The q-Onsager Algebra and the Universal Askey–Wilson Algebra

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Abstract. Recently Pascal Baseilhac and Stefan Kolb obtained a PBW basis for the q-Onsager algebra $\mathcal{O}_q$. They defined the PBW basis elements recursively, and it is obscure how to express them in closed form. To mitigate the difficulty, we bring in the universal Askey–Wilson algebra Δ_q. There is a natural algebra homomorphism $\natural \colon \mathcal{O}_q \to \Delta_q$. We apply $\natural$ to the above PBW basis, and express the images in closed form. Our results make heavy use of the Chebyshev polynomials of the second kind.

Key words: q-Onsager algebra; universal Askey–Wilson algebra; Chebyshev polynomial

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1 Introduction

In the 1944 paper [25] Lars Onsager obtained the free energy of the two-dimensional Ising model in a zero magnetic field. In that paper an infinite-dimensional Lie algebra was introduced; this algebra is now called the Onsager algebra and denoted by $\mathcal{O}$. Onsager defined his algebra by giving a linear basis and the action of the Lie bracket on the basis. In [26] Perk gave a presentation of $\mathcal{O}$ by generators and relations. This presentation involves two generators and two relations, called the Dolan/Grady relations [17]. This presentation is discussed in [30, Remark 9.1]. Via this presentation, the universal enveloping algebra of $\mathcal{O}$ admits a q-deformation $\mathcal{O}_q$ called the q-Onsager algebra [4, 29]. The algebra $\mathcal{O}_q$ is associative and infinite-dimensional. It is defined by two generators and two relations called the q-Dolan/Grady relations; these are given in (2.2), (2.3) below. The q-Dolan/Grady relations first appeared in algebraic combinatorics, in the study of Q-polynomial distance-regular graphs [27, Lemma 5.4]. Shortly thereafter they appeared in physics, in the study of statistical mechanical models [4, Section 2]. Up to the present, the representation theory of $\mathcal{O}_q$ remains an active area of research in mathematics [19, 21, 22, 28, 29, 30, 31, 32, 33] and physics [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15]. This theory involves a linear algebraic object called a tridiagonal pair [20]. A finite-dimensional irreducible $\mathcal{O}_q$-module is essentially the same thing as a tridiagonal pair of q-Racah type [29, Theorem 3.10]. These tridiagonal pairs are classified up to isomorphism in [21, Theorem 3.3]. In [22, Theorem 2.1], Ito and the present author gave a linear basis for $\mathcal{O}_q$, called the zigzag basis. More information about this basis can be found in [32, Note 4.7]. In [7], Baseilhac and Belliard conjectured another linear basis for $\mathcal{O}_q$; this one is motivated by how $\mathcal{O}_q$ is related to the reflection equation algebra [11, 14]. In [13], Baseilhac and Kolb introduced two automorphisms T_0, T_1 of $\mathcal{O}_q$ that are roughly analogous to the Lusztig automorphisms of $U_q(\widehat{\mathfrak{sl}}_2)$. They used T_0, T_1 and a method of Damiani [16] to obtain a Poincaré–Birkhoff–Witt (or PBW) basis for $\mathcal{O}_q$ [13, Theorem 4.3]. In our view this PBW basis is important and worthy of further

study. In the present paper we study the following aspect. In [13, Section 3.1] the PBW basis is defined recursively, and it is obscure how to express it in closed form. In order to mitigate the difficulty, we bring in a related algebra which we now describe. In [34] Zhedanov introduced the Askey–Wilson algebra AW(3) and used it to describe the Askey–Wilson polynomials. In [31] the present author introduced a central extension of AW(3), called the universal Askey–Wilson algebra Δ_q. In [18], Hau-Wen Huang classified up to isomorphism the finite-dimensional irreducible Δ_q-modules for q not a root of unity. A linear basis for Δ_q is given in [31, Theorem 7.5]. There is a natural algebra homomorphism $\natural\colon \mathcal{O}_q \to \Delta_q$ [31, Definition 10.4]; this is described below (2.23) in the present paper. We use $\natural$ to describe the PBW basis for $\mathcal{O}_q$ in the following way. We apply $\natural$ to the PBW basis vectors and consider their images in Δ_q. We express these images explicitly in the linear basis for Δ_q mentioned above. Our main results are Theorems 5.5, 5.6. These results make heavy use of the Chebyshev polynomials of the second kind [23, 24].

2   Preliminaries

We now begin our formal argument. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbb{F}$ denote an algebraically closed field with characteristic zero. All the algebras discussed in this paper are over $\mathbb{F}$; those without the Lie prefix are associative and have a multiplicative identity. Fix a nonzero $q \in \mathbb{F}$ that is not a root of 1. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad n \in \mathbb{Z}. \tag{2.1}$$

We will be discussing the q-Onsager algebra $\mathcal{O}_q$ and the universal Askey–Wilson algebra Δ_q. We now recall these algebras.

The algebra $\mathcal{O}_q$ (see [4, Section 2], [29, Definition 3.9]) is defined by generators A, B and relations

$$A^3 B - [3]_q A^2 BA + [3]_q ABA^2 - BA^3 = \left(q^2 - q^{-2}\right)^2 (BA - AB), \tag{2.2}$$

$$B^3 A - [3]_q B^2 AB + [3]_q BAB^2 - AB^3 = \left(q^2 - q^{-2}\right)^2 (AB - BA). \tag{2.3}$$

The relations (2.2), (2.3) are called the *q-Dolan/Grady relations*. In [13], Baseilhac and Kolb introduced the automorphisms T_0, T_1 of $\mathcal{O}_q$. These automorphisms satisfy

$$T_0(A) = A, \qquad T_0(B) = B + \frac{qA^2 B - (q + q^{-1})ABA + q^{-1}BA^2}{(q - q^{-1})(q^2 - q^{-2})}, \tag{2.4}$$

$$T_1(B) = B, \qquad T_1(A) = A + \frac{qB^2 A - (q + q^{-1})BAB + q^{-1}AB^2}{(q - q^{-1})(q^2 - q^{-2})}. \tag{2.5}$$

The inverse automorphisms satisfy

$$T_0^{-1}(A) = A, \qquad T_0^{-1}(B) = B + \frac{q^{-1}A^2 B - (q + q^{-1})ABA + qBA^2}{(q - q^{-1})(q^2 - q^{-2})}, \tag{2.6}$$

$$T_1^{-1}(B) = B, \qquad T_1^{-1}(A) = A + \frac{q^{-1}B^2 A - (q + q^{-1})BAB + qAB^2}{(q - q^{-1})(q^2 - q^{-2})}. \tag{2.7}$$

In [13], Baseilhac and Kolb used T_0 and T_1 to define some elements in $\mathcal{O}_q$, denoted

$$\{B_{n\delta + \alpha_0}\}_{n=0}^{\infty}, \qquad \{B_{n\delta + \alpha_1}\}_{n=0}^{\infty}, \qquad \{B_{n\delta}\}_{n=1}^{\infty}. \tag{2.8}$$

The elements (2.8) were shown to be a PBW basis for $\mathcal{O}_q$, provided that q is transcendental over $\mathbb{F}$ [13, Theorem 4.3]. By definition

n	$B_{n\delta+\alpha_0}$	$B_{n\delta+\alpha_1}$
0	A	B
1	$T_0(B)$	$T_1^{-1}(A)$
2	$T_0 T_1(A)$	$T_1^{-1} T_0^{-1}(B)$
3	$T_0 T_1 T_0(B)$	$T_1^{-1} T_0^{-1} T_1^{-1}(A)$
4	$T_0 T_1 T_0 T_1(A)$	$T_1^{-1} T_0^{-1} T_1^{-1} T_0^{-1}(B)$
$\vdots$	$\vdots$	$\vdots$

and for $n \geq 1$,

$$B_{n\delta} = q^{-2} B_{(n-1)\delta+\alpha_1} A - AB_{(n-1)\delta+\alpha_1} + \left(q^{-2} - 1\right) \sum_{\ell=0}^{n-2} B_{\ell\delta+\alpha_1} B_{(n-\ell-2)\delta+\alpha_1}. \tag{2.9}$$

By [13, Proposition 5.12] the elements $\{B_{n\delta}\}_{n=1}^{\infty}$ mutually commute. We have $B_\delta = q^{-2} BA - AB$. Define $\tilde{B}_\delta = q^{-2} AB - BA$. By [13, Lemma 3.1] we have $T_1(B_\delta) = \tilde{B}_\delta$ and $T_0(\tilde{B}_\delta) = B_\delta$. So as noted in [13, Lemma 3.1],

$$T_0 T_1(B_\delta) = B_\delta, \qquad T_1^{-1} T_0^{-1}(B_\delta) = B_\delta. \tag{2.10}$$

Next we recall the universal Askey–Wilson algebra Δ_q [31, Definition 1.2]. This algebra is defined by generators and relations. The generators are A, B, C. The relations assert that each of the following is central in Δ_q:

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \qquad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \qquad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}.$$

For the above three central elements, multiply each by $q + q^{-1}$ to get α, β, γ. Thus

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{q + q^{-1}}, \tag{2.11}$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q + q^{-1}}, \tag{2.12}$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q + q^{-1}}. \tag{2.13}$$

Each of α, β, γ is central in Δ_q. By [31, Corollary 8.3] the center $Z(\Delta_q)$ is generated by α, β, γ, Ω where

$$\Omega = qABC + q^2 A^2 + q^{-2} B^2 + q^2 C^2 - qA\alpha - q^{-1}B\beta - qC\gamma. \tag{2.14}$$

The element Ω is called the Casimir element. By [31, Theorem 8.2] the following is a linear basis for the $\mathbb{F}$-vector space $Z(\Delta_q)$:

$$\Omega^\ell \alpha^r \beta^s \gamma^t, \qquad \ell, r, s, t \geq 0.$$

We mention two bases for Δ_q. By [31, Theorem 4.1], the following is a linear basis for the $\mathbb{F}$-vector space Δ_q:

$$A^i B^j C^k \alpha^r \beta^s \gamma^t, \qquad i, j, k, r, s, t \geq 0. \tag{2.15}$$

By [31, Theorem 7.5], the following is a linear basis for the $\mathbb{F}$-vector space Δ_q:

$$A^i B^j C^k \Omega^\ell \alpha^r \beta^s \gamma^t, \qquad i, j, k, \ell, r, s, t \geq 0, \qquad ijk = 0. \tag{2.16}$$

For convenience we will work with the basis (2.16).

Shortly we will discuss how Δ_q is related to $\mathcal{O}_q$. To aid in this discussion we recall from [31, Section 2] a second presentation of Δ_q. By (2.11)–(2.13) the algebra Δ_q is generated by A, B, γ. Moreover

$$C = \frac{\gamma}{q + q^{-1}} - \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}, \tag{2.17}$$

$$\alpha = \frac{B^2 A - \left(q^2 + q^{-2}\right)BAB + AB^2 + \left(q^2 - q^{-2}\right)^2 A + \left(q - q^{-1}\right)^2 B\gamma}{\left(q - q^{-1}\right)\left(q^2 - q^{-2}\right)}, \tag{2.18}$$

$$\beta = \frac{A^2 B - \left(q^2 + q^{-2}\right)ABA + BA^2 + \left(q^2 - q^{-2}\right)^2 B + \left(q - q^{-1}\right)^2 A\gamma}{\left(q - q^{-1}\right)\left(q^2 - q^{-2}\right)}. \tag{2.19}$$

By [31, Theorem 2.2] the algebra Δ_q has a presentation by generators A, B, γ and relations

$$A^3 B - [3]_q A^2 BA + [3]_q ABA^2 - BA^3 = \left(q^2 - q^{-2}\right)^2 (BA - AB), \tag{2.20}$$

$$B^3 A - [3]_q B^2 AB + [3]_q BAB^2 - AB^3 = \left(q^2 - q^{-2}\right)^2 (AB - BA), \tag{2.21}$$

$$A^2 B^2 - B^2 A^2 + \left(q^2 + q^{-2}\right)(BABA - ABAB) = \left(q - q^{-1}\right)^2 (BA - AB)\gamma, \tag{2.22}$$

$$\gamma A = A\gamma, \qquad \gamma B = B\gamma. \tag{2.23}$$

The relations (2.20), (2.21) are the q-Dolan/Grady relations. Consequently there exists an algebra homomorphism $\natural \colon \mathcal{O}_q \to \Delta_q$ that sends $A \mapsto A$ and $B \mapsto B$. This homomorphism is not injective by [31, Theorem 10.9].

In order to clarify the nature of T_0, T_1, $\natural$ we now introduce some automorphisms t_0, t_1 of Δ_q such that $t_0 \natural = \natural T_0$ and $t_1 \natural = \natural T_1$. To this end, we recall from [31, Section 3] how the modular group $\mathrm{PSL}_2(\mathbb{Z})$ acts on Δ_q as a group of automorphisms. By [1] the group $\mathrm{PSL}_2(\mathbb{Z})$ has a presentation by generators ρ, σ and relations $\rho^3 = 1$, $\sigma^2 = 1$. Earlier in this section we gave two presentations of Δ_q. Using these presentations we find that $\mathrm{PSL}_2(\mathbb{Z})$ acts on Δ_q as a group of automorphisms in the following way:

u	A	B	C	α	β	γ
$\rho(u)$	B	C	A	β	γ	α
$\sigma(u)$	B	A	$C + \frac{AB - BA}{q - q^{-1}}$	β	α	γ

(2.24)

This action is faithful by [31, Theorem 3.13]. From the table (2.24) we see that the $\mathrm{PSL}_2(\mathbb{Z})$-generators ρ, σ each permute α, β, γ. This gives a group homomorphism from $\mathrm{PSL}_2(\mathbb{Z})$ onto the symmetric group S_3. Let $\mathbb{P}$ denote the kernel of the homomorphism. Thus $\mathbb{P}$ is a normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, and the quotient group $\mathrm{PSL}_2(\mathbb{Z})/\mathbb{P}$ is isomorphic to S_3. The cosets of $\mathbb{P}$ in $\mathrm{PSL}_2(\mathbb{Z})$ are

$$\mathbb{P}, \qquad \rho\mathbb{P}, \qquad \rho^2\mathbb{P}, \qquad \sigma\mathbb{P}, \qquad \rho\sigma\mathbb{P}, \qquad \rho^2\sigma\mathbb{P}.$$

We remark that in the literature the groups $\mathrm{PSL}_2(\mathbb{Z})$ and $\mathbb{P}$ are often denoted by Γ and $\Gamma(2)$, respectively; see for example [1, 2]. Define

$$t_0 = \left(\rho^2 \sigma\right)^2 = (\sigma\rho)^{-2}, \qquad t_1 = \left(\sigma\rho^2\right)^2 = (\rho\sigma)^{-2}. \tag{2.25}$$

Using (2.24), (2.25) we obtain $t_0, t_1 \in \mathbb{P}$. By [2, Proposition 4] the group $\mathbb{P}$ is freely generated by $t_0^{\pm 1}$, $t_1^{\pm 1}$. Using (2.17), (2.24), (2.25) we obtain

$$t_0(A) = A, \qquad t_0(B) = B + \frac{qA^2 B - \left(q + q^{-1}\right)ABA + q^{-1}BA^2}{\left(q - q^{-1}\right)\left(q^2 - q^{-2}\right)}, \tag{2.26}$$

$$t_1(B) = B, \qquad t_1(A) = A + \frac{qB^2 A - \left(q + q^{-1}\right)BAB + q^{-1}AB^2}{\left(q - q^{-1}\right)\left(q^2 - q^{-2}\right)} \tag{2.27}$$

and

$$t_0^{-1}(A) = A, \qquad t_0^{-1}(B) = B + \frac{q^{-1}A^2B - (q + q^{-1})ABA + qBA^2}{(q - q^{-1})(q^2 - q^{-2})}, \tag{2.28}$$

$$t_1^{-1}(B) = B, \qquad t_1^{-1}(A) = A + \frac{q^{-1}B^2A - (q + q^{-1})BAB + qAB^2}{(q - q^{-1})(q^2 - q^{-2})}. \tag{2.29}$$

The actions (2.26)–(2.29) match (2.4)–(2.7). Consequently the following diagrams commute:

$$
\begin{array}{ccc}
\mathcal{O}_q & \xrightarrow{\natural} & \Delta_q \\
{\scriptstyle T_1^{\pm 1}}\downarrow & & \downarrow{\scriptstyle t_1^{\pm 1}} \\
\mathcal{O}_q & \xrightarrow{\natural} & \Delta_q,
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{O}_q & \xrightarrow{\natural} & \Delta_q \\
{\scriptstyle T_0^{\pm 1}}\downarrow & & \downarrow{\scriptstyle t_0^{\pm 1}} \\
\mathcal{O}_q & \xrightarrow{\natural} & \Delta_q.
\end{array}
\tag{2.30}
$$

Let $\mathrm{Aut}(\mathcal{O}_q)$ denote the automorphism group of $\mathcal{O}_q$. Let G denote the subgroup of $\mathrm{Aut}(\mathcal{O}_q)$ generated by $T_0^{\pm 1}$, $T_1^{\pm 1}$. Since $\mathbb{P}$ is freely generated by $t_0^{\pm 1}$, $t_1^{\pm 1}$ there exists a group homomorphism $\varepsilon \colon \mathbb{P} \to G$ that sends $t_0^{\pm 1} \mapsto T_0^{\pm 1}$ and $t_1^{\pm 1} \mapsto T_1^{\pm 1}$. Using the commuting diagrams (2.30) one finds that for $\pi \in \mathbb{P}$ the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{O}_q & \xrightarrow{\natural} & \Delta_q \\
{\scriptstyle \varepsilon(\pi)}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathcal{O}_q & \xrightarrow{\natural} & \Delta_q.
\end{array}
\tag{2.31}
$$

We now prove that ε is an isomorphism. By construction ε is surjective. We show that ε is injective. Given an element r in the kernel of ε, we show that r is the identity in $\mathbb{P}$. To this end, we show that r fixes the generators A, B, γ of Δ_q. The map $\varepsilon(r)$ is the identity in G, so $\varepsilon(r)$ fixes the elements A, B of $\mathcal{O}_q$. By the commuting diagram (2.31) the map r fixes the elements A, B of Δ_q. Also r fixes γ since $r \in \mathbb{P}$ and everything in $\mathbb{P}$ fixes γ. We have shown that r fixes the generators A, B, γ of Δ_q so r is the identity in $\mathbb{P}$. Consequently ε is injective and hence an isomorphism.

It is mentioned in [13, Section 2.3] that one expects G to be freely generated by $T_0^{\pm 1}$, $T_1^{\pm 1}$. This is now easily proven as follows. The group $\mathbb{P}$ is freely generated by $t_0^{\pm 1}$, $t_1^{\pm 1}$. Applying the isomorphism $\varepsilon \colon \mathbb{P} \to G$ we find that G is freely generated by $T_0^{\pm 1}$, $T_1^{\pm 1}$.

Next we consider how the map $\natural \colon \mathcal{O}_q \to \Delta_q$ acts on the elements (2.8). For these elements we retain the same notation for their images under $\natural$. Our goal is to obtain these images in closed form, in terms of the basis (2.16). In order to obtain these images, it is convenient to bring in the Chebyshev polynomials of the second kind. These polynomials are reviewed in the next section.

3 The Chebyshev polynomials

In this section we review the Chebyshev polynomials of the second kind; see [23, 24] for further details. Let x denote an indeterminate. Let $\mathbb{F}[x]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in x that have all coefficients in $\mathbb{F}$.

Definition 3.1 (see [24, p. 4]). For $n \in \mathbb{N}$ define $U_n \in \mathbb{F}[x]$ by

$$U_0 = 1, \qquad U_1 = x, \qquad xU_n = U_{n+1} + U_{n-1}, \qquad n \geq 1.$$

The polynomial U_n is monic and degree n. We call U_n the nth *Chebyshev polynomial of the second kind*. For notational convenience define $U_n = 0$ for all integers $n < 0$.

Note 3.2. The above polynomials U_n are normalized to be monic. This normalization differs from the one in [23, Section 9.8.2]. To go from our normalization to the one in [23, Section 9.8.2], replace x by $2x$.

In the table below we display U_n for $0 \leq n \leq 9$.

n	U_n
0	1
1	x
2	$x^2 - 1$
3	$x^3 - 2x$
4	$x^4 - 3x^2 + 1$
5	$x^5 - 4x^3 + 3x$
6	$x^6 - 5x^4 + 6x^2 - 1$
7	$x^7 - 6x^5 + 10x^3 - 4x$
8	$x^8 - 7x^6 + 15x^4 - 10x^2 + 1$
9	$x^9 - 8x^7 + 21x^5 - 20x^3 + 5x$

By [24, pp. 332–333],

$$U_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} x^{n-2i}, \qquad n \in \mathbb{N}.$$

Next we express the polynomials U_n in a more closed form. Let z denote an indeterminate. Let $\mathbb{F}[z, z^{-1}]$ denote the $\mathbb{F}$-algebra consisting of the Laurent polynomials in z that have all coefficients in $\mathbb{F}$. This algebra has an automorphism that sends $z \mapsto z^{-1}$. An element of $\mathbb{F}[z, z^{-1}]$ that is fixed by the automorphism is called *symmetric*. The symmetric elements form a subalgebra of $\mathbb{F}[z, z^{-1}]$ called its *symmetric part*. There exists an injective algebra homomorphism $\iota \colon \mathbb{F}[x] \to \mathbb{F}[z, z^{-1}]$ that sends $x \mapsto z + z^{-1}$. The image of $\mathbb{F}[x]$ under ι is the symmetric part of $\mathbb{F}[z, z^{-1}]$. Via ι we identify $\mathbb{F}[x]$ with the symmetric part of $\mathbb{F}[z, z^{-1}]$. So for $n \in \mathbb{N}$ we view

$$\frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} = z^n + z^{n-2} + \cdots + z^{2-n} + z^{-n}$$

as an element of $\mathbb{F}[x]$.

Lemma 3.3 (see [24, p. 326]). *For $n \in \mathbb{N}$ we have*

$$U_n(x) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}},$$

where we recall $x = z + z^{-1}$.

In this paper, on several occasions we will consider generating functions in an indeterminate t. These generating functions involve a formal power series; issues of convergence are not considered. The following generating function will be useful.

Lemma 3.4 (see [23, p. 227]). *For an indeterminate t,*

$$\sum_{n \in \mathbb{N}} t^n U_n(x) = \frac{1}{1 - tx + t^2}. \tag{3.1}$$

Proof. Using Definition 3.1 one finds that the product $\left(\sum_{n \in \mathbb{N}} t^n U_n(x) \right) \left(1 - tx + t^2 \right)$ is equal to 1. Alternatively, use Lemma 3.3. ∎

The following variations on Lemma 3.4 will be used repeatedly.

Lemma 3.5. *For an indeterminate t,*

$$\sum_{n\in\mathbb{N}}(-1)^n q^n t^n U_{n-1}(x) = \frac{-1}{qt + q^{-1}t^{-1} + x},$$

$$\sum_{n\in\mathbb{N}}(-1)^n q^{-n} t^n U_{n-1}(x) = \frac{-1}{q^{-1}t + qt^{-1} + x}.$$

Lemma 3.6. *For an indeterminate t,*

$$\sum_{n\in\mathbb{N}}(-1)^n t^n [n]_q U_{n-1}(x) = \frac{t - t^{-1}}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)}.$$

Proof. Observe that

$$
\begin{aligned}
\sum_{n\in\mathbb{N}}(-1)^n t^n [n]_q U_{n-1}(x) &= \sum_{n\in\mathbb{N}}(-1)^n t^n \frac{q^n - q^{-n}}{q - q^{-1}} U_{n-1}(x) \\
&= \sum_{n\in\mathbb{N}}\frac{(-1)^n t^n q^n U_{n-1}(x)}{q - q^{-1}} - \sum_{n\in\mathbb{N}}\frac{(-1)^n t^n q^{-n} U_{n-1}(x)}{q - q^{-1}} \\
&= \frac{1}{q - q^{-1}}\frac{-1}{qt + q^{-1}t^{-1} + x} - \frac{1}{q - q^{-1}}\frac{-1}{q^{-1}t + qt^{-1} + x} \\
&= \frac{t - t^{-1}}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)}.
\end{aligned}
$$
∎

4   Some identities

In this section we give some identities for later use.

Lemma 4.1. *For $r \in \mathbb{Z}$,*

$$[r - 1]_q - \left(q + q^{-1}\right)[r]_q + [r + 1]_q = 0.$$

Proof. Use (2.1). ∎

Lemma 4.2. *For $r, s \in \mathbb{Z}$ we have*

$$[r - 1]_q[s - 1]_q[r - s]_q + [r]_q[s]_q[r - s]_q = [r - 1]_q[s]_q[r - s + 1]_q + [r]_q[s - 1]_q[r - s - 1]_q.$$

Proof. Expand each side using (2.1). ∎

Lemma 4.3. *For an indeterminate t,*

$$\sum_{\ell\in\mathbb{N}} t^{2\ell} = \frac{-t^{-1}}{t - t^{-1}}, \qquad \sum_{\ell\in\mathbb{N}} \ell t^{2\ell} = \frac{1}{\left(t - t^{-1}\right)^2},$$

$$\sum_{\ell\in\mathbb{N}} \ell^2 t^{2\ell} = -\frac{t + t^{-1}}{\left(t - t^{-1}\right)^3}, \qquad \sum_{\ell\in\mathbb{N}} \binom{\ell + 1}{2} t^{2\ell+1} = \frac{-1}{\left(t - t^{-1}\right)^3}.$$

Proof. These are readily checked. ∎

5   The main results

In this section we express the images (2.8) in the basis (2.16). In what follows, the notation $[u, v]$ means $uv - vu$. We will use a recursion found in [13]; we give a short proof for the sake of completeness.

Lemma 5.1 (see [13, Section 3.1])**.** *In the algebra* $\mathcal{O}_q$,

$$B_{\alpha_0} = A, \qquad B_{\delta+\alpha_0} = B + \frac{q[B_\delta, A]}{(q - q^{-1})(q^2 - q^{-2})}, \tag{5.1}$$

$$B_{n\delta+\alpha_0} = B_{(n-2)\delta+\alpha_0} + \frac{q[B_\delta, B_{(n-1)\delta+\alpha_0}]}{(q - q^{-1})(q^2 - q^{-2})}, \qquad n \geq 2 \tag{5.2}$$

and also

$$B_{\alpha_1} = B, \qquad B_{\delta+\alpha_1} = A - \frac{q[B_\delta, B]}{(q - q^{-1})(q^2 - q^{-2})}, \tag{5.3}$$

$$B_{n\delta+\alpha_1} = B_{(n-2)\delta+\alpha_1} - \frac{q[B_\delta, B_{(n-1)\delta+\alpha_1}]}{(q - q^{-1})(q^2 - q^{-2})}, \qquad n \geq 2. \tag{5.4}$$

Proof. We show that

$$T_0(B) = B + \frac{q[B_\delta, A]}{(q - q^{-1})(q^2 - q^{-2})}, \tag{5.5}$$

$$T_1^{-1}(A) = A - \frac{q[B_\delta, B]}{(q - q^{-1})(q^2 - q^{-2})}. \tag{5.6}$$

To verify (5.5) (resp. (5.6)) eliminate B_δ using $B_\delta = q^{-2}BA - AB$ and compare the result with (2.4) (resp. (2.7)). Lines (5.1), (5.3) follow from (5.5), (5.6) and the construction. Now consider (5.2), (5.4). First assume that $n = 2r + 1$ is odd. To verify (5.2), apply $(T_0T_1)^r$ to each side of (5.5), and use (2.10) along with $T_1(B) = B$. To verify (5.4), apply $(T_0T_1)^{-r}$ to each side of (5.6), and use (2.10) along with $T_0(A) = A$. Next assume that $n = 2r$ is even. To verify (5.2), apply $(T_0T_1)^r$ to each side of (5.6), and use (2.10) along with $T_0(A) = A$, $T_1(B) = B$. To verify (5.4), apply $(T_0T_1)^{-r}$ to each side of (5.5), and use (2.10) along with $T_0(A) = A$, $T_1(B) = B$. ∎

Lemma 5.2. *In the algebra* Δ_q,

$$B_\delta = q^{-1}(q^2 - q^{-2})C - q^{-1}(q - q^{-1})\gamma. \tag{5.7}$$

Proof. Simplify (2.13) using $qAB - q^{-1}BA = -qB_\delta$. ∎

Lemma 5.3. *In the algebra* Δ_q,

$$B_{\alpha_0} = A, \qquad B_{\delta+\alpha_0} = B + \frac{[C, A]}{q - q^{-1}}, \tag{5.8}$$

$$B_{n\delta+\alpha_0} = B_{(n-2)\delta+\alpha_0} + \frac{[C, B_{(n-1)\delta+\alpha_0}]}{q - q^{-1}}, \qquad n \geq 2 \tag{5.9}$$

and also

$$B_{\alpha_1} = B, \qquad B_{\delta+\alpha_1} = A - \frac{[C, B]}{q - q^{-1}}, \tag{5.10}$$

$$B_{n\delta+\alpha_1} = B_{(n-2)\delta+\alpha_1} - \frac{[C, B_{(n-1)\delta+\alpha_1}]}{q - q^{-1}}, \qquad n \geq 2. \tag{5.11}$$

Proof. Evaluate (5.1)–(5.4) using (5.7) and the fact that γ is central in Δ_q. ∎

Lemma 5.4. *In the algebra* Δ_q,

$$\frac{[C, A]}{q - q^{-1}} = -q^{-1}AC - q^{-1}(q + q^{-1})B + q^{-1}\beta, \tag{5.12}$$

$$\frac{[C, B]}{q - q^{-1}} = qBC + q(q + q^{-1})A - q\alpha. \tag{5.13}$$

Proof. These equations are a reformulation of (2.11), (2.12). ∎

The following is our first main result.

Theorem 5.5. *For* $n \geq 0$ *the following hold in* Δ_q:

$$B_{n\delta + \alpha_0} = (-1)^n q^{-n} A U_n(C) + (-1)^n q^{-n-1} B U_{n-1}(C) + (-1)^n \alpha \sum_{j \in \mathbb{N}} q^{2j-n+1} U_{n-2j-2}(C)$$

$$+ (-1)^{n-1} \beta \sum_{j \in \mathbb{N}} q^{2j-n} U_{n-2j-1}(C),$$

$$B_{n\delta + \alpha_1} = (-1)^n q^n B U_n(C) + (-1)^n q^{n+1} A U_{n-1}(C) + (-1)^n \beta \sum_{j \in \mathbb{N}} q^{n-2j-1} U_{n-2j-2}(C)$$

$$+ (-1)^{n-1} \alpha \sum_{j \in \mathbb{N}} q^{n-2j} U_{n-2j-1}(C).$$

Proof. By a routine induction on n, using Lemmas 5.3, 5.4. ∎

The following is our second main result.

Theorem 5.6. *In the algebra* Δ_q, *for* $n \geq 1$ *the element* $B_{n\delta}$ *is equal to* $(-1)^n (1 - q^{-2})$ *times a weighted sum with the following terms and coefficients:*

term	coefficient
Ω	$\sum_{\ell \in \mathbb{N}} [n - 2\ell - 1]_q U_{n-2\ell-2}(C)$
$\alpha\beta$	$\sum_{\ell \in \mathbb{N}} \ell^2 [n - 2\ell]_q U_{n-2\ell-1}(C)$
$\alpha^2 + \beta^2$	$-\sum_{\ell \in \mathbb{N}} \binom{\ell+1}{2} [n - 2\ell - 1]_q U_{n-2\ell-2}(C)$
γ	$[n]_q U_{n-1}(C) + 2 \sum_{\ell \in \mathbb{N}} [n - 2\ell - 2]_q U_{n-2\ell-3}(C)$
1	$-[n+1]_q U_n(C) - [3]_q [n-1]_q U_{n-2}(C) - [2]_q^2 \sum_{\ell \in \mathbb{N}} [n - 2\ell - 3]_q U_{n-2\ell-4}(C)$

Proof. We have some preliminary comments. Using (2.12), (2.13),

$$BA = q^2 AB + q(q^2 - q^{-2})C - q(q - q^{-1})\gamma,$$
$$CA = q^{-2}AC - q^{-1}(q^2 - q^{-2})B + q^{-1}(q - q^{-1})\beta,$$
$$CA^2 = q^{-4}A^2C - q^{-1}(q^4 - q^{-4})AB + q^{-2}(q^2 - q^{-2})A\beta$$
$$\qquad - (q^2 - q^{-2})^2 C + (q - q^{-1})(q^2 - q^{-2})\gamma.$$

By [31, Lemma 6.1],

$$BAC = q\Omega - q^3 A^2 - q^{-1}B^2 - q^{-1}C^2 + q^2 A\alpha + B\beta + C\gamma,$$
$$CAB = q^{-1}\Omega - q^{-3}A^2 - qB^2 - qC^2 + q^{-2}A\alpha + B\beta + C\gamma.$$

We are done with the preliminary comments. We now define some generating functions in an indeterminate t:

$$\Phi(t) = \sum_{n=0}^{\infty} t^n B_{n\delta+\alpha_1}, \qquad \Psi(t) = \sum_{n=1}^{\infty} t^n B_{n\delta}. \tag{5.14}$$

By (2.9),

$$\Psi(t) = q^{-2} t \Phi(t) A - t A \Phi(t) + (q^{-2} - 1) t^2 (\Phi(t))^2. \tag{5.15}$$

By (5.10), (5.11),

$$\frac{[C, \Phi(t)]}{q - q^{-1}} = A + t^{-1} B + (t - t^{-1}) \Phi(t). \tag{5.16}$$

We next consider what the second equation in Theorem 5.5 implies about $\Phi(t)$. Using Lemma 3.4,

$$\sum_{n\in\mathbb{N}} (-1)^n q^n t^n U_n(x) = \frac{q^{-1} t^{-1}}{qt + q^{-1} t^{-1} + x}.$$

Using Lemma 3.5,

$$\sum_{n\in\mathbb{N}} (-1)^n q^{n+1} t^n U_{n-1}(x) = \frac{-q}{qt + q^{-1} t^{-1} + x}.$$

We have

$$\sum_{n\in\mathbb{N}} (-1)^n t^n \sum_{j\in\mathbb{N}} q^{n-2j-1} U_{n-2j-2}(x)$$

$$= \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{N}} (-1)^n t^n q^{n-2j-1} U_{n-2j-2}(x)$$

$$= -\sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{N}} (-1)^{n-2j-1} t^{n-2j-1} q^{n-2j-1} U_{n-2j-2}(x) t^{2j+1}$$

$$= -\sum_{N\in\mathbb{N}} \sum_{j\in\mathbb{N}} (-1)^N t^N q^N U_{N-1}(x) t^{2j+1} \qquad \text{(change var. } N = n - 2j - 1\text{)}$$

$$= -\left(\sum_{N\in\mathbb{N}} (-1)^N t^N q^N U_{N-1}(x)\right)\left(\sum_{j\in\mathbb{N}} t^{2j+1}\right)$$

$$= \frac{-1}{qt + q^{-1} t^{-1} + x} \frac{1}{t - t^{-1}}$$

$$= \frac{-1}{(t - t^{-1})(qt + q^{-1} t^{-1} + x)}.$$

Similarly,

$$\sum_{n\in\mathbb{N}} (-1)^{n-1} t^n \sum_{j\in\mathbb{N}} q^{n-2j} U_{n-2j-1}(x) = \frac{-t^{-1}}{(t - t^{-1})(qt + q^{-1} t^{-1} + x)}.$$

By these comments and the second equation in Theorem 5.5,

$$\Phi(t)(qt + q^{-1} t^{-1} + C) = q^{-1} t^{-1} B - qA - \frac{\beta}{t - t^{-1}} - \frac{t^{-1}\alpha}{t - t^{-1}}. \tag{5.17}$$

By (5.16) and (5.17),

$$\left(q^{-1}t + qt^{-1} + C\right)\Phi(t) = qt^{-1}B - q^{-1}A - \frac{\beta}{t - t^{-1}} - \frac{t^{-1}\alpha}{t - t^{-1}}. \tag{5.18}$$

In (5.15), we multiply each side on the left by $q^{-1}t + qt^{-1} + C$ and on the right by $qt + q^{-1}t^{-1} + C$. We evaluate the result using (5.17), (5.18) to obtain

$$\begin{aligned}
\left(q^{-1}t + qt^{-1} + C\right)&\Psi(t)\left(qt + q^{-1}t^{-1} + C\right) \\
&= q^{-2}t\left(qt^{-1}B - q^{-1}A - \frac{\beta}{t - t^{-1}} - \frac{t^{-1}\alpha}{t - t^{-1}}\right)A\left(qt + q^{-1}t^{-1} + C\right) \\
&\quad - t\left(q^{-1}t + qt^{-1} + C\right)A\left(q^{-1}t^{-1}B - qA - \frac{\beta}{t - t^{-1}} - \frac{t^{-1}\alpha}{t - t^{-1}}\right) \\
&\quad + \left(q^{-2} - 1\right)t^2\left(qt^{-1}B - q^{-1}A - \frac{\beta}{t - t^{-1}} - \frac{t^{-1}\alpha}{t - t^{-1}}\right) \\
&\quad \times \left(q^{-1}t^{-1}B - qA - \frac{\beta}{t - t^{-1}} - \frac{t^{-1}\alpha}{t - t^{-1}}\right).
\end{aligned}$$

Evaluating the above equation using the preliminary comments, we find that

$$\left(q^{-1}t + qt^{-1} + C\right)\Psi(t)\left(qt + q^{-1}t^{-1} + C\right) \tag{5.19}$$

is equal to $1 - q^{-2}$ times

$$\Omega - \frac{\left(t + t^{-1}\right)\alpha\beta}{\left(t - t^{-1}\right)^2} - \frac{\alpha^2 + \beta^2}{\left(t - t^{-1}\right)^2} - \left(t + t^{-1}\right)\gamma + \left(q + q^{-1}\right)\left(t + t^{-1}\right)C + C^2.$$

Consequently $\Psi(t)$ is equal to $1 - q^{-2}$ times

$$F_1(t, C)\Omega + F_2(t, C)\alpha\beta + F_3(t, C)\left(\alpha^2 + \beta^2\right) + F_4(t, C)\gamma + F_5(t, C),$$

where

$$F_1(t, x) = \frac{1}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)},$$

$$F_2(t, x) = -\frac{t + t^{-1}}{\left(t - t^{-1}\right)^2\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)},$$

$$F_3(t, x) = \frac{-1}{\left(t - t^{-1}\right)^2\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)},$$

$$F_4(t, x) = -\frac{t + t^{-1}}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)},$$

$$F_5(t, x) = \frac{\left(q + q^{-1}\right)\left(t + t^{-1}\right)x + x^2}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)}.$$

We now compare the $\{F_i\}_{i=1}^5$ with the coefficients shown in the table of the theorem statement. Concerning F_1,

$$\sum_{n=1}^{\infty}(-1)^n t^n \sum_{\ell \in \mathbb{N}}[n - 2\ell - 1]_q U_{n-2\ell-2}(x)$$

$$= \sum_{n=1}^{\infty}\sum_{\ell \in \mathbb{N}}(-1)^n t^n[n - 2\ell - 1]_q U_{n-2\ell-2}(x)$$

$$= -\sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{N}}(-1)^{n-2\ell-1}t^{n-2\ell-1}[n-2\ell-1]_q U_{n-2\ell-2}(x)t^{2\ell+1}$$

$$= -\sum_{N\in\mathbb{N}}\sum_{\ell\in\mathbb{N}}(-1)^{N}t^{N}[N]_q U_{N-1}(x)t^{2\ell+1} \qquad \text{(change var. } N = n - 2\ell - 1\text{)}$$

$$= -\left(\sum_{N\in\mathbb{N}}(-1)^{N}t^{N}[N]_q U_{N-1}(x)\right)\left(\sum_{\ell\in\mathbb{N}}t^{2\ell+1}\right)$$

$$= \frac{t - t^{-1}}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)}\frac{1}{t - t^{-1}}$$

$$= \frac{1}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)}$$

$$= F_1(t, x).$$

Concerning F_2,

$$\sum_{n=1}^{\infty}(-1)^{n}t^{n}\sum_{\ell\in\mathbb{N}}\ell^2[n-2\ell]_q U_{n-2\ell-1}(x)$$

$$= \sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{N}}(-1)^{n}t^{n}[n-2\ell]_q U_{n-2\ell-1}(x)\ell^2$$

$$= \sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{N}}(-1)^{n-2\ell}t^{n-2\ell}[n-2\ell]_q U_{n-2\ell-1}(x)\ell^2 t^{2\ell}$$

$$= \sum_{N\in\mathbb{N}}\sum_{\ell\in\mathbb{N}}(-1)^{N}t^{N}[N]_q U_{N-1}(x)\ell^2 t^{2\ell} \qquad \text{(change var. } N = n - 2\ell\text{)}$$

$$= \left(\sum_{N\in\mathbb{N}}(-1)^{N}t^{N}[N]_q U_{N-1}(x)\right)\left(\sum_{\ell\in\mathbb{N}}\ell^2 t^{2\ell}\right)$$

$$= -\frac{t - t^{-1}}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)}\frac{t + t^{-1}}{\left(t - t^{-1}\right)^3}$$

$$= -\frac{t + t^{-1}}{\left(t - t^{-1}\right)^2\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)}$$

$$= F_2(t, x).$$

Concerning F_3,

$$-\sum_{n=1}^{\infty}(-1)^{n}t^{n}\sum_{\ell\in\mathbb{N}}\binom{\ell+1}{2}[n-2\ell-1]_q U_{n-2\ell-2}(x)$$

$$= -\sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{N}}(-1)^{n}t^{n}[n-2\ell-1]_q U_{n-2\ell-2}(x)\binom{\ell+1}{2}$$

$$= \sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{N}}(-1)^{n-2\ell-1}t^{n-2\ell-1}[n-2\ell-1]_q U_{n-2\ell-2}(x)\binom{\ell+1}{2}t^{2\ell+1}$$

$$= \sum_{N\in\mathbb{N}}\sum_{\ell\in\mathbb{N}}(-1)^{N}t^{N}[N]_q U_{N-1}(x)\binom{\ell+1}{2}t^{2\ell+1} \qquad \text{(change var. } N = n - 2\ell - 1\text{)}$$

$$= \left(\sum_{N\in\mathbb{N}}(-1)^{N}t^{N}[N]_q U_{N-1}(x)\right)\left(\sum_{\ell\in\mathbb{N}}\binom{\ell+1}{2}t^{2\ell+1}\right)$$

$$= \frac{t - t^{-1}}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)} \frac{-1}{\left(t - t^{-1}\right)^3}$$

$$= \frac{-1}{\left(t - t^{-1}\right)^2 \left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)}$$

$$= F_3(t, x).$$

Concerning F_4,

$$\sum_{n=1}^{\infty} (-1)^n t^n [n]_q U_{n-1}(x)$$

$$= \sum_{n \in \mathbb{N}} (-1)^n t^n [n]_q U_{n-1}(x)$$

$$= \frac{t - t^{-1}}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)} \tag{5.20}$$

and also

$$\sum_{n=1}^{\infty} (-1)^n t^n \sum_{\ell \in \mathbb{N}} [n - 2\ell - 2]_q U_{n-2\ell-3}(x)$$

$$= \sum_{n=1}^{\infty} \sum_{\ell \in \mathbb{N}} (-1)^n t^n [n - 2\ell - 2]_q U_{n-2\ell-3}(x)$$

$$= \sum_{n=1}^{\infty} \sum_{\ell \in \mathbb{N}} (-1)^{n-2\ell-2} t^{n-2\ell-2} [n - 2\ell - 2]_q U_{n-2\ell-3}(x) t^{2\ell+2}$$

$$= \sum_{N \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} (-1)^N t^N [N]_q U_{N-1}(x) t^{2\ell+2} \qquad \text{(change var. } N = n - 2\ell - 2\text{)}$$

$$= \left(\sum_{N \in \mathbb{N}} (-1)^N t^N [N]_q U_{N-1}(x)\right) \left(\sum_{\ell \in \mathbb{N}} t^{2\ell+2}\right)$$

$$= -\frac{t - t^{-1}}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)} \frac{t}{t - t^{-1}}$$

$$= -\frac{t}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)}. \tag{5.21}$$

Note that (5.20) plus twice (5.21) is equal to $F_4(t, x)$.

Concerning F_5,

$$\sum_{n=1}^{\infty} (-1)^n t^n [n + 1]_q U_n(x)$$

$$= -1 + \sum_{n \in \mathbb{N}} (-1)^n t^n [n + 1]_q U_n(x)$$

$$= -1 - t^{-1} \sum_{n \in \mathbb{N}} (-1)^{n+1} t^{n+1} [n + 1]_q U_n(x)$$

$$= -1 - t^{-1} \sum_{N \in \mathbb{N}} (-1)^N t^N [N]_q U_{N-1}(x) \qquad \text{(change var. } N = n + 1\text{)}$$

$$= -1 - \frac{t^{-1}\left(t - t^{-1}\right)}{\left(qt + q^{-1}t^{-1} + x\right)\left(q^{-1}t + qt^{-1} + x\right)} \tag{5.22}$$

and also

$$\sum_{n=1}^{\infty}(-1)^n t^n[n-1]_q U_{n-2}(x)$$

$$= -t\sum_{n=1}^{\infty}(-1)^{n-1}t^{n-1}[n-1]_q U_{n-2}(x)$$

$$= -t\sum_{N\in\mathbb{N}}(-1)^N t^N[N]_q U_{N-1}(x) \qquad (\text{change var. } N=n-1)$$

$$= -\frac{t\big(t-t^{-1}\big)}{\big(qt+q^{-1}t^{-1}+x\big)\big(q^{-1}t+qt^{-1}+x\big)}, \tag{5.23}$$

and also

$$\sum_{n=1}^{\infty}(-1)^n t^n \sum_{\ell\in\mathbb{N}}[n-2\ell-3]_q U_{n-2\ell-4}(x)$$

$$= \sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{N}}(-1)^n t^n[n-2\ell-3]_q U_{n-2\ell-4}(x)$$

$$= -\sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{N}}(-1)^{n-2\ell-3}t^{n-2\ell-3}[n-2\ell-3]_q U_{n-2\ell-4}(x)t^{2\ell+3}$$

$$= -\sum_{N\in\mathbb{N}}\sum_{\ell\in\mathbb{N}}(-1)^N t^N[N]_q U_{N-1}(x)t^{2\ell+3} \qquad (\text{change var. } N=n-2\ell-3)$$

$$= -\left(\sum_{N\in\mathbb{N}}(-1)^N t^N[N]_q U_{N-1}(x)\right)\left(\sum_{\ell\in\mathbb{N}}t^{2\ell+3}\right)$$

$$= \frac{t-t^{-1}}{\big(qt+q^{-1}t^{-1}+x\big)\big(q^{-1}t+qt^{-1}+x\big)}\frac{t^2}{t-t^{-1}}$$

$$= \frac{t^2}{\big(qt+q^{-1}t^{-1}+x\big)\big(q^{-1}t+qt^{-1}+x\big)}. \tag{5.24}$$

Note that (-1) times (5.22) minus $[3]_q$ times (5.23) minus $[2]_q^2$ times (5.24) is equal to $F_5(t,x)$. The result follows from the above comments. ∎

Recall the center $Z(\Delta_q)$.

Corollary 5.7. *For $n\geq 1$ the element $B_{n\delta}$ is contained in the subalgebra of Δ_q generated by C and $Z(\Delta_q)$.*

We finish the paper with some comments.
Here is another version of Theorem 5.5.

Proposition 5.8. *For $n\geq 0$ the following hold in Δ_q:*

$$B_{n\delta+\alpha_0} = (-1)^n q^n U_n(C)A + (-1)^n q^{n+1}U_{n-1}(C)B + (-1)^n\alpha\sum_{j\in\mathbb{N}}q^{n-2j-1}U_{n-2j-2}(C)$$

$$+ (-1)^{n-1}\beta\sum_{j\in\mathbb{N}}q^{n-2j}U_{n-2j-1}(C),$$

$$B_{n\delta+\alpha_1} = (-1)^n q^{-n}U_n(C)B + (-1)^n q^{-n-1}U_{n-1}(C)A + (-1)^n\beta\sum_{j\in\mathbb{N}}q^{2j-n+1}U_{n-2j-2}(C)$$

$$+ (-1)^{n-1}\alpha\sum_{j\in\mathbb{N}}q^{2j-n}U_{n-2j-1}(C).$$

Proof. Similar to the proof of Theorem 5.5.   ■

The following result might be of independent interest.

Proposition 5.9. *For $n \geq 1$ the following holds in Δ_q:*

$$U_n(C)A = q^{-2n}AU_n(C) - q^2(q - q^{-1})A\sum_{\ell \in \mathbb{N}}[2n - 4\ell - 2]_q U_{n-2\ell-2}(C)$$

$$- q^{-1}(q - q^{-1})B\sum_{\ell \in \mathbb{N}}[2n - 4\ell]_q U_{n-2\ell-1}(C)$$

$$+ (q - q^{-1})^2\alpha\sum_{\ell \in \mathbb{N}}[n - 2\ell - 1]_q[\ell + 1]_q[n - \ell]_q U_{n-2\ell-2}(C)$$

$$+ (q - q^{-1})\beta\sum_{\ell \in \mathbb{N}}[n - 2\ell]_q\big(q^{\ell-n}[\ell + 1]_q - q^{n-\ell+1}[\ell]_q\big)U_{n-2\ell-1}(C)$$

and also

$$U_n(C)B = q^{2n}BU_n(C) + q^{-2}(q - q^{-1})B\sum_{\ell \in \mathbb{N}}[2n - 4\ell - 2]_q U_{n-2\ell-2}(C)$$

$$+ q(q - q^{-1})A\sum_{\ell \in \mathbb{N}}[2n - 4\ell]_q U_{n-2\ell-1}(C)$$

$$+ (q - q^{-1})^2\beta\sum_{\ell \in \mathbb{N}}[n - 2\ell - 1]_q[\ell + 1]_q[n - \ell]_q U_{n-2\ell-2}(C)$$

$$- (q - q^{-1})\alpha\sum_{\ell \in \mathbb{N}}[n - 2\ell]_q\big(q^{n-\ell}[\ell + 1]_q - q^{\ell-n-1}[\ell]_q\big)U_{n-2\ell-1}(C).$$

Proof. We use induction on n. For $n = 1$ the equations in the proposition statement are reformulations of (2.11), (2.12). For $n \geq 2$ we proceed as follows. To obtain the first (resp. second) equation in the proposition statement, multiply each side of (2.12) (resp. (2.11)) on the left by $U_{n-1}(C)$, and evaluate the result using $CU_{n-1}(C) = U_n(C) + U_{n-2}(C)$ along with induction and Lemmas 4.1, 4.2.   ■

In the algebra $\mathcal{O}_q$ the elements $\{B_{n\delta}\}_{n=1}^{\infty}$ are defined using the formula (2.9). This formula is not symmetric in α_0, α_1. As shown in [13], there is another formula for $\{B_{n\delta}\}_{n=1}^{\infty}$ that interchanges the roles of α_0, α_1. According to [13, Section 5.2] the following holds in $\mathcal{O}_q$ for $n \geq 1$:

$$B_{n\delta} = q^{-2}BB_{(n-1)\delta+\alpha_0} - B_{(n-1)\delta+\alpha_0}B + (q^{-2} - 1)\sum_{\ell=0}^{n-2}B_{\ell\delta+\alpha_0}B_{(n-\ell-2)\delta+\alpha_0}. \tag{5.25}$$

We now sketch a proof of Theorem 5.6 that uses (5.25) instead of (2.9). Following (5.14), for the algebra Δ_q we define

$$\tilde{\Phi}(t) = \sum_{n=0}^{\infty}t^n B_{n\delta+\alpha_0}. \tag{5.26}$$

By (5.14), (5.25), (5.26) we obtain

$$\Psi(t) = q^{-2}tB\tilde{\Phi}(t) - t\tilde{\Phi}(t)B + (q^{-2} - 1)t^2\big(\tilde{\Phi}(t)\big)^2. \tag{5.27}$$

By (5.8), (5.9),

$$\frac{[\tilde{\Phi}(t), C]}{q - q^{-1}} = t^{-1}A + B + (t - t^{-1})\tilde{\Phi}(t). \tag{5.28}$$

From the first equation in Theorem 5.5 we obtain

$$\tilde{\Phi}(t)\big(q^{-1}t + qt^{-1} + C\big) = qt^{-1}A - q^{-1}B - \frac{\alpha}{t - t^{-1}} - \frac{t^{-1}\beta}{t - t^{-1}}. \tag{5.29}$$

By (5.28) and (5.29),

$$\big(qt + q^{-1}t^{-1} + C\big)\tilde{\Phi}(t) = q^{-1}t^{-1}A - qB - \frac{\alpha}{t - t^{-1}} - \frac{t^{-1}\beta}{t - t^{-1}}. \tag{5.30}$$

In (5.27), we multiply each side on the left by $qt + q^{-1}t^{-1} + C$ and on the right by $q^{-1}t + qt^{-1} + C$. We evaluate the result using (5.29), (5.30) to obtain

$$\big(qt + q^{-1}t^{-1} + C\big)\Psi(t)\big(q^{-1}t + qt^{-1} + C\big)$$
$$= q^{-2}t\big(qt + q^{-1}t^{-1} + C\big)B\left(qt^{-1}A - q^{-1}B - \frac{\alpha}{t - t^{-1}} - \frac{t^{-1}\beta}{t - t^{-1}}\right)$$
$$- t\left(q^{-1}t^{-1}A - qB - \frac{\alpha}{t - t^{-1}} - \frac{t^{-1}\beta}{t - t^{-1}}\right)B\big(q^{-1}t + qt^{-1} + C\big)$$
$$+ \big(q^{-2} - 1\big)t^2\left(q^{-1}t^{-1}A - qB - \frac{\alpha}{t - t^{-1}} - \frac{t^{-1}\beta}{t - t^{-1}}\right)$$
$$\times \left(qt^{-1}A - q^{-1}B - \frac{\alpha}{t - t^{-1}} - \frac{t^{-1}\beta}{t - t^{-1}}\right).$$

Evaluating this equation using

$$BA = q^2 AB + q\big(q^2 - q^{-2}\big)C - q\big(q - q^{-1}\big)\gamma,$$
$$CB = q^2 BC + q\big(q^2 - q^{-2}\big)A - q\big(q - q^{-1}\big)\alpha,$$
$$CB^2 = q^4 B^2 C + q^3\big(q^4 - q^{-4}\big)AB - q^2\big(q^2 - q^{-2}\big)B\alpha$$
$$+ q^4\big(q^2 - q^{-2}\big)^2 C - q^4\big(q - q^{-1}\big)\big(q^2 - q^{-2}\big)\gamma$$

and

$$ABC = q^{-1}\Omega - qA^2 - q^{-3}B^2 - qC^2 + A\alpha + q^{-2}B\beta + C\gamma,$$
$$CBA = q\Omega - q^{-1}A^2 - q^3 B^2 - q^{-1}C^2 + A\alpha + q^2 B\beta + C\gamma$$

we find that

$$\big(qt + q^{-1}t^{-1} + C\big)\Psi(t)\big(q^{-1}t + qt^{-1} + C\big)$$

is equal to $1 - q^{-2}$ times

$$\Omega - \frac{\big(t + t^{-1}\big)\alpha\beta}{\big(t - t^{-1}\big)^2} - \frac{\alpha^2 + \beta^2}{\big(t - t^{-1}\big)^2} - \big(t + t^{-1}\big)\gamma + \big(q + q^{-1}\big)\big(t + t^{-1}\big)C + C^2.$$

After this point, the present proof is the same as the original proof.

Acknowledgements

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