Integrable $\mathcal{E}$-Models, 4d Chern–Simons Theory and Affine Gaudin Models. I. Lagrangian Aspects

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Abstract. We construct the actions of a very broad family of 2d integrable σ-models. Our starting point is a universal 2d action obtained in [arXiv:2008.01829] using the framework of Costello and Yamazaki based on 4d Chern–Simons theory. This 2d action depends on a pair of 2d fields h and $\mathcal{L}$, with $\mathcal{L}$ depending rationally on an auxiliary complex parameter, which are tied together by a constraint. When the latter can be solved for $\mathcal{L}$ in terms of h this produces a 2d integrable field theory for the 2d field h whose Lax connection is given by $\mathcal{L}(h)$. We construct a general class of solutions to this constraint and show that the resulting 2d integrable field theories can all naturally be described as $\mathcal{E}$-models.

Key words: 4d Chern–Simons theory; $\mathcal{E}$-models; affine Gaudin models; integrable σ-models

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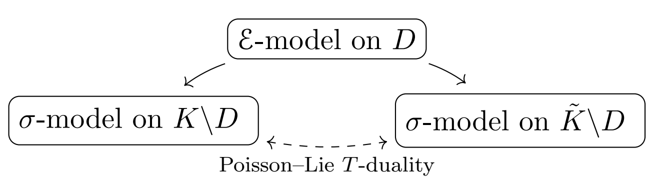
1 Introduction

The $\mathcal{E}$-model, introduced by Klimčík and Ševera [37, 39, 40], makes manifest the duality between pairs of σ-models related by Poisson–Lie T-duality. Let D be an even dimensional real Lie group whose Lie algebra $\mathfrak{d}$ is equipped with a non-degenerate symmetric invariant bilinear form $\langle\langle\cdot,\cdot\rangle\rangle_{\mathfrak{d}}$, i.e., $(\mathfrak{d}, \langle\langle\cdot,\cdot\rangle\rangle_{\mathfrak{d}})$ is a quadratic Lie algebra. The $\mathcal{E}$-model describes the dynamics of a D-valued field $l \in C^\infty(\Sigma, D)$ on a 2d worldsheet which we take here to be $\Sigma = \mathbb{R}^2$. The key ingredient entering the action of the $\mathcal{E}$-model, and which gives the model its name, is an invertible linear operator $\mathcal{E}\colon \mathfrak{d} \to \mathfrak{d}$ which is symmetric with respect to the bilinear form $\langle\langle\cdot,\cdot\rangle\rangle_{\mathfrak{d}}$. One often also assumes that $\mathcal{E}$ is an involution, i.e., that $\mathcal{E}^2 = \mathrm{id}$, which is related to the relativistic invariance of the σ-models.

Given a maximal isotropic subalgebra $\mathfrak{k} \subset \mathfrak{d}$ with respect to $\langle\langle\cdot,\cdot\rangle\rangle_{\mathfrak{d}}$ (throughout the paper will refer to maximal isotropic subalgebras as *Lagrangian* subalgebras) with corresponding connected Lie subgroup $K \subset D$ one can associate with the $\mathcal{E}$-model on D a σ-model on the left coset $K\backslash D$. In particular, if $\mathfrak{k}, \tilde{\mathfrak{k}} \subset \mathfrak{d}$ is a pair of complementary Lagrangian subalgebras with corresponding connected Lie subgroups $K, \tilde{K} \subset D$ then the associated σ-models on $K\backslash D$ and $\tilde{K}\backslash D$ are Poisson–Lie T-dual. The situation is summarised in the diagram



in which each arrow represents a canonical transformation relating the phase spaces and Hamiltonians of the respective theories [37, 40, 54, 55]. It is in this sense that the σ-models on $K\backslash D$ and $\tilde{K}\backslash D$ are often referred to as $\mathcal{E}$-models themselves.

Even though the $\mathcal{E}$-model was devised as a means of understanding Poisson–Lie T-duality, it turns out that many of the known integrable σ-models can be described as $\mathcal{E}$-models [28, 32, 33, 36] or in terms of their close relatives called dressing cosets or degenerate $\mathcal{E}$-models [38], as in [35].[1] In fact, the prototypical class of integrable deformations, given by the Yang–Baxter σ-model, was originally conceived in [30] as an example of a σ-model exhibiting Poisson–Lie symmetry which allows it to be T-dualised. It was shown only some years later that it was integrable in [31]. Another important class of integrable deformations, given by the λ-model and constructed in [57], is related to the Poisson–Lie T-dual of the Yang–Baxter σ-model by analytic continuation [29, 32, 63].

In light of the above observations, it is natural to ask under which conditions a given $\mathcal{E}$-model is also integrable or, conversely, under which conditions an integrable σ-model can be recast as an $\mathcal{E}$-model. A natural starting point is to recall that the equations of motion of the $\mathcal{E}$-model are equivalent to the flatness of a $\mathfrak{d}$-valued current $\mathcal{J}$. It was observed in [53] that if one can find a linear map $p_z\colon \mathfrak{d} \to \mathfrak{g}^{\mathbb{C}}$, with $\mathfrak{g}^{\mathbb{C}}$ a complex Lie algebra, depending rationally on a complex parameter z and satisfying a certain algebraic property reviewed in Section 4.5, then it can be used to lift the on-shell flat connection $\mathcal{J}$ to an on-shell flat meromorphic $\mathfrak{g}^{\mathbb{C}}$-valued connection $p_z(\mathcal{J})$ thus defining a Lax connection for the model. In particular, this observation was used to rederive the integrability of the Yang–Baxter σ-model and the λ-model in the framework of $\mathcal{E}$-models. However, the problem of constructing more general families of suitable maps p_z so as to produce new examples of integrable σ-models from $\mathcal{E}$-models was left open in [53]. One upshot of the present work is a systematic construction of such maps p_z.

The purpose of this paper is to construct a very general family of 2d integrable field theories, or more precisely integrable σ-models, and show that they all naturally admit descriptions as $\mathcal{E}$-models. In fact, the identification of a suitable symmetric invertible linear operator $\mathcal{E}\colon \mathfrak{d} \to \mathfrak{d}$ on the relevant quadratic Lie algebra $(\mathfrak{d}, \langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{d}})$ is a key part of our construction of these integrable σ-models.

In order to construct this broad family of integrable $\mathcal{E}$-models, we shall put to full use two general frameworks for describing 2d classical integrable field theories which have emerged over the last couple of years. The first, initially proposed in [64] and then further developed in [16, 41], is based on classical dihedral affine Gaudin models. The second, proposed by Costello and Yamazaki [13], is based instead on 4d Chern–Simons theory which was originally developed for describing integrable spin chains in [8, 9, 11, 12, 67]. See also [1, 4, 5, 6, 7, 10, 17, 26, 27, 47, 51, 60, 61] for further recent developments in the field theory setting. Although the frameworks of [64] and [13] are very different in flavour, they are in fact intimately related [65]: they are based on the Hamiltonian and Lagrangian formalisms respectively. In this paper we will construct the actions of integrable $\mathcal{E}$-models starting from the action of 4d Chern–Simons theory[2] but using also input from the theory of affine Gaudin models. The Hamiltonian analysis of these actions and their relation to classical dihedral affine Gaudin models will be considered elsewhere [43].

The Lagrangian of 4d Chern–Simons theory is proportional to $\omega \wedge \mathrm{CS}(A)$, where $\mathrm{CS}(A)$ is the Chern–Simons 3-form for a $\mathfrak{g}^{\mathbb{C}}$-valued 1-form A on $\Sigma \times \mathbb{C}P^1$ and ω is a meromorphic 1-form on $\mathbb{C}P^1$. In the setup of [13], 2d integrable field theories are described by introducing surface defects along Σ at the poles of ω on $\mathbb{C}P^1$. When ω has at most double poles, this approach was used in [17] to construct a unifying 2d action for many known 2d integrable σ-models. The

[1]See also [18, 19, 20, 58] for related works on integrable aspects of $\mathcal{E}$-models.

[2]Note that a relation between 4d Chern–Simons theory and $\mathcal{E}$-models was already described in [16]. However, this relation is very different in nature from the one described in the present article since it applies to all $\mathcal{E}$-models, regardless of whether they are integrable or not.

generalisation of this 2d action for an arbitrary meromorphic 1-form ω was obtained in [3], where the passage from 4d Chern–Simons theory to 2d integrable σ-models was streamlined and put on a firm mathematical footing using methods from homotopical algebra.

More precisely, the 2d actions derived in [17] and [3] are both actions for a certain group valued field h living on Σ but which also depend on a 1-form $\mathcal{L}$ on Σ that depends meromorphically on $\mathbb{C}P^1$. In order to obtain a 2d action for the field h alone one still needs to solve a certain boundary condition, or constraint, relating $\mathcal{L}$ to h and depending on a choice of Lagrangian subalgebra $\mathfrak{k}$ of a certain quadratic Lie algebra $\mathfrak{d}$ determined by ω. Given any solution $\mathcal{L} = \mathcal{L}(h)$ of this constraint, one obtains a 2d integrable field theory for the 2d field h. The connection $\mathcal{L}(h)$ then plays the role of the Lax connection of this 2d integrable field theory. The main purpose of this paper is to solve the boundary condition relating $\mathcal{L}$ and h in the general setting of [3]. In doing so, we are naturally led to introduce a linear operator $\mathcal{E}$ on the Lie algebra $\mathfrak{d}$, with all the properties required to define an $\mathcal{E}$-model. In fact, this linear operator has a very natural origin from the point of view of affine Gaudin models and our construction of this operator is motivated by [16]. We find that, upon solving the constraint, the 2d action of [3] coincides with that of the σ-model on the coset $K\backslash D$ associated with an $\mathcal{E}$-model.

At this point it is useful to highlight the various levels of generality of our setup.

Firstly, all the integrable σ-models which have so far been constructed using either the framework of 4d Chern–Simons theory or that of dihedral affine Gaudin models, with the exception of an example considered recently in [4], start from a choice of meromorphic 1-form ω which has at most double poles. By exploiting the results of [3], in the present work we build integrable σ-models starting from a completely general 1-form ω (although for technical reasons to be discussed in the main text, we require ω to have one double pole at infinity). To illustrate the effect of higher order poles in ω we give an explicit example in which ω has a fourth order pole.

Secondly, the constraint on $\mathcal{L}$ and h, which depends on a Lagrangian subalgebra $\mathfrak{k}$ of $\mathfrak{d}$, has so far only been solved for a limited number of concrete examples. In the present article, we solve this constraint for arbitrary Lagrangian subalgebras $\mathfrak{k}$. Even in the case when ω has at most double poles, this generality on $\mathfrak{k}$ allows us, for instance, to obtain the non-abelian T-dual of the principal chiral model from 4d Chern–Simons theory, as anticipated in [17].

To end this introduction, we will illustrate in a simple case the general family of integrable $\mathcal{E}$-models constructed in the main text.

Let $\mathfrak{g}$ be a real Lie algebra equipped with a non-degenerate symmetric invariant bilinear form $\langle\cdot,\cdot\rangle\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$. Consider the meromorphic 1-form

$$\omega = -\ell_1^\infty \frac{\prod_{i=1}^N (z-\zeta_i)}{\prod_{i=1}^N (z-z_i)} \mathrm{d}z, \tag{1.1}$$

with distinct real poles and zeroes $z_i, \zeta_i \in \mathbb{R}$ for $i = 1,\dots,N$. Note, in particular, that ω has a double pole at infinity. In the main text we shall take ω to have an arbitrary number of finite poles of arbitrary order, but also restrict to the case of a double pole at infinity.

First, we define the quadratic Lie algebra $(\mathfrak{d}, \langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}})$. As we shall see, since the 1-form ω in (1.1) has only simple poles along the real axis, the associated Lie algebra $\mathfrak{d}$ is given in this case by the direct sum of Lie algebras $\mathfrak{d} = \mathfrak{g}^{\oplus N}$, which is the Lie algebra of $D = G^{\times N}$. Moreover, $\mathfrak{d}$ comes equipped with a natural non-degenerate symmetric invariant bilinear form defined in terms of ω by

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}\colon \quad \mathfrak{d}\times\mathfrak{d}\longrightarrow\mathbb{R}, \qquad \Big\langle\!\!\Big\langle (\mathsf{u}^i)_{i=1}^N, (\mathsf{v}^j)_{j=1}^N \Big\rangle\!\!\Big\rangle_{\mathfrak{d}} = \sum_{i=1}^N (\mathrm{res}_{z_i}\,\omega)\langle \mathsf{u}^i, \mathsf{v}^i\rangle \tag{1.2}$$

for any pair of elements $(\mathsf{u}^i)_{i=1}^N, (\mathsf{v}^j)_{j=1}^N \in \mathfrak{d}$. More generally, if ω has higher order poles then the corresponding copies of $\mathfrak{g}$ in $\mathfrak{d}$ are replaced by truncated loop algebras over $\mathfrak{g}$ (or its complexification if the pole is not real) and the bilinear form (1.2) is replaced by one involving all coefficients in the partial fraction decomposition of ω.

Next, we define the linear map $\mathcal{E}\colon \mathfrak{d} \to \mathfrak{d}$. We associate to each zero ζ_i of ω an $\epsilon_i \in \mathbb{R} \setminus \{0\}$. We then define

$$\mathcal{E}(\mathsf{u}^i)_{i=1}^N = \left(\sum_{j,k=1}^N \frac{\prod_{r\neq j}(\zeta_r - z_k) \prod_{r\neq i}(z_r - \zeta_j)}{\prod_{r\neq k}(z_r - z_k) \prod_{r\neq j}(\zeta_r - \zeta_j)} \epsilon_j \mathsf{u}^k \right)_{i=1}^N, \tag{1.3}$$

for every $(\mathsf{u}^i)_{i=1}^N \in \mathfrak{d}$. This operator is invertible because $\epsilon_i \neq 0$ and it is symmetric with respect to (1.2). Moreover, if $\mathfrak{g}$ is compact and the sign of each ϵ_i is chosen to coincide with the sign of $-\varphi'(\zeta_i)$, where $\varphi(z)$ is the twist function defined by $\omega = \varphi(z)\mathrm{d}z$, then $\mathcal{E}$ is positive with respect to (1.2). This ensures that the Hamiltonian is positive. Furthermore, if $\epsilon_i^2 = 1$ for $i = 1, \ldots, N$ then $\mathcal{E}^2 = \mathrm{id}$ which ensures that the model is relativistic invariant.

Finally, let $\mathfrak{k}$ be a Lagrangian subalgebra of $\mathfrak{d}$ and K the associated connected Lie subgroup of D. (We also require that $\mathfrak{k}$ satisfies a technical condition together with the operator $\mathcal{E}$, which can be easily ensured for instance if $\mathcal{E}$ is positive; we refer to the main text for details.) We can then construct the σ-model for a field $l \in C^\infty(\Sigma, D)$ with a gauge symmetry by K, from the $\mathcal{E}$-model associated with the above data. By our construction, this σ-model on $K\backslash D$ is integrable and its Lax connection, depending rationally on the spectral parameter z, is given explicitly in the present case by

$$\mathcal{L}(z) = \sum_{i,j=1}^N \frac{\prod_{r\neq i}(\zeta_r - z_j) \prod_r (z_r - \zeta_i)}{\prod_{r\neq j}(z_r - z_j) \prod_{r\neq i}(\zeta_r - \zeta_i)} \frac{\mathcal{J}^j}{z - \zeta_i} \tag{1.4}$$

for a certain $\mathfrak{g}$-valued 1-form $\mathcal{J}^j$ on Σ for each $j = 1, \ldots, N$ depending on the D-valued field l. We refer to Section 5.3 for details. The equations of motion of the $\mathcal{E}$-model, given in this case by the flatness of $\mathcal{L}(z_i) = \mathcal{J}^i$ for each $i = 1, \ldots, N$, is equivalent to the flatness of the above Lax connection for all z.

The above example contains the Yang–Baxter σ-model, the λ-model and more generally the family of integrable σ-models constructed in [2] which couple together $N_1 \in \mathbb{Z}_{\geq 0}$ copies of the Yang–Baxter σ-model and $N_2 \in \mathbb{Z}_{\geq 0}$ copies of the λ-model, where $2N_1 + 2N_2 = N$.

The plan of the paper is as follows. We begin in Section 2 by reviewing the definition of the $\mathcal{E}$-model and the construction of the associated σ-model on $K\backslash D$. In particular, we discuss the properties of $\mathcal{E}$ and various projectors that will be relevant for our purposes. In Section 3 we review the general 2d action constructed in [3] and in Section 3.6 we derive from it the 2d action that serves as the starting point for our analysis. In Section 4 we construct a solution to the constraint from [3] and relate the resulting 2d integrable field theory to an $\mathcal{E}$-model. Finally, we give a number of simple examples in Section 5 to illustrate the general construction before concluding in Section 6.

List of notations

For the reader's convenience we gather here a list of notations used throughout the paper.

- $\mathfrak{g}$, G – real finite-dimensional Lie algebra and corresponding Lie group,

  $\langle \cdot, \cdot \rangle$ – non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$,
- $\boldsymbol{z}$ – set of independent poles of ω,

  $[\boldsymbol{z}]$ – independent poles of ω counting multiplicities,

  $\mathfrak{g}^{[\boldsymbol{z}]}$, $G^{[\boldsymbol{z}]}$ – associated defect Lie algebra and Lie group,

$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{g}^{[z]}}$ – non-degenerate invariant symmetric bilinear form on $\mathfrak{g}^{[z]}$,

$\mathfrak{f}$, F – Lagrangian subalgebra of $\mathfrak{g}^{[z]}$ and corresponding Lie subgroup of $G^{[z]}$,

- $\boldsymbol{z}'$ – set of independent finite poles of ω,

  $[\boldsymbol{z}']$ – independent finite poles of ω counting multiplicities,

  $\mathfrak{d}$, D – associated defect Lie algebra and Lie group,

  $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}$ – non-degenerate invariant symmetric bilinear form on $\mathfrak{d}$,

  $\mathfrak{k}$, K – Lagrangian subalgebra of $\mathfrak{d}$ and corresponding Lie subgroup of D,

- $\boldsymbol{\zeta}$ – set of independent zeroes of ω,

  $(\boldsymbol{\zeta})$ – independent zeroes of ω counting multiplicities,

  $\mathfrak{g}^{(\boldsymbol{\zeta})}$ – associated vector space,

  $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{g}^{(\boldsymbol{\zeta})}}$ – non-degenerate symmetric bilinear form on $\mathfrak{g}^{(\boldsymbol{\zeta})}$.

2 Background on the $\mathcal{E}$-model

Throughout this section we let $\mathfrak{d}$ denote an arbitrary real even dimensional Lie algebra equipped with a non-degenerate ad-invariant symmetric bilinear form

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}\colon\ \mathfrak{d}\times\mathfrak{d}\longrightarrow\mathbb{R}.$$

We let D be a real Lie group with Lie algebra $\mathfrak{d}$. In Section 3.6 below we shall introduce a specific real even dimensional Lie algebra $\mathfrak{d}$ and corresponding Lie group D, to which the results of the present section will apply verbatim.

For any linear operator $\mathcal{O}\in\operatorname{End}\mathfrak{d}$ we denote by ${}^t\mathcal{O}$ its transpose with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}$, namely such that $\langle\!\langle\mathsf{U},\mathcal{O}\mathsf{V}\rangle\!\rangle_{\mathfrak{d}}=\langle\!\langle{}^t\mathcal{O}\mathsf{U},\mathsf{V}\rangle\!\rangle_{\mathfrak{d}}$, for any $\mathsf{U},\mathsf{V}\in\mathfrak{d}$. We also let Ad_l denote the adjoint action $\operatorname{Ad}_l\mathsf{U}\coloneqq l\mathsf{U}l^{-1}$ of $l\in D$ on $\mathfrak{d}$.

2.1 The operators $\mathcal{E}$ and $\mathcal{P}_l$

We fix an invertible operator $\mathcal{E}\in\operatorname{End}\mathfrak{d}$, symmetric with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}$, i.e., ${}^t\mathcal{E}=\mathcal{E}$, and let

$$\langle\!\langle\mathsf{U},\mathsf{V}\rangle\!\rangle_{\mathfrak{d},\mathcal{E}}\coloneqq\langle\!\langle\mathsf{U},\mathcal{E}^{-1}\mathsf{V}\rangle\!\rangle_{\mathfrak{d}} \tag{2.1}$$

for every $\mathsf{U},\mathsf{V}\in\mathfrak{d}$. This defines another non-degenerate bilinear form on $\mathfrak{d}$.

We suppose that $\mathfrak{d}$ admits a Lagrangian subalgebra $\mathfrak{k}$ with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}$, which thus satisfies $\dim\mathfrak{k}=\frac{1}{2}\dim\mathfrak{d}$.

We shall need to make another important assumption on the operator $\mathcal{E}$ and the Lagrangian subalgebra $\mathfrak{k}\subset\mathfrak{d}$. Namely, we will suppose that for any $l\in D$ we have

$$\operatorname{Ad}_l^{-1}\mathfrak{k}\cap\mathcal{E}\operatorname{Ad}_l^{-1}\mathfrak{k}=\{0\}. \tag{2.2}$$

Before exploring the consequences of this assumption, we give a sufficient condition on the operator $\mathcal{E}$ for the condition (2.2) to hold.

Remark 2.1. As we shall see later in Section 2.5, in particular Remark 2.8, this sufficient condition is actually quite natural in the study of σ-models, as it will be related to the property of the Hamiltonian being bounded below in these models.

Lemma 2.2. *If $\mathcal{E}$ is such that $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d},\mathcal{E}}$ is positive-definite, any Lagrangian subalgebra $\mathfrak{k}\subset\mathfrak{d}$ satisfies (2.2).*

Proof. Let us fix $l \in D$. By applying successively the definition (2.1) of $\langle\langle \cdot, \cdot \rangle\rangle_{\mathfrak{d},\mathcal{E}}$, the ad-invariance of $\langle\langle \cdot, \cdot \rangle\rangle_{\mathfrak{d}}$ and the isotropy of $\mathfrak{k}$, we have

$$\langle\langle \mathrm{Ad}_l^{-1} \mathfrak{k}, \mathcal{E} \, \mathrm{Ad}_l^{-1} \mathfrak{k} \rangle\rangle_{\mathfrak{d},\mathcal{E}} = \langle\langle \mathrm{Ad}_l^{-1} \mathfrak{k}, \mathrm{Ad}_l^{-1} \mathfrak{k} \rangle\rangle_{\mathfrak{d}} = \langle\langle \mathfrak{k}, \mathfrak{k} \rangle\rangle_{\mathfrak{d}} = 0.$$

Hence the subspaces $\mathrm{Ad}_l^{-1} \mathfrak{k}$ and $\mathcal{E} \, \mathrm{Ad}_l^{-1} \mathfrak{k}$ are orthogonal with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{\mathfrak{d},\mathcal{E}}$. If the latter is positive-definite, these subspaces then have trivial intersection. ■

Since the operators $\mathcal{E}$ and Ad_l^{-1} are both invertible, we have

$$\dim \mathrm{Ad}_l^{-1} \mathfrak{k} = \dim \mathcal{E} \, \mathrm{Ad}_l^{-1} \mathfrak{k} = \dim \mathfrak{k} = \frac{1}{2} \dim \mathfrak{d}.$$

By the assumption (2.2), we thus have the vector space direct sum decomposition (here we explicitly use the assumption that $\mathfrak{k} \subset \mathfrak{d}$ is Lagrangian)

$$\mathfrak{d} = \mathrm{Ad}_l^{-1} \mathfrak{k} \dotplus \mathcal{E} \, \mathrm{Ad}_l^{-1} \mathfrak{k}. \tag{2.3}$$

As observed in the proof of Lemma 2.2, this direct sum is orthogonal with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{\mathfrak{d},\mathcal{E}}$. We define the projector $\mathcal{P}_l$ relative to (2.3) with kernel and image

$$\ker \mathcal{P}_l = \mathrm{Ad}_l^{-1} \mathfrak{k} \qquad \text{and} \qquad \mathrm{im} \, \mathcal{P}_l = \mathcal{E} \, \mathrm{Ad}_l^{-1} \mathfrak{k}. \tag{2.4}$$

It will also be convenient to introduce the operator

$$\overline{\mathcal{P}}_l \coloneqq \mathrm{id} - {}^t\mathcal{P}_l, \tag{2.5}$$

which is easily seen to define another projector with kernel and image

$$\ker \overline{\mathcal{P}}_l = \mathrm{Ad}_l^{-1} \mathfrak{k} \qquad \text{and} \qquad \mathrm{im} \, \overline{\mathcal{P}}_l = \mathcal{E}^{-1} \, \mathrm{Ad}_l^{-1} \mathfrak{k}. \tag{2.6}$$

To see the first equation, observe that $\ker \overline{\mathcal{P}}_l = \ker(\mathrm{id} - {}^t\mathcal{P}_l) = \mathrm{im} \, {}^t\mathcal{P}_l$ which is equal to the subspace orthogonal to $\ker \mathcal{P}_l = \mathrm{Ad}_k^{-1} \mathfrak{k}$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{\mathfrak{d}}$ and therefore to $\mathrm{Ad}_l^{-1} \mathfrak{k}$ itself since it is Lagrangian. Similarly, one finds that $\mathrm{im} \, \overline{\mathcal{P}}_l$ is the subspace orthogonal to $\mathrm{im} \, \mathcal{P}_l = \mathcal{E} \, \mathrm{Ad}_l^{-1} \mathfrak{k}$ which one checks is given by $\mathcal{E}^{-1} \, \mathrm{Ad}_l^{-1} \mathfrak{k}$.

We shall need the following technical properties of the projectors $\mathcal{P}_l$ and $\overline{\mathcal{P}}_l$, whose proof we give in Appendix A.

Proposition 2.3. *The projectors $\mathcal{P}_l$ and $\overline{\mathcal{P}}_l$ have the following properties:*

(i) ${}^t\mathcal{P}_l = \mathcal{E}^{-1} \mathcal{P}_l \mathcal{E}$,

(ii) $\mathcal{P}_l \mathcal{E} + \mathcal{E} \overline{\mathcal{P}}_l = \mathcal{E}$ *and* $\mathcal{E}^{-1} \mathcal{P}_l + \overline{\mathcal{P}}_l \mathcal{E}^{-1} = \mathcal{E}^{-1}$,

(iii) ${}^t\mathcal{P}_l \overline{\mathcal{P}}_l = {}^t\overline{\mathcal{P}}_l \mathcal{P}_l = 0$,

(iv) $\mathcal{P}_l - \overline{\mathcal{P}}_l = {}^t\mathcal{P}_l \mathcal{P}_l = -{}^t\overline{\mathcal{P}}_l \overline{\mathcal{P}}_l$,

(v) $\mathcal{P}_l {}^t\mathcal{P}_l = \overline{\mathcal{P}}_l {}^t\overline{\mathcal{P}}_l = 0$,

(vi) $\overline{\mathcal{P}}_l = \mathcal{P}_l$ *if* $\mathcal{E}^2 = \mathrm{id}$.

2.2 The $\mathcal{E}$-model action

Define $\Sigma \coloneqq \mathbb{R}^2$ on which we fix coordinates (τ, σ) for convenience. The $\mathcal{E}$-model describes the dynamics of a D-valued field $\ell \in C^\infty(\Sigma, D)$ on Σ with the first order action [37, 39, 40]

$$S_{\mathcal{E}}(\ell) = \frac{1}{2}\int_\Sigma \big(\langle\!\langle \ell^{-1}\partial_\tau \ell, \ell^{-1}\partial_\sigma \ell\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle \ell^{-1}\partial_\sigma \ell, \mathcal{E}\,\ell^{-1}\partial_\sigma \ell\rangle\!\rangle_{\mathfrak{d}}\big)\mathrm{d}\sigma\wedge\mathrm{d}\tau - \frac{1}{2}I^{\mathrm{WZ}}_{\mathfrak{d}}[\ell]. \tag{2.7}$$

Here we introduce the standard WZ-term for ℓ as[3]

$$I^{\mathrm{WZ}}_{\mathfrak{d}}[\ell] \coloneqq -\frac{1}{6}\int_{\Sigma\times I} \big\langle\!\big\langle \widehat{\ell}^{-1}\mathrm{d}\widehat{\ell}, \big[\widehat{\ell}^{-1}\mathrm{d}\widehat{\ell}, \widehat{\ell}^{-1}\mathrm{d}\widehat{\ell}\big]\big\rangle\!\big\rangle_{\mathfrak{d}}, \tag{2.8}$$

where $I \coloneqq [0,1]$ and $\widehat{\ell} \in C^\infty(\Sigma\times I, D)$ is any smooth extension of ℓ to $\Sigma\times I$ with the property that $\widehat{\ell} = \ell$ near $\Sigma\times\{0\} \subset \Sigma\times I$ and $\widehat{\ell} = \mathrm{id}$ near $\Sigma\times\{1\}\subset\Sigma\times I$. The WZ-term $I^{\mathrm{WZ}}_{\mathfrak{d}}[\ell]$ is independent of the choice of extension $\widehat{\ell}$; see, e.g., [3].

Let K be the connected Lie subgroup of D corresponding to the Lie subalgebra $\mathfrak{k}\subset\mathfrak{d}$ from Section 2.1. In this section we recall the derivation of the action for the σ-model on the left coset $K\backslash D$ starting from the $\mathcal{E}$-model action (2.7).

We begin by introducing a new D-valued field $l \in C^\infty(\Sigma, D)$ and a K-valued field $b \in C^\infty(\Sigma, K)$, then define the action $S'_{\mathcal{E},\mathfrak{k}}(l, b) \coloneqq S_{\mathcal{E}}(bl)$. Of course, the latter is invariant under the gauge transformation

$$l \longmapsto kl, \qquad b \longmapsto bk^{-1}, \tag{2.9}$$

with local parameter $k \in C^\infty(\Sigma, K)$, and fixing this gauge invariance by imposing the gauge condition $b = \mathrm{id}$ we recover the original action (2.7) for the field l. However, since we would like to keep the gauge invariance (2.9), so as to obtain a model on $K\backslash D$, we will eliminate b in a different way.

To compute the action $S'_{\mathcal{E},\mathfrak{k}}(l, b)$ explicitly, we make use of the Polyakov–Wiegmann identity [48]

$$I^{\mathrm{WZ}}_{\mathfrak{d}}[bl] = I^{\mathrm{WZ}}_{\mathfrak{d}}[b] + I^{\mathrm{WZ}}_{\mathfrak{d}}[l] + \int_\Sigma \big(\langle\!\langle b^{-1}\partial_\tau b, \partial_\sigma l l^{-1}\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle b^{-1}\partial_\sigma b, \partial_\tau l l^{-1}\rangle\!\rangle_{\mathfrak{d}}\big)\mathrm{d}\sigma\wedge\mathrm{d}\tau \tag{2.10}$$

and of the fact that

$$(bl)^{-1}\partial_\mu(bl) = \mathrm{Ad}_l^{-1}\, b^{-1}\partial_\mu b + l^{-1}\partial_\mu l$$

for $\mu = \tau, \sigma$. By the isotropy of the subalgebra $\mathfrak{k}\subset\mathfrak{d}$, the WZ-term for the K-valued field b and $\langle\!\langle \mathrm{Ad}_l^{-1}\, b^{-1}\partial_\tau b, \mathrm{Ad}_l^{-1}\, b^{-1}\partial_\sigma b\rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle b^{-1}\partial_\tau b, b^{-1}\partial_\sigma b\rangle\!\rangle_{\mathfrak{d}}$ both vanish. After a few manipulations, one observes that $S'_{\mathcal{E},\mathfrak{k}}(l, b)$ depends on the field b only through $Y = b^{-1}\partial_\sigma b$. More precisely, we have

$$\begin{aligned} S'_{\mathcal{E},\mathfrak{k}}(l, b) &= \frac{1}{2}\int_\Sigma \big(\langle\!\langle l^{-1}\partial_\tau l, l^{-1}\partial_\sigma l\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle l^{-1}\partial_\sigma l, \mathcal{E}\,l^{-1}\partial_\sigma l\rangle\!\rangle_{\mathfrak{d}}\big)\mathrm{d}\sigma\wedge\mathrm{d}\tau - \frac{1}{2}I^{\mathrm{WZ}}_{\mathfrak{d}}[l] \\ &\quad + \int_\Sigma \left(\langle\!\langle Y, \mathrm{Ad}_l(l^{-1}\partial_\tau l - \mathcal{E}\,l^{-1}\partial_\sigma l)\rangle\!\rangle_{\mathfrak{d}} - \frac{1}{2}\langle\!\langle Y, \mathrm{Ad}_l\,\mathcal{E}\,\mathrm{Ad}_l^{-1}\, Y\rangle\!\rangle_{\mathfrak{d}}\right)\mathrm{d}\sigma\wedge\mathrm{d}\tau, \end{aligned} \tag{2.11}$$

which is quadratic and algebraic in Y. We can therefore integrate out the degrees of freedom in the field b, or equivalently in Y. For that, we first determine its equation of motion by computing the variation of the action under an infinitesimal variation $\delta Y \in C^\infty(\Sigma, \mathfrak{k})$ of Y, which reads

$$\delta S'_{\mathcal{E},\mathfrak{k}}(l, b) = \int_\Sigma \big(\langle\!\langle \delta Y, \mathrm{Ad}_l\,\big(l^{-1}\partial_\tau l - \mathcal{E}\,l^{-1}\partial_\sigma l - \mathcal{E}\,\mathrm{Ad}_l^{-1}\, Y\big)\rangle\!\rangle_{\mathfrak{d}}\big)\mathrm{d}\sigma\wedge\mathrm{d}\tau.$$

[3] Here, we follow the conventions of [16, 17] for the definition of WZ-terms.

The vanishing of the above variation for any $\mathfrak{k}$-valued field δY requires that

$$Z \coloneqq \mathrm{Ad}_l\big(l^{-1}\partial_\tau l - \mathcal{E}\, l^{-1}\partial_\sigma l - \mathcal{E}\,\mathrm{Ad}_l^{-1}\, Y\big) \tag{2.12}$$

belongs to the subspace orthogonal to $\mathfrak{k}$ with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}$, which coincides with $\mathfrak{k}$ itself since $\mathfrak{k}$ is Lagrangian.

To solve the equation of motion $Z \in \mathfrak{k}$, we first rewrite (2.12) as

$$\mathcal{E}\,\mathrm{Ad}_l^{-1} Y + \mathrm{Ad}_l^{-1} Z = l^{-1}\partial_\tau l - \mathcal{E}\, l^{-1}\partial_\sigma l. \tag{2.13}$$

Recall the projector $\mathcal{P}_l$ introduced in Section 2.1 through its kernel and image (2.4). By definition of Y, the quantity $\mathcal{E}\,\mathrm{Ad}_l^{-1} Y$ is valued in $\mathrm{im}\,\mathcal{P}_l = \mathcal{E}\,\mathrm{Ad}_l^{-1}\mathfrak{k}$. Moreover, the equation of motion $Z \in \mathfrak{k}$ is equivalent to $\mathrm{Ad}_l^{-1} Z$ belonging to $\ker\mathcal{P}_l = \mathrm{Ad}_l^{-1}\mathfrak{k}$. Applying $\mathcal{P}_l$ to equation (2.13), we thus get $\mathcal{E}\,\mathrm{Ad}_l^{-1} Y = \mathcal{P}_l\big(l^{-1}\partial_\tau l - \mathcal{E}\, l^{-1}\partial_\sigma l\big)$, hence

$$Y = \mathrm{Ad}_l\, \mathcal{E}^{-1}\mathcal{P}_l\big(l^{-1}\partial_\tau l - \mathcal{E}\, l^{-1}\partial_\sigma l\big).$$

Note that in the above derivation, we have used the existence of the projector $\mathcal{P}_l$ and thus the assumption (2.2) made in Section 2.1. Reinserting the above expression for Y in the action (2.11), we finally arrive at an action for the field l alone. After a few manipulations, using part (ii) of Proposition 2.3, we find

$$\begin{aligned} S_{\mathcal{E},\mathfrak{k}}(l) \coloneqq \frac{1}{2}\int_\Sigma \Big(&\langle\!\langle l^{-1}\partial_\tau l, \mathcal{E}^{-1}\mathcal{P}_l\big(l^{-1}\partial_\tau l\big)\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle l^{-1}\partial_\sigma l, \mathcal{E}\overline{\mathcal{P}}_l\big(l^{-1}\partial_\sigma l\big)\rangle\!\rangle_{\mathfrak{d}} \\ &+ \langle\!\langle l^{-1}\partial_\tau l, \big(\overline{\mathcal{P}}_l - {}^t\mathcal{P}_l\big)\big(l^{-1}\partial_\sigma l\big)\rangle\!\rangle_{\mathfrak{d}}\Big)\mathrm{d}\sigma\wedge\mathrm{d}\tau - \frac{1}{2}I_{\mathfrak{d}}^{\mathrm{WZ}}[l]. \end{aligned} \tag{2.14}$$

Written in this form, the model is relativistic if and only if $\mathcal{E}^{-1}\mathcal{P}_l = \mathcal{E}\overline{\mathcal{P}}_l$ and $\overline{\mathcal{P}}_l - {}^t\mathcal{P}_l$ is skew-symmetric. By part (vi) of Proposition 2.3 we deduce that these conditions are trivially satisfied if $\mathcal{E}^2 = \mathrm{id}$. In the latter case the action (2.14) can be rewritten, making multiple use of properties from Proposition 2.3, in the more familiar form

$$S(f) \coloneqq S_{\mathcal{E},\mathfrak{k}}(f^{-1}) = -\frac{1}{2}\int_\Sigma \big\langle\!\big\langle f^{-1}\partial_- f, \big(\mathrm{id} - 2P_f(\mathcal{E})\big)\big(f^{-1}\partial_+ f\big)\big\rangle\!\big\rangle_{\mathfrak{d}} \mathrm{d}\sigma^+\wedge\mathrm{d}\sigma^- + \frac{1}{2}I_{\mathfrak{d}}^{\mathrm{WZ}}[f],$$

where $\partial_\pm \coloneqq \partial_\tau \pm \partial_\sigma$ and $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$. Here we have introduced the new D-valued field $f \coloneqq l^{-1}$, to match the notation of [32], and defined

$$P_f(\mathcal{E}) \coloneqq \mathrm{Ad}_f^{-1}\,\mathcal{E}\mathcal{P}_{f^{-1}}(\mathcal{E} - \mathrm{id})\,\mathrm{Ad}_f \tag{2.15}$$

which is easily seen to be a projector. Moreover, its kernel and image can be deduced from those of $\mathcal{P}_{f^{-1}}$ in (2.4) to be given by

$$\ker P_f(\mathcal{E}) = \big(\mathrm{id} + \mathrm{Ad}_f^{-1}\,\mathcal{E}\,\mathrm{Ad}_f\big)\mathfrak{d} \qquad \text{and} \qquad \mathrm{im}\, P_f(\mathcal{E}) = \mathfrak{k},$$

which coincide with those given, for instance, in [32]. Note that it follows from the decomposition (2.3) that $\big(\mathrm{id} + \mathrm{Ad}_f^{-1}\,\mathcal{E}\,\mathrm{Ad}_f\big)\mathfrak{d} = \big(\mathrm{id} + \mathrm{Ad}_f^{-1}\,\mathcal{E}\,\mathrm{Ad}_f\big)\mathfrak{k}$. Indeed, (2.3) implies that $\mathfrak{d} = \mathfrak{k} \dotplus \big(\mathrm{Ad}_f^{-1}\,\mathcal{E}\,\mathrm{Ad}_f - \mathrm{id}\big)\mathfrak{k}$. The result then follows by acting on both sides with $\big(\mathrm{id} + \mathrm{Ad}_f^{-1}\,\mathcal{E}\,\mathrm{Ad}_f\big)$ and noting that $(\mathcal{E} + \mathrm{id})(\mathcal{E} - \mathrm{id}) = 0$.

In what follows we shall only use the alternative form (2.14) of the action (note that this was previously used in [28]).

2.3 Gauge invariance

The action $S_{\mathcal{E},\mathfrak{k}}(l)$ was obtained in Section 2.2 by integrating out the field b from the action $S'_{\mathcal{E},\mathfrak{k}}(l,b)$, which is invariant under the gauge transformation (2.9). Therefore, by construction, $S_{\mathcal{E},\mathfrak{k}}(l)$ should be invariant under the residual gauge transformation $l \mapsto kl$ with $k \in C^\infty(\Sigma, K)$. In this section we check this statement explicitly using the expression (2.14) of $S_{\mathcal{E},\mathfrak{k}}(l)$.

We will need the gauge transformation of the projectors $\mathcal{P}_l$ and $\overline{\mathcal{P}}_l$. By equation (2.4), the kernel of $\mathcal{P}_{kl}$ is given by

$$\ker \mathcal{P}_{kl} = \mathrm{Ad}_{kl}^{-1} \mathfrak{k} = \mathrm{Ad}_l^{-1} \mathrm{Ad}_k^{-1} \mathfrak{k} = \mathrm{Ad}_l^{-1} \mathfrak{k} = \ker \mathcal{P}_l.$$

Similarly, one finds $\mathrm{im}\, \mathcal{P}_{kl} = \mathrm{im}\, \mathcal{P}_l$. We thus have $\mathcal{P}_{kl} = \mathcal{P}_l$, hence also $\overline{\mathcal{P}}_{kl} = \overline{\mathcal{P}}_l$, i.e., the projectors $\mathcal{P}_l$ and $\overline{\mathcal{P}}_l$ are gauge invariant. Moreover, we have

$$\mathcal{P}_l\big((kl)^{-1}\partial_\tau(kl)\big) = \mathcal{P}_l\big(l^{-1}\partial_\tau l + \mathrm{Ad}_l^{-1} k^{-1}\partial_\tau k\big) = \mathcal{P}_l\big(l^{-1}\partial_\tau l\big),$$

where we have used the fact that $\mathrm{Ad}_l^{-1} k^{-1}\partial_\tau k$ belongs to $\mathrm{Ad}_l^{-1} \mathfrak{k} = \ker \mathcal{P}_l$. Therefore we deduce $\langle\!\langle l^{-1}\partial_\tau l, \mathcal{E}^{-1}\mathcal{P}_l(l^{-1}\partial_\tau l)\rangle\!\rangle_{\mathfrak{d}}$ is gauge invariant, using the fact that ${}^t\big(\mathcal{E}^{-1}\mathcal{P}_l\big) = \mathcal{E}^{-1}\mathcal{P}_l$ by part (i) in Proposition 2.3.

Similarly, one finds that

$$\overline{\mathcal{P}}_l\big((kl)^{-1}\partial_\sigma(kl)\big) = \overline{\mathcal{P}}_l\big(l^{-1}\partial_\sigma l + \mathrm{Ad}_l^{-1} k^{-1}\partial_\sigma k\big) = \overline{\mathcal{P}}_l(l^{-1}\partial_\sigma l),$$

using $\ker \overline{\mathcal{P}}_l = \mathrm{Ad}_l^{-1} \mathfrak{k}$ from (2.6). The second term in the action (2.14) is thus also gauge invariant (using the symmetry of $\mathcal{E}\overline{\mathcal{P}}_l$).

Rewriting the third term in (2.14) as $\langle\!\langle l^{-1}\partial_\tau l, \overline{\mathcal{P}}_l(l^{-1}\partial_\sigma l)\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle \mathcal{P}_l(l^{-1}\partial_\tau l), l^{-1}\partial_\sigma l\rangle\!\rangle_{\mathfrak{d}}$, using the gauge invariance of $\mathcal{P}_l(l^{-1}\partial_\tau l)$ and $\overline{\mathcal{P}}_l(l^{-1}\partial_\sigma l)$ derived above and the fact that $(kl)^{-1}\partial_\mu(kl) = l^{-1}\partial_\mu l + \mathrm{Ad}_l^{-1} k^{-1}\partial_\mu k$, one deduces that

$$\begin{aligned} S_{\mathcal{E},\mathfrak{k}}(kl) &= S_{\mathcal{E},\mathfrak{k}}(l) - \frac{1}{2} I_{\mathfrak{d}}^{\mathrm{WZ}}[kl] + \frac{1}{2} I_{\mathfrak{d}}^{\mathrm{WZ}}[l] \\ &\quad + \frac{1}{2}\int_\Sigma \Big(\langle\!\langle \mathrm{Ad}_l^{-1} k^{-1}\partial_\tau k, \overline{\mathcal{P}}_l\big(l^{-1}\partial_\sigma l\big)\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle \mathcal{P}_l\big(l^{-1}\partial_\tau l\big), \mathrm{Ad}_l^{-1} k^{-1}\partial_\sigma k\rangle\!\rangle_{\mathfrak{d}}\Big)\, \mathrm{d}\sigma\wedge\mathrm{d}\tau. \end{aligned}$$

Using the Polyakov–Wiegmann identity (2.10) (with b replaced by k and noting that $I_{\mathfrak{d}}^{\mathrm{WZ}}[k]$ vanishes as $\mathfrak{k}$ is isotropic) and the facts that ${}^t\overline{\mathcal{P}}_l = \mathrm{id} - \mathcal{P}_l$ and ${}^t\mathcal{P}_l = \mathrm{id} - \overline{\mathcal{P}}_l$, one rewrites the above equation as

$$\begin{aligned} S_{\mathcal{E},\mathfrak{k}}(kl) &= S_{\mathcal{E},\mathfrak{k}}(l) - \frac{1}{2}\int_\Sigma \Big(\langle\!\langle \mathcal{P}_l \,\mathrm{Ad}_l^{-1} k^{-1}\partial_\tau k, l^{-1}\partial_\sigma l\rangle\!\rangle_{\mathfrak{d}}\Big)\, \mathrm{d}\sigma\wedge\mathrm{d}\tau \\ &\quad + \frac{1}{2}\int_\Sigma \Big(\langle\!\langle l^{-1}\partial_\tau l, \overline{\mathcal{P}}_l \,\mathrm{Ad}_l^{-1} k^{-1}\partial_\sigma k\rangle\!\rangle_{\mathfrak{d}}\Big)\, \mathrm{d}\sigma\wedge\mathrm{d}\tau. \end{aligned}$$

Finally, since $\mathrm{Ad}_l^{-1} k^{-1}\partial_\mu k$ belongs to $\mathrm{Ad}_l^{-1} \mathfrak{k} = \ker \mathcal{P}_l = \ker \overline{\mathcal{P}}_l$, we simply obtain $S_{\mathcal{E},\mathfrak{k}}(kl) = S_{\mathcal{E},\mathfrak{k}}(l)$, as expected.

2.4 Equations of motion

In this subsection, we derive the equations of motion of the field l coming from the $\mathcal{E}$-model action (2.14). It will be useful to introduce a $\mathfrak{d}$-valued 1-form $\mathcal{J} := \mathcal{J}_\sigma \mathrm{d}\sigma + \mathcal{J}_\tau \mathrm{d}\tau$ with components

$$\mathcal{J}_\sigma := \overline{\mathcal{P}}_l\big(l^{-1}\partial_\sigma l\big) + \mathcal{E}^{-1}\mathcal{P}_l\big(l^{-1}\partial_\tau l\big), \tag{2.16a}$$

$$\mathcal{J}_\tau := \mathcal{E}\overline{\mathcal{P}}_l\big(l^{-1}\partial_\sigma l\big) + \mathcal{P}_l\big(l^{-1}\partial_\tau l\big), \tag{2.16b}$$

such that $\mathcal{J}_\tau = \mathcal{E}\mathcal{J}_\sigma$. It follows from the computations in Section 2.3 that these expressions are invariant under the local symmetry $l \mapsto kl$ for arbitrary $k \in C^\infty(\Sigma, K)$. One also checks directly from the definitions of the projectors $\mathcal{P}_l$ and $\overline{\mathcal{P}}_l$ in Section 2.1 that the gauge transformation of $\mathcal{J}$ by l is valued in the subalgebra $\mathfrak{k}$, namely

$$ {}^l\mathcal{J} := -\mathrm{d}l l^{-1} + \mathrm{Ad}_l\, \mathcal{J} \in \mathfrak{k}. \tag{2.17}$$

In terms of $\mathcal{J}$, the action (2.14) can be rewritten in the simple form

$$ S_{\mathcal{E},\mathfrak{k}}(l) = -\frac{1}{2}\int_\Sigma \langle\!\langle l^{-1}\mathrm{d}l, \mathcal{J}\rangle\!\rangle_{\mathfrak{d}} - \frac{1}{2} I^{\mathrm{WZ}}_{\mathfrak{d}}[l]. \tag{2.18}$$

Here, and in the rest of the paper, we extend bilinear pairings such as $\langle\!\langle \cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}$ to Lie algebra valued forms using the exterior product. We will then derive the equations of motion of this action by varying the field l by an infinitesimal right multiplication. We will need the following lemma, which describes the transformation of $\mathcal{J}$ under this transformation.

Lemma 2.4. *Under an infinitesimal multiplication $\delta l = l\epsilon$ of l, where $\epsilon \in C^\infty(\Sigma, \mathfrak{d})$, the variation of $\mathcal{J}_\sigma$ is given by*

$$ \delta\mathcal{J}_\sigma = \overline{\mathcal{P}}_l\big(\partial_\sigma\epsilon + [\mathcal{J}_\sigma, \epsilon]\big) + \mathcal{E}^{-1}\mathcal{P}_l\big(\partial_\tau\epsilon + [\mathcal{J}_\tau, \epsilon]\big). $$

Moreover, the variation of $\mathcal{J}_\tau$ is given by $\delta\mathcal{J}_\tau = \mathcal{E}\delta\mathcal{J}_\sigma$.

Proof. Let us consider the gauge transformation (2.17) of $\mathcal{J}$ by l. Its variation under an infinitesimal multiplication $\delta l = l\epsilon$ is given in terms of the variation of $\mathcal{J}$ by

$$ \delta({}^l\mathcal{J}) = \mathrm{Ad}_l\,\big(\delta\mathcal{J} + [\epsilon, \mathcal{J}] - d\epsilon\big). $$

Rewriting this equation in components and acting with Ad_l^{-1}, we get

$$ \delta\mathcal{J}_\sigma = \mathrm{Ad}_l^{-1}\,\delta\big({}^l\mathcal{J}_\sigma\big) + \partial_\sigma\epsilon + [\mathcal{J}_\sigma, \epsilon] \qquad \text{and} \qquad \mathcal{E}\,\delta\mathcal{J}_\sigma = \mathrm{Ad}_l^{-1}\,\delta\big({}^l\mathcal{J}_\tau\big) + \partial_\tau\epsilon + [\mathcal{J}_\tau, \epsilon], $$

where we have used the fact that $\mathcal{J}_\tau = \mathcal{E}\mathcal{J}_\sigma$. Recall from (2.17) that ${}^l\mathcal{J}_\sigma$ and ${}^l\mathcal{J}_\tau$, and thus also their variations $\delta({}^l\mathcal{J}_\sigma)$ and $\delta({}^l\mathcal{J}_\tau)$, are valued in $\mathfrak{k}$. The first terms in the right-hand sides of the above equations are thus valued in $\mathrm{Ad}_l^{-1}\,\mathfrak{k} = \ker\mathcal{P}_l = \ker\overline{\mathcal{P}}_l$. Applying $\overline{\mathcal{P}}_l$ to the first equation and $\mathcal{P}_l$ to the second one, we thus get:

$$ \overline{\mathcal{P}}_l(\delta\mathcal{J}_\sigma) = \overline{\mathcal{P}}_l\big(\partial_\sigma\epsilon + [\mathcal{J}_\sigma, \epsilon]\big) \qquad \text{and} \qquad \mathcal{P}_l\mathcal{E}(\delta\mathcal{J}_\sigma) = \mathcal{P}_l\big(\partial_\tau\epsilon + [\mathcal{J}_\tau, \epsilon]\big). $$

Using the fact that $\mathcal{P}_l\mathcal{E} = \mathcal{E} - \mathcal{E}\overline{\mathcal{P}}_l$ (see part (*ii*) of Proposition 2.3) and taking the sum of the first equation above and the action of $\mathcal{E}^{-1}$ on the second, we thus get

$$ \delta\mathcal{J}_\sigma = \overline{\mathcal{P}}_l\big(\partial_\sigma\epsilon + [\mathcal{J}_\sigma, \epsilon]\big) + \mathcal{E}^{-1}\mathcal{P}_l\big(\partial_\tau\epsilon + [\mathcal{J}_\tau, \epsilon]\big). $$

This ends the proof of the lemma (noting that the variation of $\mathcal{J}_\tau = \mathcal{E}\mathcal{J}_\sigma$ directly follows from the variation of $\mathcal{J}_\sigma$). ■

Using Lemma 2.4, one can compute the variation of the action (2.18) and derive the equations of motion of the model.

Proposition 2.5. *The equations of motion of the action (2.18) take the form of the zero curvature equation:*

$$ \mathrm{d}\mathcal{J} + \frac{1}{2}[\mathcal{J}, \mathcal{J}] = 0. $$

Proof. Combining Lemma 2.4 with the facts that ${}^t\overline{\mathcal{P}}_l = \mathrm{id} - \mathcal{P}_l$ and ${}^t(\mathcal{E}^{-1}\mathcal{P}_l) = \mathcal{E}^{-1}\mathcal{P}_l$ (see Section 2.1), one has

$$\langle\!\langle l^{-1}\partial_\tau l, \delta\mathcal{J}_\sigma\rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle(\mathrm{id} - \mathcal{P}_l)l^{-1}\partial_\tau l, \partial_\sigma\epsilon + [\mathcal{J}_\sigma, \epsilon]\rangle\!\rangle_{\mathfrak{d}} + \langle\!\langle\mathcal{E}^{-1}\mathcal{P}_l\, l^{-1}\partial_\tau l, \partial_\tau\epsilon + [\mathcal{J}_\tau, \epsilon]\rangle\!\rangle_{\mathfrak{d}}.$$

Similarly, using ${}^t\mathcal{P}_l = \mathrm{id} - \overline{\mathcal{P}}_l$ and ${}^t(\mathcal{E}\overline{\mathcal{P}}_l) = \mathcal{E}\overline{\mathcal{P}}_l$ we get

$$\langle\!\langle l^{-1}\partial_\sigma l, \delta\mathcal{J}_\tau\rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle\mathcal{E}\overline{\mathcal{P}}_l\, l^{-1}\partial_\sigma l, \partial_\sigma\epsilon + [\mathcal{J}_\sigma, \epsilon]\rangle\!\rangle_{\mathfrak{d}} + \langle\!\langle(\mathrm{id} - \overline{\mathcal{P}}_l)l^{-1}\partial_\sigma l, \partial_\tau\epsilon + [\mathcal{J}_\tau, \epsilon]\rangle\!\rangle_{\mathfrak{d}}.$$

Taking the difference of the above two equations, we obtain

$$\begin{aligned}\langle\!\langle l^{-1}\partial_\sigma l, \delta\mathcal{J}_\tau\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle l^{-1}\partial_\tau l, \delta\mathcal{J}_\sigma\rangle\!\rangle_{\mathfrak{d}} &= \langle\!\langle(\mathrm{id} - \overline{\mathcal{P}}_l)l^{-1}\partial_\sigma l - \mathcal{E}^{-1}\mathcal{P}_l\, l^{-1}\partial_\tau l, \partial_\tau\epsilon + [\mathcal{J}_\tau, \epsilon]\rangle\!\rangle_{\mathfrak{d}} \\ &\quad - \langle\!\langle(\mathrm{id} - \mathcal{P}_l)l^{-1}\partial_\tau l - \mathcal{E}\overline{\mathcal{P}}_l\, l^{-1}\partial_\sigma l, \partial_\sigma\epsilon + [\mathcal{J}_\sigma, \epsilon]\rangle\!\rangle_{\mathfrak{d}} \\ &= \langle\!\langle l^{-1}\partial_\sigma l - \mathcal{J}_\sigma, \partial_\tau\epsilon + [\mathcal{J}_\tau, \epsilon]\rangle\!\rangle_{\mathfrak{d}} \\ &\quad - \langle\!\langle l^{-1}\partial_\tau l - \mathcal{J}_\tau, \partial_\sigma\epsilon + [\mathcal{J}_\sigma, \epsilon]\rangle\!\rangle_{\mathfrak{d}},\end{aligned}$$

where in the last equality we have used the definition (2.16) of $\mathcal{J}_\sigma$ and $\mathcal{J}_\tau$. In terms of forms, we can rewrite the above equation as

$$\langle\!\langle l^{-1}\mathrm{d}l, \delta\mathcal{J}\rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle\mathrm{d}\epsilon + [\mathcal{J}, \epsilon], \mathcal{J} - l^{-1}\mathrm{d}l\rangle\!\rangle_{\mathfrak{d}}.$$

Under the infinitesimal multiplication $\delta l = l\epsilon$, the variation of the 1-form $l^{-1}\mathrm{d}l$ is given by

$$\delta\big(l^{-1}\mathrm{d}l\big) = \mathrm{d}\epsilon + \big[l^{-1}\mathrm{d}l, \epsilon\big],$$

while the variation of the WZ-term $I^{\mathrm{WZ}}_{\mathfrak{d}}[l]$ follows from the Polyakov–Wiegmann identity [48] and reads

$$\delta I^{\mathrm{WZ}}_{\mathfrak{d}}[l] = \int_\Sigma \langle\!\langle\mathrm{d}\epsilon, l^{-1}\mathrm{d}l\rangle\!\rangle_{\mathfrak{d}}.$$

Combining all the above, we then determine the variation of the action (2.18) to be

$$\begin{aligned}\delta S_{\mathcal{E},\mathfrak{k}}(l) &= -\frac{1}{2}\int_\Sigma \big(\langle\!\langle\delta(l^{-1}\mathrm{d}l), \mathcal{J}\rangle\!\rangle_{\mathfrak{d}} + \langle\!\langle l^{-1}\mathrm{d}l, \delta\mathcal{J}\rangle\!\rangle_{\mathfrak{d}}\big) - \frac{1}{2}\delta I^{\mathrm{WZ}}_{\mathfrak{d}}[l] \\ &= -\int_\Sigma \bigg(\langle\!\langle\mathrm{d}\epsilon, \mathcal{J}\rangle\!\rangle_{\mathfrak{d}} + \frac{1}{2}\langle\!\langle[\mathcal{J}, \epsilon], \mathcal{J}\rangle\!\rangle_{\mathfrak{d}} + \frac{1}{2}\langle\!\langle[l^{-1}\mathrm{d}l, \epsilon], \mathcal{J}\rangle\!\rangle_{\mathfrak{d}} + \frac{1}{2}\langle\!\langle l^{-1}\mathrm{d}l, [\mathcal{J}, \epsilon]\rangle\!\rangle_{\mathfrak{d}}\bigg).\end{aligned}$$

Using the ad-invariance of $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}$ and integration by part, we finally get

$$\delta S_{\mathcal{E},\mathfrak{k}}(l) = \int_\Sigma \left\langle\!\!\left\langle \epsilon, d\mathcal{J} + \frac{1}{2}[\mathcal{J}, \mathcal{J}]\right\rangle\!\!\right\rangle_{\mathfrak{d}}.$$

The result now follows by requiring that $\delta S_{\mathcal{E},\mathfrak{k}}(l) = 0$ for every ϵ. ■

Remark 2.6. In the relativistic case when $\mathcal{E}^2 = \mathrm{id}$, see Section 2.5 below and in particular Remark 2.8, one can rewrite (2.16a) in terms of the projector (2.15) as

$$\mathcal{J}_\sigma = -\partial_\sigma f f^{-1} + \frac{1}{2}\,\mathrm{Ad}_f\,\big(P_f(\mathcal{E})\big(f^{-1}\partial_+ f\big) - P_f(-\mathcal{E})\big(f^{-1}\partial_- f\big)\big).$$

This coincides with [32, equation (17)] up to an overall sign, which is due to a difference in conventions. Indeed, the equations of motion from Proposition 2.5 can be written in components as

$$\partial_\tau\mathcal{J}_\sigma - \partial_\sigma(\mathcal{E}\mathcal{J}_\sigma) + [\mathcal{E}\mathcal{J}_\sigma, \mathcal{J}_\sigma] = 0$$

which are to be compared with the equations of motion in [32, equation (9)].

2.5 Energy-momentum tensor

The following proposition will be useful in the discussion of Section 4.6. We give its proof in Appendix B.

Proposition 2.7. *The components of the energy-momentum tensor of the* $\mathcal{E}$*-model* (2.14) *are given by*

$$T^\tau{}_\tau = -T^\sigma{}_\sigma = \frac{1}{2}\langle\langle \mathcal{J}_\sigma, \mathcal{E}\mathcal{J}_\sigma \rangle\rangle_{\mathfrak{d}}, \qquad T^\tau{}_\sigma = \frac{1}{2}\langle\langle \mathcal{J}_\sigma, \mathcal{J}_\sigma \rangle\rangle_{\mathfrak{d}}, \qquad T^\sigma{}_\tau = -\frac{1}{2}\langle\langle \mathcal{J}_\sigma, \mathcal{E}^2\mathcal{J}_\sigma \rangle\rangle_{\mathfrak{d}},$$

where $\mathcal{J}_\sigma$ *is the* $\mathfrak{d}$*-valued field defined in* (2.16)*.*

Remark 2.8. The relativistic invariance of the $\mathcal{E}$-model can be deduced immediately from Proposition 2.7. Defining the 2d Minkowski metric $\eta_{\mu\nu}$ by $\eta_{\tau\tau} = -\eta_{\sigma\sigma} = 1$ and $\eta_{\tau\sigma} = -\eta_{\sigma\tau} = 0$, we can lower the indices of the energy-momentum tensor and define $T_{\mu\nu} \coloneqq \eta_{\mu\rho} T^\rho{}_\nu$. In particular, we get

$$T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2}\langle\langle \mathcal{J}_\sigma, \mathcal{E}\mathcal{J}_\sigma \rangle\rangle_{\mathfrak{d}}, \qquad T_{\tau\sigma} = \frac{1}{2}\langle\langle \mathcal{J}_\sigma, \mathcal{J}_\sigma \rangle\rangle_{\mathfrak{d}}, \qquad T_{\sigma\tau} = \frac{1}{2}\langle\langle \mathcal{J}_\sigma, \mathcal{E}^2\mathcal{J}_\sigma \rangle\rangle_{\mathfrak{d}}.$$

It follows that if $\mathcal{E}^2 = \mathrm{id}$ then the energy-momentum tensor $T_{\mu\nu}$ is symmetric, which implies the relativistic invariance of the model.

Moreover, if $\mathcal{E}$ is positive with respect to the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_{\mathfrak{d}}$ then the Hamiltonian $\int_{\mathbb{R}} \mathrm{d}\sigma\, T^\tau{}_\tau$ is positive.

3 2d integrable field theories from 4d Chern–Simons theory

The general 2d action obtained in [3] will serve as the starting point of our analysis in Section 4 below, so in this section we begin by reviewing the results of [3], referring the reader to the latter for details and proofs.

In order to ensure that the 2d action is real, we will also impose reality conditions following [17], see also [16, 64] in the context of affine Gaudin models. Although this was not directly considered in [3], the analysis there readily applies to the real setting.

3.1 Surface defects

Let ω be a meromorphic 1-form on $\mathbb{C}P^1$. We denote its set of poles by $\boldsymbol{\mathfrak{z}} \subset \mathbb{C}P^1$ and denote by $n_x \in \mathbb{Z}_{\geq 1}$ the order of a pole $x \in \boldsymbol{\mathfrak{z}}$. Although this was not necessary in the analysis of [3], we shall assume here that ω has a double pole at infinity, namely $\infty \in \boldsymbol{\mathfrak{z}}$ with $n_\infty = 2$. Let us fix a coordinate z on $\mathbb{C} \subset \mathbb{C}P^1$ so that ω can be written explicitly as

$$\omega = \left(\sum_{x \in \boldsymbol{\mathfrak{z}}'} \sum_{p=0}^{n_x - 1} \frac{\ell_p^x}{(z-x)^{p+1}} - \ell_1^\infty \right) \mathrm{d}z =: \varphi(z) \mathrm{d}z, \tag{3.1}$$

where $\boldsymbol{\mathfrak{z}}' \coloneqq \boldsymbol{\mathfrak{z}} \setminus \{\infty\}$, for some $\ell_p^x \in \mathbb{C}$ which we refer to as the *levels*. We also define $\ell_0^\infty \coloneqq \mathrm{res}_\infty\, \omega = -\sum_{x \in \boldsymbol{\mathfrak{z}}'} \ell_0^x$.

We impose reality conditions on each pole $x \in \boldsymbol{\mathfrak{z}}$ and its corresponding set of levels ℓ_p^x, $p = 0, \ldots, n_x - 1$ by requiring that $\overline{\varphi(z)} = \varphi(\bar{z})$. In particular, we define the subset of real poles $\boldsymbol{z}_{\mathrm{r}} \coloneqq \boldsymbol{z}'_{\mathrm{r}} \sqcup \{\infty\}$, where $\boldsymbol{z}'_{\mathrm{r}} \coloneqq \boldsymbol{\mathfrak{z}}' \cap \mathbb{R}$. By the above assumption on φ the associated levels are real, i.e., $\ell_p^x \in \mathbb{R}$. The remaining poles come in complex conjugate pairs and we define $\boldsymbol{z}_{\mathrm{c}} \coloneqq \{x \in \boldsymbol{\mathfrak{z}} \mid \Im x > 0\}$ so that $\boldsymbol{\mathfrak{z}} = \boldsymbol{z}_{\mathrm{r}} \sqcup \boldsymbol{z}_{\mathrm{c}} \sqcup \bar{\boldsymbol{z}}_{\mathrm{c}}$. For every $x \in \boldsymbol{z}_{\mathrm{c}} \sqcup \bar{\boldsymbol{z}}_{\mathrm{c}}$ we have $n_{\bar{x}} = n_x$ and

$\overline{\ell^x_p} = \ell^{\bar{x}}_p$ for $p = 0, \dots, n_x - 1$. It is convenient to introduce the set $\boldsymbol{z} := \boldsymbol{z}_{\rm r} \sqcup \boldsymbol{z}_{\rm c}$ of independent poles. We also introduce the subset $\boldsymbol{z}' := \boldsymbol{z}'_{\rm r} \sqcup \boldsymbol{z}_{\rm c} \subset \boldsymbol{z}$ of finite independent poles in $\boldsymbol{z}$.

The set of zeroes of ω can be similarly decomposed as $\boldsymbol{\zeta}_{\rm r} \sqcup \boldsymbol{\zeta}_{\rm c} \sqcup \bar{\boldsymbol{\zeta}}_{\rm c}$ with $\boldsymbol{\zeta}_{\rm r} \subset \mathbb{R}$ the subset of real zeroes and $\boldsymbol{\zeta}_{\rm c} \subset \{z \in \mathbb{C} \mid \Im z > 0\}$ the subset of complex zeroes. We introduce the set $\boldsymbol{\zeta} := \boldsymbol{\zeta}_{\rm r} \sqcup \boldsymbol{\zeta}_{\rm c}$ of independent zeroes and let $m_y \in \mathbb{Z}_{\geq 1}$ denote the order of the zero $y \in \boldsymbol{\zeta}$. For $y \in \boldsymbol{\zeta}_{\rm c}$, ω also has a zero of order $m_{\bar{y}} := m_y$ at $\bar{y} \in \bar{\boldsymbol{\zeta}}_{\rm c}$.

It will be convenient to introduce the group $\Pi = \{\mathrm{id}, \mathsf{t}\} \cong \mathbb{Z}_2$ which acts on $\mathbb{C}P^1$ by letting t act by complex conjugation $\mu_{\mathsf{t}} \colon z \mapsto \bar{z}$. Note that we can then write $\boldsymbol{\mathcal{Z}} = \Pi \boldsymbol{z}$ and we have $\Pi \boldsymbol{\zeta} = \boldsymbol{\zeta}_{\rm r} \sqcup \boldsymbol{\zeta}_{\rm c} \sqcup \bar{\boldsymbol{\zeta}}_{\rm c}$. Let $\Pi_x \subset \Pi$ denote the stabiliser subgroup of a point $x \in \mathbb{C}$, so that $\Pi_x = \{\mathrm{id}\}$ is the trivial group for $x \in \boldsymbol{z}_{\rm c} \sqcup \bar{\boldsymbol{z}}_{\rm c}$ and $\Pi_x = \Pi$ for $x \in \boldsymbol{z}_{\rm r}$. In particular, $|\Pi_x| = 2$ if $x \in \boldsymbol{z}_{\rm r}$ and $|\Pi_x| = 1$ if $x \in \boldsymbol{z}_{\rm c}$. The analogous statements hold for the stabilisers Π_y of zeroes $y \in \boldsymbol{\zeta}_{\rm r} \sqcup \boldsymbol{\zeta}_{\rm c} \sqcup \bar{\boldsymbol{\zeta}}_{\rm c}$ of ω.

Let $C := \mathbb{C}P^1 \setminus \boldsymbol{\zeta}$, $\Sigma := \mathbb{R}^2$ and $X := \Sigma \times C$. We will always think of ω as defining a 1-form on X with singularities along the disjoint union

$$\mathcal{D} := \bigsqcup_{x \in \boldsymbol{z}} \Sigma_x$$

of *surface defects* $\Sigma_x := \Sigma \times \{x\} \subset X$, each trivially homeomorphic to Σ, i.e., $\Sigma_x \cong \Sigma$. We denote the embedding of the individual surface defects by

$$\iota_x \colon \ \Sigma_x \hookrightarrow X.$$

To account for the fact that poles and zeroes of ω may not be simple, it will be convenient to let $[\boldsymbol{z}]$ denote the set of pairs $[x, p]$ with $x \in \boldsymbol{z}$ and $p = 0, \dots, n_x - 1$, and likewise, let $(\boldsymbol{\zeta})$ be the set of pairs (y, q) with $y \in \boldsymbol{\zeta}$ and $q = 0, \dots, m_y - 1$. We think of the collection $[x, p]$ for $p = 0, \dots, n_x - 1$ (resp. (y, q) for $q = 0, \dots, m_y - 1$) as an infinitesimal "thickening" of the pole x (resp. the zero y).

3.2 Defect Lie algebra

Let G be a real simply connected Lie group. We suppose that its Lie algebra $\mathfrak{g}$ is equipped with a non-degenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

Let $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of $\mathfrak{g}$, which comes equipped with an anti-linear involution $\tau \colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ given by complex conjugation in the second tensor factor. We extend the bilinear form on $\mathfrak{g}$ to a bilinear form $\langle \cdot, \cdot \rangle \colon \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \to \mathbb{C}$ by complex linearity, so that $\langle \tau \mathsf{u}, \tau \mathsf{v} \rangle = \overline{\langle \mathsf{u}, \mathsf{v} \rangle}$ for any $\mathsf{u}, \mathsf{v} \in \mathfrak{g}^{\mathbb{C}}$. Let $\tau \colon G^{\mathbb{C}} \to G^{\mathbb{C}}$ be the lift of $\tau \colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ to an involutive automorphism of $G^{\mathbb{C}}$. The real Lie group G can then be identified as the subgroup of fixed points of τ.

Let $\mathcal{T}^{n_x}_x := \mathbb{R}[\varepsilon_x]/(\varepsilon^{n_x}_x)$ for real poles $x \in \boldsymbol{z}_{\rm r}$ and $\mathcal{T}^{n_x}_x := \mathbb{C}[\varepsilon_x]/(\varepsilon^{n_x}_x)$ for complex poles $x \in \boldsymbol{z}_{\rm c}$. We define the *defect Lie algebra* as the real Lie algebra

$$\mathfrak{g}^{[\boldsymbol{z}]} := \bigoplus_{x \in \boldsymbol{z}_{\rm r}} \mathfrak{g} \otimes_{\mathbb{R}} \mathcal{T}^{n_x}_x \oplus \bigoplus_{x \in \boldsymbol{z}_{\rm c}} \big(\mathfrak{g}^{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{T}^{n_x}_x \big)_{\mathbb{R}}, \tag{3.2}$$

where $(\mathfrak{g}^{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{T}^{n_x}_x)_{\mathbb{R}}$ is the realification of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{T}^{n_x}_x$, i.e., $\mathfrak{g}^{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{T}^{n_x}_x$ regarded as a Lie algebra over $\mathbb{R}$. We use the notation $\mathsf{u}^{[x,p]} := \mathsf{u} \otimes \varepsilon^p_x \in \mathfrak{g}^{[\boldsymbol{z}]}$ for any $\mathsf{u} \in \mathfrak{g}$ and $[x, p] \in [\boldsymbol{z}_{\rm r}]$ or $\mathsf{u} \in \mathfrak{g}^{\mathbb{C}}$ and $[x, p] \in [\boldsymbol{z}_{\rm c}]$. The Lie algebra relations of $\mathfrak{g}^{[\boldsymbol{z}]}$ are given explicitly in terms of this basis as

$$\big[\mathsf{u}^{[x,p]}, \mathsf{v}^{[y,q]}\big] = \delta_{xy} [\mathsf{u}, \mathsf{v}]^{[x,p+q]}.$$

Note that this is zero if $p+q \geq n_x$. We equip $\mathfrak{g}^{[\boldsymbol{z}]}$ with a non-degenerate invariant symmetric bilinear form defined by

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{g}^{[\boldsymbol{z}]}} \colon \; \mathfrak{g}^{[\boldsymbol{z}]} \times \mathfrak{g}^{[\boldsymbol{z}]} \longrightarrow \mathbb{R}, \qquad \big\langle\!\big\langle \mathsf{u}^{[x,p]}, \mathsf{v}^{[y,q]} \big\rangle\!\big\rangle_{\mathfrak{g}^{[\boldsymbol{z}]}} = \delta_{xy} \frac{2}{|\Pi_x|} \Re\big(\ell^x_{p+q} \langle \mathsf{u}, \mathsf{v} \rangle\big). \tag{3.3}$$

Here we define $\ell^x_p = 0$ for all $p \geq n_x$. Note that for $x \in \boldsymbol{z}_{\rm r}$ we have

$$\big\langle\!\big\langle \mathsf{u}^{[x,p]}, \mathsf{v}^{[y,q]} \big\rangle\!\big\rangle_{\mathfrak{g}^{[\boldsymbol{z}]}} = \delta_{xy}\, \ell^x_{p+q} \langle \mathsf{u}, \mathsf{v} \rangle,$$

while for $x \in \boldsymbol{z}_{\rm c}$ we have

$$\big\langle\!\big\langle \mathsf{u}^{[x,p]}, \mathsf{v}^{[y,q]} \big\rangle\!\big\rangle_{\mathfrak{g}^{[\boldsymbol{z}]}} = \delta_{xy} \big(\ell^x_{p+q} \langle \mathsf{u}, \mathsf{v} \rangle + \ell^{\bar{x}}_{p+q} \langle \tau \mathsf{u}, \tau \mathsf{v} \rangle \big).$$

One can also introduce a real Lie group with Lie algebra $\mathfrak{g}^{[\boldsymbol{z}]}$ which we will call the *defect group* and denote by $G^{[\boldsymbol{z}]}$.

From now on we will assume that the real Lie algebra $\mathfrak{g}^{[\boldsymbol{z}]}$ is even dimensional. In other words, either $\mathfrak{g}$ itself is even dimensional or the number of poles of ω counting multiplicities, namely $\sum_{x \in \boldsymbol{z}} n_x$, is even. Note that since we are assuming $n_\infty = 2$ it follows that $\sum_{x \in \boldsymbol{z}'} n_x$ is also even.

3.3 The map $\boldsymbol{j}^*$

Let $\Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)$ denote the complex vector space of smooth $\mathfrak{g}^{\mathbb{C}}$-valued 1-forms on X. We can define two actions of the group Π on $\Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)$: we can let $\mathsf{t} \in \Pi$ act as the pullback by complex conjugation $\mu_{\mathsf{t}} \colon z \mapsto \bar{z}$ or we can let it act as τ on $\mathfrak{g}^{\mathbb{C}}$. We let $\Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)^\Pi$ denote the real vector space consisting of 1-forms $\eta \in \Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)$ on which these two actions agree, namely such that $\mu_{\mathsf{t}}^* \eta = \tau \eta$.

The relationship between the defect Lie algebra $\mathfrak{g}^{[\boldsymbol{z}]}$ and the surface defect $\mathcal{D}$ can be understood through the following linear map of real vector spaces

$$\boldsymbol{j}^* \colon \; \Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)^\Pi \longrightarrow \Omega^1\big(\Sigma, \mathfrak{g}^{[\boldsymbol{z}]}\big), \qquad \eta \longmapsto \left(\sum_{p=0}^{n_x - 1} \frac{1}{p!} \iota_x^* (\partial_z^p \eta) \otimes \varepsilon_x^p \right)_{x \in \boldsymbol{z}}. \tag{3.4}$$

In words, this map takes a smooth equivariant 1-form $\eta \in \Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)^\Pi$ and returns the first n_x terms in the holomorphic part of its Taylor expansion at points on the surface defect $\mathcal{D}$, keeping only the two components of the 1-form along Σ. Note that for $x \in \boldsymbol{z}_{\rm r}$ the corresponding component of (3.4) is indeed in $\mathfrak{g} \otimes_{\mathbb{R}} \mathcal{T}^{n_x}_x$ since

$$\tau\big(\iota_x^*(\partial_z^p \eta)\big) = \iota_x^*(\partial_{\bar{z}}^p (\tau \eta)) = \iota_x^*(\partial_{\bar{z}}^p (\mu_{\mathsf{t}}^* \eta)) = \iota_x^* \mu_{\mathsf{t}}^* (\partial_z^p \eta) = \iota_x^*(\partial_z^p \eta),$$

where in the first equality we used the anti-linearity of τ, in the second step the equivariance of η and in the final step the fact that $\mu_{\mathsf{t}} \circ \iota_x = \iota_x$ since $x \in \boldsymbol{z}_{\rm r}$.

In the simplest case when $n_x = 1$ for all $x \in \boldsymbol{z}$, the map $\boldsymbol{j}^*$ is simply the pullback by the embedding $\boldsymbol{j} \colon \mathcal{D} \hookrightarrow X$ since we have the canonical identification

$$\Omega^1\big(\mathcal{D}, \mathfrak{g}^{\mathbb{C}}\big)^\Pi \cong \left(\bigoplus_{x \in \boldsymbol{z}} \Omega^1\big(\Sigma, \mathfrak{g}^{\mathbb{C}}\big) \right)^\Pi \cong \Omega^1 \left(\Sigma, \left(\bigoplus_{x \in \boldsymbol{z}} \mathfrak{g}^{\mathbb{C}} \right)^\Pi \right) = \Omega^1\big(\Sigma, \mathfrak{g}^{[\boldsymbol{z}]}\big). \tag{3.5}$$

In this case, the map (3.4) is given simply by $\boldsymbol{j}^* \eta = (\iota_x^* \eta)_{x \in \boldsymbol{z}}$, namely the collection of pullbacks of η to each surface defect Σ_x.

3.4 4d Chern–Simons theory with edge modes

The action of 4-dimensional Chern–Simons theory [13] for an equivariant $\mathfrak{g}^{\mathbb{C}}$-valued 1-form $A \in \Omega^1(X, \mathfrak{g}^{\mathbb{C}})^{\Pi}$ is given by integrating $\frac{\mathrm{i}}{4\pi}\omega \wedge \mathrm{CS}(A) \in \Omega^4(X)$ over X, where $\mathrm{CS}(A) := \langle A, \mathrm{d}A + \frac{1}{3}[A, A]\rangle$ is the Chern–Simons 3-form. By [17, Lemma 2.4] this action is real.

Strictly speaking, the 4-form $\omega \wedge \mathrm{CS}(A)$ is not integrable in the neighbourhood of a surface defect Σ_x corresponding to a higher order pole $x \in \boldsymbol{z}$ with $n_x > 1$. For this reason, one needs to introduce a suitable regularisation of the action [3], which we denote by $S_{4\mathrm{d}}(A)$. The proof of [17, Lemma 2.4] generalises to this regularised action, showing that it is also real. The behaviour of $S_{4\mathrm{d}}(A)$ under gauge transformations

$$A \longmapsto {}^{g}A := -\mathrm{d}g g^{-1} + gAg^{-1}$$

by $g \in C^\infty(X, G^{\mathbb{C}})^{\Pi}$ was studied in [3], where it was shown that gauge invariance can be achieved in two separate but equivalent ways: either by imposing boundary conditions on the field $A \in \Omega^1(X, \mathfrak{g}^{\mathbb{C}})^{\Pi}$, or by coupling A to a new field localised in the formal neighbourhood of the surface defect $\mathcal{D}$ which amounts to a field $h \in C^\infty(\Sigma, G^{[\boldsymbol{z}]})$.

To describe a general class of boundary conditions on A, note that by applying the map (3.4) to $A \in \Omega^1(X, \mathfrak{g}^{\mathbb{C}})^{\Pi}$ we obtain a 1-form $\boldsymbol{j}^*A \in \Omega^1(\Sigma, \mathfrak{g}^{[\boldsymbol{z}]})$. Let $\mathfrak{f} \subset \mathfrak{g}^{[\boldsymbol{z}]}$ be a Lagrangian subalgebra of the defect Lie algebra $\mathfrak{g}^{[\boldsymbol{z}]}$ and let F be the corresponding connected real Lie subgroup of $G^{[\boldsymbol{z}]}$. It was shown in [3] that the regularised action $S_{4\mathrm{d}}(A)$ becomes gauge invariant if we restrict attention to fields $A \in \Omega^1(X, \mathfrak{g}^{\mathbb{C}})^{\Pi}$ for which $\boldsymbol{j}^*A \in \Omega^1(\Sigma, \mathfrak{f})$. Correspondingly, gauge transformation parameters $g \in C^\infty(X, G^{\mathbb{C}})^{\Pi}$ are restricted to be such that their "pullback" to the formal neighbourhood of $\mathcal{D}$ is F-valued.

An alternative way of ensuring gauge-invariance of 4d Chern–Simons theory, which provides a direct route to the action of the 2d integrable field theory [3], requires introducing a new field $h \in C^\infty(\Sigma, G^{[\boldsymbol{z}]})$ called the *edge mode*. In the simplest case when ω has only simple poles, i.e., $n_x = 1$ for all $x \in \boldsymbol{z}$, we have a canonical isomorphism $C^\infty(\mathcal{D}, G^{\mathbb{C}})^{\Pi} \cong C^\infty(\Sigma, G^{[\boldsymbol{z}]})$ by the same line of reasoning as in (3.5), allowing us to view the edge mode in this case as a $G^{\mathbb{C}}$-valued function on $\mathcal{D}$. In the presence of higher order poles of ω, one can think of the edge mode as a $G^{\mathbb{C}}$-valued field localised in a formal neighbourhood of the defect $\mathcal{D}$. Its role is to witness the boundary condition on A. Specifically, rather than imposing boundary conditions on A *strictly*, as in the previous paragraph, we require A to satisfy these boundary conditions only up to a gauge transformation by the edge mode, namely

$$^{h}(\boldsymbol{j}^*A) \in \Omega^1(\Sigma, \mathfrak{f}). \tag{3.6}$$

Remark 3.1. Here we depart slightly from the conventions used in [3], where the condition (3.6) was written as $^{h^{-1}}(\boldsymbol{j}^*A) \in \Omega^1(\Sigma, \mathfrak{f})$. Effectively, our edge mode coincides with the inverse of the edge mode in [3].

We can now ensure gauge invariance of 4d Chern–Simons theory by coupling the bulk field A to the edge mode h, through its "pullback" $\boldsymbol{j}^*A$. Explicitly, we introduce the extended action [3]

$$S^{\mathrm{ext}}_{4\mathrm{d}}(A, h) = S_{4\mathrm{d}}(A) - \frac{1}{2}\int_\Sigma \langle\!\langle h^{-1}\mathrm{d}h, \boldsymbol{j}^*A\rangle\!\rangle_{\mathfrak{g}^{[\boldsymbol{z}]}} - \frac{1}{2} I^{\mathrm{WZ}}_{\mathfrak{g}^{[\boldsymbol{z}]}}[h], \tag{3.7}$$

where we use the standard WZ-term defined as in (2.8) but with the group $G^{[\boldsymbol{z}]}$ replacing the role of D. The action (3.7) and the constraint (3.6) are invariant under the gauge transformation

$$A \longmapsto {}^{g}A, \qquad h \longmapsto h(\boldsymbol{j}^*g)^{-1} \tag{3.8a}$$

for any $g \in C^\infty\big(X, G^{\mathbb{C}}\big)^\Pi$. There is also a further gauge transformation acting on the edge mode alone as

$$h \longmapsto fh \tag{3.8b}$$

for any $f \in C^\infty(\Sigma, F)$. The invariance of the action (3.7) under these gauge transformations follows using the Polyakov–Wiegmann identity on the WZ-term.

3.5 Reduction to 2d integrable field theories

Having introduced edge modes in the extended action (3.7), the passage to 2d integrable field theories is now fairly direct. Indeed, the edge mode $h \in C^\infty\big(\Sigma, G^{[\boldsymbol{z}]}\big)$ will ultimately play the role of the collection of fields of the 2d integrable field theory. The gauge field $A \in \Omega^1(X, \mathfrak{g})$, on the other hand, will become the Lax connection $\mathcal{L}$ of the integrable field theory. For this to happen, however, we have to restrict attention to 1-forms A which only have components along $\Sigma \subset X$ and which depend holomorphically on the complex direction $C \subset X$. More precisely, this can be done by focusing on a certain class of solutions to part of the equations of motion for the extended action (3.7), which we now describe.

For a complex vector space V we let $R^\infty_{\Pi\boldsymbol{\zeta}}(V)$ denote the space of V-valued rational functions with poles at each $y \in \Pi\boldsymbol{\zeta}$ of order at most m_y, the order of the zero y of ω. If V is equipped with an anti-linear involution $\tau\colon V \to V$ then we can define an action of Π on V by letting $\mathsf{t} \in \Pi$ act as τ. This then also lifts to an action of Π on $R^\infty_{\Pi\boldsymbol{\zeta}}(V)$. We can also define an action of Π on $R^\infty_{\Pi\boldsymbol{\zeta}}(V)$ by letting $\mathsf{t} \in \Pi$ act as the pullback by complex conjugation $\mu_{\mathsf{t}}\colon z \mapsto \bar{z}$. We let $R^\infty_{\Pi\boldsymbol{\zeta}}(V)^\Pi$ denote the real vector space of rational functions in $R^\infty_{\Pi\boldsymbol{\zeta}}(V)$ on which these two actions coincide. In what follows we will either take $V = \mathfrak{g}^{\mathbb{C}}$ or $V = C^\infty\big(\Sigma, \mathfrak{g}^{\mathbb{C}}\big)$, where the action of Π on the latter is induced from the action of Π on $\mathfrak{g}^{\mathbb{C}}$.

Following [3], we will restrict attention to *admissible* 1-forms $\mathcal{L} \in \Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)^\Pi$ with the following properties:

(a) We have $\mathcal{L} = \mathcal{L}_\sigma \mathrm{d}\sigma + \mathcal{L}_\tau \mathrm{d}\tau$ with both components $\mathcal{L}_\sigma, \mathcal{L}_\tau \in R^\infty_{\Pi\boldsymbol{\zeta}}\big(C^\infty\big(\Sigma, \mathfrak{g}^{\mathbb{C}}\big)\big)^\Pi$. Explicitly, this means that we can write, for $\mu = \sigma, \tau$,

$$\mathcal{L}_\mu = \mathcal{L}_{\mathrm{c},\mu} + \sum_{(y,q)\in(\Pi\boldsymbol{\zeta})} \frac{\mathcal{L}^{(y,q)}_\mu}{(z-y)^{q+1}},$$

for some $\mathcal{L}_{\mathrm{c},\mu} \in C^\infty(\Sigma, \mathfrak{g})$ and $\mathcal{L}^{(y,q)}_\mu \in C^\infty\big(\Sigma, \mathfrak{g}^{\mathbb{C}}\big)$. In the case when $y \in \boldsymbol{\zeta}_{\mathrm{r}}$ we have $\mathcal{L}^{(y,q)}_\mu \in C^\infty(\Sigma, \mathfrak{g})$ and for $y \in \boldsymbol{\zeta}_{\mathrm{c}} \sqcup \bar{\boldsymbol{\zeta}}_{\mathrm{c}}$ we have $\tau\mathcal{L}^{(y,q)}_\mu = \mathcal{L}^{(\bar{y},q)}_\mu$ for all $q = 0, \dots, m_y = m_{\bar{y}}$.

(b) The single component of the curvature $\mathrm{d}\mathcal{L} + \frac{1}{2}[\mathcal{L}, \mathcal{L}] = F(\mathcal{L})_{\sigma\tau}\mathrm{d}\sigma \wedge \mathrm{d}\tau$ is also such that $F(\mathcal{L})_{\sigma\tau} \in R^\infty_{\Pi\boldsymbol{\zeta}}\big(C^\infty\big(\Sigma, \mathfrak{g}^{\mathbb{C}}\big)\big)^\Pi$. Given property ($a$), this is equivalent to the commutator term in $F(\mathcal{L})_{\sigma\tau}$ having no poles of order greater than m_y at each $y \in \boldsymbol{\zeta}$. Explicitly, we can write this as

$$\sum_{q=p-m_y+1}^{m_y-1} \big[\mathcal{L}^{(y,q)}_\sigma, \mathcal{L}^{(y,p-q)}_\tau\big] = 0$$

for every $y \in \boldsymbol{\zeta}$ and every $p = m_y - 1, \dots, 2m_y - 2$.

Let us now *suppose* that for every $h \in C^\infty\big(\Sigma, G^{[\boldsymbol{z}]}\big)$ there exists an admissible 1-form $\mathcal{L} = \mathcal{L}(h) \in \Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)^\Pi$ such that the condition (3.6) holds, namely

$${}^{h}(\boldsymbol{j}^*\mathcal{L}) \in \Omega^1(\Sigma, \mathfrak{f}). \tag{3.9}$$

Moreover, we require that the collection of solutions $\mathcal{L}(h)$ for every $h \in C^\infty\big(\Sigma, G^{[\boldsymbol{z}]}\big)$ is equivariant under those gauge transformations of the form (3.8) which preserve the class of admissible 1-forms. Specifically, for every $g \in C^\infty(\Sigma, G)$ and $f \in C^\infty(\Sigma, F)$ we should have

$$^{\Delta(g)^{-1}}\big(\boldsymbol{j}^*\mathcal{L}\big(fh\Delta(g)^{-1}\big)\big) = \boldsymbol{j}^*\mathcal{L}(h), \tag{3.10}$$

with $\Delta\colon G \to G^{\times|\boldsymbol{z}|} \subset G^{[\boldsymbol{z}]}$ the diagonal map. (At the Lie algebra level, the latter is given explicitly by the diagonal embedding $\mathfrak{g} \to \mathfrak{g}^{\oplus|\boldsymbol{z}|} \subset \mathfrak{g}^{[\boldsymbol{z}]}$, $\mathsf{u} \to (\mathsf{u}^{[x,0]})_{x\in\boldsymbol{z}}$.) Note that (3.10) is compatible with the constraint (3.9).

In the terminology of [3, Remark 5.10] this amounts to specifying a section of a certain surjective map. This section can then be used to pull back the action of 4d Chern–Simons theory in the presence of edge modes (3.7) to the action of a 2d integrable field theory with Lax connection $\mathcal{L}(h)$. More explicitly, given an admissible 1-form $\mathcal{L}(h)$ with the properties described above, if we substitute $A = \mathcal{L}(h)$ in the action (3.7) then the first term $S_{4\mathrm{d}}(\mathcal{L}(h))$ vanishes using both admissibility properties (a) and (b) and we are left with the action

$$S_{2\mathrm{d}}(h) = -\frac{1}{2}\int_\Sigma \langle\!\langle h^{-1}\mathrm{d}h, \boldsymbol{j}^*\mathcal{L}(h)\rangle\!\rangle_{\mathfrak{g}^{[\boldsymbol{z}]}} - \frac{1}{2} I^{\mathrm{WZ}}_{\mathfrak{g}^{[\boldsymbol{z}]}}[h] \tag{3.11}$$

for the field $h \in C^\infty\big(\Sigma, G^{[\boldsymbol{z}]}\big)$. In particular, computing the variation of the action (3.11) with respect to the fields h and $\mathcal{L}(h)$, taking into account the constraint (3.9) relating these two fields, see [3, equations (5.2)–(5.5)] for details, we find that the equations of motion take the form

$$\mathrm{d}\boldsymbol{j}^*\mathcal{L}(h) + \frac{1}{2}[\boldsymbol{j}^*\mathcal{L}(h), \boldsymbol{j}^*\mathcal{L}(h)] = 0.$$

By the admissibility of the 1-form $\mathcal{L}(h)$ it then follows from [3, Proposition 5.6], see also the related discussion in Section 4.5 below, that the above equation of motion is equivalent to the zero-curvature equation for $\mathcal{L}(h)$ itself, namely

$$\mathrm{d}\mathcal{L}(h) + \frac{1}{2}[\mathcal{L}(h), \mathcal{L}(h)] = 0.$$

Furthermore, because of the behaviour (3.10) of $\mathcal{L}(h)$ under gauge transformations, the action (3.11) is invariant under

$$h \longmapsto fh\Delta(g)^{-1} \tag{3.12}$$

for any $g \in C^\infty(\Sigma, G)$ and $f \in C^\infty(\Sigma, F)$.

3.6 Removing the edge mode at infinity

In order to obtain the $\mathcal{E}$-model from the action (3.11) we will need to make one further reduction. Specifically, we shall partially fix the gauge invariance (3.12) by setting the component of the edge mode $h \in C^\infty\big(\Sigma, G^{[\boldsymbol{z}]}\big)$ at infinity to the identity.

Consider the real Lie subalgebra of the defect Lie algebra $\mathfrak{g}^{[\boldsymbol{z}]}$ defined by

$$\mathfrak{d} := \bigoplus_{x\in\boldsymbol{z}'_{\mathrm{r}}} \mathfrak{g} \otimes_{\mathbb{R}} \mathcal{T}^{n_x}_x \oplus \bigoplus_{x\in\boldsymbol{z}_{\mathrm{c}}} \big(\mathfrak{g}^{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{T}^{n_x}_x\big)_{\mathbb{R}}. \tag{3.13}$$

Notice that in comparing this definition with that of $\mathfrak{g}^{[\boldsymbol{z}]}$ in (3.2) we have simply removed the factor $\mathfrak{g} \otimes_{\mathbb{R}} \mathcal{T}^2_\infty$ corresponding to the pole at infinifty. We let D denote the corresponding connected Lie subgroup of the defect group $G^{[\boldsymbol{z}]}$. Recall from Section 3.2 that we are assuming

$\dim \mathfrak{g}^{[\boldsymbol{z}]}$ is even, meaning that either $\mathfrak{g}$ is even dimensional or $\sum_{x\in\boldsymbol{z}'} n_x$ is even. Hence $\dim \mathfrak{d}$ is also even.

The non-degenerate bilinear form on $\mathfrak{g}^{[\boldsymbol{z}]}$ defined in (3.3) restricts to the subalgebra $\mathfrak{d} \subset \mathfrak{g}^{[\boldsymbol{z}]}$. We denote this restriction by

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{d}} \colon \ \mathfrak{d} \times \mathfrak{d} \longrightarrow \mathbb{R}. \tag{3.14}$$

Remark 3.2. A more natural notation for the Lie group and Lie algebra in (3.13) would be $G^{[\boldsymbol{z}']}$ and $\mathfrak{g}^{[\boldsymbol{z}']}$. We could also keep calling the induced bilinear form (3.14) as $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{g}^{[\boldsymbol{z}]}}$. The reason for using the above notation is that these will correspond to the standard notation for the Lie group on which the $\mathcal{E}$-model is defined.

Recalling from Section 3.1 that we are assuming $n_\infty = 2$, there is an obvious Lagrangian subalgebra of the factor $\mathfrak{g} \otimes_{\mathbb{R}} \mathcal{T}^2_\infty$ of $\mathfrak{g}^{[\boldsymbol{z}]}$ at infinity given by the abelian subalgebra

$$\mathfrak{g}_{\rm ab} \coloneqq \mathfrak{g} \otimes_{\mathbb{R}} \varepsilon_\infty \mathbb{R}[\varepsilon_\infty]/\big(\varepsilon_\infty^2\big).$$

For any choice of Lagrangian subalgebra $\mathfrak{k} \subset \mathfrak{d}$ we can then take $\mathfrak{f} = \mathfrak{g}_{\rm ab} \oplus \mathfrak{k} \subset \mathfrak{g}^{[\boldsymbol{z}]}$ for the Lagrangian subalgebra used in Section 3.4.

We denote by $G\big(\mathcal{T}^2_\infty\big)$ the factor of the Lie group $G^{[\boldsymbol{z}]}$ corresponding to the point at infinity. Concretely it is given by the tangent bundle TG and as a Lie group it is isomorphic to $G \ltimes \mathfrak{g}$. By a slight abuse of notation we will still denote by G the Lie subgroup of $G\big(\mathcal{T}^2_\infty\big)$ identified with the subgroup $G \times \{0\}$ of $G \ltimes \mathfrak{g}$. Letting $G_{\rm ab} \subset G\big(\mathcal{T}^2_\infty\big)$, identified as $\{\mathrm{id}\} \times \mathfrak{g} \subset G \ltimes \mathfrak{g}$, and $K \subset D$ denote the connected Lie subgroups corresponding to the Lagrangian subalgebras $\mathfrak{g}_{\rm ab} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathcal{T}^2_\infty$ and $\mathfrak{k} \subset \mathfrak{d}$, then we also have the corresponding Lie subgroup $F = G_{\rm ab} \times K \subset G^{[\boldsymbol{z}]} = G\big(\mathcal{T}^2_\infty\big) \times D$.

Let $h_\infty \in C^\infty\big(\Sigma, G\big(\mathcal{T}^2_\infty\big)\big)$ be the component of the edge mode $h \in C^\infty\big(\Sigma, G^{[\boldsymbol{z}]}\big)$ at infinity. It can be factorised uniquely as $h_\infty = vu$ for some $u \in C^\infty(\Sigma, G)$ and $v \in C^\infty(\Sigma, G_{\rm ab})$ relative to the global decomposition $G\big(\mathcal{T}^2_\infty\big) = G_{\rm ab} G$. Using the transformation (3.12) with $f = \big(v^{-1}, \mathrm{id}_K\big)$ and $g = u$, we can then bring h_∞ to the identity element.

Let $l \in C^\infty(\Sigma, D)$ denote the remaining components of the edge mode in D, so that we can write $h = \big(\mathrm{id}_{G(\mathcal{T}^2_\infty)}, l\big)$. The component of the condition (3.9) at infinity then says that $(\boldsymbol{\jmath}^* \mathcal{L})|_{\mathfrak{g}\otimes_{\mathbb{R}}\mathcal{T}^2_\infty} \in \Omega^1(\Sigma, \mathfrak{g}_{\rm ab})$, which is equivalent to saying that the 1-form $\mathcal{L}$ vanishes at infinity. In terms of the notation introduced in the admissibility condition (a) we therefore have $\mathcal{L}_{{\rm c},\mu} = 0$ for $\mu = \sigma, \tau$. In other words, having fixed the component of the edge mode at infinity to the identity, we will now focus on admissible 1-forms $\mathcal{L} \in \Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)^\Pi$ of the form

$$\mathcal{L}_\mu = \sum_{(y,q)\in(\Pi\boldsymbol{\zeta})} \frac{\mathcal{L}_\mu^{(y,q)}}{(z-y)^{q+1}}. \tag{3.15}$$

The remaining components of the constraint (3.9) read

$${}^{l}(\tilde{\boldsymbol{\jmath}}^* \mathcal{L}) \in \Omega^1(\Sigma, \mathfrak{k}), \tag{3.16}$$

where the map $\tilde{\boldsymbol{\jmath}}^*$ is defined as in (3.4) but with infinity removed, namely

$$\tilde{\boldsymbol{\jmath}}^* \colon \quad \Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)^\Pi \longrightarrow \Omega^1(\Sigma, \mathfrak{d}), \qquad \eta \longmapsto \left(\sum_{p=0}^{n_x-1} \frac{1}{p!} \iota_x^*\big(\partial_z^p \eta\big) \otimes \varepsilon_x^p \right)_{x\in\boldsymbol{z}'}. \tag{3.17}$$

Recall that in Section 3.5 we assumed the existence of a collection of admissible 1-forms $\mathcal{L}(h) \in \Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)^\Pi$ for each $h \in C^\infty\big(\Sigma, G^{[\boldsymbol{z}]}\big)$ satisfying the constraint (3.9). This allowed us to obtain the action (3.11) of a 2d integrable field theory for the field $h \in C^\infty\big(\Sigma, G^{[\boldsymbol{z}]}\big)$

with associated Lax connection $\mathcal{L}(h)$. Moreover, we supposed that the 1-forms $\mathcal{L}(h)$ behave as (3.10) under the gauge transformations (3.12) of h, ensuring that these transformations define local symmetries of the action. In the present subsection, we used part of these gauge symmetries to fix $h = (\mathrm{id}_{G(\mathcal{T}^2_\infty)}, l)$, with $l \in C^\infty(\Sigma, D)$. Through this gauge fixing, finding admissible solutions $\mathcal{L}(h)$ of the constraint (3.9) for each $h \in C^\infty\big(\Sigma, G^{[\boldsymbol{z}]}\big)$, behaving under gauge transformations as in (3.10), then becomes equivalent to finding admissible 1-forms $\mathcal{L}(l) \in \Omega^1\big(X, \mathfrak{g}^\mathbb{C}\big)^\Pi$ of the form (3.15) for each $l \in C^\infty(\Sigma, D)$, which solve the contraint equation (3.16) and with the property that

$$\tilde{\boldsymbol{\jmath}}^* \mathcal{L}(kl) = \tilde{\boldsymbol{\jmath}}^* \mathcal{L}(l) \tag{3.18}$$

for all $k \in C^\infty(\Sigma, K)$. This last property follows from (3.10) and describes the behaviour of the collection of 1-forms $\mathcal{L}(l)$ under what remains of the gauge symmetries (3.12), namely the transformations $l \mapsto kl$ for $k \in C^\infty(\Sigma, K)$. Performing the gauge fixing $h = (\mathrm{id}_{G(\mathcal{T}^2_\infty)}, l)$ in the action (3.11), we then obtain

$$S_{2\mathrm{d}}(l) = -\frac{1}{2} \int_\Sigma \langle\!\langle l^{-1} \mathrm{d}l, \tilde{\boldsymbol{\jmath}}^* \mathcal{L}(l) \rangle\!\rangle_{\mathfrak{d}} - \frac{1}{2} I^{\mathrm{WZ}}_{\mathfrak{d}}[l], \tag{3.19}$$

where the WZ-term $I^{\mathrm{WZ}}_{\mathfrak{d}}[l]$ is defined in the same way as in (2.8). By construction, the action (3.19) is invariant under the residual gauge symmetry $l \mapsto kl$ with $k \in C^\infty(\Sigma, K)$, and defines an integrable field theory with Lax connection $\mathcal{L}(l)$.

4 Integrable $\mathcal{E}$-models from 4d Chern–Simons theory

In Section 3 we reviewed the results of [3] and arrived at the final expression (3.19) for the action of a 2d integrable field theory in the case when ω has a second order pole at infinity, i.e., ω is of the form (3.1). Comparing the form of the action (3.19) with that of the $\mathcal{E}$-model written in the form (2.18) strongly suggests that the 2d integrable field theories described by (3.19) correspond to integrable $\mathcal{E}$-models.

Recall, however, that the derivation of the action (3.19) hinges on the assumption made in Section 3.6 that the constraint (3.16) admits a solution $\mathcal{L} = \mathcal{L}(l) \in \Omega^1\big(X, \mathfrak{g}^\mathbb{C}\big)^\Pi$ in the subspace of admissible $\mathfrak{g}^\mathbb{C}$-valued 1-forms, for every $l \in C^\infty(\Sigma, D)$, with the property (3.18). In order to complete the description of the 2d integrable field theory, it therefore remains to verify this assumption and explicitly construct solutions of the constraint equation (3.16) within the admissible class of 1-forms.

In order to construct a general class of solutions to the constraint (3.16) in Section 4.4, we will see that we are naturally led to introduce an operator $\mathcal{E} \colon \mathfrak{d} \to \mathfrak{d}$ in Section 4.2 which will correspond to the operator of the same name in the $\mathcal{E}$-model. The relationship between the actions (2.18) and (3.19) will then be made explicit in Section 4.4.

4.1 The maps $\boldsymbol{j}_{z'}$ and $\boldsymbol{\pi}_\zeta$

Since the admissibility conditions (*a*) and (*b*) from Section 3.5 are formulated in terms of the components $\mathcal{L}_\sigma$ and $\mathcal{L}_\tau$, it will be more convenient to express the constraint (3.16) in terms of these components as well.

Recall from Section 3.5 the definition of the real vector space of Π-equivariant V-valued rational functions $R^\infty_{\Pi\boldsymbol{\zeta}}(V)^\Pi$ for any complex vector space V equipped with an action of Π. Having removed the component of the edge mode at infinity in Section 3.6, we are now working with admissible 1-forms $\mathcal{L} \in \Omega^1\big(X, \mathfrak{g}^\mathbb{C}\big)^\Pi$ of the form (3.15). It is therefore convenient to introduce the subspace $R_{\Pi\boldsymbol{\zeta}}(V) \subset R^\infty_{\Pi\boldsymbol{\zeta}}(V)$ of V-valued rational functions which vanish at infinity.

This subspace is clearly stable under the action of Π so that we may form the real vector space of Π-equivariants $R_{\Pi\boldsymbol{\zeta}}(V)^\Pi \subset R^\infty_{\Pi\boldsymbol{\zeta}}(V)^\Pi$. In terms of this notation, we are therefore focusing on the class of admissible 1-form with components $\mathcal{L}_\sigma, \mathcal{L}_\tau \in R_{\Pi\boldsymbol{\zeta}}\big(C^\infty\big(\Sigma, \mathfrak{g}^{\mathbb{C}}\big)\big)^\Pi$.

We define, cf. (3.17),

$$\boldsymbol{j}_{\boldsymbol{z}'}\colon \quad R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^\Pi \longrightarrow \mathfrak{d}, \qquad f \longmapsto \Bigg(\sum_{p=0}^{n_x-1} \frac{1}{p!} (\partial_z^p f)|_x \otimes \varepsilon_x^p \Bigg)_{x\in\boldsymbol{z}'}, \tag{4.1}$$

which returns the first n_x terms in the Taylor expansion of the rational function at the set of finite poles $\boldsymbol{z}'$ of ω. This extends component-wise to a morphism

$$\boldsymbol{j}_{\boldsymbol{z}'}\colon\ R_{\Pi\boldsymbol{\zeta}}\big(C^\infty\big(\Sigma, \mathfrak{g}^{\mathbb{C}}\big)\big)^\Pi \longrightarrow C^\infty(\Sigma, \mathfrak{d}).$$

Note that for an admissible 1-form $\mathcal{L} = \mathcal{L}_\sigma \mathrm{d}\sigma + \mathcal{L}_\tau \mathrm{d}\tau \in \Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)$ we have the relation $\tilde{\boldsymbol{j}}^*\mathcal{L} = \boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma \mathrm{d}\sigma + \boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\tau \mathrm{d}\tau$ with the map $\tilde{\boldsymbol{j}}^*$ in (3.17). We can then rewrite (3.16) equivalently in components as

$$-\partial_\sigma l l^{-1} + \mathrm{Ad}_l(\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma) \in C^\infty(\Sigma, \mathfrak{k}), \qquad -\partial_\tau l l^{-1} + \mathrm{Ad}_l(\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\tau) \in C^\infty(\Sigma, \mathfrak{k}). \tag{4.2}$$

We associate with the zeroes of ω the real vector space

$$\mathfrak{g}^{(\boldsymbol{\zeta})} \coloneqq \bigoplus_{(y,q)\in(\boldsymbol{\zeta}_{\mathrm{r}})} \mathfrak{g} \oplus \bigoplus_{(y,q)\in(\boldsymbol{\zeta}_{\mathrm{c}})} \mathfrak{g}^{\mathbb{C}}, \tag{4.3}$$

where $\mathfrak{g}^{\mathbb{C}}$ is regarded as a real vector space. Recall that $(\boldsymbol{\zeta})$ is the set of pairs (y,q) with $y \in \boldsymbol{\zeta}$ and $q = 0, \dots, m_y - 1$. Notice that the definition of $\mathfrak{g}^{(\boldsymbol{\zeta})}$ in (4.3) is very similar to that of $\mathfrak{g}^{[\boldsymbol{z}]}$ in (3.2). However, it is important to note that the former is only a vector space while the latter is a Lie algebra.

We shall also make use of the isomorphism

$$\boldsymbol{\pi}_{\boldsymbol{\zeta}}\colon \quad R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^\Pi \xrightarrow{\cong} \mathfrak{g}^{(\boldsymbol{\zeta})}, \qquad \sum_{(y,q)\in(\Pi\boldsymbol{\zeta})} \frac{\mathsf{u}^{(y,q)}}{(z-y)^{q+1}} \longmapsto \big(\mathsf{u}^{(y,q)}\big)_{(y,q)\in(\boldsymbol{\zeta})} \tag{4.4}$$

which, as in the case of (4.1), extends component-wise to an isomorphism

$$\boldsymbol{\pi}_{\boldsymbol{\zeta}}\colon\ R_{\Pi\boldsymbol{\zeta}}\big(C^\infty\big(\Sigma, \mathfrak{g}^{\mathbb{C}}\big)\big)^\Pi \xrightarrow{\cong} C^\infty\big(\Sigma, \mathfrak{g}^{(\boldsymbol{\zeta})}\big). \tag{4.5}$$

Applied explicitly to the components of the Lax connection in (3.15) this gives

$$\boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\mu = \big(\mathcal{L}_\mu^{(y,q)}\big)_{(y,q)\in\boldsymbol{\zeta}}.$$

In particular, since (4.5) is an isomorphism, the field content of the Lax connection $\mathcal{L}$ in (3.15) is completely encoded in the collection of coefficients $\boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\mu \in C^\infty\big(\Sigma, \mathfrak{g}^{(\boldsymbol{\zeta})}\big)$ in the partial fraction decomposition of its components at the set of zeroes $\boldsymbol{\zeta}$ of ω.

4.2 Admissible 1-forms for the $\mathcal{E}$-model

The admissibility condition (b) can easily be solved by choosing an $\epsilon_y \in \mathbb{R}\setminus\{0\}$ for each $y \in \boldsymbol{\zeta}_{\mathrm{r}}$ and an $\epsilon_y \in \mathbb{C}\setminus\{0\}$ for each $y \in \boldsymbol{\zeta}_{\mathrm{c}}$, and requiring that

$$\mathcal{L}_\tau^{(y,q)} = \epsilon_y \mathcal{L}_\sigma^{(y,q)} \tag{4.6}$$

for all $(y, q) \in (\Pi\boldsymbol{\zeta})$, where we define $\epsilon_{\bar{y}} := \overline{\epsilon_y}$ for $\bar{y} \in \bar{\boldsymbol{\zeta}}_{\rm c}$. This condition, which also appeared in [3, 13, 17, 65], is motivated by the expression for the Lax connection in affine Gaudin models. In the case when all the zeroes of ω are simple it takes the form [16, equations (2.39)–(2.40)], which is to be compared with the expressions in the admissibility condition (3.15) for $m_y = 1$, combined with (4.6). In order to make use of the condition (4.6) to solve the constraint (4.2), it will be convenient to first reformulate it as a relation between $\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma$ and $\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\tau$.

Consider the linear isomorphism

$$\widetilde{\mathcal{E}}\colon\ \mathfrak{g}^{(\boldsymbol{\zeta})} \xrightarrow{\cong} \mathfrak{g}^{(\boldsymbol{\zeta})}, \qquad \big(\mathsf{u}^{(y,q)}\big)_{(y,q)\in(\boldsymbol{\zeta})} \longmapsto \big(\epsilon_y \mathsf{u}^{(y,q)}\big)_{(y,q)\in(\boldsymbol{\zeta})}. \tag{4.7}$$

We may then rewrite (4.6) as

$$\boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\tau = \widetilde{\mathcal{E}}(\boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\sigma). \tag{4.8}$$

Lemma 4.1. *We have an isomorphism of real vector spaces*

$$\mathcal{C} := \boldsymbol{j}_{\boldsymbol{z}'} \circ \boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\colon\ \mathfrak{g}^{(\boldsymbol{\zeta})} \xrightarrow{\cong} \mathfrak{d}.$$

In particular, the linear map $\boldsymbol{j}_{\boldsymbol{z}'}$ defined in (4.1) is also an isomorphism.

Proof. Let $\mathsf{U} = \big(\mathsf{u}^{(y,q)}\big)_{(y,q)\in(\boldsymbol{\zeta})} \in \mathfrak{g}^{(\boldsymbol{\zeta})}$. The components of $\boldsymbol{j}_{\boldsymbol{z}'}\big(\boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\mathsf{U}\big) \in \mathfrak{d}$ are obtained by taking the first n_x terms in the Taylor expansion of the rational function $\boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\mathsf{U}$ at each $x \in \boldsymbol{z}'$. For the purpose of this proof, it is necessary to also consider separately the Taylor expansions at the conjugate poles $\bar{x} \in \bar{\boldsymbol{z}}_{\rm c}$ for each $x \in \boldsymbol{z}_{\rm c}$, even though these are related to the Taylor expansions at x by the automorphism τ. Explicitly, the coefficients of the expansions at all the poles $x \in \boldsymbol{\mathfrak{Z}}'$ are given by

$$\frac{1}{p!}\big(\partial_z^p(\boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\mathsf{U})\big)\big|_x = \sum_{(y,q)\in(\Pi\boldsymbol{\zeta})} C^{[x,p]}{}_{(y,q)}\mathsf{u}^{(y,q)}$$

for all $[x, p] \in [\boldsymbol{\mathfrak{Z}}']$, where we have introduced the notation $\mathsf{u}^{(\bar{y},q)} := \tau\big(\mathsf{u}^{(y,q)}\big)$ for any $y \in \boldsymbol{\zeta}_{\rm c}$ and

$$C^{[x,p]}{}_{(y,q)} := \binom{p+q}{p}\frac{(-1)^p}{(x-y)^{p+q+1}} \tag{4.9}$$

for all $[x, p] \in [\boldsymbol{\mathfrak{Z}}']$ and $(y, q) \in (\Pi\boldsymbol{\zeta})$. The expressions in (4.9) are the components of what is known as a confluent Cauchy matrix, see for instance [62, equation (13)].

The map $\mathcal{C}\colon \mathfrak{g}^{(\boldsymbol{\zeta})} \to \mathfrak{d}$ is then defined for any

$$\mathsf{U} = \big(\mathsf{u}^{(y,q)}\big)_{(y,q)\in(\boldsymbol{\zeta})} \in \mathfrak{g}^{(\boldsymbol{\zeta})} \text{ by } \mathcal{C}(\mathsf{U}) = \big(\mathcal{C}(\mathsf{U})^{x,p} \otimes \varepsilon_x^p\big)_{[x,p]\in[\boldsymbol{z}']},$$

where

$$\mathcal{C}(\mathsf{U})^{x,p} = \sum_{(y,q)\in(\boldsymbol{\zeta}_{\rm r})} C^{[x,p]}{}_{(y,q)}\mathsf{u}^{(y,q)} + \sum_{(y,q)\in(\boldsymbol{\zeta}_{\rm c})} \Big(C^{[x,p]}{}_{(y,q)}\mathsf{u}^{(y,q)} + C^{[x,p]}{}_{(\bar{y},q)}\tau\big(\mathsf{u}^{(y,q)}\big)\Big), \tag{4.10}$$

in terms of the Cauchy matrix (4.9).

On the other hand, since ω is a meromorphic differential with zeroes at each $y \in \Pi\boldsymbol{\zeta}$ of order m_y and poles at each $x \in \boldsymbol{\mathfrak{Z}}$ of order n_x, we have

$$\sum_{y\in\Pi\boldsymbol{\zeta}} m_y = \sum_{x\in\boldsymbol{\mathfrak{Z}}} n_x - 2.$$

In other words, since we are assuming that $n_\infty = 2$, this yields

$$\sum_{y \in \Pi\boldsymbol{\zeta}} m_y = \sum_{x \in \boldsymbol{\mathcal{Z}}'} n_x.$$

It follows that (4.9) are the components of a *square* confluent Cauchy matrix which is known to be invertible [62, Corollary 10].

The inverse $\mathcal{C}^{-1}\colon \mathfrak{d} \to \mathfrak{g}^{(\boldsymbol{\zeta})}$ is then given explicitly, for any $\mathsf{V} = \big(\mathsf{v}^{x,p} \otimes \varepsilon_x^p\big)_{[x,p]\in[\boldsymbol{z}']} \in \mathfrak{d}$, by $\mathcal{C}^{-1}(\mathsf{V}) = \big(\mathcal{C}^{-1}(\mathsf{V})^{(y,q)}\big)_{(y,q)\in(\boldsymbol{\zeta})}$, where

$$\begin{aligned}\mathcal{C}^{-1}(\mathsf{V})^{(y,q)} = {} & \sum_{[x,p]\in[\boldsymbol{z}_{\rm r}]} \big(C^{-1}\big)^{(y,q)}{}_{[x,p]}\mathsf{v}^{x,p} \\ & + \sum_{[x,p]\in[\boldsymbol{z}_{\rm c}]} \Big(\big(C^{-1}\big)^{(y,q)}{}_{[x,p]}\mathsf{v}^{x,p} + \big(C^{-1}\big)^{(y,q)}{}_{[\bar{x},p]}\tau\big(\mathsf{v}^{x,p}\big)\Big),\end{aligned} \tag{4.11}$$

in terms of the inverse $\big(C^{-1}\big)^{(y,q)}{}_{[x,p]}$ of the Cauchy matrix (4.9). ■

Recall from Section 4.1 that the field content of the Lax connection $\mathcal{L}$ in (3.15) is encoded in the collection of coefficients $\boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\mu \in C^\infty\big(\Sigma, \mathfrak{g}^{(\boldsymbol{\zeta})}\big)$. By Lemma 4.1, the field content of $\mathcal{L}$ is equivalently encoded in the jets $\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\mu \in C^\infty(\Sigma, \mathfrak{d})$ of its components at the set of finite poles $\boldsymbol{z}'$ of ω.

Applying the linear isomorphism $\mathcal{C}$ from Lemma 4.1 to both sides of the relation (4.8) we may obtain

$$\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\tau = \mathcal{E}(\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma), \tag{4.12}$$

where we have defined the linear isomorphism

$$\mathcal{E} := \mathcal{C} \circ \widetilde{\mathcal{E}} \circ \mathcal{C}^{-1}\colon\ \mathfrak{d} \xrightarrow{\cong} \mathfrak{d}. \tag{4.13}$$

Using the explicit forms (4.7), (4.10) and (4.11) for the linear maps $\widetilde{\mathcal{E}}$, $\mathcal{C}$ and $\mathcal{C}^{-1}$, we may express the linear isomorphism (4.13) in components as follows. For every $\mathsf{U} = \big(\mathsf{u}^{y,q} \otimes \varepsilon_y^q\big)_{[y,q]\in[\boldsymbol{z}']} \in \mathfrak{d}$ we have $\mathcal{E}(\mathsf{U}) = \big(\mathcal{E}(\mathsf{U})^{x,p} \otimes \varepsilon_x^p\big)_{[x,p]\in[\boldsymbol{z}']}$, where

$$\mathcal{E}(\mathsf{U})^{x,p} = \sum_{[y,q]\in[\boldsymbol{z}_{\rm r}]} E^{[x,p]}{}_{[y,q]}\mathsf{u}^{y,q} + \sum_{[y,q]\in[\boldsymbol{z}_{\rm c}]} \Big(E^{[x,p]}{}_{[y,q]}\mathsf{u}^{y,q} + E^{[x,p]}{}_{[\bar{y},q]}\tau\big(\mathsf{u}^{y,q}\big)\Big), \tag{4.14}$$

for some coefficients $E^{[x,p]}{}_{[y,q]}$ expressible in terms of the Cauchy matrix (4.9), its inverse and the choice of ϵ_y for $y \in \boldsymbol{\zeta}$.

4.3 Properties of $\mathcal{E}$

Recall that the Lie algebra $\mathfrak{d}$ defined in Section 3.6 comes equipped with a non-degenerate symmetric invariant bilinear form $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\mathfrak{d}}$. To study the symmetry property of $\mathcal{E}$ with respect to the latter, it is convenient to pull back this bilinear form to the vector space $\mathfrak{g}^{(\boldsymbol{\zeta})}$ along the linear isomorphism from Lemma 4.1 since the action of $\widetilde{\mathcal{E}}$ is much simpler. It is useful to do this in two steps, by first pulling back $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\mathfrak{d}}$ to $R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^\Pi$ along the isomorphism (4.1), and then to $\mathfrak{g}^{(\boldsymbol{\zeta})}$ along the inverse of the isomorphism $\boldsymbol{\pi}_{\boldsymbol{\zeta}}$ given in (4.4).

We define the non-degenerate symmetric bilinear form

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_\omega\colon\ R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^\Pi \times R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^\Pi \longrightarrow \mathbb{R}, \tag{4.15a}$$

defined for any $f, g \in R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^{\Pi}$ by

$$\langle\!\langle f, g \rangle\!\rangle_\omega \coloneqq \sum_{x \in \boldsymbol{z}'} \frac{2}{|\Pi_x|} \Re\big(\mathrm{res}_x \langle f, g\rangle \omega\big) = -\sum_{y \in \boldsymbol{\zeta}} \frac{2}{|\Pi_y|} \Re\big(\mathrm{res}_y \langle f, g\rangle \omega\big). \tag{4.15b}$$

The equality here follows from the vanishing of the sum of residues, after observing that the poles of $\langle f, g\rangle\omega$ belong to the set $\mathfrak{Z}' \sqcup \Pi\boldsymbol{\zeta} = \Pi\boldsymbol{z}' \sqcup \Pi\boldsymbol{\zeta}$. (Observe that infinity is not a pole of $\langle f, g\rangle\omega$ because the double pole of ω at infinity is compensated by the simple zeroes of f and g there.) Note in particular that

$$\sum_{x \in \boldsymbol{z}'} \frac{2}{|\Pi_x|} \Re\big(\mathrm{res}_x \langle f, g\rangle \omega\big) = \sum_{x \in \mathfrak{Z}'} \mathrm{res}_x \langle f, g\rangle \omega$$

and similarly for the sum of residues at the zeroes $\boldsymbol{\zeta}$.

Lemma 4.2. *For any $f, g \in R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^{\Pi}$, we have $\langle\!\langle \boldsymbol{j}_{\boldsymbol{z}'} f, \boldsymbol{j}_{\boldsymbol{z}'} g \rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle f, g \rangle\!\rangle_\omega$.*

Proof. Let $f, g \in R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^{\Pi}$. By definition we have

$$\begin{aligned}
\langle\!\langle f, g \rangle\!\rangle_\omega &= \sum_{x \in \boldsymbol{z}'} \frac{2}{|\Pi_x|} \Re\big(\mathrm{res}_x \langle f, g\rangle \omega\big) = \sum_{x \in \boldsymbol{z}'} \sum_{p=0}^{n_x - 1} \frac{2}{|\Pi_x|} \Re\bigg(\mathrm{res}_x \langle f, g\rangle \frac{\ell^x_p \mathrm{d}z}{(z-x)^{p+1}}\bigg) \\
&= \sum_{x \in \boldsymbol{z}'} \sum_{p=0}^{n_x - 1} \frac{2}{|\Pi_x|} \Re\bigg(\frac{\ell^x_p}{p!} \big(\partial_z^p \langle f, g\rangle\big)\big|_x\bigg) \\
&= \sum_{x \in \boldsymbol{z}'} \sum_{p=0}^{n_x - 1} \sum_{q=0}^{p} \frac{2}{|\Pi_x|} \Re\bigg(\ell^x_p \bigg\langle \frac{1}{q!} (\partial_z^q f)|_x, \frac{1}{(p-q)!} (\partial_z^{p-q} g)|_x \bigg\rangle\bigg) \\
&= \sum_{x \in \boldsymbol{z}'} \sum_{q,r=0}^{n_x - 1} \frac{2}{|\Pi_x|} \Re\bigg(\ell^x_{q+r} \bigg\langle \frac{1}{q!} (\partial_z^q f)|_x, \frac{1}{r!} (\partial_z^r g)|_x \bigg\rangle\bigg) = \langle\!\langle \boldsymbol{j}_{\boldsymbol{z}'} f, \boldsymbol{j}_{\boldsymbol{z}'} g \rangle\!\rangle_{\mathfrak{d}},
\end{aligned}$$

where in the second equality we used the explicit expression (3.1) for ω, dropping the term with ℓ^∞_1 since it does not contribute to the residue at any of the finite poles $x \in \boldsymbol{z}'$. In the second last step we changed variable from p to $r = p - q$ and used the convention that $\ell^x_p = 0$ for $p \geq n_x$. The last equality is by definition (3.3) of the induced bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{d}}$ on $\mathfrak{d} \subset \mathfrak{g}^{[\boldsymbol{z}]}$ and of the map $\boldsymbol{j}_{\boldsymbol{z}'}$ in (4.1). ■

We introduce the symmetric bilinear form

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{g}^{(\boldsymbol{\zeta})}} \colon\ \mathfrak{g}^{(\boldsymbol{\zeta})} \times \mathfrak{g}^{(\boldsymbol{\zeta})} \longrightarrow \mathbb{R}, \tag{4.16a}$$

defined for any $\mathsf{U} = (\mathsf{u}^{(x,p)})_{(x,p) \in (\boldsymbol{\zeta})}$, $\mathsf{V} = (\mathsf{v}^{(y,q)})_{(y,q) \in (\boldsymbol{\zeta})} \in \mathfrak{g}^{(\boldsymbol{\zeta})}$ by

$$\langle\!\langle \mathsf{U}, \mathsf{V} \rangle\!\rangle_{\mathfrak{g}^{(\boldsymbol{\zeta})}} \coloneqq -\sum_{y \in \boldsymbol{\zeta}} \sum_{\substack{p,q=0 \\ p+q \geq m_y - 1}}^{m_y - 1} \frac{2}{|\Pi_y|} \Re\bigg(\frac{\big(\partial_z^{p+q+1-m_y} \psi_y\big)(y)}{(p+q+1-m_y)!} \big\langle \mathsf{u}^{(y,p)}, \mathsf{v}^{(y,q)} \big\rangle\bigg). \tag{4.16b}$$

Here we wrote the twist function $\varphi(z)$ defined in (3.1) as $\varphi(z) = \psi_y(z)(z-y)^{m_y}$ with $\psi_y(y) \neq 0$ using the fact that it has a zero of order m_y at $y \in \boldsymbol{\zeta}$. This definition is motivated by the following lemma.

Lemma 4.3. *For any $f, g \in R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^{\Pi}$, we have $\langle\!\langle \boldsymbol{\pi}_{\boldsymbol{\zeta}} f, \boldsymbol{\pi}_{\boldsymbol{\zeta}} g \rangle\!\rangle_{\mathfrak{g}^{(\boldsymbol{\zeta})}} = \langle\!\langle f, g \rangle\!\rangle_\omega$. In particular, for any $\mathsf{U}, \mathsf{V} \in \mathfrak{g}^{(\boldsymbol{\zeta})}$ we have $\langle\!\langle \mathsf{U}, \mathsf{V} \rangle\!\rangle_{\mathfrak{g}^{(\boldsymbol{\zeta})}} = \langle\!\langle \mathcal{C}\mathsf{U}, \mathcal{C}\mathsf{V} \rangle\!\rangle_{\mathfrak{d}}$.*

Proof. Let $f, g \in R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^{\Pi}$ which we can write out explicitly as

$$f(z) = \sum_{(y,p)\in(\Pi\boldsymbol{\zeta})} \frac{\mathsf{u}^{(y,p)}}{(z-y)^{p+1}}, \qquad g(z) = \sum_{(x,q)\in(\Pi\boldsymbol{\zeta})} \frac{\mathsf{v}^{(x,q)}}{(z-x)^{q+1}}.$$

Using the second expression for the bilinear form in (4.15) we then find

$$\begin{aligned}
\langle\!\langle f, g\rangle\!\rangle_\omega &= -\sum_{y\in\boldsymbol{\zeta}} \frac{2}{|\Pi_y|} \Re\big(\operatorname{res}_y \langle f, g\rangle\omega\big) = -\sum_{y\in\boldsymbol{\zeta}} \sum_{p=0}^{m_y-1} \frac{2}{|\Pi_y|} \Re\bigg(\operatorname{res}_y \bigg\langle \frac{\mathsf{u}^{(y,p)}}{(z-y)^{p+1}}, \varphi g\bigg\rangle \mathrm{d}z\bigg)\\
&= -\sum_{y\in\boldsymbol{\zeta}} \sum_{p=0}^{m_y-1} \frac{2}{|\Pi_y|} \Re\bigg\langle \mathsf{u}^{(y,p)}, \frac{1}{p!}\big(\partial_z^p(\varphi g)\big)\big|_y\bigg\rangle\\
&= -\sum_{y\in\boldsymbol{\zeta}} \sum_{p,q=0}^{m_y-1} \frac{2}{|\Pi_y|} \Re\bigg(\frac{1}{p!}\partial_z^p\bigg(\frac{\varphi(z)}{(z-y)^{q+1}}\bigg)\bigg|_y \big\langle \mathsf{u}^{(y,p)}, \mathsf{v}^{(y,q)}\big\rangle\bigg).
\end{aligned}$$

In the second equality we wrote $\omega = \varphi \mathrm{d}z$. In the third equality we used the fact that φg is regular at $\boldsymbol{\zeta}$, so that only the poles from f contribute to the residue at each $y \in \boldsymbol{\zeta}$, and took the residue. In the last equality we have used the fact that the terms in $\partial_z^q(\varphi g)$ coming from the poles of g at $x \neq y$ all vanish at y. Using $\varphi(z) = \psi_y(z)(z-y)^{m_y}$ we find

$$\frac{1}{p!}\partial_z^p\bigg(\frac{\varphi(z)}{(z-y)^{q+1}}\bigg)\bigg|_y = \frac{1}{p!}\partial_z^p\big(\psi_y(z)(z-y)^{m_y-q-1}\big)\big|_y = \frac{\big(\partial_z^{p+q+1-m_y}\psi_y\big)(y)}{(p+q+1-m_y)!},$$

where in the last step we have used the Leibniz rule and the fact that if any term still contains a factor of $(z-y)$ it will vanish upon setting $z = y$.

The final statement follows form the above, Lemma 4.2 and the definition of $\mathcal{C}$ in Lemma 4.1. ■

Corollary 4.4. *$\mathcal{E}$ is symmetric with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}$.*

Proof. Let $\mathsf{U}, \mathsf{V} \in \mathfrak{d}$. Using 4.3 we have

$$\langle\!\langle \mathsf{U}, \mathcal{E}\mathsf{V}\rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle \mathcal{C}^{-1}\mathsf{U}, \mathcal{C}^{-1}\mathcal{E}\mathsf{V}\rangle\!\rangle_{\mathfrak{g}^{(\boldsymbol{\zeta})}} = \langle\!\langle \mathcal{C}^{-1}\mathsf{U}, \widetilde{\mathcal{E}}\mathcal{C}^{-1}\mathsf{V}\rangle\!\rangle_{\mathfrak{g}^{(\boldsymbol{\zeta})}}.$$

Since $\widetilde{\mathcal{E}}$ is clearly symmetric with respect to (4.16) the claim follows. ■

Remark 4.5. When all the zeroes of ω are simple and real, i.e., $m_y = 1$ for every $y \in \boldsymbol{\zeta}$ and $\boldsymbol{\zeta} = \boldsymbol{\zeta}_{\mathrm{r}}$, a simple condition for $\mathcal{E}$ to be positive (i.e., $\langle\!\langle\cdot, \mathcal{E}\cdot\rangle\!\rangle_{\mathfrak{d}}$ to be positive-definite) can be given in the case when $\mathfrak{g}$ is compact, in which case we can choose the bilinear form $\langle\cdot,\cdot\rangle\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ to be positive definite. Specifically, in this case the bilinear form (4.16) on $\mathfrak{g}^{(\boldsymbol{\zeta})}$ reduces simply to

$$\langle\!\langle \mathsf{U}, \mathsf{V}\rangle\!\rangle_{\mathfrak{g}^{(\boldsymbol{\zeta})}} = -\sum_{y\in\boldsymbol{\zeta}_{\mathrm{r}}} \psi_y(y)\big\langle \mathsf{u}^{(y,p)}, \mathsf{v}^{(y,q)}\big\rangle.$$

It then follows directly from the proof of Corollary 4.4 that $\mathcal{E}$ is positive with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{d}}$ if and only if $\widetilde{\mathcal{E}}$ is positive with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{g}^{(\boldsymbol{\zeta})}}$, i.e., if and only if $-\epsilon_y\psi_y(y) > 0$ for every $y \in \boldsymbol{\zeta}$. Noting that $\psi_y(y) = \varphi'(y)$ this means $\varphi'(y)$ and ϵ_y should have opposite signs.

4.4 Recovering the $\mathcal{E}$-model

In Section 4.2 we described a very simple class of admissible $\mathfrak{g}^{\mathbb{C}}$-valued 1-form $\mathcal{L} \in \Omega^1\big(X, \mathfrak{g}^{\mathbb{C}}\big)^{\Pi}$, namely ones satisfying the condition (4.6). We showed that the latter could be rewritten in the form (4.12) in terms of the linear isomorphism $\mathcal{E} \in \operatorname{End}\mathfrak{d}$ defined in (4.13). We will now show that, assuming $\mathcal{E}$ and $\mathfrak{k} \subset \mathfrak{d}$ satisfy the condition (2.2), there exists a *unique* solution to the constraint (4.2) for $\mathcal{L}$ in terms of $l \in C^\infty(\Sigma, D)$ within this class of admissible 1-forms. In Section 4.3, specifically Remark 4.5, we gave sufficient conditions for (2.2) to hold in view of Lemma 2.2.

Let us define $B_\sigma := -\partial_\sigma l l^{-1} + \operatorname{Ad}_l(\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma)$ and $B_\tau := -\partial_\tau l l^{-1} + \operatorname{Ad}_l(\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\tau)$, which both belong to $C^\infty(\Sigma, \mathfrak{k})$ by (4.2). Then using the relation (4.12) we deduce

$$\operatorname{Ad}_l^{-1} B_\tau = \mathcal{E}\operatorname{Ad}_l^{-1} B_\sigma + \mathcal{E}\big(l^{-1}\partial_\sigma l\big) - l^{-1}\partial_\tau l. \tag{4.17}$$

The left hand side takes value in $\operatorname{Ad}_l^{-1}\mathfrak{k} = \ker \mathcal{P}_l$ while the first term on the right hand side is valued in $\mathcal{E}\operatorname{Ad}_l^{-1}\mathfrak{k} = \operatorname{im}\mathcal{P}_l$. Here $\mathcal{P}_l$ is the projector with kernel and image (2.4) defined in Section 2.1, where now $l \in C^\infty(\Sigma, D)$ is the edge mode from Section 3.6. Note that the existence of this projector is ensured by the condition (2.2) which we are assuming holds.

Applying $\mathcal{P}_l$ to both sides of the equation (4.17) we then obtain

$$0 = \mathcal{E}\big(\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma - l^{-1}\partial_\sigma l\big) + \mathcal{P}_l\big(\mathcal{E}\big(l^{-1}\partial_\sigma l\big) - l^{-1}\partial_\tau l\big).$$

We can now solve this for $\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma$ and then substitute the result into the relation (4.12) to find $\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\tau$, yielding

$$\begin{aligned}
\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma &= \big(\mathrm{id} - \mathcal{E}^{-1}\mathcal{P}_l\mathcal{E}\big)\big(l^{-1}\partial_\sigma l\big) + \mathcal{E}^{-1}\mathcal{P}_l\big(l^{-1}\partial_\tau l\big),\\
\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\tau &= (\mathcal{E} - \mathcal{P}_l\mathcal{E})\big(l^{-1}\partial_\sigma l\big) + \mathcal{P}_l\big(l^{-1}\partial_\tau l\big).
\end{aligned}$$

Finally, note that using part (ii) of Proposition 2.3 we can rewrite these as

$$\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma = \overline{\mathcal{P}}_l\big(l^{-1}\partial_\sigma l\big) + \mathcal{E}^{-1}\mathcal{P}_l\big(l^{-1}\partial_\tau l\big), \tag{4.18a}$$

$$\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\tau = \mathcal{E}\overline{\mathcal{P}}_l\big(l^{-1}\partial_\sigma l\big) + \mathcal{P}_l\big(l^{-1}\partial_\tau l\big). \tag{4.18b}$$

Since $\boldsymbol{j}_{\boldsymbol{z}'}$ is invertible by Lemma 4.1, this gives the desired unique solution $\mathcal{L} = \mathcal{L}(l)$ of the constraint (3.16). See Section 4.5 below.

We observe that the expressions (4.18) coincide with those in (2.16) for the current $\mathcal{J}$ in the $\mathcal{E}$-model. It is now clear, as advertised at the start of this section, that the action (3.19) for the solution $\mathcal{L} = \mathcal{L}(l)$ to the constraint equation (3.16) which we have obtained in this section coincides exactly with the action of the $\mathcal{E}$-model written in the form (2.18). Explicitly, we then have

$$\begin{aligned}
S_{2\mathrm{d}}(l) = \frac{1}{2}\int_\Sigma \Big(&\langle\!\langle l^{-1}\partial_\tau l, \mathcal{E}^{-1}\mathcal{P}_l\big(l^{-1}\partial_\tau l\big)\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle l^{-1}\partial_\sigma l, \mathcal{E}\overline{\mathcal{P}}_l\big(l^{-1}\partial_\sigma l\big)\rangle\!\rangle_{\mathfrak{d}}\\
&+ \langle\!\langle l^{-1}\partial_\tau l, \big(\overline{\mathcal{P}}_l - {}^t\mathcal{P}_l\big)\big(l^{-1}\partial_\sigma l\big)\rangle\!\rangle_{\mathfrak{d}}\Big)\mathrm{d}\sigma\wedge\mathrm{d}\tau - \frac{1}{2}I^{\mathrm{WZ}}_{\mathfrak{d}}[l],
\end{aligned} \tag{4.19}$$

which coincides with $S_{\mathcal{E},\mathfrak{k}}(l)$ defined in (2.14). We also note that, as required from Section 3.6, the solution $\mathcal{L}(l)$ of the constraint (3.16) satisfies (3.18) for any $k \in C^\infty(\Sigma, K)$ since the expressions for $\mathcal{J}$ were noted in Section 2.4 to have this property.

4.5 The inverse of $\boldsymbol{j}_{\boldsymbol{z}'}$

The admissibility condition (b) from Section 3.5 plays a central role in the passage from 4d Chern–Simons theory to 2d integrable field theories in the approach described in [3]. In particular, by [3, Proposition 5.6] it allows one to lift the flatness equation for the 1-form $\mathcal{J} = \mathcal{J}_\sigma \mathrm{d}\sigma + \mathcal{J}_\tau \mathrm{d}\tau$ with components $\mathcal{J}_\mu \coloneqq \boldsymbol{j}_{\boldsymbol{z}'} \mathcal{L}_\mu$, i.e.,

$$\mathrm{d}\mathcal{J} + \frac{1}{2}[\mathcal{J}, \mathcal{J}] = 0, \tag{4.20}$$

which is essentially the boundary equation of motion for the extended action (3.7), to the flatness of the Lax connection $\mathcal{L}$ itself, namely

$$\mathrm{d}\mathcal{L} + \frac{1}{2}[\mathcal{L}, \mathcal{L}] = 0. \tag{4.21}$$

In this section we will give a different perspective on the above passage from (4.20) to (4.21) in the case when the admissibility condition (b) is ensured by the $\mathcal{E}$-model condition (4.12). It follows from Section 4.4 that in this case the flatness equation (4.20) coincides with the equations of motion for the $\mathcal{E}$-model by Proposition 2.5.

Recall from Lemma 4.1 that the map $\boldsymbol{j}_{\boldsymbol{z}'}$ defined in (4.1) is an isomorphism. We denote its inverse by

$$\boldsymbol{p}\colon\ \mathfrak{d} \longrightarrow R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^{\Pi}. \tag{4.22}$$

Applying this map to both equations in (4.18) it follows that the components of the Lax connection $\mathcal{L} = \mathcal{L}_\sigma \mathrm{d}\sigma + \mathcal{L}_\tau \mathrm{d}\tau$ of the $\mathcal{E}$-model are given by

$$\mathcal{L}_\sigma = \boldsymbol{p}\mathcal{J}_\sigma, \qquad \mathcal{L}_\tau = \boldsymbol{p}\mathcal{J}_\tau. \tag{4.23}$$

Lemma 4.6. *For any* $\mathsf{U} \in \mathfrak{d}$ *we have* $\boldsymbol{p}([\mathsf{U}, \mathcal{E}\mathsf{U}]) = [\boldsymbol{p}\mathsf{U}, \boldsymbol{p}\mathcal{E}\mathsf{U}]$.

Proof. Let $\mathsf{U} \in \mathfrak{d}$. Since $\boldsymbol{j}_{\boldsymbol{z}'}$ is an isomorphism we can write it as $\mathsf{U} = \boldsymbol{j}_{\boldsymbol{z}'} f$ for some $f \in R_{\Pi\boldsymbol{\zeta}}(\mathfrak{g})^{\Pi}$. Then we have

$$[\boldsymbol{p}\mathsf{U}, \boldsymbol{p}\mathcal{E}\mathsf{U}] = \big[f, \boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\widetilde{\mathcal{E}}\boldsymbol{\pi}_{\boldsymbol{\zeta}} f\big] = \boldsymbol{p}\boldsymbol{j}_{\boldsymbol{z}'}\big(\big[f, \boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\widetilde{\mathcal{E}}\boldsymbol{\pi}_{\boldsymbol{\zeta}} f\big]\big),$$

where in the first equality we substituted $\mathcal{E} = \boldsymbol{j}_{\boldsymbol{z}'}\boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\widetilde{\mathcal{E}}\boldsymbol{\pi}_{\boldsymbol{\zeta}}\boldsymbol{j}_{\boldsymbol{z}'}^{-1}$ and $\mathsf{U} = \boldsymbol{j}_{\boldsymbol{z}'} f$ and used the fact that $\boldsymbol{p}\boldsymbol{j}_{\boldsymbol{z}'} = \mathrm{id}$. The second equality follows from noting that

$$\big[f, \boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\widetilde{\mathcal{E}}\boldsymbol{\pi}_{\boldsymbol{\zeta}} f\big] \in R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^{\Pi}, \tag{4.24}$$

i.e., that the order of the pole of $\big[f, \boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\widetilde{\mathcal{E}}\boldsymbol{\pi}_{\boldsymbol{\zeta}} f\big]$ at each $y \in \Pi\boldsymbol{\zeta}$ is of order at most m_y, and inserting the identity on $R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^{\Pi}$ in the form $\mathrm{id} = \boldsymbol{p}\boldsymbol{j}_{\boldsymbol{z}'}$. Indeed, by definitions (4.4) and (4.7) of the operators $\boldsymbol{\pi}_{\boldsymbol{\zeta}}$ and $\widetilde{\mathcal{E}}$, we have that

$$\begin{aligned} \boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\widetilde{\mathcal{E}}\boldsymbol{\pi}_{\boldsymbol{\zeta}}\colon\ & R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^{\Pi} \longrightarrow R_{\Pi\boldsymbol{\zeta}}\big(\mathfrak{g}^{\mathbb{C}}\big)^{\Pi}, \\ & \sum_{(y,q)\in(\Pi\boldsymbol{\zeta})} \frac{\mathsf{u}^{(y,q)}}{(z-y)^{q+1}} \longmapsto \sum_{(y,q)\in(\Pi\boldsymbol{\zeta})} \frac{\epsilon_y \mathsf{u}^{(y,q)}}{(z-y)^{q+1}}. \end{aligned}$$

We thus see that (4.24) is true for precisely the same reason that the relation (4.6) we imposed on the components of $\mathcal{L}$ in Section 4.2 solves the admissibility condition (b).

Noting that the operation of taking jets of $\mathfrak{g}$-valued functions is a morphism of Lie algebras, we then obtain that

$$[\boldsymbol{p}\mathsf{U}, \boldsymbol{p}\mathcal{E}\mathsf{U}] = \boldsymbol{p}\big(\big[\boldsymbol{j}_{\boldsymbol{z}'} f, \boldsymbol{j}_{\boldsymbol{z}'}\boldsymbol{\pi}_{\boldsymbol{\zeta}}^{-1}\widetilde{\mathcal{E}}\boldsymbol{\pi}_{\boldsymbol{\zeta}} f\big]\big) = \boldsymbol{p}([\mathsf{U}, \mathcal{E}\mathsf{U}]),$$

where the last step is by definition of $\mathcal{E}$ in (4.13) and the fact that $\mathsf{U} = \boldsymbol{j}_{\boldsymbol{z}'} f$. ■

Remark 4.7. In the relativistic case $\mathcal{E}^2 = \mathrm{id}$, we have the following direct comparison with the results of [53]. Let $\mathfrak{d}_\pm := \ker(\mathcal{E} \mp \mathrm{id}) = \mathrm{im}(\mathcal{E} \pm \mathrm{id}) \subset \mathfrak{d}$ denote the ± 1 eigenspaces of $\mathcal{E}$ in $\mathfrak{d}$, so that we have a direct sum decomposition $\mathfrak{d} = \mathfrak{d}_+ \dot{+} \mathfrak{d}_-$. The statement of Lemma 4.6 can then be rephrased as follows: for any $\mathsf{U}_\pm \in \mathfrak{d}_\pm$ we have

$$\boldsymbol{p}\big([\mathsf{U}_+, \mathsf{U}_-]\big) = \big[\boldsymbol{p}(\mathsf{U}_+), \boldsymbol{p}(\mathsf{U}_-)\big].$$

This is exactly the property considered in [53]. Note, however, that our linear map p in (4.22) already takes values in $\mathfrak{g}^{\mathbb{C}}$-valued rational functions, rather than just $\mathfrak{g}^{\mathbb{C}}$ itself as in [53]. Thus our map (4.22) plays the role of the spectral parameter dependent map $p_\lambda \colon \mathfrak{d} \to \mathfrak{g}^{\mathbb{C}}$ from [53], where λ there is the spectral parameter.

We can now give an alternative derivation of (4.21) from (4.20) by using Lemma 4.6. Specifically, writing $\mathcal{J} \in \Omega^1(\Sigma, \mathfrak{d})$ in components as $\mathcal{J} = \mathcal{J}_\sigma \mathrm{d}\sigma + \mathcal{J}_\tau \mathrm{d}\tau$ we note that $\frac{1}{2}[\mathcal{J}, \mathcal{J}] = [\mathcal{J}_\sigma, \mathcal{J}_\tau] \mathrm{d}\sigma \wedge \mathrm{d}\tau = [\mathcal{J}_\sigma, \mathcal{E}\mathcal{J}_\sigma] \mathrm{d}\sigma \wedge \mathrm{d}\tau$, where in the last step we used the condition (4.12). It follows that

$$\begin{aligned} \frac{1}{2}\boldsymbol{p}([\mathcal{J}, \mathcal{J}]) &= \boldsymbol{p}([\mathcal{J}_\sigma, \mathcal{E}\mathcal{J}_\sigma])\mathrm{d}\sigma \wedge \mathrm{d}\tau = [\boldsymbol{p}\mathcal{J}_\sigma, \boldsymbol{p}\mathcal{E}\mathcal{J}_\sigma]\mathrm{d}\sigma \wedge \mathrm{d}\tau = [\boldsymbol{p}\mathcal{J}_\sigma, \boldsymbol{p}\mathcal{J}_\tau]\mathrm{d}\sigma \wedge \mathrm{d}\tau \\ &= \frac{1}{2}[\boldsymbol{p}\mathcal{J}, \boldsymbol{p}\mathcal{J}], \end{aligned} \tag{4.25}$$

where in the second equality we used Lemma 4.6. Applying the linear map (4.22) to (4.20) we thus obtain

$$\mathrm{d}(\boldsymbol{p}\mathcal{J}) + \frac{1}{2}[\boldsymbol{p}\mathcal{J}, \boldsymbol{p}\mathcal{J}] = 0,$$

where in the first term we used the linearity of $\boldsymbol{p}$ and in the second term we used (4.25). This is equivalent to (4.21) by definition of $\mathcal{J}$. The above derivation of the flatness equation for $\boldsymbol{p}\mathcal{J}$ from that of $\mathcal{J}$ is analogous to [53, Proposition 1].

4.6 Energy-momentum tensor

In Section 2.5 we derived expressions for the components of the energy-momentum tensor of the $\mathcal{E}$-model in terms of the $\mathfrak{d}$-valued field $\mathcal{J}_\sigma \in C^\infty(\Sigma, \mathfrak{d})$, the linear operator $\mathcal{E} \colon \mathfrak{d} \to \mathfrak{d}$ and the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{d}}$ on $\mathfrak{d}$, see Proposition 2.7. Having identified $\mathcal{J}_\sigma$ with the image under $\boldsymbol{j}_{\boldsymbol{z}'}$ of the Lax matrix $\mathcal{L}_\sigma$ in Section 4.4, we may re-express the components of the energy-momentum tensor of the integrable $\mathcal{E}$-models we have constructed in terms of the Lax matrix $\mathcal{L}_\sigma$ itself.

Specifically, we may rewrite the expressions in Proposition 2.7 as

$$T^\tau{}_\sigma = \frac{1}{2}\langle\!\langle \boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma, \boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma \rangle\!\rangle_{\mathfrak{d}} = \frac{1}{2}\langle\!\langle \boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\sigma, \boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\sigma \rangle\!\rangle_{\mathfrak{g}(\boldsymbol{\zeta})}, \tag{4.26a}$$

$$T^\tau{}_\tau = -T^\sigma{}_\sigma = \frac{1}{2}\langle\!\langle \boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma, \mathcal{E}\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma \rangle\!\rangle_{\mathfrak{d}} = \frac{1}{2}\langle\!\langle \boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\sigma, \widetilde{\mathcal{E}}\boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\sigma \rangle\!\rangle_{\mathfrak{g}(\boldsymbol{\zeta})}, \tag{4.26b}$$

$$T^\sigma{}_\tau = -\frac{1}{2}\langle\!\langle \boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma, \mathcal{E}^2\boldsymbol{j}_{\boldsymbol{z}'}\mathcal{L}_\sigma \rangle\!\rangle_{\mathfrak{d}} = -\frac{1}{2}\langle\!\langle \boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\sigma, \widetilde{\mathcal{E}}^2\boldsymbol{\pi}_{\boldsymbol{\zeta}}\mathcal{L}_\sigma \rangle\!\rangle_{\mathfrak{g}(\boldsymbol{\zeta})}, \tag{4.26c}$$

where in each case we used Lemmas 4.2 and 4.3 in the last equality.

The expressions (4.26) can be directly compared with those in [16, Proposition 2.4] for the energy-momentum tensor of an affine Gaudin model which were derived in the case when ω has only simple zeroes. Indeed, in the present notation, the expressions in [16, Proposition 2.4] read

$$T^\tau{}_\sigma = \sum_{y \in \boldsymbol{\zeta}} q_y, \qquad T^\tau{}_\tau = -T^\sigma{}_\sigma = \sum_{y \in \boldsymbol{\zeta}} \epsilon_y q_y, \qquad T^\sigma{}_\tau = -\sum_{y \in \boldsymbol{\zeta}} \epsilon_y^2 q_y, \tag{4.27}$$

where $q_y := -\frac{1}{2}\varphi'(y)\langle \mathcal{L}_\sigma^{(y,0)}, \mathcal{L}_\sigma^{(y,0)}\rangle$ for each simple zero $y \in \boldsymbol{\zeta}$. Using the definition (4.16) of the bilinear form $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\mathfrak{g}^{(\boldsymbol{\zeta})}}$ on $\mathfrak{g}^{(\boldsymbol{\zeta})}$ along with Remark 4.5 about the simple zero case, and the definition of the operator $\widetilde{\mathcal{E}}$ in (4.7), we see that (4.27) coincides exactly with the expressions in (4.26). In particular, the relativistic invariance of the affine Gaudin model was shown in [16] to be ensured by $\epsilon_y^2 = 1$ for all $y \in \boldsymbol{\zeta}$. We see that this coincides with the condition $\mathcal{E}^2 = \mathrm{id}$ for the relativistic invariance of the $\mathcal{E}$-model, see Remark 2.8.

In Remark 4.5, we also gave a simple condition for the operator $\mathcal{E}$ to be positive in the case when ω has simple real poles and $\mathfrak{g}$ is compact, namely that $\varphi'(y)$ and ϵ_y should have opposite signs for every $y \in \boldsymbol{\zeta}$. This corresponds to the condition given in [16, Section 2.2.3] for the Hamiltonian $\int_{\mathbb{R}} \mathrm{d}\sigma\, T^\tau{}_\tau$ to be positive. See also Section 2.5.

4.7 Symmetries of the model

4.7.1 Global G^{diag}-symmetry

In this section we show that the $\mathcal{E}$-model action (4.19) for the edge mode $l \in C^\infty(\Sigma, D)$ has a *global* diagonal G-symmetry.

Let $\Delta\colon G \to G^{\times|\boldsymbol{z}'|} \subset D$ denote the diagonal embedding of G into D. For any $g_0 \in G$ and $\mathsf{U} = (\mathsf{u}_{x,p} \otimes \varepsilon_x^p)_{[x,p]\in[\boldsymbol{z}']} \in \mathfrak{d}$, the adjoint action of $\Delta(g_0) \in D$ on U reads

$$\mathrm{Ad}_{\Delta(g_0)}\,\mathsf{U} = \big((\mathrm{Ad}_{g_0}\,\mathsf{u}_{x,p}) \otimes \varepsilon_x^p\big)_{[x,p]\in[\boldsymbol{z}']}.$$

Since $g_0 \in G$ we have $\tau\,\mathrm{Ad}_{g_0} = \mathrm{Ad}_{g_0}\,\tau$ and so it follows from the explicit form (4.14) of the linear operator $\mathcal{E}\colon \mathfrak{d} \to \mathfrak{d}$ defined in (4.13) that

$$\mathcal{E}\,\mathrm{Ad}_{\Delta(g_0)} = \mathrm{Ad}_{\Delta(g_0)}\,\mathcal{E}. \tag{4.28}$$

Proposition 4.8. *The action (4.19) is invariant under $l \mapsto l\Delta(g_0)$ for any $g_0 \in G$.*

Proof. By construction, the kernel and image of the projector $\mathcal{P}_{l\,\Delta(g_0)}$ are given by

$$\ker \mathcal{P}_{l\,\Delta(g_0)} = \mathrm{Ad}^{-1}_{l\,\Delta(g_0)}\,\mathfrak{k} = \mathrm{Ad}^{-1}_{\Delta(g_0)}\,\mathrm{Ad}_l^{-1}\,\mathfrak{k} = \mathrm{Ad}^{-1}_{\Delta(g_0)}\ker\mathcal{P}_l,$$
$$\mathrm{im}\,\mathcal{P}_{l\,\Delta(g_0)} = \mathcal{E}\,\mathrm{Ad}^{-1}_{l\,\Delta(g_0)}\,\mathfrak{k} = \mathcal{E}\,\mathrm{Ad}^{-1}_{\Delta(g_0)}\,\mathrm{Ad}_l^{-1}\,\mathfrak{k} = \mathrm{Ad}^{-1}_{\Delta(g_0)}\,\mathcal{E}\,\mathrm{Ad}_l^{-1}\,\mathfrak{k} = \mathrm{Ad}^{-1}_{\Delta(g_0)}\,\mathrm{im}\,\mathcal{P}_l,$$

where in the second line we have used (4.28) in the third step. It is a standard result on projectors that $\mathcal{P}_{l\,\Delta(g_0)}$ is thus given by

$$\mathcal{P}_{l\,\Delta(g_0)} = \mathrm{Ad}^{-1}_{\Delta(g_0)}\,\mathcal{P}_l\,\mathrm{Ad}_{\Delta(g_0)}\,.$$

Similar equalities also hold for $\overline{\mathcal{P}}_{l\,\Delta(g_0)}$ and the transpose of $\mathcal{P}_{l\,\Delta(g_0)}$ and $\overline{\mathcal{P}}_{l\,\Delta(g_0)}$.

Moreover, under $l \mapsto l\Delta(g_0)$, the Maurer–Cartan current $l^{-1}\partial_\mu l$ transforms as $l^{-1}\partial_\mu l \mapsto \mathrm{Ad}^{-1}_{\Delta(g_0)}\,l^{-1}\partial_\mu l$. Putting all of the above together it now follows that the first term in the action (4.19) is invariant under $l \mapsto l\Delta(g_0)$.

Finally, since the WZ-term in (4.19) is independent of the choice of extension $\widehat{l}^{-1}\mathrm{d}\widehat{l}$ of the 1-form $l^{-1}\mathrm{d}l \in \Omega^1(\Sigma, \mathfrak{d})$ to the bulk $\Sigma \times I$, we can choose this extension for the transformed 1-form $\mathrm{Ad}^{-1}_{\Delta(g_0)}\,l^{-1}\mathrm{d}l$ to be $\mathrm{Ad}^{-1}_{\Delta(g_0)}\,\widehat{l}^{-1}\mathrm{d}\widehat{l}$, from which it follows that the WZ-term is also invariant under the transformation $l \mapsto l\Delta(g_0)$. ■

Remark 4.9. The only property of the element $\Delta(g_0) \in D$ which we used in the proof of Proposition 4.8 is (4.28). Therefore, the statement of the proposition would also hold for any other Lie subgroup of D with the property that all its elements $d \in D$ are such that $\mathrm{Ad}_d\colon \mathfrak{d} \to \mathfrak{d}$ commutes with $\mathcal{E}$.

4.7.2 Global symmetries for $\mathfrak{k}$ an ideal

We now consider the $\mathcal{E}$-model action (4.19) in a more specific setup.

Proposition 4.10. *If the Lagrangian subalgebra $\mathfrak{k} \subset \mathfrak{d}$ is an ideal, then the $\mathcal{E}$-model action (4.19) is invariant under $l \mapsto al$ for any $a \in D$.*

Remark 4.11. In fact, since the $\mathcal{E}$-model action has a gauge invariance $l \mapsto kl$ for $k \in C^\infty(\Sigma, K)$, see Section 2.3, the global symmetry in Proposition 4.10 is really only an additional symmetry by the Lie group $K\backslash D$.

Proof. Proposition 4.10 The currents $l^{-1}\partial_\mu l$ and the WZ-term in (4.19) are both invariant under the transformation $l \mapsto al$. It remains to check that $\mathcal{P}_{al} = \mathcal{P}_l$ and $\overline{\mathcal{P}}_{al} = \overline{\mathcal{P}}_l$. Since $\mathfrak{k}$ is an ideal of $\mathfrak{d}$ we have $\mathrm{Ad}_l^{-1} \mathfrak{k} = \mathfrak{k}$ and thus

$$\operatorname{im} \mathcal{P}_l = \mathcal{E}^{-1}\mathfrak{k}, \qquad \ker \mathcal{P}_l = \mathfrak{k}.$$

We see that the projector $\mathcal{P}_l$, and also $\overline{\mathcal{P}}_l$, is in fact independent of l. It is therefore invariant under $l \mapsto al$, as required. ■

We can construct simple examples of Lagrangian ideals $\mathfrak{k} \subset \mathfrak{d}$ in the case when all the multiplicities n_x of the poles $x \in \boldsymbol{z}$ of ω are even, i.e., $n_x = 2r_x$ for some $r_x \in \mathbb{Z}_{\geq 1}$. Recall the definition (3.13) of the Lie algebra $\mathfrak{d}$, namely

$$\mathfrak{d} = \bigoplus_{x \in \boldsymbol{z}'_{\mathrm{r}}} \mathfrak{g} \otimes_{\mathbb{R}} \mathcal{T}_x^{2r_x} \oplus \bigoplus_{x \in \boldsymbol{z}_{\mathrm{c}}} \big(\mathfrak{g}^{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{T}_x^{2r_x}\big)_{\mathbb{R}}.$$

Consider the ideals $\mathcal{T}_x^{\geq r_x} := \varepsilon_x^{r_x}\mathbb{R}[\varepsilon_x]/\big(\varepsilon_x^{2r_x}\big) \subset \mathcal{T}_x^{2r_x}$ for real finite poles $x \in \boldsymbol{z}'_{\mathrm{r}}$ and $\mathcal{T}_x^{\geq r_x} := \varepsilon_x^{r_x}\mathbb{C}[\varepsilon_x]/\big(\varepsilon_x^{2r_x}\big) \subset \mathcal{T}_x^{2r_x}$ for complex poles $x \in \boldsymbol{z}_{\mathrm{c}}$. It is easy to check that

$$\mathfrak{k} := \bigoplus_{x \in \boldsymbol{z}'_{\mathrm{r}}} \mathfrak{g} \otimes_{\mathbb{R}} \mathcal{T}_x^{\geq r_x} \oplus \bigoplus_{x \in \boldsymbol{z}_{\mathrm{c}}} \big(\mathfrak{g}^{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{T}_x^{\geq r_x}\big)_{\mathbb{R}}$$

is a Lagrangian ideal of $\mathfrak{d}$.

5 Examples

In this section we give a few examples of the above general construction, including details of the one mentioned in the introduction. In each case, we make a choice of meromorphic 1-form ω, Lagrangian subalgebra $\mathfrak{k} \subset \mathfrak{d}$ and set of parameters ϵ_y associated with each zero $y \in \boldsymbol{\zeta}$ of ω. This is then fed into the general construction to produce the action and Lax connection of the corresponding integrable σ-models.

5.1 Principal chiral model and non-abelian T-dual

We rederive the well-known actions of the principal chiral model and its non-abelian T-dual as an application of our general construction. The principal chiral model was already derived in [13, Section 10.2] and then again in [17, Section 5.1] using the unifying 2d action valid when ω has at most double poles. We derive it again in Section 5.1.3 below since it provides the simplest illustration of our construction.

However, the derivation of the non-abelian T-dual in Section 5.1.4 below from 4d Chern–Simons theory is new. This was conjectured but could not be derived in [17] since the formula for the unifying 2d action there was only applicable under a technical condition on the Lagrangian subalgebra $\mathfrak{k} \subset \mathfrak{d}$, see [17, equation (4.16)]. The latter is not satisfied by the choice of Lagrangian subalgebra used in Section 5.1.4 to derive the non-abelian T-dual.

We begin by setting up the formalism to discuss both the principal chiral model and its non-abelian T-dual. We let $a > 0$ and consider the 1-form

$$\omega = \frac{a^2 \mathrm{d}z}{z^2} - \mathrm{d}z. \tag{5.1}$$

In the notation of Section 3.1 we have the set of poles $\boldsymbol{z} = \{0, \infty\}$ and the set of zeroes is $\boldsymbol{\zeta} = \{a, -a\}$. Both poles are double poles so that $n_0 = n_\infty = 2$ and both zeroes are simple so that $m_a = m_{-a} = 1$. Moreover, all the zeroes and poles are real so here $\boldsymbol{z} = \boldsymbol{z}_{\mathrm{r}}$ and $\boldsymbol{\zeta} = \boldsymbol{\zeta}_{\mathrm{r}}$. The levels are read off from ω to be $\ell^0_0 = 0$, $\ell^0_1 = a^2$ and $\ell^\infty_1 = 1$.

We also choose the parameters associated to the set of zeroes in (4.6) to be

$$\epsilon_{\pm a} = \pm 1. \tag{5.2}$$

Below we shall construct all the data associated with the choice of 1-form (5.1) and parameters (5.2), and then use this data to build the actions for the principal chiral model in Section 5.1.3 and its non-abelian T-dual in Section 5.1.4.

5.1.1 Lie groups D, K and $\tilde{K}$

The defect Lie algebra (3.13) is given here by

$$\mathfrak{d} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{R}[\varepsilon_0]/\big(\varepsilon_0^2\big) = \mathfrak{g} \ltimes \mathfrak{g}_{\mathrm{ab}},$$

where $\mathfrak{g}_{\mathrm{ab}} := \mathfrak{g} \otimes_{\mathbb{R}} \varepsilon_0 \mathbb{R}[\varepsilon_0]/\big(\varepsilon_0^2\big)$ is isomorphic to the vector space $\mathfrak{g}$ equipped with the trivial Lie bracket and the adjoint action of $\mathfrak{g}$. By using the abbreviated notation $\mathsf{u}^p := \mathsf{u}^{[0,p]} = \mathsf{u} \otimes \varepsilon_0^p$ for any $\mathsf{u} \in \mathfrak{g}$ and $p \in \{0, 1\}$, the Lie algebra relations in $\mathfrak{d}$ read

$$\big[\mathsf{u}^0, \mathsf{v}^0\big] = [\mathsf{u}, \mathsf{v}]^0, \qquad \big[\mathsf{u}^0, \mathsf{v}^1\big] = \big[\mathsf{u}^1, \mathsf{v}^0\big] = [\mathsf{u}, \mathsf{v}]^1, \qquad \big[\mathsf{u}^1, \mathsf{v}^1\big] = 0, \tag{5.3}$$

for any $\mathsf{u}, \mathsf{v} \in \mathfrak{g}$.

The associated Lie group D is the tangent bundle TG of G, which in the right trivialisation is isomorphic to the Lie group $G \ltimes \mathfrak{g}$ with product and inverse

$$(g, \mathsf{u})(h, \mathsf{v}) = \big(gh, \mathsf{u} + \mathrm{Ad}_g \mathsf{v}\big), \qquad (g, \mathsf{u})^{-1} = \big(g^{-1}, -\mathrm{Ad}_g^{-1} \mathsf{u}\big)$$

for every $g, h \in G$ and $\mathsf{u}, \mathsf{v} \in \mathfrak{g}$.

We have the two obvious Lie subalgebras

$$\tilde{\mathfrak{k}} := \mathfrak{g} \oplus \{0\} = \big\{\mathsf{u}^0 \mid \mathsf{u} \in \mathfrak{g}\big\}, \qquad \mathfrak{k} := \{0\} \oplus \mathfrak{g}_{\mathrm{ab}} = \big\{\mathsf{u}^1 \mid \mathsf{u} \in \mathfrak{g}\big\} \tag{5.4}$$

of $\mathfrak{d}$. These are complementary since we have the direct sum decomposition

$$\mathfrak{d} = \mathfrak{k} \dotplus \tilde{\mathfrak{k}}. \tag{5.5}$$

Let $\tilde{K}, K \subset D$ denote the corresponding connected Lie subgroups of D, which are isomorphic to $G \times \{0\}$ and $\{\mathrm{id}\} \times \mathfrak{g}$, respectively. In particular, it is clear that K is normal in D. We have the global decomposition $D = K\tilde{K} = \tilde{K}K$.

The bilinear form (3.3) on $\mathfrak{d}$ is given explicitly by

$$\langle\!\langle \mathsf{u}^0, \mathsf{v}^0 \rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle \mathsf{u}^1, \mathsf{v}^1 \rangle\!\rangle_{\mathfrak{d}} = 0, \qquad \langle\!\langle \mathsf{u}^0, \mathsf{v}^1 \rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle \mathsf{u}^1, \mathsf{v}^0 \rangle\!\rangle_{\mathfrak{d}} = a^2 \langle \mathsf{u}, \mathsf{v} \rangle, \tag{5.6}$$

for any $\mathsf{u}, \mathsf{v} \in \mathfrak{g}$ so that $\mathfrak{k}$ and $\tilde{\mathfrak{k}}$ are both Lagrangian subalgebras of $\mathfrak{d}$.

5.1.2 Linear operator $\mathcal{E}$

The real vector space (4.3) associated with the two simple zeroes of ω is given here by $\mathfrak{g}^{(\boldsymbol{\zeta})} = \mathfrak{g}\oplus\mathfrak{g}$. One checks by computing the Cauchy matrix (4.9) that the isomorphism $\mathcal{C}\colon \mathfrak{g}^{(\boldsymbol{\zeta})} \to \mathfrak{d}$ from Lemma 4.1 (see also (4.10) and (4.11)) and its inverse are given here by

$$\mathcal{C}(\mathsf{u},\mathsf{v}) = -\frac{\mathsf{u}^0}{a} + \frac{\mathsf{v}^0}{a} - \frac{\mathsf{u}^1}{a^2} - \frac{\mathsf{v}^1}{a^2}, \qquad \mathcal{C}^{-1}\big(\mathsf{u}^0 + \mathsf{v}^1\big) = \left(-\frac{1}{2}a\mathsf{u} - \frac{1}{2}a^2\mathsf{v}, \frac{1}{2}a\mathsf{u} - \frac{1}{2}a^2\mathsf{v}\right)$$

for every $\mathsf{u},\mathsf{v}\in\mathfrak{g}$. Given the choice of parameters ϵ_y for $y\in\boldsymbol{\zeta}$ we made in (5.2), the linear isomorphism (4.7) then reads $\widetilde{\mathcal{E}}(\mathsf{u},\mathsf{v}) = (\mathsf{u},-\mathsf{v})$ and the linear operator (4.13) is found to act in the present case as

$$\mathcal{E}\big(\mathsf{u}^0 + \mathsf{v}^1\big) = a\mathsf{v}^0 + \frac{\mathsf{u}^1}{a} \tag{5.7}$$

for every $\mathsf{u},\mathsf{v}\in\mathfrak{g}$. We have $\mathcal{E}^2 = \mathrm{id}$, corresponding to the fact that the principal chiral model and its non-abelian T-dual are relativistic.

The Lie algebra relations (5.3), bilinear form (5.6) and linear operator (5.7) agree with those for the principal chiral model given in [32, equations (19)–(21)].

5.1.3 Principal chiral model

Here we apply the general construction of Section 4 with $\mathfrak{k}$ defined in (5.4) as the Lagrangian subalgebra. Since we have the global factorisation $D = K\tilde{K}$ we can factorise the field $l\in C^\infty(\Sigma,D)$ uniquely as $l = kg$ for some $k\in C^\infty(\Sigma,K)$ and $g\in C^\infty\big(\Sigma,\tilde{K}\big)$. Using the gauge invariance of the action (2.14) under $l\mapsto k^{-1}l$ from Section 2.3, we can then fix $l = g$. In particular, the action (2.14) now reads

$$\begin{aligned} S_{2\mathrm{d}}(g) = \frac{1}{2}\int_\Sigma \big(&\langle\!\langle g^{-1}\partial_\tau g, \mathcal{E}\mathcal{P}_g\big(g^{-1}\partial_\tau g\big)\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle g^{-1}\partial_\sigma g, \mathcal{E}\mathcal{P}_g\big(g^{-1}\partial_\sigma g\big)\rangle\!\rangle_{\mathfrak{d}} \\ &+ \langle\!\langle g^{-1}\partial_\tau g, \big(\mathcal{P}_g - {}^t\mathcal{P}_g\big)\big(g^{-1}\partial_\sigma g\big)\rangle\!\rangle_{\mathfrak{d}}\big)\mathrm{d}\sigma\wedge\mathrm{d}\tau - \frac{1}{2}I^{\mathrm{WZ}}_{\tilde{\mathfrak{k}}}[g] \end{aligned} \tag{5.8}$$

for the group valued field $g\in C^\infty(\Sigma,\tilde{K})$, where we have used the fact that $\mathcal{E}^2 = \mathrm{id}$ together with part (vi) of Proposition 2.3. Here $\mathcal{P}_g$ denotes the projector defined by the relations (2.4), namely

$$\ker\mathcal{P}_g = \mathrm{Ad}_g^{-1}\mathfrak{k}, \qquad \mathrm{im}\,\mathcal{P}_g = \mathcal{E}\,\mathrm{Ad}_g^{-1}\mathfrak{k}.$$

Since $[\tilde{\mathfrak{k}},\mathfrak{k}]\subset\mathfrak{k}$ we have $\mathrm{Ad}_g^{-1}\mathfrak{k} = \mathfrak{k}$ and hence

$$\ker\mathcal{P}_g = \mathfrak{k}, \qquad \mathrm{im}\,\mathcal{P}_g = \mathcal{E}\mathfrak{k} = \tilde{\mathfrak{k}},$$

where the last equality uses the explicit forms (5.4) and (5.7) of the two subalgebras $\mathfrak{k},\tilde{\mathfrak{k}}\subset\mathfrak{d}$ and of $\mathcal{E}$. Thus $\mathcal{P}_g$ is simply the projection onto $\tilde{\mathfrak{k}}$ along $\mathfrak{k}$, relative to the direct sum decomposition (5.5). In particular, it acts as the identity on $g^{-1}\partial_\mu g\in C^\infty(\Sigma,\tilde{\mathfrak{k}})$. Moreover, since $\tilde{\mathfrak{k}}$ is isotropic with respect to the bilinear form (5.6), the WZ-term in the action (5.8) vanishes.

Putting all of the above together and noting the identity $\langle\!\langle\mathsf{u}^0,\mathcal{E}\mathsf{v}^0\rangle\!\rangle_{\mathfrak{d}} = a\langle\mathsf{u},\mathsf{v}\rangle$ for any $\mathsf{u},\mathsf{v}\in\mathfrak{g}$, the action (5.8) reduces to the usual principal chiral model action

$$S_{2\mathrm{d}}(g) = \frac{1}{2}a\int_\Sigma \langle g^{-1}\partial_+ g, g^{-1}\partial_- g\rangle\mathrm{d}\sigma\wedge\mathrm{d}\tau,$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$, for the Lie group valued field $g \in C^\infty(\Sigma, \tilde{K}) \cong C^\infty(\Sigma, G)$. The global $G^{\rm diag}$-symmetry from Proposition 4.8 corresponds here to the right G-symmetry of the principal chiral model. Since the Lagrangian subalgebra $\mathfrak{k} \subset \mathfrak{d}$ is an ideal we also have the global left symmetry of Proposition 4.10 by the Lie group $K \backslash D \simeq \tilde{K} \simeq G$, corresponding to the left G-symmetry of the principal chiral model.

Let $j \coloneqq g^{-1} \mathrm{d} g$. Using $*\mathrm{d}\sigma = -\mathrm{d}\tau$ and $*\mathrm{d}\tau = -\mathrm{d}\sigma$, the Lax connection (4.23) is given by $\mathcal{L} = \boldsymbol{p}\big(j^0 - \frac{1}{a} * j^1\big)$, where the map $\boldsymbol{p}$ in (4.22) is the inverse of the map $\boldsymbol{j}_{\boldsymbol{z}'}$ defined in (4.1) and given explicitly in the present case by

$$\boldsymbol{j}_{\boldsymbol{z}'}\colon\ \frac{\mathsf{u}_a}{z-a} + \frac{\mathsf{u}_{-a}}{z+a} \longmapsto \frac{1}{a}(\mathsf{u}_{-a} - \mathsf{u}_a)^0 + \frac{1}{a^2}(-\mathsf{u}_{-a} - \mathsf{u}_a)^1.$$

Its inverse is then given explicitly by

$$\boldsymbol{p}\colon\ \mathsf{v}^0 + \mathsf{w}^1 \longmapsto \frac{a\mathsf{v} + a^2 \mathsf{w}}{2(a-z)} + \frac{a\mathsf{v} - a^2\mathsf{w}}{2(a+z)}. \tag{5.9}$$

We therefore obtain the Lax connection of the principal chiral model

$$\mathcal{L} = \boldsymbol{p}\big(j^0 - *j^1\big) = a\frac{aj - z * j}{a^2 - z^2} = \frac{aj_+}{a-z}\mathrm{d}\sigma^+ + \frac{aj_-}{a+z}\mathrm{d}\sigma^-, \tag{5.10}$$

where $j_\pm \coloneqq g^{-1}\partial_\pm g$. This coincides with the usual Lax connection of the principal chiral model after rescaling the spectral parameter as $z \mapsto az$.

5.1.4 Non-abelian *T*-dual

We will now use the reverse factorisation $D = \tilde{K}K$, treating $\tilde{K} \subset D$ as the Lie subgroup which we quotient by in Section 2.2. As in Section 5.1.3, we can factorise our field $l \in C^\infty(\Sigma, D)$ uniquely as $l = \tilde{k}p$ for some $\tilde{k} \in C^\infty(\Sigma, \tilde{K})$ and $p \in C^\infty(\Sigma, K)$. We may therefore use the gauge invariance by the subgroup $\tilde{K}$ from Section 2.3 to fix $l = p$, obtaining the action (2.14) for $p \in C^\infty(\Sigma, K)$, where $\tilde{\mathfrak{k}}$ now plays the role of $\mathfrak{k}$. As in Section 5.1.3, the WZ-term in this action vanishes since the Lie subalgebra $\mathfrak{k}$ is isotropic. Furthermore, by definition we can write $p = \big(\mathrm{id}, \frac{1}{a}\mathsf{m}\big)$ for some $\mathsf{m} \in C^\infty(\Sigma, \mathfrak{g})$, where the factor of $\frac{1}{a}$ is introduced for later convenience. In particular, we then have $p^{-1}\partial_\mu p = \frac{1}{a}\partial_\mu \mathsf{m}^1 \in C^\infty(\Sigma, \mathfrak{k})$. In the present case, the action (2.14) therefore simplifies to

$$\begin{aligned} S_{2\rm d}(\mathsf{m}) = \frac{1}{2}a^{-2}\int_\Sigma \big(&\langle\!\langle \partial_\tau \mathsf{m}^1, \mathcal{E}\tilde{\mathcal{P}}_p(\partial_\tau \mathsf{m}^1)\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle \partial_\sigma \mathsf{m}^1, \mathcal{E}\tilde{\mathcal{P}}_p(\partial_\sigma \mathsf{m}^1)\rangle\!\rangle_{\mathfrak{d}} \\ &+ \langle\!\langle \partial_\tau \mathsf{m}^1, \big(\tilde{\mathcal{P}}_p - {}^t\tilde{\mathcal{P}}_p\big)\big(\partial_\sigma \mathsf{m}^1\big)\rangle\!\rangle_{\mathfrak{d}}\big)\mathrm{d}\sigma\wedge\mathrm{d}\tau, \end{aligned} \tag{5.11}$$

where now $\tilde{\mathcal{P}}_p$ denotes the projector defined in the same way as in (2.4) but relative to the Lagrangian subalgebra $\tilde{\mathfrak{k}}$, namely

$$\begin{aligned} \ker\tilde{\mathcal{P}}_p &= \mathrm{Ad}_p^{-1}\tilde{\mathfrak{k}} = \left(\mathrm{id} - \frac{1}{a}\mathrm{ad}_{\mathsf{m}^1}\right)\tilde{\mathfrak{k}} = \left\{\mathsf{u}^0 - \frac{1}{a}[\mathsf{m},\mathsf{u}]^1 \,\middle|\, \mathsf{u} \in \mathfrak{g}\right\}, \\ \mathrm{im}\,\tilde{\mathcal{P}}_p &= \mathcal{E}\,\mathrm{Ad}_p^{-1}\tilde{\mathfrak{k}} = \mathcal{E}\left(\left(\mathrm{id} - \frac{1}{a}\mathrm{ad}_{\mathsf{m}^1}\right)\tilde{\mathfrak{k}}\right) = \left\{\frac{1}{a}\mathsf{u}^1 - [\mathsf{m},\mathsf{u}]^0 \,\middle|\, \mathsf{u}\in\mathfrak{g}\right\}. \end{aligned}$$

It is straightforward to check that the projector $\tilde{\mathcal{P}}_p\colon \mathfrak{d} \to \mathfrak{d}$ with the above kernel and image is given by

$$\tilde{\mathcal{P}}_p\big(\mathsf{u}^0 + \mathsf{v}^1\big) = -\left(\frac{\mathrm{ad}_{\mathsf{m}}^2}{\mathrm{id} - \mathrm{ad}_{\mathsf{m}}^2}\mathsf{u}\right)^0 - \left(\frac{a\,\mathrm{ad}_{\mathsf{m}}}{\mathrm{id} - \mathrm{ad}_{\mathsf{m}}^2}\mathsf{v}\right)^0 + \left(\frac{a^{-1}\,\mathrm{ad}_{\mathsf{m}}}{\mathrm{id} - \mathrm{ad}_{\mathsf{m}}^2}\mathsf{u}\right)^1 + \left(\frac{1}{\mathrm{id} - \mathrm{ad}_{\mathsf{m}}^2}\mathsf{v}\right)^1$$

for any $\mathsf{u}, \mathsf{v} \in \mathfrak{g}$. In particular, applying this to $\partial_\mu \mathsf{m}^1$ we find

$$\tilde{\mathcal{P}}_p(\partial_\mu \mathsf{m}^1) = \bigg(- \frac{a\, \mathrm{ad}_\mathsf{m}}{\mathrm{id} - \mathrm{ad}_\mathsf{m}^2} \partial_\mu \mathsf{m} \bigg)^0 + \bigg(\frac{1}{\mathrm{id} - \mathrm{ad}_\mathsf{m}^2} \partial_\mu \mathsf{m} \bigg)^1. \tag{5.12}$$

Substituting this into the action (5.11) and using the expressions (5.6) and (5.7) of the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\mathfrak{d}$ and the linear operator $\mathcal{E}$ we arrive at the standard action of the non-abelian T-dual of the principal chiral model [24, 25], namely

$$S_{2\mathrm{d}}(\mathsf{m}) = \frac{1}{2} a \int_\Sigma \Big\langle \partial_+ \mathsf{m}, \frac{1}{\mathrm{id} - \mathrm{ad}_\mathsf{m}} \partial_- \mathsf{m} \Big\rangle \mathrm{d}\sigma \wedge \mathrm{d}\tau.$$

The Lax connection (4.23) is now given by

$$\mathcal{L} = \boldsymbol{p}\bigg(\frac{1}{a} \tilde{\mathcal{P}}_p(\mathrm{d}\mathsf{m}^1) - \frac{1}{a} \mathcal{E} \tilde{\mathcal{P}}_p(*\mathrm{d}\mathsf{m}^1) \bigg).$$

Using the explicit form of the inverse $\boldsymbol{p}$ in (5.9) and using (5.12) we find

$$\mathcal{L} = \frac{a}{a - z} \frac{1}{\mathrm{id} + \mathrm{ad}_\mathsf{m}} \partial_+ \mathsf{m}\, \mathrm{d}\sigma^+ - \frac{a}{a + z} \frac{1}{\mathrm{id} - \mathrm{ad}_\mathsf{m}} \partial_- \mathsf{m}\, \mathrm{d}\sigma^-.$$

This becomes the usual Lax connection of the non-abelian T-dual of the principal chiral model after the rescaling $z \mapsto az$.

5.2 Fourth order pole

In this section we give an example of our construction in the case when ω has a pole of order 4. We will let $a > b > 0$ and take

$$\omega = \frac{(z^2 - a^2)(b^2 - z^2)}{z^4} \mathrm{d}z. \tag{5.13}$$

The set of poles is $\boldsymbol{z} = \boldsymbol{z}_\mathrm{r} = \{0, \infty\}$ with orders $n_0 = 4$ and $n_\infty = 2$. The associated levels are $\ell^0_3 = -a^2 b^2$, $\ell^0_2 = 0$, $\ell^0_1 = a^2 + b^2$, $\ell^0_0 = 0$ and $\ell^\infty_1 = 1$. The set of zeroes is $\boldsymbol{\zeta} = \boldsymbol{\zeta}_\mathrm{r} = \{a, -a, b, -b\}$ with all zeroes being simple.

We let the parameters in (4.6) associated with the set $\boldsymbol{\zeta}$ of zeroes of ω be

$$\epsilon_{\pm a} = \pm 1, \qquad \epsilon_{\pm b} = \mp 1. \tag{5.14}$$

This choice ensures that $\mathcal{E}$ will be positive (for compact $\mathfrak{g}$) and such that $\mathcal{E}^2 = \mathrm{id}$. Indeed, the latter condition follows since $\epsilon_y^2 = 1$ for all $y \in \boldsymbol{\zeta}$ and positivity follows from Remark 4.5 after noting that $-\epsilon_y \varphi'(y) > 0$ for each $y \in \boldsymbol{\zeta}$. In what follows we shall construct all the necessary data associated with the choice of 1-form (5.13) and parameters (5.14). We then extract from this data the action and Lax connection of a new 2d integrable field theory.

5.2.1 Lie groups D and K

Since the pole 0 is real, the defect Lie algebra (3.13) is

$$\mathfrak{d} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{R}[\varepsilon_0]/(\varepsilon_0^4).$$

As in Section 5.1 we use the abbreviated notation $\mathsf{u}^p := \mathsf{u}^{[0,p]} = \mathsf{u} \otimes \varepsilon_0^p$ for any $\mathsf{u} \in \mathfrak{g}$ and $p \in \{0, 1, 2, 3\}$. The Lie algebra relations in $\mathfrak{d}$ read

$$[\mathsf{u}^p, \mathsf{v}^q] = [\mathsf{u}, \mathsf{v}]^{p+q},$$

for any $\mathsf{u}, \mathsf{v} \in \mathfrak{g}$ and $p, q \in \{0, 1, 2, 3\}$. In particular, recall this vanishes for $p + q \geq 4$.

The Lie group D is given by the 3rd order jet bundle J^3G of the Lie group G. In the right trivialisation it is isomorphic to $G\times\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}$ equipped with the Lie group product and inverse [66]

$$\begin{aligned}(g,\mathsf{u},\mathsf{v},\mathsf{w})(h,\mathsf{x},\mathsf{y},\mathsf{z}) &= \big(gh,\mathsf{u}+\mathrm{Ad}_g\,\mathsf{x},\mathsf{v}+\mathrm{Ad}_g\,\mathsf{y}+[\mathsf{u},\mathrm{Ad}_g\,\mathsf{x}],\\ &\qquad \mathsf{w}+\mathrm{Ad}_g\,\mathsf{z}+2[\mathsf{u},\mathrm{Ad}_g\,\mathsf{y}]+[\mathsf{v},\mathrm{Ad}_g\,\mathsf{x}]+[\mathsf{u},[\mathsf{u},\mathrm{Ad}_g\,\mathsf{x}]]\big),\\ (g,\mathsf{u},\mathsf{v},\mathsf{w})^{-1} &= \big(g^{-1},-\mathrm{Ad}_g^{-1}\,\mathsf{u},-\mathrm{Ad}_g^{-1}\,\mathsf{v},-\mathrm{Ad}_g^{-1}\,\mathsf{w}+\mathrm{Ad}_g^{-1}[\mathsf{u},\mathsf{v}]\big)\end{aligned}$$

for every $g,h\in G$ and $\mathsf{u},\mathsf{v},\mathsf{w},\mathsf{x},\mathsf{y},\mathsf{z}\in\mathfrak{g}$.

We consider the ideal $\mathfrak{k}\subset\mathfrak{d}$ defined in Section 4.7.2, which is given here by

$$\mathfrak{k}=\mathfrak{g}\otimes_{\mathbb{R}}\varepsilon_0^2\mathbb{R}[\varepsilon_0]/\big(\varepsilon_0^4\big)=\big\{\mathsf{u}^2+\mathsf{v}^3\mid \mathsf{u},\mathsf{v}\in\mathfrak{g}\big\}.$$

Let $K\subset D$ denote the corresponding connected Lie subgroup of D. It is isomorphic to $\{\mathrm{id}\}\times\{0\}\times\mathfrak{g}\times\mathfrak{g}$ which is normal in $G\times\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}$ since

$$(g,\mathsf{u},\mathsf{v},\mathsf{w})(\mathrm{id},0,\mathsf{y},\mathsf{z})(g,\mathsf{u},\mathsf{v},\mathsf{w})^{-1}=\big(\mathrm{id},0,\mathrm{Ad}_g\,\mathsf{y},\mathrm{Ad}_g\,\mathsf{z}+3[\mathsf{u},\mathrm{Ad}_g\,\mathsf{y}]\big),$$

for any $g\in G$ and $\mathsf{u},\mathsf{v},\mathsf{w},\mathsf{y},\mathsf{z}\in\mathfrak{g}$. In particular, the left coset $K\backslash D$ is naturally a Lie group which, as a manifold, is diffeomorphic to $G\times\mathfrak{g}\times\{0\}\times\{0\}$.

The bilinear form (3.3) on $\mathfrak{d}$ is given explicitly by

$$\langle\!\langle \mathsf{u}^p,\mathsf{v}^q\rangle\!\rangle_{\mathfrak{d}}=\begin{cases}-a^2b^2\langle\mathsf{u},\mathsf{v}\rangle, & \text{if}\quad p+q=3,\\ \big(a^2+b^2\big)\langle\mathsf{u},\mathsf{v}\rangle, & \text{if}\quad p+q=1,\\ 0, & \text{otherwise}\end{cases}$$

for any $\mathsf{u},\mathsf{v}\in\mathfrak{g}$. In particular, we see that $\mathfrak{k}$ is indeed a Lagrangian subalgebra of $\mathfrak{d}$.

5.2.2 Linear operator $\mathcal{E}$

The real vector space (4.3) associated with the zeroes of ω is given here by $\mathfrak{g}^{(\boldsymbol{\zeta})}=\mathfrak{g}^{\oplus 4}$. With the choice of parameters ϵ_y for $y\in\boldsymbol{\zeta}$ in (5.14) we find by explicitly computing the Cauchy matrix (4.9) that the linear operator $\mathcal{E}\colon\mathfrak{d}\to\mathfrak{d}$ in (4.13) is given by

$$\begin{aligned}\mathcal{E}\big(\mathsf{u}^0+\mathsf{v}^1+\mathsf{w}^2+\mathsf{x}^3\big) &= \frac{a^2-ab+b^2}{a-b}\mathsf{v}^0-\frac{a^2b^2}{a-b}\mathsf{x}^0+\frac{1}{a-b}\mathsf{u}^1-\frac{ab}{a-b}\mathsf{w}^1\\ &\quad+\frac{1}{a-b}\mathsf{v}^2-\frac{ab}{a-b}\mathsf{x}^2+\frac{1}{ab(a-b)}\mathsf{u}^3-\frac{a^2-ab+b^2}{ab(a-b)}\mathsf{w}^3,\end{aligned}\tag{5.15}$$

for every $\mathsf{u},\mathsf{v},\mathsf{w},\mathsf{x}\in\mathfrak{g}$. As previously noted after (5.14) we have $\mathcal{E}^2=\mathrm{id}$. Since $\mathfrak{k}\subset\mathfrak{d}$ is an ideal we have $\mathrm{Ad}_l^{-1}\,\mathfrak{k}=\mathfrak{k}$ for any $l\in D$ and therefore the projector $\mathcal{P}_l$ defined as in (2.4) has kernel and image

$$\begin{aligned}&\ker\mathcal{P}_l=\mathfrak{k}=\big\{\mathsf{u}^2+\mathsf{v}^3\mid\mathsf{u},\mathsf{v}\in\mathfrak{g}\big\},\\ &\operatorname{im}\mathcal{P}_l=\mathcal{E}\mathfrak{k}=\big\{a^3b^3\mathsf{u}^0+a^2b^2\mathsf{v}^1+a^2b^2\mathsf{u}^2+\big(a^2-ab+b^2\big)\mathsf{v}^3\mid\mathsf{u},\mathsf{v}\in\mathfrak{g}\big\},\end{aligned}$$

where we used the explicit form of $\mathcal{E}$ in (5.15). Note that $\mathcal{P}_l$ is therefore independent of l. Explicitly, we have

$$\begin{aligned}&\mathcal{P}_l\big(\mathsf{u}^0+\mathsf{v}^1+\mathsf{w}^2+\mathsf{x}^3\big)=\mathsf{u}^0+\mathsf{v}^1+\frac{1}{ab}\mathsf{u}^2+\frac{a^2-ab+b^2}{a^2b^2}\mathsf{v}^3,\\ &\mathcal{E}\mathcal{P}_l\big(\mathsf{u}^0+\mathsf{v}^1+\mathsf{w}^2+\mathsf{x}^3\big)=\frac{b-a}{ab}\mathsf{v}^2+\frac{b-a}{a^2b^2}\mathsf{u}^3,\end{aligned}\tag{5.16}$$

for any $\mathsf{u},\mathsf{v},\mathsf{w},\mathsf{x}\in\mathfrak{g}$.

5.2.3 Action

Let $p = (g, \mathsf{u}, 0, 0) \in C^\infty(\Sigma, D)$ be a representative of a class in $K\backslash D$ in D. We would like to explicitly determine the corresponding action (2.14), namely

$$S_{2\mathrm{d}}(p) = \frac{1}{2}\int_\Sigma \big(\langle\!\langle p^{-1}\mathrm{d}p, \mathcal{E}\mathcal{P}_p(*p^{-1}\mathrm{d}p)\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle p^{-1}\mathrm{d}p, \mathcal{P}_p(p^{-1}\mathrm{d}p)\rangle\!\rangle_{\mathfrak{d}}\big) - \frac{1}{2} I_{\mathfrak{d}}^{\mathrm{WZ}}[p]. \tag{5.17}$$

First, we note that [66]

$$p^{-1}\mathrm{d}p = \big(g^{-1}\mathrm{d}g\big)^0 + \big(\operatorname{Ad}_g^{-1}\mathrm{d}\mathsf{u}\big)^1 - \frac{1}{2}\big(\operatorname{Ad}_g^{-1}[\mathsf{u}, \mathrm{d}\mathsf{u}]\big)^2 + \frac{1}{6}\big(\operatorname{Ad}_g^{-1}[\mathsf{u}, [\mathsf{u}, \mathrm{d}\mathsf{u}]]\big)^3.$$

Applying the operators (5.16) we then find

$$\begin{aligned}
\mathcal{P}_p\big(p^{-1}\mathrm{d}p\big) &= \big(g^{-1}\mathrm{d}g\big)^0 + \big(\operatorname{Ad}_g^{-1}\mathrm{d}\mathsf{u}\big)^1 + \frac{1}{ab}\big(g^{-1}\mathrm{d}g\big)^2 + \frac{a^2 - ab + b^2}{a^2b^2}\big(\operatorname{Ad}_g^{-1}\mathrm{d}\mathsf{u}\big)^3,\\
\mathcal{E}\mathcal{P}_p\big(*p^{-1}\mathrm{d}p\big) &= \frac{b-a}{ab}\big(\operatorname{Ad}_g^{-1}*\mathrm{d}\mathsf{u}\big)^2 + \frac{b-a}{a^2b^2}\big(*g^{-1}\mathrm{d}g\big)^3.
\end{aligned}$$

The first two terms in the action (5.17) then take the form

$$\begin{aligned}
\langle\!\langle p^{-1}\mathrm{d}p, \mathcal{E}\mathcal{P}_p\big(*p^{-1}\mathrm{d}p\big)\rangle\!\rangle_{\mathfrak{d}} &= (a-b)\langle \mathrm{d}gg^{-1}, *\mathrm{d}gg^{-1}\rangle + ab(a-b)\langle \mathrm{d}\mathsf{u}, *\mathrm{d}\mathsf{u}\rangle,\\
\langle\!\langle p^{-1}\mathrm{d}p, \mathcal{P}_p\big(p^{-1}\mathrm{d}p\big)\rangle\!\rangle_{\mathfrak{d}} &= -(a-b)^2\langle \mathrm{d}gg^{-1}, \mathrm{d}\mathsf{u}\rangle + \frac{a^2b^2}{2}\Big\langle [\mathsf{u}, \mathrm{d}\mathsf{u}], \mathrm{d}\mathsf{u} + \frac{1}{3}\big[\mathsf{u}, \mathrm{d}gg^{-1}\big]\Big\rangle.
\end{aligned}$$

On the other hand, we find that the Wess–Zumino 3-form is exact since

$$\begin{aligned}
\langle\!\langle \widehat{p}^{-1}\mathrm{d}\widehat{p}, \big[\widehat{p}^{-1}\mathrm{d}\widehat{p}, \widehat{p}^{-1}\mathrm{d}\widehat{p}\big]\rangle\!\rangle_{\mathfrak{d}} = \mathrm{d}\Big(&\big\langle \mathrm{d}\widehat{g}\widehat{g}^{-1}, 6\big(a^2 + b^2\big)\mathrm{d}\widehat{\mathsf{u}} - a^2b^2[\widehat{\mathsf{u}}, [\widehat{\mathsf{u}}, \mathrm{d}\widehat{\mathsf{u}}]]\big\rangle\\
&- a^2b^2\big\langle [\widehat{\mathsf{u}}, [\mathrm{d}\widehat{\mathsf{u}}, \mathrm{d}\widehat{\mathsf{u}}]]\big\rangle\Big).
\end{aligned}$$

Putting all of the above together we then arrive at the action

$$\begin{aligned}
S_{2\mathrm{d}}(g, \mathsf{u}) = \int_\Sigma \bigg(&\frac{1}{2}(a-b)\langle \mathrm{d}gg^{-1}, *\mathrm{d}gg^{-1}\rangle + \frac{1}{2}ab(a-b)\langle \mathrm{d}\mathsf{u}, *\mathrm{d}\mathsf{u}\rangle - ab\langle \mathrm{d}gg^{-1}, \mathrm{d}\mathsf{u}\rangle\\
&- \frac{1}{6}a^2b^2\langle \mathsf{u}, [\mathrm{d}\mathsf{u}, \mathrm{d}\mathsf{u}]\rangle\bigg).
\end{aligned} \tag{5.18}$$

It is interesting to note that in the limit $b \to 0$ we recover the principal chiral model action. In particular, the model with action (5.18) can be seen as a deformation of the principal chiral model to which a new $\mathfrak{g}$-valued field u is added. In fact, removing all the terms involving the field g from the above action we are left with the action of the pseudo-dual of the principal chiral model for the field u [14, 46, 68]. One may therefore view the action (5.18) as coupling together a principal chiral model field g and a pseudo-dual principal chiral model field u in an integrable way.

Note that the pseudo-dual of the principal chiral model was derived very recently in [4] starting from 6d holomorphic Chern–Simons theory. It was argued there that such an action could also be derived directly from 4d Chern–Simons theory where ω is taken to have a fourth order pole but is regular at infinity. By contrast, in the present work we explicitly required ω to have a double pole at infinity in (3.1) and then used the right diagonal gauge invariance by G in (3.12) to fix the corresponding edge modes at infinity in Section 3.6. We expect that by starting instead from a meromorphic 1-form ω with a fourth order pole at the origin and which is regular at infinity we would obtain a gauged version of the action (5.18). Moreover, after fixing the gauge invariance by setting $g = \mathrm{id}$ this action should reduce to that of the pseudo-dual of the principal chiral model field, as in [4].

5.2.4 Lax connection

The Lax connection (4.23) takes the form

$$\begin{aligned}\mathcal{L} &= \boldsymbol{p}\big(\mathcal{P}_p\big(p^{-1}\mathrm{d}p\big) - \mathcal{E}\mathcal{P}_p\big(*p^{-1}\mathrm{d}p\big)\big)\\ &= \boldsymbol{p}\bigg(j^0 + \mathrm{Ad}_g^{-1}\,\mathsf{du}^1 + \frac{1}{ab}j^2 + \frac{a-b}{ab}\,\mathrm{Ad}_g^{-1} *\mathsf{du}^2 + \frac{a^2-ab+b^2}{a^2b^2}\,\mathrm{Ad}_g^{-1}\,\mathsf{du}^3 + \frac{a-b}{a^2b^2} * j^3\bigg),\end{aligned}$$

where we have introduced the shorthand $j \coloneqq g^{-1}\mathrm{d}g$ and $\boldsymbol{p}$ is the inverse (4.22) of $\boldsymbol{j}_{\boldsymbol{z}'}$ defined in (4.1) given explicitly here by

$$\begin{aligned}\boldsymbol{p}\colon\ \mathsf{u}^0 + \mathsf{v}^1 + \mathsf{w}^2 + \mathsf{x}^3 \longmapsto{}& \frac{a^3(\mathsf{u}+a\mathsf{v}-b^2(\mathsf{w}+a\mathsf{x}))}{2(b^2-a^2)(z-a)} + \frac{a^3(-\mathsf{u}+a\mathsf{v}+b^2(\mathsf{w}-a\mathsf{x}))}{2(b^2-a^2)(z+a)}\\ &- \frac{b^3(\mathsf{u}+b\mathsf{v}-a^2(\mathsf{w}+b\mathsf{x}))}{2(b^2-a^2)(z-b)} - \frac{b^3(-\mathsf{u}+b\mathsf{v}+a^2(\mathsf{w}-b\mathsf{x}))}{2(b^2-a^2)(z+b)}.\end{aligned}$$

The above Lax connection therefore explicitly reads

$$\begin{aligned}\mathcal{L} &= \bigg(a^2\frac{j_+ + b\,\mathrm{Ad}_g^{-1}\,\partial_+\mathsf{u}}{(a+b)(a-z)} + b^2\frac{j_+ - a\,\mathrm{Ad}_g^{-1}\,\partial_+\mathsf{u}}{(a+b)(b+z)}\bigg)\mathrm{d}\sigma^+\\ &\quad + \bigg(a^2\frac{j_- - b\,\mathrm{Ad}_g^{-1}\,\partial_-\mathsf{u}}{(a+b)(a+z)} + b^2\frac{j_- + a\,\mathrm{Ad}_g^{-1}\,\partial_-\mathsf{u}}{(a+b)(b-z)}\bigg)\mathrm{d}\sigma^-.\end{aligned}$$

Note that in the limit $b \to 0$ we recover the Lax connection of the principal chiral model, in the form given in (5.10). This is in agreement with the observation made above about the action (5.18).

The flatness of $\mathcal{L}$ is equivalent to

$$\mathrm{d}j + \frac{1}{2}[j,j] = 0, \tag{5.19a}$$

$$\mathrm{d}\big(\mathrm{Ad}_g^{-1}\,\mathsf{du}\big) + \mathrm{Ad}_g^{-1}\big[\mathsf{du}, \mathrm{d}gg^{-1}\big] = 0, \tag{5.19b}$$

$$\mathrm{d} * \mathsf{du} + \frac{ab[\mathsf{du},\mathsf{du}]}{2(a-b)} + \frac{\big[\mathrm{d}gg^{-1}, \mathrm{d}gg^{-1}\big]}{2(a-b)} = 0, \tag{5.19c}$$

$$\mathrm{d}\big(*\mathrm{d}gg^{-1}\big) + \frac{ab}{a-b}\big[\mathsf{du}, \mathrm{d}gg^{-1}\big] = 0. \tag{5.19d}$$

The equations (5.19a) and (5.19b) are both identically true off-shell. The first is the Maurer–Cartan equation for j and the second holds because

$$\begin{aligned}\mathrm{d}\big(\mathrm{Ad}_g^{-1}\,\mathsf{du}\big) &= \mathrm{d}\big(g^{-1}\mathsf{du}g\big) = -g^{-1}\mathrm{d}gg^{-1}\wedge\mathsf{du}g - g^{-1}\mathsf{du}\wedge\mathrm{d}g\\ &= -\frac{1}{2}\,\mathrm{Ad}_g^{-1}\big(\big[\mathsf{du},\mathrm{d}gg^{-1}\big] + \big[\mathrm{d}gg^{-1},\mathsf{du}\big]\big) = -\,\mathrm{Ad}_g^{-1}\big[\mathsf{du},\mathrm{d}gg^{-1}\big].\end{aligned}$$

The equations (5.19c) and (5.19d) coincide with the equations of motion obtained from the action (5.18), as expected.

5.3 Real simple zeroes and poles

In this final section we discuss the example mentioned in the introduction. Let ω be given as in (1.1), namely

$$\omega = -\ell_1^\infty \frac{\prod_{i=1}^N (z-\zeta_i)}{\prod_{i=1}^N (z-z_i)}\mathrm{d}z,$$

where the poles and zeroes are all real and distinct. In the notation of Section 3.1 we then have $\boldsymbol{z} = \boldsymbol{z}_{\rm r} = \{z_i\}_{i=1}^N$ and $\boldsymbol{\zeta} = \boldsymbol{\zeta}_{\rm r} = \{\zeta_i\}_{i=1}^N$. And since all the poles and zeroes are simple we have $n_x = 1$ for all $x \in \boldsymbol{z}$ and $m_y = 1$ for all $y \in \boldsymbol{\zeta}$.

As in the introduction, we shall use the shorthand notation $\epsilon_i \coloneqq \epsilon_{\zeta_i}$ for every $i = 1, \ldots, N$ but leave these real non-zero parameters arbitrary.

Since all the poles of ω are real and simple, the defect Lie algebra (3.13) is simply given by the direct sum of Lie algebras $\mathfrak{d} = \mathfrak{g}^{\oplus N}$. The corresponding Lie group is $D = G^{\times N}$. As in the introduction, we shall leave the choice of Lagrangian subalgebra $\mathfrak{k} \subset \mathfrak{d}$ unspecified, and only assume that it satisfies the technical condition (2.3). We denote the corresponding connected Lie subgroup by $K \subset D$.

Since all the zeroes of ω are real and simple, the real vector space (4.3) associated with these zeroes is given here by $\mathfrak{g}^{(\boldsymbol{\zeta})} = \mathfrak{g}^{\oplus N}$.

The linear isomorphism $\mathcal{C} \colon \mathfrak{g}^{(\boldsymbol{\zeta})} = \mathfrak{g}^{\oplus N} \to \mathfrak{d} = \mathfrak{g}^{\oplus N}$ from Lemma 4.1 is given in components by (4.9), namely

$$C^i{}_j \coloneqq C^{[z_i,0]}{}_{(\zeta_j,0)} = \frac{1}{z_i - \zeta_j} \tag{5.20}$$

for $i, j = 1, \ldots, N$. These are simply the components of the usual Cauchy matrix. It is well know that the inverse of the Cauchy matrix (5.20) has components

$$\left(C^{-1}\right)^i{}_j = \frac{\prod_{r\neq i}(\zeta_r - z_j) \prod_r (z_r - \zeta_i)}{\prod_{r\neq j}(z_r - z_j) \prod_{r\neq i}(\zeta_r - \zeta_i)} \tag{5.21}$$

for $i, j = 1, \ldots, N$. In particular, since $\widetilde{\mathcal{E}} \colon \mathfrak{g}^{(\boldsymbol{\zeta})} = \mathfrak{g}^{\oplus N} \to \mathfrak{g}^{(\boldsymbol{\zeta})} = \mathfrak{g}^{\oplus N}$ defined in (4.7) is given in components by the diagonal matrix $\operatorname{diag}(\epsilon_1, \ldots, \epsilon_N)$, the components of the linear operator $\mathcal{E} \colon \mathfrak{d} = \mathfrak{g}^{\oplus N} \to \mathfrak{d} = \mathfrak{g}^{\oplus N}$ defined in (4.13) are given by

$$\sum_{j=1}^N C^i{}_j \epsilon_j \left(C^{-1}\right)^j{}_k = \sum_{j=1}^N \epsilon_j \frac{\prod_{r\neq j}(\zeta_r - z_k) \prod_{r\neq i}(z_r - \zeta_j)}{\prod_{r\neq k}(z_r - z_k) \prod_{r\neq j}(\zeta_r - \zeta_j)}.$$

These coincide with the components given in (1.3).

Finally, to compute the Lax connection using the general formula (4.23) we need to compute the inverse $\boldsymbol{p}$ in (4.22) of the map $\boldsymbol{j}_{\boldsymbol{z}'}$ defined in (4.1). The latter reads

$$\boldsymbol{j}_{\boldsymbol{z}'} \colon \sum_{j=1}^N \frac{\mathsf{u}_j}{z - \zeta_j} \longmapsto \left(\sum_{j=1}^N \frac{\mathsf{u}_j}{z_i - \zeta_j} \right)_{i=1}^N = \left(\sum_{j=1}^N C^i{}_j \mathsf{u}_j \right)_{i=1}^N,$$

where the equality is by definition (5.20) of the Cauchy matrix. Its inverse is then clearly given by

$$\boldsymbol{p} \colon (\mathsf{v}_i)_{i=1}^N \longmapsto \sum_{i,j=1}^N \frac{\left(C^{-1}\right)^i{}_j \mathsf{v}_j}{z - \zeta_i}. \tag{5.22}$$

According to (4.23), the Lax connection is now given by $\mathcal{L} = \boldsymbol{p}(\mathcal{J}_\sigma \mathrm{d}\sigma + \mathcal{J}_\tau \mathrm{d}\tau)$, where the $\mathfrak{d}$-valued fields $\mathcal{J}_\sigma, \mathcal{J}_\tau \in C^\infty(\Sigma, \mathfrak{d})$ are given in components by

$$\begin{aligned}
\mathcal{J}_\sigma &= (\mathcal{J}_\sigma^i)_{i=1}^N = l^{-1}\partial_\sigma l - \mathcal{E}^{-1}\mathcal{P}_l\big(\mathcal{E}\big(l^{-1}\partial_\sigma l\big) - l^{-1}\partial_\tau l\big), \\
\mathcal{J}_\tau &= (\mathcal{J}_\tau^i)_{i=1}^N = \mathcal{E}\big(l^{-1}\partial_\sigma l\big) - \mathcal{P}_l\big(\mathcal{E}\big(l^{-1}\partial_\sigma l\big) - l^{-1}\partial_\tau l\big).
\end{aligned}$$

In other words, using the explicit form (5.22) of the linear map $\boldsymbol{p}$ we deduce that the Lax connection reads

$$\mathcal{L} = \sum_{i,j=1}^{N} \frac{\left(C^{-1}\right)^{i}{}_{j}\left(\mathcal{J}^{j}_{\sigma}\mathrm{d}\sigma + \mathcal{J}^{j}_{\tau}\mathrm{d}\tau\right)}{z-\zeta_i},$$

which corresponds to the expression (1.4) from the introduction using the explicit inverse of the Cauchy matrix in (5.21).

6 Outlook

In this work we constructed a very broad family of integrable $\mathcal{E}$-models using the formalism of Costello–Yamazaki [13] by starting from the general 2d action obtained in [3]. There are a number of interesting open problems.

6.1 Hamiltonian formalism

In this work we focused entirely on constructing the actions of the new family of 2d integrable field theories. Indeed, the formalism of Costello–Yamazaki [13] is most convenient for describing integrable field theories in the Lagrangian formalism.

By contrast, 2d integrable field theories can be best described in the Hamiltonian formalism using the framework of classical dihedral affine Gaudin models proposed in [64], and further developed in [16, 41]. The formalisms of [13] and [64] were shown to be intimately related in [65] by performing a Hamiltonian analysis of 4d Chern–Simons theory. It would therefore be interesting to perform the Hamiltonian analysis of the family of integrable $\mathcal{E}$-model actions described in the present work. In particular, one should show that the Poisson bracket of the Lax matrix is of the Maillet r/s-form [44, 45] with twist function, which is equivalent to describing these models as classical dihedral affine Gaudin models. We will come back to this in a forthcoming paper [43].

6.2 Degenerate $\mathcal{E}$-model

An important restriction we imposed on the general setting of [3] is that the 1-form ω had a double pole at infinity. This allowed us in Section 3.6 to partially fix the gauge invariance of the 2d action of [3], by bringing the component of the edge mode at infinity to the identity.

It would be natural to try to extend our construction to the general setting of [3] by allowing arbitrary orders at all the poles of ω. The resulting 2d integrable field theory would have an additional gauge invariance and so it is natural to expect that this generalisation would lead to an integrable family of the class of degenerate $\mathcal{E}$-models introduced in [38], see also [34, 35, 56, 59], which, in particular, would include by [35] the bi-Yang–Baxter model with WZ-term [15].

6.3 Integrable $\mathcal{E}$-model hierarchy

A crucial step in our analysis was imposing the condition (4.6) on the coefficients in the partial fraction decomposition of the components of the Lax connection $\mathcal{L}$. Indeed, this condition gave a particular way of satisfying the admissibility condition (b) from Section 3.5 and we showed that within this class of admissible 1-forms $\mathcal{L}$ there was a unique solution $\mathcal{L} = \mathcal{L}(l)$ to the boundary equation (3.16) relating $\mathcal{L}$ to the edge mode $l \in C^{\infty}(\Sigma, D)$. Moreover, the condition (4.6) is at the origin of the introduction of the operator $\mathcal{E}$ in our construction.

However, (4.6) is by no means the only way to solve the admissibility condition (b), and it would be very interesting to explore other classes of admissible 1-forms $\mathcal{L}$. In the case when the zeroes of ω are all simple, an obvious alternative way to solve the admissibility condition (b) is to work in a representation of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, pick $n \in \mathbb{Z}_{\geq 1}$ and set

$$\mathcal{L}_{t_n}^{(y,0)} = \epsilon_{n,y} \big(\mathcal{L}_{\sigma}^{(y,0)} \big)^n, \tag{6.1}$$

for every $y \in \boldsymbol{\zeta}$ and some choice of $\epsilon_{n,y}$. On the left hand side we used the notation t_n instead of τ for the time coordinate since we expect the corresponding model to be related to a different flow in the same hierarchy. Indeed, the above solution (6.1) of the admissibility condition (b) is motivated by the expressions for the Lax matrices inducing higher flows in the integrable hierarchies of affine Gaudin models [42]. Explicitly, when $\mathfrak{g}^{\mathbb{C}}$ is of type B, for instance, to each simple zero $y \in \boldsymbol{\zeta}$ and every odd positive integer n is associated a higher flow ∂_{t_n} with corresponding Lax matrix

$$\mathcal{L}_{t_n} = \frac{\big(\mathcal{L}_{\sigma}^{(y,0)} \big)^n}{z - y}.$$

We therefore expect from [42] that the Lax connection defined by imposing (6.1) corresponds to a higher flow of the same integrable $\mathcal{E}$-model hierarchy. It would be very interesting to investigate this further. In particular, we expect from [42], see also [21, 22, 23], that when $\mathfrak{g}^{\mathbb{C}}$ is of type B, C or D the condition (6.1) should give non-trivial commuting flows for all odd $n \in \mathbb{Z}_{\geq 1}$, corresponding to the set of exponents of the (untwisted) affine Kac–Moody algebra associated with $\mathfrak{g}^{\mathbb{C}}$. In type A we also expect that one should have to modify the ansatz (6.1) accordingly to produce commuting flows [21, 22, 23, 42].

6.4 3d Chern–Simons theory

The $\mathcal{E}$-model on the infinite cyclinder $S^1 \times \mathbb{R}$ was shown in [52] to arise from 3d Chern–Simons theory for the Lie group D on the solid cylinder $\bigcirc \times \mathbb{R}$, with $\bigcirc$ a disc, by imposing a suitable boundary condition on the gauge field at the boundary $\partial \bigcirc \simeq S^1$. Moreover, the σ-model on $K \backslash D$ was also obtained from 3d Chern–Simons theory on a hollowed out cylinder $\circledcirc \times \mathbb{R}$, with $\circledcirc$ an annulus, by imposing the same boundary condition as before on the gauge field at the outer boundary and another topological boundary condition depending on the choice of Lagrangian subalgebra $\mathfrak{k} \subset \mathfrak{d}$ at the inner boundary.

It would be interesting to understand if, in the integrable case, there is a relation between the above description of the σ-model on $K \backslash D$ from 3d Chern–Simons derived in [52] and the description from 4d Chern–Simons theory obtained here.

Another possible connection to 3d Chern–Simons theory is suggested by the results of [49, 50] where the action of the λ-deformation [57] of the principal chiral model, in the form of the universal 2d action (3.19), was obtained from a certain "doubled" version of 3d Chern–Simons theory on $\bigcirc \times \mathbb{R}$.

A Proof of Proposition 2.3

As noted in Section 2.1, the direct sum (2.3) is orthogonal with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{d},\mathcal{E}}$. The corresponding projector $\mathcal{P}_l$ is then symmetric with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{d},\mathcal{E}}$. Hence, for any $\mathsf{U}, \mathsf{V} \in \mathfrak{d}$, we have

$$\langle\!\langle \mathsf{U}, {}^t\mathcal{P}_l \mathsf{V} \rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle \mathcal{P}_l \mathsf{U}, \mathsf{V} \rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle \mathcal{P}_l \mathsf{U}, \mathcal{E} \mathsf{V} \rangle\!\rangle_{\mathfrak{d},\mathcal{E}} = \langle\!\langle \mathsf{U}, \mathcal{P}_l \mathcal{E} \mathsf{V} \rangle\!\rangle_{\mathfrak{d},\mathcal{E}} = \langle\!\langle \mathsf{U}, \mathcal{E}^{-1} \mathcal{P}_l \mathcal{E} \mathsf{V} \rangle\!\rangle_{\mathfrak{d}},$$

where we used the symmetry of $\mathcal{P}_l$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{d},\mathcal{E}}$ in the third step. Part (i) now follows from the non-degeneracy of $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{d}}$.

Using this result and the definition (2.5) of $\overline{\mathcal{P}}_l$, we then get

$$\mathcal{P}_l \mathcal{E} + \mathcal{E}\overline{\mathcal{P}}_l = \mathcal{P}_l \mathcal{E} - \mathcal{E}\,{}^t\mathcal{P}_l + \mathcal{E} = \mathcal{E}.$$

This is the first equation in (ii). The second one is simply obtained by multiplying it on both sides by $\mathcal{E}^{-1}$.

Part (iii) is easily proved by observing that ${}^t\overline{\mathcal{P}}_l\,\mathcal{P}_l = (\mathrm{id} - \mathcal{P}_l)\mathcal{P}_l = 0$ since $\mathcal{P}_l$ is a projector, and similarly for ${}^t\mathcal{P}_l\overline{\mathcal{P}}_l = 0$.

Let us now prove (iv). Let $\mathsf{U}, \mathsf{V} \in \mathfrak{d}$. From the definition (2.5) of $\overline{\mathcal{P}}_l$, we have

$$\begin{aligned}
\langle\!\langle \big(\mathcal{P}_l - \overline{\mathcal{P}}_l\big)\mathsf{U}, \mathsf{V}\rangle\!\rangle_{\mathfrak{d}} &= \langle\!\langle \mathcal{P}_l\mathsf{U}, \mathsf{V}\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle \mathsf{U}, (\mathrm{id} - \mathcal{P}_l)\mathsf{V}\rangle\!\rangle_{\mathfrak{d}} \\
&= \langle\!\langle \mathcal{P}_l\mathsf{U}, \mathcal{P}_l\mathsf{V} + (\mathrm{id} - \mathcal{P}_l)\mathsf{V}\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle \mathcal{P}_l\mathsf{U} + (\mathrm{id} - \mathcal{P}_l)\mathsf{U}, (\mathrm{id} - \mathcal{P}_l)\mathsf{V}\rangle\!\rangle_{\mathfrak{d}} \\
&= \langle\!\langle \mathcal{P}_l\mathsf{U}, \mathcal{P}_l\mathsf{V}\rangle\!\rangle_{\mathfrak{d}} - \langle\!\langle (\mathrm{id} - \mathcal{P}_l)\mathsf{U}, (\mathrm{id} - \mathcal{P}_l)\mathsf{V}\rangle\!\rangle_{\mathfrak{d}}.
\end{aligned}$$

Clearly $(\mathrm{id} - \mathcal{P}_l)\mathsf{U}$ and $(\mathrm{id} - \mathcal{P}_l)\mathsf{V}$ belong to $\ker \mathcal{P}_l = \mathrm{Ad}_l^{-1}\,\mathfrak{k}$. By the ad-invariance of $\langle\!\langle \cdot, \cdot\rangle\!\rangle_{\mathfrak{d}}$ and the isotropy of $\mathfrak{k}$, we thus get $\langle\!\langle (\mathrm{id} - \mathcal{P}_l)\mathsf{U}, (\mathrm{id} - \mathcal{P}_l)\mathsf{V}\rangle\!\rangle_{\mathfrak{d}} = 0$, leaving only the first term in the above equation. Moreover, a similar computation can be performed with $\mathcal{P}_l$ and $\overline{\mathcal{P}}_l$ exchanged. In the end, we get

$$\langle\!\langle (\mathcal{P}_l - \overline{\mathcal{P}}_l)\mathsf{U}, \mathsf{V}\rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle \mathcal{P}_l\mathsf{U}, \mathcal{P}_l\mathsf{V}\rangle\!\rangle_{\mathfrak{d}} = -\langle\!\langle \overline{\mathcal{P}}_l\mathsf{U}, \overline{\mathcal{P}}_l\mathsf{V}\rangle\!\rangle_{\mathfrak{d}}. \tag{A.1}$$

Bringing all operators on the left-hand side of the bilinear form $\langle\!\langle \cdot, \cdot\rangle\!\rangle_{\mathfrak{d}}$ and using its non-degeneracy, we get $\mathcal{P}_l - \overline{\mathcal{P}}_l = {}^t\mathcal{P}_l\mathcal{P}_l = -{}^t\overline{\mathcal{P}}_l\overline{\mathcal{P}}_l$, proving part (iv).

Part (v) then follows from

$$\mathcal{P}_l{}^t\mathcal{P}_l = \big(1 - {}^t\overline{\mathcal{P}}_l\big)\big(1 - \overline{\mathcal{P}}_l\big) = 1 - {}^t\overline{\mathcal{P}}_l - \overline{\mathcal{P}}_l + {}^t\overline{\mathcal{P}}_l\overline{\mathcal{P}}_l = \mathcal{P}_l - \overline{\mathcal{P}}_l + \overline{\mathcal{P}}_l - \mathcal{P}_l = 0, \tag{A.2}$$

and from a similar computation for $\overline{\mathcal{P}}_l{}^t\overline{\mathcal{P}}_l$.

Finally, we note that $\mathcal{P}_l\mathsf{U}$ and $\mathcal{P}_l\mathsf{V}$ in (A.1) belong to $\mathrm{im}\,\mathcal{P}_l = \mathcal{E}\,\mathrm{Ad}_l^{-1}\,\mathfrak{k}$. If $\mathcal{E}^2 = \mathrm{id}$, the latter is an isotropic subspace and hence $\langle\!\langle \big(\mathcal{P}_l - \overline{\mathcal{P}}_l\big)\mathsf{U}, \mathsf{V}\rangle\!\rangle_{\mathfrak{d}} = \langle\!\langle \mathcal{P}_l\mathsf{U}, \mathcal{P}_l\mathsf{V}\rangle\!\rangle_{\mathfrak{d}} = 0$. So (vi) follows from the non-degeneracy of $\langle\!\langle \cdot, \cdot\rangle\!\rangle_{\mathfrak{d}}$.

B Proof of Proposition 2.7

It is convenient to first consider a general model describing a field $l \in C^\infty(\Sigma, D)$, with an action of the form

$$\begin{aligned}
S(l) = \int_\Sigma \bigg(&\frac{1}{2}\langle\!\langle l^{-1}\partial_\tau l, \mathcal{O}^{\tau\tau} l^{-1}\partial_\tau l\rangle\!\rangle_{\mathfrak{d}} - \frac{1}{2}\langle\!\langle l^{-1}\partial_\sigma l, \mathcal{O}^{\sigma\sigma} l^{-1}\partial_\sigma l\rangle\!\rangle_{\mathfrak{d}} \\
&+ \langle\!\langle l^{-1}\partial_\tau l, \mathcal{O}^{\tau\sigma} l^{-1}\partial_\sigma l\rangle\!\rangle_{\mathfrak{d}} \bigg) \mathrm{d}\sigma \wedge \mathrm{d}\tau - \frac{1}{2} I_{\mathfrak{d}}^{\mathrm{WZ}}[l],
\end{aligned} \tag{B.1}$$

where $\mathcal{O}^{\tau\tau}$, $\mathcal{O}^{\sigma\sigma}$ and $\mathcal{O}^{\tau\sigma}$ are linear operators on $\mathfrak{d}$, which can depend on the field l but not on its derivatives. Without loss of generality, we can suppose $\mathcal{O}^{\tau\tau}$ and $\mathcal{O}^{\sigma\sigma}$ symmetric with respect to $\langle\!\langle \cdot, \cdot\rangle\!\rangle_{\mathfrak{d}}$. Note that the model (B.1) is relativistic if and only if $\mathcal{O}^{\tau\tau} = \mathcal{O}^{\sigma\sigma}$ and $\mathcal{O}^{\tau\sigma}$ is skew-symmetric with respect to $\langle\!\langle \cdot, \cdot\rangle\!\rangle_{\mathfrak{d}}$. The goal of this appendix is to prove the following result.

Lemma B.1. *The components of the energy-momentum tensor of the model* (B.1) *are given by*

$$\begin{aligned}
T^\tau{}_\tau &= -T^\sigma{}_\sigma = \frac{1}{2}\langle\!\langle l^{-1}\partial_\tau l, \mathcal{O}^{\tau\tau} l^{-1}\partial_\tau l\rangle\!\rangle_{\mathfrak{d}} + \frac{1}{2}\langle\!\langle l^{-1}\partial_\sigma l, \mathcal{O}^{\sigma\sigma} l^{-1}\partial_\sigma l\rangle\!\rangle_{\mathfrak{d}}, \\
T^\tau{}_\sigma &= \langle\!\langle l^{-1}\partial_\tau l, \mathcal{O}^{\tau\tau} l^{-1}\partial_\sigma l\rangle\!\rangle_{\mathfrak{d}} + \langle\!\langle l^{-1}\partial_\sigma l, \mathcal{O}^{\tau\sigma} l^{-1}\partial_\sigma l\rangle\!\rangle_{\mathfrak{d}}, \\
T^\sigma{}_\tau &= -\langle\!\langle l^{-1}\partial_\tau l, \mathcal{O}^{\sigma\sigma} l^{-1}\partial_\sigma l\rangle\!\rangle_{\mathfrak{d}} + \langle\!\langle l^{-1}\partial_\tau l, \mathcal{O}^{\tau\sigma} l^{-1}\partial_\tau l\rangle\!\rangle_{\mathfrak{d}}.
\end{aligned}$$

Proof. Let us fix a basis $\{I_A\}_{A=1,\dots,\dim\mathfrak{d}}$ of the Lie algebra $\mathfrak{d}$ and a choice of local coordinates $\{y^M\}_{M=1,\dots,\dim\mathfrak{d}}$ on the group manifold D, parametrising the element l in D. We define the vielbeins ${e^A}_M$ through the decomposition

$$l^{-1}\frac{\partial l}{\partial y^M} = {e^A}_M I_A.$$

We may then express the components $l^{-1}\partial_\mu l$ for $\mu=\tau,\sigma$ of the Maurer–Cartan current as

$$l^{-1}\partial_\mu l = {e^A}_M\, \partial_\mu y^M\, I_A. \tag{B.2}$$

For any linear operator $\mathcal{O}$ on $\mathfrak{d}$, let us also define

$$\mathcal{O}_{AB} \coloneqq \langle\langle I_A, \mathcal{O} I_B\rangle\rangle_{\mathfrak{d}}.$$

The symmetry of $\mathcal{O}^{\tau\tau}$ and $\mathcal{O}^{\sigma\sigma}$ then translates to the fact that $\mathcal{O}^{\tau\tau}_{AB}=\mathcal{O}^{\tau\tau}_{BA}$ and $\mathcal{O}^{\sigma\sigma}_{AB}=\mathcal{O}^{\sigma\sigma}_{BA}$ for every $A,B=1,\dots,\dim\mathfrak{d}$. Using the above definitions, we then rewrite the action (B.1) as the integral over Σ of the Lagrangian density

$$L = {e^A}_M {e^B}_N \bigg(\frac{1}{2}\mathcal{O}^{\tau\tau}_{AB}\, \partial_\tau y^M \partial_\tau y^N - \frac{1}{2}\mathcal{O}^{\sigma\sigma}_{AB}\, \partial_\sigma y^M \partial_\sigma y^N + (\mathcal{O}^{\tau\sigma}_{AB} + W_{AB})\, \partial_\tau y^M \partial_\sigma y^N \bigg),$$

where $W_{AB}=-W_{BA}$ describes the contribution of the Wess–Zumino term to the action (such a rewriting of the Wess–Zumino term as a two-dimensional integral is always possible, at least locally — as we shall see, an explicit expression for W_{AB} will not be needed in what follows).

The energy-momentum tensor of the model can be computed explicitly in terms of the Lagrangian density L as

$$T^\mu{}_\nu = \frac{\partial L}{\partial(\partial_\mu y^M)}\partial_\nu y^M - \delta^\mu{}_\nu\, L. \tag{B.3}$$

From the above expression of L, the symmetry of $\mathcal{O}^{\tau\tau}_{AB}$ and $\mathcal{O}^{\sigma\sigma}_{AB}$ and the skew-symmetry of W_{AB}, we get

$$\frac{\partial L}{\partial(\partial_\tau y^M)} = {e^A}_M {e^B}_N \big(\mathcal{O}^{\tau\tau}_{AB}\partial_\tau y^N + (\mathcal{O}^{\tau\sigma}_{AB} + W_{AB})\partial_\sigma y^N \big), \tag{B.4a}$$

$$\frac{\partial L}{\partial(\partial_\sigma y^M)} = {e^A}_M {e^B}_N \big(-\mathcal{O}^{\sigma\sigma}_{AB}\partial_\sigma y^N + (\mathcal{O}^{\tau\sigma}_{BA} - W_{AB})\partial_\tau y^N \big). \tag{B.4b}$$

We then deduce that

$$T^\tau{}_\tau = \frac{1}{2}{e^A}_M {e^B}_N \big(\mathcal{O}^{\tau\tau}_{AB}\, \partial_\tau y^M \partial_\tau y^N + \mathcal{O}^{\sigma\sigma}_{AB}\, \partial_\sigma y^M \partial_\sigma y^N \big).$$

Note in particular that the parts proportional to $\mathcal{O}^{\tau\sigma}_{AB}+W_{AB}$ coming from the two terms in the right-hand side of (B.3) cancel for $\mu=\nu=\tau$. A similar computation yields $T^\sigma{}_\sigma = -T^\tau{}_\tau$. In terms of the Maurer–Cartan currents (B.2), we find

$$T^\tau{}_\tau = -T^\sigma{}_\sigma = \frac{1}{2}\langle\langle l^{-1}\partial_\tau l, \mathcal{O}^{\tau\tau} l^{-1}\partial_\tau l\rangle\rangle_{\mathfrak{d}} + \frac{1}{2}\langle\langle l^{-1}\partial_\sigma l, \mathcal{O}^{\sigma\sigma} l^{-1}\partial_\sigma l\rangle\rangle_{\mathfrak{d}}.$$

Let us now compute the crossed-term $T^\tau{}_\sigma$. Substituting the expression (B.4a) in (B.3), we obtain

$$T^\tau{}_\sigma = {e^A}_M {e^B}_N \big(\mathcal{O}^{\tau\tau}_{AB}\, \partial_\sigma y^M \partial_\tau y^N + (\mathcal{O}^{\tau\sigma}_{AB} + W_{AB})\, \partial_\sigma y^M \partial_\sigma y^N \big).$$

The term containing W_{AB} vanishes, as it is given by the contraction of the symmetric tensor $e^A{}_M e^B{}_N \partial_\sigma y^M \partial_\sigma y^N$ with the skew-symmetric tensor W_{AB}. One then rewrites the resulting expression in terms of the Maurer–Cartan currents (B.2) as

$$T^\tau{}_\sigma = \langle\!\langle l^{-1}\partial_\tau l, \mathcal{O}^{\tau\tau} l^{-1}\partial_\sigma l \rangle\!\rangle_{\mathfrak{d}} + \langle\!\langle l^{-1}\partial_\sigma l, \mathcal{O}^{\tau\sigma} l^{-1}\partial_\sigma l \rangle\!\rangle_{\mathfrak{d}}.$$

A similar computation leads to the announced expression of $T^\tau{}_\sigma$. ■

Let us now turn to the proof of Proposition 2.7. In the above notations, the action (2.14) corresponds to the choice of operators

$$\mathcal{O}^{\tau\tau} = \mathcal{E}^{-1}\mathcal{P}_l, \qquad \mathcal{O}^{\sigma\sigma} = \mathcal{E}\overline{\mathcal{P}}_l \qquad \text{and} \qquad \mathcal{O}^{\tau\sigma} = \overline{\mathcal{P}}_l - {}^t\mathcal{P}_l. \tag{B.5}$$

Applying Lemma B.1, we then get an explicit expression of the components $T^\mu{}_\nu$ of the energy-momentum tensor. For instance, we have

$$T^\tau{}_\tau = -T^\sigma{}_\sigma = \frac{1}{2}\langle\!\langle l^{-1}\partial_\tau l, \mathcal{E}^{-1}\mathcal{P}_l\, l^{-1}\partial_\tau l \rangle\!\rangle_{\mathfrak{d}} + \frac{1}{2}\langle\!\langle l^{-1}\partial_\sigma l, \mathcal{E}\overline{\mathcal{P}}_l\, l^{-1}\partial_\sigma l \rangle\!\rangle_{\mathfrak{d}}.$$

On the other hand, from the definition (2.16) of $\mathcal{J}_\sigma$ and the symmetry of $\mathcal{E}$, we get

$$\begin{aligned}\frac{1}{2}\langle\!\langle \mathcal{J}_\sigma, \mathcal{E}\mathcal{J}_\sigma \rangle\!\rangle_{\mathfrak{d}} = {}& \frac{1}{2}\langle\!\langle \mathcal{P}_l\, l^{-1}\partial_\tau l, \mathcal{E}^{-1}\mathcal{P}_l\, l^{-1}\partial_\tau l \rangle\!\rangle_{\mathfrak{d}} + \frac{1}{2}\langle\!\langle \overline{\mathcal{P}}_l\, l^{-1}\partial_\sigma l, \mathcal{E}\overline{\mathcal{P}}_l\, l^{-1}\partial_\sigma l \rangle\!\rangle_{\mathfrak{d}} \\ &+ \langle\!\langle \mathcal{P}_l\, l^{-1}\partial_\tau l, \overline{\mathcal{P}}_l\, l^{-1}\partial_\sigma l \rangle\!\rangle_{\mathfrak{d}}.\end{aligned}$$

The last line vanishes due to part (iii) of Proposition 2.3. Moreover, from part (i), we have

$${}^t\mathcal{P}_l\mathcal{E}^{-1}\mathcal{P}_l = \mathcal{E}^{-1}\mathcal{P}_l^2 = \mathcal{E}^{-1}\mathcal{P}_l,$$

where we have used the fact that $\mathcal{P}_l$ is a projector and thus that $\mathcal{P}_l^2 = \mathcal{P}_l$. A similar computation yields ${}^t\overline{\mathcal{P}}_l\mathcal{E}\overline{\mathcal{P}}_l = \mathcal{E}\overline{\mathcal{P}}_l$. Thus, we get

$$\frac{1}{2}\langle\!\langle \mathcal{J}_\sigma, \mathcal{E}\mathcal{J}_\sigma \rangle\!\rangle_{\mathfrak{d}} = T^\tau{}_\tau = -T^\sigma{}_\sigma.$$

Let us now turn our attention to $T^\tau{}_\sigma$. Applying Lemma B.1 with the operators (B.5), we get

$$T^\tau{}_\sigma = \langle\!\langle l^{-1}\partial_\tau l, \mathcal{E}^{-1}\mathcal{P}_l\, l^{-1}\partial_\sigma l \rangle\!\rangle_{\mathfrak{d}} + \langle\!\langle l^{-1}\partial_\sigma l, \big(\overline{\mathcal{P}}_l - \mathcal{P}_l\big) l^{-1}\partial_\sigma l \rangle\!\rangle_{\mathfrak{d}},$$

where in particular we transposed the operator ${}^t\mathcal{P}_l$ in the last term. On the other hand, it follows from (2.16) that

$$\begin{aligned}\frac{1}{2}\langle\!\langle \mathcal{J}_\sigma, \mathcal{J}_\sigma \rangle\!\rangle_{\mathfrak{d}} = {}& \frac{1}{2}\langle\!\langle \mathcal{E}^{-1}\mathcal{P}_l\, l^{-1}\partial_\tau l, \mathcal{E}^{-1}\mathcal{P}_l\, l^{-1}\partial_\tau l \rangle\!\rangle_{\mathfrak{d}} + \frac{1}{2}\langle\!\langle \overline{\mathcal{P}}_l\, l^{-1}\partial_\sigma l, \overline{\mathcal{P}}_l\, l^{-1}\partial_\sigma l \rangle\!\rangle_{\mathfrak{d}} \\ &+ \langle\!\langle \mathcal{E}^{-1}\mathcal{P}_l\, l^{-1}\partial_\tau l, \overline{\mathcal{P}}_l\, l^{-1}\partial_\sigma l \rangle\!\rangle_{\mathfrak{d}}.\end{aligned}$$

Note from part (i) of Proposition 2.3 that $\mathcal{E}^{-1}\mathcal{P}_l$ is symmetric. Using also part (v) we get

$${}^t\big(\mathcal{E}^{-1}\mathcal{P}_l\big)\mathcal{E}^{-1}\mathcal{P}_l = \mathcal{E}^{-1}\mathcal{P}_l\,{}^t\big(\mathcal{E}^{-1}\mathcal{P}_l\big) = \mathcal{E}^{-1}\mathcal{P}_l{}^t\mathcal{P}_l\mathcal{E}^{-1} = 0,$$

so that the first term in $\frac{1}{2}\langle\!\langle \mathcal{J}_\sigma, \mathcal{J}_\sigma \rangle\!\rangle_{\mathfrak{d}}$ vanishes. Using the same properties, we also get

$${}^t\big(\mathcal{E}^{-1}\mathcal{P}_l\big)\overline{\mathcal{P}}_l = \mathcal{E}^{-1}\mathcal{P}_l\overline{\mathcal{P}}_l = \mathcal{E}^{-1}\big(\mathcal{P}_l - \mathcal{P}_l{}^t\mathcal{P}_l\big) = \mathcal{E}^{-1}\mathcal{P}_l.$$

Finally, using ${}^t\overline{\mathcal{P}}_l\overline{\mathcal{P}}_l = \overline{\mathcal{P}}_l - \mathcal{P}_l$ (see part (iv) of Proposition 2.3), we find that

$$\frac{1}{2}\langle\!\langle \mathcal{J}_\sigma, \mathcal{J}_\sigma \rangle\!\rangle_{\mathfrak{d}} = T^\tau{}_\sigma.$$

A similar computation yields $-\frac{1}{2}\langle\!\langle \mathcal{J}_\sigma, \mathcal{E}^2\mathcal{J}_\sigma \rangle\!\rangle_{\mathfrak{d}} = T^\sigma{}_\tau$.

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