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Deformations of Pre-Symplectic Structures: a Dirac Geometry Approach

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Abstract. We explain the geometric origin of the L_∞-algebra controlling deformations of pre-symplectic structures.

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1 Introduction

A pre-symplectic form is just a closed 2-form of constant rank. For instance, the restriction of a symplectic form to a coisotropic submanifold (such as the zero level set of a moment map) is pre-symplectic. Given a pre-symplectic from η of rank k, we constructed in [7] an algebraic structure that encodes the deformations of η, i.e., the 2-forms nearby η (in the C^0-sense) which are both closed and of constant rank k. As in many deformation problems, this algebraic structure is an L_∞-algebra, which we call the *Koszul* L_∞*-algebra* of η. Its construction – which is somewhat involved due to the simultaneous presence of the closedness and constant rank conditions – relies on a certain BV_∞-algebra structure on the differential forms and builds on the work of Fiorenza–Manetti [1]. The Koszul L_∞-algebra has the property that its Maurer–Cartan elements are in bijection with the pre-symplectic deformations of η.

Given that pre-symplectic forms are geometric objects, it is natural to ask for a geometric derivation of the algebraic structure that governs their deformations (the Koszul L_∞-algebra). The present note provides an answer to this question. The idea is the following: instead of restricting oneself to the realm of 2-forms, work in the larger class of almost Dirac structures, and consider deformations of

$$\mathrm{graph}(\eta) := \{(v, \eta(v,\cdot)) \,|\, v \in TM\} \subset TM \oplus T^*M$$

within the Dirac structures satisfying a constant rank condition. This is explained in Section 3.2, which is the heart of this note.

The first step in [7] is to provide a parametrization of the constant rank forms nearby η in terms of (an open subset in) a vector space. This parametrization is obtained naturally by taking the point of view of Dirac linear algebra in Section 3.3.

The second step in [7] was to show that the closedness condition translates into a Maurer–Cartan equation for a suitable L_∞-algebra. In Section 3.4 we re-obtain this result, and further we improve slightly a result of [7], see our Corollary 2.9.

Combining these results, in Section 3.5 we recover the fact that the L_∞-algebra governing deformations of Dirac structures, in the case at hand and upon a suitable restriction, is the Koszul L_∞-algebra.

The Koszul L_∞-algebra depends on an auxiliary choice of a distribution transverse to $\ker(\eta)$. In the Dirac-geometric interpretation, this translates into a suitable choice of a complement of $\mathrm{graph}(\eta)$ in $TM \oplus T^*M$. One of the achievements of [3] is to establish a general framework to control the effects of changing the complement, exhibiting explicit canonical L_∞-isomorphisms between the corresponding L_∞-algebras. A consequence of this note and of [3] is that the Koszul L_∞-algebra of (M, η) is well-defined up to L_∞-isomorphisms.

2 Review: deformations of pre-symplectic structures

We review the results on deformations of pre-symplectic structures obtained in the first three sections of [7].

2.1 Pre-symplectic structures

Fix a smooth manifold M.

Definition 2.1. A 2-form η on M is called *pre-symplectic* if

1) η is closed,
2) the vector bundle map $\eta^\sharp \colon TM \to T^*M, v \mapsto \iota_v\eta = \eta(v, \cdot)$ has constant rank.

A *pre-symplectic manifold* is a pair (M, η) consisting of a manifold M and a pre-symplectic structure η on M. We denote the space of all pre-symplectic structures of rank k on M by $\mathsf{Pre\text{-}Sym}^k(M)$.

A pre-symplectic manifold (M, η) gives rise to a distribution

$$K := \ker\left(\eta^\sharp\right).$$

This distribution is involutive since η is closed, hence K is tangent to a foliation of M. Denote by $r \colon \Omega(M) \to \Gamma(\wedge K^*)$ the restriction map. We define the horizontal differential forms as the elements of

$$\Omega_{\mathrm{hor}}(M) := \ker(r).$$

They form a subcomplex of the de Rham complex $\Omega(M)$, since the de Rham differential commutes with the pullback of differential forms. The subcomplex $\Omega_{\mathrm{hor}}(M)$ is the multiplicative ideal of $\Omega(M)$ generated by $\Gamma(K^\circ)$, where $K^\circ \subset T^*M$ denotes the annihilator of K.

2.2 A parametrization of constant rank 2-forms

In this subsection we fix a finite-dimensional, real vector space V. Recall that a bivector $Z \in \wedge^2 V$ is encoded by the induced linear map

$$Z^\sharp \colon\ V^* \to V, \qquad \xi \mapsto \iota_\xi Z = Z(\xi, \cdot).$$

Define

$$\mathcal{I}_Z := \left\{\beta \in \wedge^2 V^* \colon \mathrm{id}_V + Z^\sharp\beta^\sharp \text{ is invertible}\right\},$$

an open neighborhood of the origin in $\wedge^2 V^*$. Let $F\colon \mathcal{I}_Z \to \wedge^2 V^*$ be the map determined by

$$(F(\beta))^\sharp = \beta^\sharp\big(\mathrm{id} + Z^\sharp\beta^\sharp\big)^{-1}. \tag{2.1}$$

The map F is non-linear and smooth. It is a diffeomorphism from $\mathcal{I}_Z$ to $\mathcal{I}_{-Z}$, which keeps the origin fixed.

Fix $\eta \in \wedge^2 V^*$ of rank k. We now use F to construct submanifold charts for the space $\big(\wedge^2 V^*\big)_k$ of skew-symmetric bilinear forms on V of rank k. Fix a subspace $G \subset V$ such that $K \oplus G = V$, where $K = \ker\big(\eta^\sharp\big)$. The restriction of η to G is a non-degenerate skew bilinear form, therefore there is a unique $Z \in \wedge^2 G \subset \wedge^2 V$ such that

$$Z^\sharp\colon\ G^* \to G, \qquad \xi \mapsto \iota_\xi Z = Z(\xi,\cdot)$$

equals $-\big(\eta|_G^\sharp\big)^{-1}$.

Definition 2.2. The *Dirac exponential map* $\exp_\eta$ of η (and for fixed G) is the mapping

$$\exp_\eta\colon\ \mathcal{I}_Z \to \wedge^2 V^*, \qquad \beta \mapsto \eta + F(\beta).$$

Let $r\colon \wedge^2 V^* \to \wedge^2 K^*$ be the restriction map; we have the natural identification $\ker(r) \cong \wedge^2 G^* \oplus (G^* \otimes K^*)$. By the following theorem [7, Theorem 2.6], the restriction of $\exp_\eta$ to $\ker(r)$ is a submanifold chart for $\big(\wedge^2 V^*\big)_k \subset \wedge^2 V^*$.

Theorem 2.3 (parametrizing constant rank forms)**.**

(i) *Let $\beta \in \mathcal{I}_Z$. Then $\exp_\eta(\beta)$ lies in $\big(\wedge^2 V^*\big)_k$ if, and only if, β lies in* $\ker(r) = (K^* \otimes G^*) \oplus \wedge^2 G^*$.

(ii) *Let $\beta = (\mu,\sigma) \in \mathcal{I}_Z \cap \big((K^* \otimes G^*) \oplus \wedge^2 G^*\big)$. Then $\exp_\eta(\beta)$ is the unique skew-symmetric bilinear form on V with the following properties:*

- *its restriction to G equals $(\eta + F(\sigma))|_{\wedge^2 G}$;*
- *its kernel is the graph of the map $Z^\sharp\mu^\sharp = -\big(\eta|_G^\sharp\big)^{-1}\mu^\sharp\colon K \to G$.*

(iii) *The Dirac exponential map $\exp_\eta\colon \mathcal{I}_Z \to \wedge^2 V^*$ restricts to a diffeomorphism*

$$\mathcal{I}_Z \cap \big((K^* \otimes G^*) \oplus \wedge^2 G^*\big) \xrightarrow{\cong} \big\{\eta' \in \big(\wedge^2 V^*\big)_k \mid \ker\big((\eta')^\sharp\big) \text{ is transverse to } G\big\}$$

onto an open neighborhood of η in $\big(\wedge^2 V^*\big)_k$.

Remark 2.4. By the above linear algebra construction, given a pre-symplectic manifold (M,η), choosing a subbundle G complementary to $K = \ker\big(\eta^\sharp\big)$, one obtains a map

$$\exp_\eta\colon\ \mathcal{I}_Z \cap \big((K^* \otimes G^*) \oplus \big(\wedge^2 G^*\big)\big) \to \wedge^2 T^*M. \tag{2.2}$$

It is a not a vector bundle morphism but just a smooth fiberwise map. It maps the zero section to η, and its image is an open neighborhood of η in the space of 2-forms having the same rank as η. The map $\exp_\eta$ allows to parametrize deformations of η inside $\mathsf{Pre\text{-}Sym}^k(M)$ by means of sections $(\mu,\sigma) \in \Gamma(K^* \otimes G^*) \oplus \Gamma\big(\wedge^2 G^*\big) \cong \Omega^2_{\mathrm{hor}}(M)$ which are sufficiently small in the C^0-sense and for which the 2-form $(\exp_\eta)(\mu,\sigma)$ is a closed.

2.3 An L_∞-algebra associated to a bivector field

In this subsection we canonically associate an L_∞-algebra to any bivector field Z on a manifold M.

Definition 2.5. Let Z be a bivector field on M. The *Koszul bracket* associated to Z is the operation

$$[\cdot,\cdot]_Z\colon\ \Omega^r(M)\times\Omega^s(M)\to\Omega^{r+s-1}(M),$$
$$[\alpha,\beta]_Z := (-1)^{|\alpha|+1}\big(\mathcal{L}_Z(\alpha\wedge\beta)-\mathcal{L}_Z(\alpha)\wedge\beta-(-1)^{|\alpha|}\alpha\wedge\mathcal{L}_Z(\beta)\big).$$

Here $\mathcal{L}_Z=\iota_Z\circ d-d\circ\iota_Z$, where ι_Z denotes contraction with Z and d is the de Rham differential. On 1-forms α and β, the Koszul bracket reads $[\alpha,\beta]_Z=\mathcal{L}_{Z^\sharp\alpha}\beta-\mathcal{L}_{Z^\sharp\beta}\alpha-d\langle Z,\alpha\wedge\beta\rangle$.

In general the Koszul bracket of Z does not satisfy the graded Jacobi identity (it does only when Z is a Poisson bivector-field). We will see in Proposition 2.7 that nevertheless there is a well-behaved algebraic structure associated to Z. To this aim, recall that a differential form $\alpha\in\Omega^r(M)$ induces by contraction a linear map

$$\alpha^\sharp\colon\ TM\to\wedge^{r-1}T^*M,\qquad v\mapsto\iota_v\alpha,$$

and, following [2, Section 2.3], we extend this definition to a collection of forms $\alpha_1,\dots,\alpha_n$ by setting

$$\alpha_1^\sharp\wedge\cdots\wedge\alpha_n^\sharp\colon\ \wedge^n TM\to\wedge^{|\alpha_1|+\cdots+|\alpha_n|-n}T^*M,$$
$$v_1\wedge\cdots\wedge v_n\mapsto\sum_{\sigma\in S_n}(-1)^{|\sigma|}\alpha_1^\sharp(v_{\sigma(1)})\wedge\cdots\wedge\alpha_n^\sharp(v_{\sigma(n)}).$$

Definition 2.6. We define the trinary bracket $[\cdot,\cdot,\cdot]_Z\colon\Omega^r(M)\times\Omega^s(M)\times\Omega^k(M)\to\Omega^{r+s+k-3}(M)$ associated to the bivector field Z to be

$$[\alpha,\beta,\gamma]_Z := \big(\alpha^\sharp\wedge\beta^\sharp\wedge\gamma^\sharp\big)\big(\tfrac{1}{2}[Z,Z]\big).$$

These brackets endow $\Omega(M)[2]$ with an $L_\infty[1]$-algebra structure, extending the results of Fiorenza and Manetti [5]. The following is [7, Proposition 3.5]:

Proposition 2.7 (the $L_\infty[1]$-algebra $\Omega(M)[2]$)**.** *Let Z be a bivector field on M. The multilinear maps λ_1, λ_2, λ_3 on the graded vector space $\Omega(M)[2]$ given by*

1) *λ_1 the de Rham differential d,*
2) *$\lambda_2(\alpha[2]\odot\beta[2])=-\big(\mathcal{L}_Z(\alpha\wedge\beta)-\mathcal{L}_Z(\alpha)\wedge\beta-(-1)^{|\alpha|}\alpha\wedge\mathcal{L}_Z(\beta)\big)[2]=(-1)^{|\alpha|}([\alpha,\beta]_Z)[2]$, and*
3) *$\lambda_3(\alpha[2]\odot\beta[2]\odot\gamma[2])=(-1)^{|\beta|+1}\big(\alpha^\sharp\wedge\beta^\sharp\wedge\gamma^\sharp\big(\tfrac{1}{2}[Z,Z]\big)\big)[2]$,*

define the structure of an $L_\infty[1]$-algebra on $\Omega(M)[2]$.

We now explain the geometric relevance of the $L_\infty[1]$-algebra $(\Omega(M)[2],\lambda_1,\lambda_2,\lambda_3)$. As for any $L_\infty[1]$-algebra, it comes with distinguished elements:

Definition 2.8. An element $\beta\in\Omega^2(M)$ is a *Maurer–Cartan element* of $(\Omega(M)[2],\lambda_1,\lambda_2,\lambda_3)$ if it satisfies the *Maurer–Cartan equation*

$$d(\beta[2])+\tfrac{1}{2}\lambda_2(\beta[2]\odot\beta[2])+\tfrac{1}{6}\lambda_3(\beta[2]\odot\beta[2]\odot\beta[2])=0.$$

Recall that at the beginning of Section 2.2 we defined an open subset $\mathcal{I}_Z\subset\wedge^2T^*M$ and a map $F\colon\mathcal{I}_Z\to\wedge^2T^*M$. The following is [7, Corollary 3.9].

Corollary 2.9 (Maurer–Cartan elements of $\Omega(M)[2]$)**.** *There is an open subset $\mathcal{U}\subset\mathcal{I}_Z$, which contains the zero section of $\wedge^2T^*M$, such that a 2-form $\beta\in\Gamma(\mathcal{U})$ is a Maurer–Cartan element of $(\Omega(M)[2],\lambda_1,\lambda_2,\lambda_3)$ if, and only if, the 2-form $F(\beta)$ is closed.*

In Section 3.4 we will show that for the open subset $\mathcal{U}$ one can choose the whole of $\mathcal{I}_Z$.

2.4 The Koszul L_∞-algebra of a pre-symplectic manifold

Let again η be a pre-symplectic structure on a manifold M. Fix a subbundle $G \subset TM$ which is complementary to the kernel K of η. Consider the bivector field Z satisfying $Z^\sharp = -\big(\eta|_G^\sharp\big)^{-1}$. The following is [7, Theorem 3.17].

Theorem 2.10 (the Koszul $L_\infty[1]$-algebra)**.** *The $L_\infty[1]$-algebra structure on $\Omega(M)[2]$ associated to the bivector field Z, see Proposition 2.7, maps $\Omega_{\mathrm{hor}}(M)[2]$ to itself. The subcomplex $\Omega_{\mathrm{hor}}(M)[2] \subset \Omega(M)[2]$ therefore inherits the structure of an $L_\infty[1]$-algebra, which we call the Koszul $L_\infty[1]$-algebra of (M, η).*

We denote by $\mathsf{MC}(\eta)$ the set of Maurer–Cartan elements of the Koszul $L_\infty[1]$-algebra of (M, η).

In view of the above theorem, the following result [7, Theorem 3.19] is an immediate consequence of Theorem 2.3 and Corollary 2.9.

Theorem 2.11 (Maurer–Cartan elements of the Koszul $L_\infty[1]$-algebra)**.** *Let (M, η) be a pre-symplectic manifold. The choice of a complement G to the kernel of η determines a bivector field Z by requiring $Z^\sharp = -\big(\eta|_G^\sharp\big)^{-1}$. Suppose β is a 2-form on M, which lies in $\mathcal{I}_Z$. The following statements are equivalent:*

1. *β is a Maurer–Cartan element of the Koszul $L_\infty[1]$-algebra $\Omega_{\mathrm{hor}}(M)[2]$ of (M, η), which was introduced in Theorem 2.10.*
2. *The image of β under the map $\exp_\eta$, which is introduced in Definition 2.2, is a pre-symplectic structure of the same rank as η.*

The above Theorem 2.11 is the main result of [7], as it states that the Koszul $L_\infty[1]$-algebra governs the deformations of the pre-symplectic structure η. More precisely: the fibrewise map $\exp_\eta$ as in (2.2), on the level of sections, restricts to a map

$$\boxed{\exp_\eta\colon\ \Gamma(\mathcal{I}_Z) \cap \mathsf{MC}(\eta) \to \mathsf{Pre\text{-}Sym}^k(M)}$$

which is injective and whose image consists of the pre-symplectic structures of rank equal to the rank of η and with kernel transverse to G.

3 Dirac geometric interpretation

In the remainder of this note we explain the geometric framework that underlies the results of Section 2 recalled from [7]. We recover naturally the statements made there and provide some alternative and more geometric proofs.

3.1 Background on Dirac geometry

We first review some notions from Dirac linear algebra. Let V be a finite-dimensional, real vector space. We denote by $\mathbb{V}$ the direct sum $V \oplus V^*$ and by $\langle \cdot, \cdot \rangle$ the following non-degenerate pairing on $\mathbb{V}$:

$$\langle (v, \xi), (w, \chi) \rangle := \xi(w) + \chi(v).$$

Definition 3.1. A subspace $W \subset \mathbb{V}$ is called *Lagrangian* if for all $w, w' \in W$ we have $\langle w, w' \rangle = 0$ and $\dim(W) = \dim(V)$. Two subspaces W and $\tilde{W} \subset \mathbb{V}$ are *transverse*, if $W \oplus \tilde{W} = \mathbb{V}$.

Given an element $Z \in \wedge^2 V$, we defined the linear map $Z^\sharp\colon V^* \to V$ in Section 2.2, and we can consider the Lagrangian subspace $\mathrm{graph}(Z) := \big\{\big(Z^\sharp\xi, \xi\big) \,|\, \xi \in V^*\big\} \subset \mathbb{V}$. Similarly, for $\beta \in \wedge^2 V^*$ we define $\beta^\sharp\colon V \to V^*$ and consider $\mathrm{graph}(\beta)$.

Every $\beta \in \wedge^2 V^*$ defines an orthogonal transformation $\mathfrak{t}_\beta$ of $(\mathbb{V}, \langle\cdot,\cdot\rangle)$, by

$$(v, \xi) \mapsto \big(v, \xi + \beta^\sharp(v)\big).$$

Similarly, every $Z \in \wedge^2 V$ gives rise to an orthogonal transformation $\mathfrak{t}_Z$, which takes (v, ξ) to $\big(v + Z^\sharp(\xi), \xi\big)$. In particular, elements of $\wedge^2 V^*$ and $\wedge^2 V$ act on the set of Lagrangian subspaces of $\mathbb{V}$.

Remark 3.2. Suppose L, R are transverse Lagrangian subspaces of $\mathbb{V}$. There is a canonical isomorphism

$$R \cong L^*, \qquad r \mapsto \langle r, \cdot\rangle|_L.$$

Since R is transverse to L, any subspace of $\mathbb{V}$ transverse to R is the graph of a linear map $L \to R$. Any *Lagrangian* subspace transverse to R is the graph of a linear map $L \to R$ such that, composing with the canonical isomorphism above, we obtain a *skew-symmetric* linear map $L \to L^*$ (i.e., the sharp map associated to an element of $\wedge^2 L^*$).

Let us now briefly recall the basic constituencies of Dirac geometry. Consider the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$. It comes equipped with a non-degenerate pairing

$$\langle (X, \alpha), (Y, \beta)\rangle := \alpha(Y) + \beta(X)$$

and the Dorfman bracket

$$[\![(X, \alpha), (Y, \beta)]\!] = ([X, Y], \mathcal{L}_X \beta - \iota_Y d\alpha).$$

Together with the projection to TM, this makes $\mathbb{T}M$ into an example of Courant algebroid.

Definition 3.3. An *almost Dirac structure* on M is a Lagrangian subbundle $L \subset (\mathbb{T}M, \langle\cdot,\cdot\rangle)$. A *Dirac structure* is an almost Dirac structure whose space of sections is closed with respect to the Dorfman bracket $[\![\cdot,\cdot]\!]$.

Remark 3.4. Let L, R be transverse Dirac structures on M. As seen in Remark 3.2, almost Dirac structures transverse to R are in bijection with elements of $\Gamma\big(\wedge^2 L^*\big)$. We now recall a result of Liu–Weinstein–Xu [4] establishing when such an almost Dirac structure is Dirac. Recall that every Dirac structure, with the restricted Dorfman bracket and anchor, is a Lie algebroid. Since L is a Lie algebroid, it induces a differential d_L on $\Gamma(\wedge L^*)$. Further[1], since $L^* \cong R$ is a Lie algebroid, it induces a graded Lie bracket $[\cdot,\cdot]_{L^*}$ on $\Gamma(\wedge L^*)[1]$. Together with d_L and $[\cdot,\cdot]_{L^*}$, the graded vector space $\Gamma(\wedge L^*)[1]$ becomes a differential graded Lie algebra. The main result of [4] is: for all $\varepsilon \in \Gamma\big(\wedge^2 L^*\big)$, the graph $L_\varepsilon = \{v + \iota_v \varepsilon\colon v \in L\}$ is a Dirac structure iff ε satisfies the Maurer–Cartan equation, that is

$$d_L \varepsilon + \tfrac{1}{2}[\varepsilon, \varepsilon]_{L^*} = 0.$$

[1]The Lie algebroid structures on L and L^* are compatible in the sense that the pair (L, L^*) forms a Lie bialgebroid.

3.2 Deformations of pre-symplectic structures: the point of view of Dirac geometry

In this subsection we cast the deformations of pre-symplectic forms in the framework of Dirac geometry.

Let η be a pre-symplectic form on M, with kernel K. The natural way to parametrize deformations of η is by 2-forms α such that $\eta+\alpha$ is again pre-symplectic, but this parametrization has a serious flaw: the space of such α's does not have a natural vector space structure, due to the constant rank condition. Taking the point of view of Dirac geometry, the above approach to parametrize the deformations of η amounts to deforming the Dirac structure $\operatorname{graph}(\eta)$ using $\{0\}\oplus T^*M$ as a complement.

A better way to parametrize the deformations of η in terms of Dirac geometry works as follows. Let us first choose a complement G to K. Then

$$G\oplus K^*$$

is a complement[2] of $\operatorname{graph}(\eta)$. We can now use $G\oplus K^*$ – instead of $\{0\}\oplus T^*M$ – to parametrize deformations of the Dirac structure $\operatorname{graph}(\eta)$. This choice of complement has the advantage of linearizing the constant rank condition, as we show in Proposition 3.7 below. (Notice that when η is *symplectic*, the new complement is just TM, hence we are deforming η by viewing it as a Poisson structure, just as in [7, Section 1.3].)

We first state two lemmas about the effect of applying the orthogonal transformation $\mathfrak{t}_{-\eta}$ of $TM\oplus T^*M$, given by $(v,\xi)\mapsto \big(v,\xi-\eta^\sharp(v)\big)$.

Lemma 3.5. *Denote by $Z\in\Gamma\big(\wedge^2G\big)$ the bivector field such that $Z^\sharp$ is the inverse of $-(\eta|_G)^\sharp$. Then $\mathfrak{t}_{-\eta}$ maps $G\oplus K^*$ to $\operatorname{graph}(Z)$.*

Proof. $\mathfrak{t}_\eta(\operatorname{graph}(Z))=\big\{\big(Z^\sharp\xi,\xi|_K\big)\colon \xi\in T^*M\big\}=G\oplus K^*$. ∎

Lagrangian subbundles nearby $\operatorname{graph}(\eta)$ can be written, for some $\bar\beta\in\Gamma(\wedge^2(\operatorname{graph}(\eta))^*)$, as the graph of the map

$$\bar\beta^\sharp\colon\ \operatorname{graph}(\eta)\to(\operatorname{graph}(\eta))^*\cong G\oplus K^*,$$

by Remark 3.2. We denote this graph as $\Phi_{G\oplus K^*}\big(\bar\beta\big)$. Moreover, let $\beta\in\Omega^2(M)$ be the 2-form corresponding to $\bar\beta$ under the isomorphism $\operatorname{graph}(\eta)\cong TM, v+\iota_v\eta\mapsto v$ and denote by $\Phi_Z(\beta)$ the graph of the map $\beta^\sharp\colon TM\to T^*M\cong\operatorname{graph}(Z)$.

Lemma 3.6. *$\mathfrak{t}_{-\eta}$ maps $\Phi_{G\oplus K^*}\big(\bar\beta\big)$ to $\Phi_Z(\beta)$.*

Proof. $\mathfrak{t}_{-\eta}$ preserves the pairing on $TM\oplus T^*M$, clearly maps $\operatorname{graph}(\eta)$ to TM, and maps $G\oplus K^*$ to $\operatorname{graph}(Z)$ by Lemma 3.5. Therefore the statement follows by functoriality. ∎

Now we can explain why the choice of $G\oplus K^*$ as a complement is a good one to describe pre-symplectic deformations.

Proposition 3.7. *Let $\bar\beta\in\Gamma\big(\wedge^2(\operatorname{graph}(\eta))^*\big)$.*

(i) *The rank of*

$$\Phi_{G\oplus K^*}\big(\bar\beta\big)\cap TM \tag{3.1}$$

equals the rank of

$$\{v\in K\colon \iota_v\beta\in G^*\}. \tag{3.2}$$

[2] Indeed, for every $v\in TM$ we have $\iota_v\eta\in K^\circ=G^*$, so requiring that $\iota_v\eta$ lies in K^* implies $\iota_v\eta=0$. This means that $v\in K$, so requiring that v lies in G implies $v=0$.

(ii) Assume that $\Phi_{G\oplus K^*}(\bar{\beta})$ is the graph of a 2-form. Then the rank of this 2-form equals rank(η) *iff β lies in the vector space $\Omega^2_{\mathrm{hor}}(M)$ of horizontal 2-forms.*

Proof. (i) Applying the transformation $\mathfrak{t}_{-Z} \circ \mathfrak{t}_{-\eta}$ to $\Phi_{G\oplus K^*}(\bar{\beta})$, by Lemma 3.6 we obtain $\mathfrak{t}_{-Z}(\Phi_Z(\beta)) = \mathrm{graph}(\beta)$. Applying it to TM we obtain $\{(v + Z^\sharp \iota_v \eta, -\iota_v \eta) \mid v \in TM\} = K \oplus G^*$.

Hence applying the transformation to the intersection (3.1) we obtain

$$\mathrm{graph}(\beta) \cap (K \oplus G^*),$$

which is isomorphic to (3.2).

(ii) Denote by η' the 2-form whose graph is $\Phi_{G\oplus K^*}(\bar{\beta})$. The kernel of η' is given by (3.1), and the assertion follows immediately from (i). Recall that the vector space $\Omega^2_{\mathrm{hor}}(M)$ of horizontal 2-forms was defined in Section 2.1, as the space of 2-forms that vanish on $\wedge^2 K$. ■

Remark 3.8. Since $\mathfrak{t}_{-\eta}$ is actually an automorphism of the standard Courant algebroid $TM \oplus T^*M$, the following two deformation problems of Dirac structures are equivalent:

- deformations of graph(η), using the complement $G \oplus K^*$,
- deformations of TM, using the complement graph(Z).

The latter deformation problem is easier to handle, and the $L_\infty[1]$-algebra structure governing it will be recovered in Section 3.4.

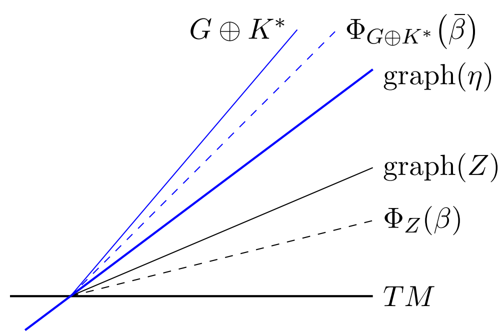

Figure 1. The Dirac structures graph(η) and TM, together with the complementary Lagrangian subbundles we use to deform them.

3.3 Dirac-geometric interpretation of Section 2.2

Using Dirac linear algebra, we explain and re-prove the results recalled in Section 2.2.

3.3.1 Revisiting the map F from formula (2.1)

Let V be a finite-dimensional, real vector space. We fix a bivector $Z \in \wedge^2 V$. Recall that $\mathcal{I}_Z$ consists of elements $\beta \in \wedge^2 V^*$ such that $\mathrm{id} + Z^\sharp \beta^\sharp$ is invertible. In formula (2.1), we defined the map $F\colon \mathcal{I}_Z \to \wedge^2 V^*$ given by

$$(F(\beta))^\sharp = \beta^\sharp \left(\mathrm{id} + Z^\sharp \beta^\sharp\right)^{-1}.$$

The following lemma provides a geometric explanation of the map F.

Lemma 3.9. *Fix $Z \in \wedge^2 V$.*

(*i*) *Taking graphs with respect to the decompositions $\mathbb{V} = V \oplus V^*$ resp. $\mathbb{V} = V \oplus \mathrm{graph}(Z)$, yields bijections*

$$\begin{aligned}\Phi_0\colon\ \wedge^2 V^* &\xrightarrow{\cong} \{\textit{Lagrangian subspace of } \mathbb{V} \textit{ transverse to } V^*\},\\ \alpha &\mapsto \{(v, \iota_v\alpha) \,|\, v \in V\},\\ \Phi_Z\colon\ \wedge^2 V^* &\xrightarrow{\cong} \{\textit{Lagrangian subspace of } \mathbb{V} \textit{ transverse to } \mathrm{graph}(Z)\},\\ \beta &\mapsto \big\{\big(v + Z^\sharp(\iota_v\beta), \iota_v\beta\big) \,|\, v \in V\big\}.\end{aligned}$$

(*ii*) *Given $\beta \in \wedge^2 V^*$, the Lagrangian subspace $\Phi_Z(\beta)$ is transverse to $V^* \subset \mathbb{V}$ if, and only if $\beta \in \mathcal{I}_Z$.*

(*iii*) *The map*

$$\Phi_0^{-1} \circ \Phi_Z\colon\ \mathcal{I}_Z \to \wedge^2 V^*$$

is well-defined and coincides with F.

In particular, the map F is characterized by the property that

$$\mathrm{graph}(F(\beta)) = \Phi_Z(\beta) \tag{3.3}$$

for all $\beta \in \mathcal{I}_Z$. In other words, $F(\beta)$ is obtained taking the graph of β w.r.t. the splitting $\mathbb{V} = V \oplus \mathrm{graph}(Z)$.

Proof. (i) According to Remark 3.2, any Lagrangian subspace transverse to V^* is the graph of a skew-symmetric linear map $V \to V^*$, and therefore can be written as $\{(v, \iota_v\alpha) \,|\, v \in V\}$ for some $\alpha \in \wedge^2 V^*$. Similarly, $\mathrm{graph}(Z)$ is transverse to V and the induced isomorphism $\mathrm{graph}(Z) \cong V^*$ is just $(Z^\sharp(\xi), \xi) \mapsto \xi$. Hence any Lagrangian subspace transverse to $\mathrm{graph}(Z)$ can be written as $\{(v, 0) + (Z^\sharp(\iota_v\beta), \iota_v\beta) \,|\, v \in V\}$ for some $\beta \in \wedge^2 V^*$.

(ii) The expression for $\Phi_Z(\beta)$ in item (i) shows that $\Phi_Z(\beta) \cap V^* = \{(0, \iota_v\beta) \,|\, v \in V, v + Z^\sharp(\iota_v\beta) = 0\}$. This intersection is trivial iff $\ker\big(\mathrm{id} + Z^\sharp\beta^\sharp\big) \subseteq \ker\big(\beta^\sharp\big)$. In turn, this condition is equivalent to $\big(\mathrm{id} + Z^\sharp\beta^\sharp\big)$ being injective, and thus invertible.

(iii) Finally, if $\mathrm{id} + Z^\sharp\beta^\sharp$ is invertible, $\Phi_Z(\beta)$ is transverse to V^* by item (ii). By item (i) the element $\Phi_0^{-1}(\Phi_Z(\beta))$ is well-defined. In concrete terms, it is given by $\alpha \in \wedge^2 V^*$ such that for all $v \in V$, there is $w \in V$ for which

$$\big(v + Z^\sharp\beta^\sharp(v), \beta^\sharp(v)\big) = \big(w, \alpha^\sharp(w)\big)$$

holds. Equivalently, this means that $\alpha^\sharp\big(\mathrm{id} + Z^\sharp\beta^\sharp\big)(v) = \beta^\sharp(v)$ for all $v \in V$. This shows that $\Phi_0^{-1} \circ \Phi_Z$ agrees with F. ■

3.3.2 Revisiting Theorem 2.3 (parametrizing constant rank forms)

Now let $\eta \in \wedge^2 V^*$ be of rank k, fix a complement G to $K := \ker(\eta)$, and denote by $Z \in \wedge^2 G$ the bivector determined by $Z^\sharp = -\big(\eta|_G^\sharp\big)^{-1}$. In Section 3.2 we considered deformations of the Dirac structure $\mathrm{graph}(\eta)$ using $G \oplus K^*$ as a complement. They are graphs of 2-forms given by the Dirac exponential map $\exp_\eta$ (see Definition 2.2). More precisely:

Lemma 3.10. *For all $\beta \in \mathcal{I}_Z$ we have*

$$\mathrm{graph}(\exp_\eta(\beta)) = \Phi_{G\oplus K^*}\big(\bar{\beta}\big). \tag{3.4}$$

Proof. We have $\mathrm{graph}(\exp_\eta(\beta)) = \mathfrak{t}_\eta(\Phi_Z(\beta)) = \Phi_{G\oplus K^*}(\bar\beta)$, where the first equality holds by equation (3.3) and the second by Lemma 3.6. ■

Using this we recover Theorem 2.3, in particular item (i) stating that $\exp_\eta(\beta)$ has rank equal to $k = \dim(K)$ iff β is horizontal.

Alternative proof of Theorem 2.3. (i) Apply Proposition 3.7(ii) together with equation (3.4).

(ii) We only prove the statement about the kernel of $\exp_\eta(\beta)$. Write $\beta = (\mu,\sigma)$. By the proof of Proposition 3.7(i), the intersection of the subspace (3.4) with V is $(\mathfrak{t}_\eta\circ\mathfrak{t}_Z)(\mathrm{graph}(\beta)\cap(K\oplus G^*))$, which is precisely the image of K under $\mathrm{id} + Z^\sharp\mu^\sharp$.

(iii) By Lemma 3.9(ii), the map Φ_Z provides a bijection between $\mathcal{I}_Z$ and Lagrangian subspaces transverse to $\mathrm{graph}(Z)$ and to V^*. Hence $\mathfrak{t}_\eta\circ\Phi_Z$ provides a bijection between $\mathcal{I}_Z$ and Lagrangian subspaces transverse to $\mathfrak{t}_\eta(\mathrm{graph}(Z)) = G\oplus K^*$ (see Lemma 3.5) and to V^*. The latter are exactly the graphs of elements $\eta'\in\wedge^2V^*$ so that the $\eta'|_{\wedge^2G}$ is non-degenerate. Hence, by the proof of Lemma 3.10, $\exp_\eta$ provides a bijection between $\mathcal{I}_Z$ and such η'. We conclude using (i). ■

3.4 Dirac-geometric interpretation of Section 2.3

Using Dirac geometry and adapting results from [2], we explain and re-prove the results recalled in Section 2.3. Fix a bivector field Z on M.

3.4.1 Revisiting Proposition 2.7 (the $L_\infty[1]$-algebra $\Omega(M)[2]$)

In Proposition 2.7, the $L_\infty[1]$-algebra $(\Omega(M)[2],\lambda_1,\lambda_2,\lambda_3)$ was constructed out of a bivector field Z. It can be recovered using Dirac geometry – or more precisely, the deformation theory of Dirac structures – as a special case of the construction from [2, Section 2.2].

Proposition 3.11. *Let L be a Dirac structure and R a complementary almost Dirac structure, i.e., we have a vector bundle decomposition $L\oplus R = \mathbb{T}M$. Then $\Gamma(\wedge L^*)[2]$ has an induced $L_\infty[1]$-algebra structure, whose only non-trivial multibrackets are μ_1, μ_2, μ_3 given as follows:*

1) *μ_1 is the differential d_L associated to the Lie algebroid L,*
2) *$\mu_2(\alpha[2]\odot\beta[2]) = -(-1)^{|\alpha|}[\alpha,\beta]_{L^*}[2]$, where $[\cdot,\cdot]_{L^*} := \mathrm{pr}_R([\![\cdot,\cdot]\!])$ denotes the (extension of) the bracket of the almost Lie algebroid $R\cong L^*$,*
3) *$\mu_3(\alpha[2]\odot\beta[2]\odot\gamma[2]) = (-1)^{|\beta|}(\alpha^\sharp\wedge\beta^\sharp\wedge\gamma^\sharp)\psi[2]$, where $\psi\in\Gamma(\wedge^3L)$ is given by $\Gamma(\wedge^3L^*)\to\mathcal{C}^\infty(M)$, $\xi_1\wedge\xi_2\wedge\xi_3\mapsto\langle\mathrm{pr}_L([\![\xi_1,\xi_2]\!]),\xi_3\rangle$, where we made use of the identification $R\cong L^*$.*

More generally, Proposition 3.11 holds if replacing $\mathbb{T}M$ by any Courant algebroid.

Proof. The proof is a minor adaptation of the first part of the proof of [2, Lemma 2.6], setting $\varphi = 0$ there. We recall briefly the idea of the latter. By [6] there is a natural description of the Courant algebroid structure on $\mathbb{T}M$ in terms of graded geometry. One can use it to apply Voronov's higher derived brackets construction (see [8, 9]) and obtain an $L_\infty[1]$-algebra structure on $\Gamma(\wedge L^*)[2]$. The multibrackets obtained are the ones in the statement of the lemma, as one checks using [6] and via computations in local coordinates. ■

Alternative proof of Proposition 2.7. Let Z be a bivector field on M. We apply Proposition 3.11 for the case $L = TM$ and $R = \mathrm{graph}(Z)$. In this case d_L is the de Rham differential, and the bracket on R is given by the formula for the Koszul bracket. One checks that ψ is the trivector field $-\frac{1}{2}[Z,Z]$, using [7, Lemma 1.6]. Hence the $L_\infty[1]$-brackets on $\Omega(M)[2]$ given by Proposition 3.11 are $\mu_1 = \lambda_1$, $\mu_2 = -\lambda_2$ and $\mu_3 = \lambda_3$. Applying the automorphism $-\mathrm{id}$ to $\Omega(M)[2]$ yields Proposition 2.7. ■

3.4.2 Revisiting Corollary 2.9 (Maurer–Cartan elements of $\Omega(M)[2]$)

We now turn to Maurer–Cartan elements. In Lemma 3.9(i), we gave a parametrization of all almost Dirac structures that are transverse to graph(Z) in terms of 2-forms β on M. This parametrization is given by

$$\beta \mapsto \Phi_Z(\beta) = \big\{\big(v + Z^\sharp(\iota_v\beta), \iota_v\beta\big) \,|\, v \in TM\big\}.$$

We present the second part of [2, Lemma 2.6], which is an extension of the work by Liu–Weinstein–Xu recalled in Remark 3.4.

Proposition 3.12. *Let L be given a Dirac structure and R a complementary almost Dirac structure. An element $\sigma \in \Gamma\big(\wedge^2 L^*\big)[2]$ is a Maurer–Cartan element of the $L_\infty[1]$-algebra structure given in Proposition 3.11 iff the graph*

$$\Gamma_\sigma := \{(X - \iota_X\sigma) \colon X \in L\} \subset L \oplus R$$

is a Dirac structure. (The above inclusion makes use of the identification $R \cong L^*$.)

Corollary 2.9 states that for $\beta \in \Omega^2(M)$ taking values in some sufficiently small neighborhood $\mathcal{U}$ of the zero section in $\wedge^2 T^*M$ – in particular taking values in $\mathcal{I}_Z$, i.e., id $+ Z^\sharp\beta^\sharp$ is invertible, – β is a Maurer–Cartan element of $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$ iff $F(\beta)$ is closed. We now provide an alternative proof of this result, which also shows that one can choose $\mathcal{U}$ to equal $\mathcal{I}_Z$.

Alternative proof of Corollary 2.9. For any $\beta \in \Omega^2(M)$, being a Maurer–Cartan element of the $L_\infty[1]$-algebra $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$ is equivalent to $\Phi_Z(\beta)$ being a Dirac structure. This follows from applying Proposition 3.12 to the Dirac structure $L = TM$ and to the almost Dirac structure $R = \text{graph}(Z)$, noticing that $\Gamma_{-\beta} = \{(v + Z^\sharp(\iota_v\beta), \iota_v\beta) \,|\, v \in TM\} = \Phi_Z(\beta)$. When $\beta \in \Gamma(\mathcal{I}_Z)$, we know that $\Phi_Z(\beta)$ can be written as the graph of the 2-form $F(\beta)$, by equation (3.3). Now use the fact that the graph of a 2-form is a Dirac structure if, and only if, the 2-form is closed. ■

Remark 3.13. In this subsection we recovered the $L_\infty[1]$-algebra $\Omega(M)[2]$ of Proposition 2.7 as the $L_\infty[1]$-algebra governing deformations of the Dirac structure TM taking graph(Z) as a complement. By Remark 3.8, this deformation problem is equivalent to the deformations of the Dirac structure graph(η) taking $G \oplus K^*$ as the complement. This explains why the $L_\infty[1]$-algebra $\Omega(M)[2]$ governs the latter deformation problem, and therefore is relevant for the deformations of pre-symplectic structures.

3.5 Dirac-geometric interpretation of Section 2.4

Theorem 2.10 can be deduced from a general statement about (almost) Dirac structures, however doing so amounts essentially to the same computations that were needed for the proof given in [7]. We include this general statement for the sake of completeness.

Proposition 3.14. *In the setting of Proposition 3.11, let K be a subbundle of L and define $\Gamma_{\text{hor}}(\wedge L^*)$ as the kernel of the restriction map $\Gamma(\wedge L^*) \to \Gamma(\wedge K^*)$. Then the multibrackets μ_1, μ_2, μ_3 preserve $\Gamma_{\text{hor}}(\wedge L^*)[2]$ iff K satisfies the following:*

- *K is a Lie subalgebroid of L,*
- *$\langle [\![\xi_1, \xi_2]\!], K + K^\circ \rangle = 0$ for all $\xi_1, \xi_2 \in \Gamma(K^\circ)$, where we use the identification $K^\circ \subset L^* \cong R$ and $[\![\cdot,\cdot]\!]$ denotes the Dorfman bracket.*

Proof. We will use the fact that μ_1, μ_2, μ_3 are derivations w.r.t. the wedge product in each entry. The Lie algebroid differential d_L preserves $\Gamma_{\mathrm{hor}}(\wedge L^*)$ iff the subbundle K is involutive. The bracket $[\cdot,\cdot]_{L^*}$ preserves $\Gamma_{\mathrm{hor}}(\wedge L^*)$ iff $\langle [\![\xi_1,\xi_2]\!], K\rangle = 0$ for all $\xi_1, \xi_2 \in \Gamma(K^\circ)$. The trinary bracket μ_3 preserves $\Gamma_{\mathrm{hor}}(\wedge L^*)$ iff $\mu_3(\xi_1,\xi_2,\xi_3) = 0$ for all $\xi_i \in \Gamma(K^\circ)$, which in turn is equivalent to $\langle [\![\xi_1,\xi_2]\!], \xi_3\rangle = 0$. ■

Finally, as mentioned earlier, Theorem 2.11 follows immediately from the other results presented.

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