation we used the fact the bilinear form is coordinated with the gradation and its invariance, in the third equality we used that fact that  $[\mathfrak{n}, e_{-\theta}] = 0$ , and finally we used the definition (2.24) of  $G_a(\lambda)$  and the fact that  $\tilde{e}_m = e_{-\theta}$ . ■

**Remark 4.12.** We note that Corollary 4.11 and the criterion in [4] lead to another proof of Theorem 4.7.

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## Note added

The collaborative research of our project aiming at generalizing the results in [3] to twisted affine Kac–Moody algebras started in 2017. The three authors communicated by email and also in person during several visits of D.V. and D.Y. to Trieste to meet with B.D. at SISSA. During these periods, we achieved the extension of the matrix-resolvent method to the DS hierarchies associated to affine Kac–Moody algebras, and a draft containing the main results of what are now Sections 1–3 was written by the three of us, while Section 4 contains further results found by D.V. and D.Y. after Boris Dubrovin passed away in March of 2019.

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