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On Free Field Realizations of $W(2,2)$-Modules

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Abstract. The aim of the paper is to study modules for the twisted Heisenberg–Virasoro algebra $\mathcal{H}$ at level zero as modules for the $W(2,2)$-algebra by using construction from [*J. Pure Appl. Algebra* **219** (2015), 4322–4342, arXiv:1405.1707]. We prove that the irreducible highest weight $\mathcal{H}$-module is irreducible as $W(2,2)$-module if and only if it has a typical highest weight. Finally, we construct a screening operator acting on the Heisenberg–Virasoro vertex algebra whose kernel is exactly $W(2,2)$ vertex algebra.

Key words: Heisenberg–Virasoro Lie algebra; vertex algebra; $W(2,2)$ algebra; screening-operators

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1 Introduction

Lie algebra $W(2,2)$ was first introduced by W. Zhang and C. Dong in [20] as part of a classification of certain simple vertex operator algebras. Its representation theory has been studied in [14, 15, 18, 19] and several other papers. Although $W(2,2)$ is an extension of the Virasoro algebra, its representation theory is very different. This is most notable with highest weight representations. It was shown in [19] that some Verma modules contain a cosingular vector.

Highest weight representation theory of the twisted Heisenberg–Virasoro Lie algebra has also been studied recently. Representations with nontrivial action of C_I have been developed in [6]. Representations at level zero, i.e., with trivial action of C_I were studied in [8] due to their importance in some constructions over the toroidal Lie algebras (see [7, 9]). In this case, a free field realization of highest weight modules along with the fusion rules for a suitable category of modules were obtained in [4].

Irreducible highest weight modules of highest weights $(0,0)$ over these algebras carry the structure of simple vertex operator algebras. Let us denote these vertex operator algebras as $L^{W(2,2)}(c_L, c_W)$ and $L^{\mathcal{H}}(c_L, c_{L,I})$. It was proved in [4] that simple vertex operator algebra $L^{W(2,2)}(c_L, c_W)$ embeds into Heisenberg–Virasoro vertex operator algebra $L^{\mathcal{H}}(c_L, c_{L,I})$ so that $c_W = -24c_{L,I}^2$. As a result each highest weight module over $\mathcal{H}$ is also a $W(2,2)$-module. In this paper we shall completely describe the structure of the irreducible highest weight $\mathcal{H}$-modules as $W(2,2)$-modules. We show that in generic case the resulting $W(2,2)$-module is irreducible. However, in case of a module of highest weight such that associated Verma module over $W(2,2)$ contains cosingular vectors (we shall call this kind of weight atypical), irreducible $\mathcal{H}$-module is reducible over $W(2,2)$. We shall denote the irreducible highest weight $\mathcal{H}$-module

$L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ shortly as $L^{\mathcal{H}}(h, h_I)$. We also use the following notation[1]

$$h_{p,r} = \left(1 - p^2\right)\frac{c_L - 2}{24} + p(p-1) + p\frac{1-r}{2}$$

for $p, r \in \mathbb{Z}_{>0}$. Define

$$\mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I}) = \{(h_{p,r}, (1 \pm p)c_{L,I}) \,|\, p, r \in \mathbb{Z}_{>0}\}.$$

We call a weight (h, h_I) *atypical* for $\mathcal{H}$ (resp. *typical*) if $(h, h_I) \in \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$ (resp. $(h, h_I) \notin \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$). We shall refer to a highest weight module over $\mathcal{H}$ as (a)typical if its highest weight is (a)typical for $\mathcal{H}$.

The next theorem gives a main result of the paper.

Theorem 1.1. *Assume that $c_{L,I} \neq 0$.*

(1) *$L^{\mathcal{H}}(h, h_I)$ is irreducible as a $W(2,2)$-module if and only if*

$$(h, h_I) \notin \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I}).$$

(2) *If $(h, h_I) \in \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$ then $L^{\mathcal{H}}(h, h_I)$ is a non-split extension of two irreducible highest weight $W(2,2)$-modules.*

We recall some aspects of representation theories of infinite-dimensional Lie algebras $\mathcal{H}$ and $W(2,2)$ in Section 2. The main results on the branching rules will be proved in Section 3. From the free field realization in [4] follows that irreducible $\mathcal{H}$-modules are pairwise contragredient. For half of these modules, proofs rely on a $W(2,2)$-homomorphism between Verma modules over $W(2,2)$ and $\mathcal{H}$ which is induced by a homomorphism of vertex operator algebras. The rest is then proved elegantly by passing to contragredients. We also prove a very interesting result that the Verma module for $\mathcal{H}$ with typical highest weight is an infinite direct sum of irreducible $W(2,2)$-modules (cf. Theorem 3.7). This result presents a $W(2,2)$-analogue of certain Feigin–Fuchs modules for the Virasoro algebra (cf. Remark 3.8).

From the results in the paper, we see that the vertex algebra $L^{W(2,2)}(c_L, c_W)$ has many properties similar to the $\mathcal{W}$-algebras appearing in logarithmic conformal field theory (LCFT):

- $L^{W(2,2)}(c_L, c_W)$ admits a free field realization inside of the Heisenberg–Virasoro vertex algebra $L^{\mathcal{H}}(c_L, c_{L,I})$.
- Typical modules are realized as irreducible modules for $L^{\mathcal{H}}(c_L, c_{L,I})$.
- In the atypical case, irreducible $L^{\mathcal{H}}(c_L, c_{L,I})$-modules as $L^{W(2,2)}(c_L, c_W)$-modules have semi–simple rank two.

The singlet vertex algebra $\overline{M(1)}$ has similar properties. $\overline{M(1)}$ is realized as kernel of a screening operator inside the Heisenberg vertex algebra $M(1)$ (cf. [1]). In Section 4 we construct the screening operator

$$S_1\colon\ L^{\mathcal{H}}(c_L, c_{L,I}) \to L^{\mathcal{H}}(1, 0),$$

which commutes with the action of $W(2,2)$-algebra such that

$$\mathrm{Ker}_{L^{\mathcal{H}}(c_L, c_{L,I})}\, S_1 \cong L^{W(2,2)}(c_L, c_W).$$

Our construction uses an extension $\mathcal{V}_{\text{ext}}$ of the vertex algebra $L^{\mathcal{H}}(c_L, c_{L,I})$ by a non-weight module for the Heisenberg–Virasoro vertex algebra. In our forthcoming paper [5], we shall present an explicit realization of $\mathcal{V}_{\text{ext}}$ and apply this construction to the study of intertwining operators and logarithmic modules.

[1] We emphasise a term $\frac{c_L - 2}{24}$ for its importance in a free field realization of $\mathcal{H}$ (see [4] for details).

2 Lie algebra $W(2,2)$ and the twisted Heisenberg–Virasoro Lie algebra at level zero

$W(2,2)$ is a Lie algebra with basis $\{L(n), W(n), C_L, C_W : n \in \mathbb{Z}\}$ over $\mathbb{C}$, and a Lie bracket

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m}\frac{n^3-n}{12}C_L,$$
$$[L(n), W(m)] = (n-m)W(n+m) + \delta_{n,-m}\frac{n^3-n}{12}C_W,$$
$$[W(n), W(m)] = [\cdot, C_L] = [\cdot, C_W] = 0.$$

Highest weight representation theory over $W(2,2)$ was studied in [14, 19]. However, representations treated in these papers have equal central charges $C_L = C_W$. These results have recently been generalised to $C_L \neq C_W$ in [15]. Here we state the most important results. Verma module with central charge (c_L, c_W) and highest weight (h, h_W) is denoted by $V^{W(2,2)}(c_L, c_W, h, h_W)$, its highest weight vector by v_{h,h_W} and irreducible quotient module by $L^{W(2,2)}(c_L, c_W, h, h_W)$.

Recall the definition of a cosingular vector. Homogeneous vector $v \in M$ is called cosingular (or subsingular) if it is not singular in M and if there is a proper submodule $N \subset M$ such that $v + N$ is a singular vector in M/N.

Theorem 2.1 ([15, 19]). *Let $c_W \neq 0$.*

(i) Verma module $V^{W(2,2)}(c_L, c_W, h, h_W)$ is reducible if and only if $h_W = \frac{1-p^2}{24}c_W$ for some $p \in \mathbb{Z}_{>0}$. In that case, there exists a singular vector $u'_p \in \mathbb{C}[W(-1), \dots, W(-p)]v_{h,h_W}$ such that $U(W(2,2))u'_p \cong V^{W(2,2)}(c_L, c_W, h+p, h_W)$.

(ii) A quotient module[2]

$$V^{W(2,2)}(c_L, c_W, h, h_W)/U(W(2,2))u'_p =: \widetilde{L}^{W(2,2)}(c_L, c_W, h_{p,r}, h_W)$$

is reducible if and only if $h = h_{p,r}$ for some $r \in \mathbb{Z}_{>0}$. In that case, there is a cosingular vector $u_{rp} \in V^{W(2,2)}(c_L, c_W, h, h_W)_{h+rp}$ such that $\overline{u_{rp}} := u_{rp} + U(W(2,2))u'_p$ is a singular vector in $\widetilde{L}^{W(2,2)}(c_L, c_W, h_{p,r}, h_W)$ which generates a submodule isomorphic to $L^{W(2,2)}(c_L, c_W, h_{p,r} + rp, h_W)$. The short sequence

$$0 \to L^{W(2,2)}(c_L, c_W, h_{p,r} + rp, h_W) \to \widetilde{L}^{W(2,2)}(c_L, c_W, h_{p,r}, h_W) \to L^{W(2,2)}(c_L, c_W, h_{p,r}, h_W) \to 0, \tag{2.1}$$

where the highest weight vector in $L^{W(2,2)}(c_L, c_W, h_{p,r} + rp, h_W)$ maps to $\overline{u_{rp}}$ is exact.

Define

$$\mathcal{AT}_{W(2,2)}(c_L, c_W) = \left\{ \left(h_{p,r}, \frac{1-p^2}{24}c_W \right) \mid p, r \in \mathbb{Z}_{>0} \right\}.$$

Remark 2.2. We will refer to the (modules of) highest weights $(h, h_W) \in \mathcal{AT}_{W(2,2)}(c_L, c_W)$ as *atypical* for $W(2,2)$, and otherwise as *typical*. Again, we refer to a highest weight $W(2,2)$-module as (a)typical depending on its highest weight. So a Verma module over $W(2,2)$ contains a nontrivial cosingular vector if and only if it is atypical.

Proposition 2.3. *Let $h_W = \frac{1-p^2}{24}c_W$, $p \in \mathbb{Z}_{>0}$.*

[2] This module is denoted by L' in [15, 19]. We change notation to $\widetilde{L}$ due to use of superscript $W(2,2)$.

(i) *Let* $(h_{p,r}, h_W)$, $r \in \mathbb{Z}_{>0}$ *be an atypical weight and* $k \in \mathbb{Z}$. *Then* $(h_{p,r} + kp, h_W)$ *is atypical if and only if* $k < \frac{r}{2}$.

(ii) *Atypical Verma module* $V^{W(2,2)}(h_{p,r}, h_W)$ *contains exactly* $\lfloor \frac{r+1}{2} \rfloor$ *cosingular vectors. The weights of these vectors are* $h_{p,r} + (r-i)p = h_{p,-r+2i}$, $i = 0, \ldots, \lfloor \frac{r-1}{2} \rfloor$.

Proof. (i) Directly from Theorem 2.1 since $h_{p,r} + kp = h_{p,r-2k}$.

(ii) Follows from (i) since $V^{W(2,2)}(h_{p,r}, h_W)$ contains an infinite chain of submodules isomorphic to Verma modules of highest weights $h_{p,r} + ip = h_{p,r-2i}$, $i > 0$. Applying Theorem 2.1 to each of these submodules we obtain cosingular vectors of weights

$$h_{p,r-2i} + (r-2i)p = h_{p,r} + (r-i)p = h_{p,-r+2i}$$

as long as $r - 2i > 0$. ■

Remark 2.4. Standard PBW basis for $V^{W(2,2)}(c_L, c_W, h, h_W)$ consists of vectors

$$W(-m_s) \cdots W(-m_1) L(-n_t) \cdots L(-n_1) v_{h,h_W}$$

such that $m_s \geq \cdots \geq m_1 \geq 1$, $n_t \geq \cdots \geq n_1 \geq 1$. The only nonzero component of u_{rp} belonging to $\mathbb{C}[L(-1), L(-2), \ldots]v$ is $L(-p)^r v_{h,h_W}$ [19].

Define $P_2(n) = \sum_{i=0}^{n} P(n-i)P(i)$ where P is a partition function with $P(0) = 1$. We have the following character formulas [19]

$$\operatorname{char} V^{W(2,2)}(c_L, c_W, h, h_W) = q^h \sum_{n \geq 0} P_2(n) q^n = q^h \prod_{k \geq 1} \left(1 - q^k\right)^{-2},$$

for all $h, h_W \in \mathbb{C}$. If $h_W = \frac{1-p^2}{24} c_W$, then

$$\operatorname{char} \widetilde{L}^{W(2,2)}(c_L, c_W, h, h_W) = q^h \left(1 - q^p\right) \sum_{n \geq 0} P_2(n) q^n = q^h \left(1 - q^p\right) \prod_{k \geq 1} \left(1 - q^k\right)^{-2}.$$

If (h, h_W) is typical for $W(2,2)$, then this is the character of an irreducible highest weight module. Finally, the character of atypical irreducible module is

$$\begin{aligned} \operatorname{char} L^{W(2,2)}(c_L, c_W, h_{p,r}, h_W) &= q^{h_{p,r}} \left(1 - q^p\right)\left(1 - q^{rp}\right) \sum_{n \geq 0} P_2(n) q^n \\ &= q^{h_{p,r}} \left(1 - q^p\right)\left(1 - q^{rp}\right) \prod_{k \geq 1} \left(1 - q^k\right)^{-2}. \end{aligned}$$

The twisted Heisenberg–Virasoro algebra $\mathcal{H}$ is the universal central extension of the Lie algebra of differential operators on a circle of order at most one. It is the infinite-dimensional complex Lie algebra with a basis

$$\{L(n), I(n) \colon n \in \mathbb{Z}\} \cup \{C_L, C_{LI}, C_I\}$$

and commutation relations

$$\begin{aligned} &[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m} \frac{n^3 - n}{12} C_L, \\ &[L(n), I(m)] = -mI(n+m) - \delta_{n,-m}\left(n^2 + n\right) C_{LI}, \\ &[I(n), I(m)] = n\delta_{n,-m} C_I, \qquad [\mathcal{H}, C_L] = [\mathcal{H}, C_{LI}] = [\mathcal{H}, C_I] = 0. \end{aligned}$$

The Lie algebra $\mathcal{H}$ admits the following triangular decomposition

$$\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^0 \oplus \mathcal{H}^+, \tag{2.2}$$
$$\mathcal{H}^\pm = \operatorname{span}_\mathbb{C}\{I(\pm n), L(\pm n) \,|\, n \in \mathbb{Z}_{>0}\}, \qquad \mathcal{H}^0 = \operatorname{span}_\mathbb{C}\{I(0), L(0), C_L, C_{L,I}, C_I\}.$$

Although they seem to be two similar extensions of the Virasoro algebra, representation theories of $W(2,2)$ and $\mathcal{H}$ are different. The main reason for that lies in the fact that $I(0)$ is a central element, while $W(0)$ is not. However, applying free field realization, we shall see that highest weight modules over the two algebras are related.

Denote by $V^\mathcal{H}(c_L, c_I, c_{L,I}, h, h_I)$ the Verma module and by v_{h,h_I} its highest weight vector. C_L, C_I, $C_{L,I}$, $L(0)$ and $I(0)$ act on v_{h,h_I} by scalars c_L, c_I, $c_{L,I}$, h and h_I, respectively. Then $(c_L, c_I, c_{L,I})$ is called a central charge, and (h, h_I) a highest weight. In this paper we consider central charges $(c_L, 0, c_{L,I})$ such that $c_{L,I} \neq 0$.

Theorem 2.5 ([8]). *Let $c_{L,I} \neq 0$. Verma module $V^\mathcal{H}(c_L, 0, c_{L,I}, h, h_I)$ is reducible if and only if $h_I = (1 \pm p)c_{L,I}$ for some $p \in \mathbb{Z}_{>0}$. In that case, there is a singular vector $v_p^\pm$ of weight p, which generates a maximal submodule in $V^\mathcal{H}(c_L, 0, c_{L,I}, h, h_I)$ isomorphic to $V^\mathcal{H}(c_L, 0, c_{L,I}, h + p, h_I)$.*

Remark 2.6. In case $h_I = (1 + p)c_{L,I}$ an explicit formula for a singular vector v_p^+ is obtained using Schur polynomials in $I(-1), \dots, I(-p)$. See [4] for details. Assume that $x \in U(W(2,2))_-$ is such that $xv_{h,h_I} \in V^\mathcal{H}(c_L, 0, c_{L,I}, h, h_I)$ lies in a maximal submodule. Then x does not have a nontrivial additive component (in PBW basis) that belongs to $\mathbb{C}[L(-1), L(-2), \dots]$ [8].

There is an infinite chain of Verma submodules generated by singular vectors $v_{kp}^\pm$, $k \in \mathbb{Z}_{>0}$, with all the subquotients being irreducible. Note that there is no mention of $\widetilde{L}^\mathcal{H}$ since there are no cosingular vectors in $V^\mathcal{H}$.

The following character formulas were obtained in [8]:

$$\operatorname{char} V^\mathcal{H}(c_L, 0, c_L, h, h_I) = q^h \sum_{n \geq 0} P_2(n) q^n = q^h \prod_{k \geq 1} \left(1 - q^k\right)^{-2},$$
$$\operatorname{char} L^\mathcal{H}(c_L, 0, c_L, h, h_I) = q^h \left(1 - q^p\right) \sum_{n \geq 0} P_2(n) q^n = q^h \left(1 - q^p\right) \prod_{k \geq 1} \left(1 - q^k\right)^{-2}.$$

Remark 2.7. Throughout the rest of the paper we work with highest weight modules over the Lie algebras $W(2,2)$ and $\mathcal{H}$ so we always denote algebra in superscript. In order to avoid too cumbersome notation, we omit central charges. Therefore, we write $V^\mathcal{H}(h, h_I)$ for Verma module over $\mathcal{H}$, $V^{W(2,2)}(h, h_W)$ for Verma module over $W(2,2)$ and so on. We always assume that c_W and $c_{L,I}$ are nonzero. Moreover, if we work with several modules over both algebras, c_L is equal for all modules.

We shall write $\langle x \rangle_{W(2,2)}$ for a cyclic submodule $U(W(2,2))x$ and $\langle x \rangle_\mathcal{H}$ for $U(\mathcal{H})x$. Finally, $\cong_{W(2,2)}$ denotes an isomorphism of $W(2,2)$-modules.

3 Irreducible highest weight modules

In this section we present main results of the paper which completely describe the structure of (irreducible) highest weight modules for $\mathcal{H}$ as $W(2,2)$-modules. The main tool is the homomorphism between $W(2,2)$ and the Heisenberg–Virasoro vertex algebras from [4].

$L^{W(2,2)}(c_L, c_W, 0, 0)$ is a simple universal vertex algebra associated to Lie algebra $W(2,2)$ (cf. [19, 20]) which we denote by $L^{W(2,2)}(c_L, c_W)$. It is generated by fields

$$L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \qquad W(z) = Y(W, z) = \sum_{n \in \mathbb{Z}} W(n) z^{-n-2},$$

where $\omega = L(-2)\mathbf{1}$ and $W = W(-2)\mathbf{1}$. Each highest weight $W(2,2)$-module is also a module over a vertex operator algebra $L^{W(2,2)}(c_L, c_W)$.

Likewise (see [7]) $L^{\mathcal{H}}(c_L, 0, c_{L,I}, 0, 0)$ is a simple Heisenberg–Virasoro vertex operator algebra, which we denote by $L^{\mathcal{H}}(c_L, c_{L,I})$. This algebra is generated by the fields

$$L(z) = Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}, \qquad I(z) = Y(I, z) = \sum_{n\in\mathbb{Z}} I(n)z^{-n-1},$$

where $\omega = L(-2)\mathbf{1}$ and $I = I(-1)\mathbf{1}$. Moreover, highest weight $\mathcal{H}$-modules are modules over a vertex operator algebra $L^{\mathcal{H}}(c_L, c_{L,I})$.

It was shown in [4] that there is a monomorphism of vertex operator algebras

$$\begin{aligned} \Psi\colon\ L^{W(2,2)}(c_L, c_W) &\to L^{\mathcal{H}}(c_L, c_{L,I}), \\ \omega &\mapsto L(-2)\mathbf{1}, \\ W &\mapsto (I(-1)^2 + 2c_{L,I}I(-2))\mathbf{1}, \end{aligned} \tag{3.1}$$

where $c_W = -24c_{L,I}^2$. By means of Ψ, each highest weight module over $\mathcal{H}$ becomes an $L^{W(2,2)}(c_L, c_W)$-module and therefore a module over $W(2,2)$. In particular, Ψ induces a nontrivial $W(2,2)$-homomorphism (which we shall denote by the same letter)

$$\Psi\colon\ V^{W(2,2)}(c_L, c_W, h, h_W) \to V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I),$$

where $c_W = -24c_{L,I}^2$ and $h_W = h_I(h_I - 2c_{L,I})$. Ψ maps the highest weight vector v_{h,h_W} to the highest weight vector v_{h,h_I} and the action of $W(-n)$ on $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ is given by

$$\begin{aligned} W(-n) &\equiv 2c_{L,I}(n-1)I(-n) + \sum_{i\in\mathbb{Z}} I(-i)I(-n+i), \\ W(-n) &\equiv 2c_{L,I}\left(n - 1 + \frac{h_I}{c_{L,I}}\right)I(-n) + \sum_{i\neq 0,n} I(-i)I(-n+i). \end{aligned} \tag{3.2}$$

Note that $h_W = \frac{1-p^2}{24}c_W$ if and only if $h_I = (1 \pm p)c_{L,I}$, so either both of these Verma modules are irreducible, or they are reducible with singular vectors at equal levels. Moreover, $(h, h_W) \in \mathcal{AT}_{W(2,2)}(c_L, c_W)$ if and only if $(h, h_I) \in \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$.

Throughout the rest of this section we assume that $c_W = -24c_{L,I}^2$.

Lemma 3.1 ([4, Lemma 7.2])**.** *Suppose that $h_I \neq (1-p)c_{L,I}$ for all $p \in \mathbb{Z}_{>0}$. Then Ψ is an isomorphism of $W(2,2)$-modules. In particular, if $h_I \neq (1 \pm p)c_{L,I}$ for $p \in \mathbb{Z}_{>0}$, then*

$$L^{\mathcal{H}}(h, h_I) \cong_{W(2,2)} L^{W(2,2)}(h, h_W),$$

where $h_W = h_I(h_I - 2c_{L,I})$.

Lemma 3.2. *Suppose that $x \in V^{\mathcal{H}}(h, h_I)$ is $\mathcal{H}$-singular. Then x is $W(2,2)$-singular as well. In particular, if y is a homogeneous vector such that $x = \Psi(y)$, then y is either singular or cosingular vector in $V^{W(2,2)}(h, h_W)$.*

Proof. Let $x \in V^{\mathcal{H}}(h, h_I)$ be a $\mathcal{H}$-singular vector, i.e., $L(k)x = I(k)x = 0$ for all $k \in \mathbb{Z}_{>0}$. From (3.2) we have

$$W(n)x = -2c_{L,I}(n+1)I(n)x + \sum_{i\in\mathbb{Z}} I(-i)I(n+i)x,$$

so $W(n)x = 0$ for all $n \in \mathbb{Z}_{>0}$. Therefore, x is $W(2,2)$-singular. If $x = \Psi(y)$, then $L(k)y, W(k)y \in \operatorname{Ker}\Psi$ for $k > 0$. Therefore $y + \operatorname{Ker}\Psi$ is a singular vector in $V^{W(2,2)}(h, h_W)/\operatorname{Ker}\Psi$. ∎

Theorem 3.3. *Let* $p \in \mathbb{Z}_{>0}$.

(i) *If* $(h, (1+p)c_{L,I})$ *is typical for* $\mathcal{H}$ *(equivalently if* $\left(h, \frac{1-p^2}{24}c_W\right)$ *is typical for* $W(2,2)$*) then*

$$L^{\mathcal{H}}(h,(1+p)c_{L,I}) \cong_{W(2,2)} L^{W(2,2)}\left(h, \frac{1-p^2}{24}c_W\right). \tag{3.3}$$

(ii) *If* $(h_{p,r}, (1+p)c_{L,I}) \in \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$ *(equivalently if* $(h_{p,r}, \frac{1-p^2}{24}c_W) \in \mathcal{AT}_{W(2,2)}(c_L, c_W)$*) then*

$$L^{\mathcal{H}}(h_{p,r},(1+p)c_{L,I}) \cong_{W(2,2)} \widetilde{L}^{W(2,2)}\left(h_{p,r}, \frac{1-p^2}{24}c_W\right)$$

and the short sequence of $W(2,2)$*-modules*

$$\begin{aligned} 0 \to L^{W(2,2)}\left(h_{p,r}+rp, \frac{1-p^2}{24}c_W\right) &\to L^{\mathcal{H}}(h_{p,r},(1+p)c_{L,I}) \\ \to L^{W(2,2)}\left(h_{p,r}, \frac{1-p^2}{24}c_W\right) &\to 0 \end{aligned} \tag{3.4}$$

is exact.

Proof. By Lemma 3.1, Ψ is an isomorphism of Verma modules and thus by Lemma 3.2 it maps a $W(2,2)$-singular vector u'_p to an $\mathcal{H}$-singular vector v_p^+. If $h \neq h_{p,r}$, both of these vectors generate maximal submodules in respective Verma modules so (3.3) follows.

Now suppose that $h = h_{p,r}$. We need to show that a cosingular vector u_{rp} is not mapped into a maximal submodule of $V^{\mathcal{H}}(h_{p,r}, h_I)$. But u_{rp} has $L(-p)^r v$ as an additive component (see Remark 2.4), and by construction (3.1), $\Psi(u_{rp})$ also must have this additive component. However, $\Psi(u_{rp})$ can not lie in a maximal $\mathcal{H}$-submodule of $V^{\mathcal{H}}(h, h_I)$ (see Remark 2.6). This means that isomorphism Ψ of Verma modules induces a $W(2,2)$-isomorphism of $\widetilde{L}^{W(2,2)}(h, h_W)$ and $L^{\mathcal{H}}(h, h_I)$ for all $h \in \mathbb{C}$. Exactness of (3.4) is just an application of (2.1). ■

Remark 3.4. Note that the image $\Psi(u_{rp})$ of a $W(2,2)$-cosingular vector is neither $\mathcal{H}$-singular, nor $\mathcal{H}$-cosingular in $V^{\mathcal{H}}(h_{p,r}, (1+p)c_{L,I})$. For example, $L(-1)v_{0,0}$ in $V^{\mathcal{H}}(0, 2c_{L,I})$ is $W(2,2)$-cosingular, but not $\mathcal{H}$-singular since $I(1)L(-1)v_{0,0} = 2c_{L,I}v_{0,0}$.

If $h_I = (1-p)c_{L,I}$, then Ψ is not an isomorphism. We shall present a $W(2,2)$-structure of Verma module later. In order to examine irreducible $W(2,2)$-modules we apply the properties of contragredient modules.

Let us recall the definition of contragredient module (see [12]). Assume that (M, Y_M) is a graded module over a vertex operator algebra V such that $M = \oplus_{n=0}^{\infty} M(n)$, $\dim M(n) < \infty$ and suppose that there is $\gamma \in \mathbb{C}$ such that $L(0)|M(n) \equiv (\gamma + n)\,\mathrm{Id}$. The contragredient module (M^*, Y_{M^*}) is defined as follows. For every $n \in \mathbb{Z}_{>0}$ let $M(n)^*$ be the dual vector space and let $M^* = \oplus_{n=0}^{\infty} M(n)^*$ be a restricted dual of M. Consider the natural pairing $\langle \cdot, \cdot \rangle : M^* \otimes M \to \mathbb{C}$. Define the linear map $Y_{M^*} : V \to \operatorname{End} M^*[[z, z^{-1}]]$ such that

$$\langle Y_{M^*}(v,z)m', m\rangle = \langle m', Y_M\big(e^{zL(1)}\big(-z^{-2}\big)^{L(0)}v, z^{-1}\big)m\rangle \tag{3.5}$$

for each $v \in V$, $m \in M$, $m' \in M^*$. Then (M^*, Y_{M^*}) is a V-module.

In particular, choosing $v = \omega = L_{-2}\mathbf{1}$ in (3.5) one gets

$$\langle L(n)m', m\rangle = \langle m', L(-n)m\rangle.$$

Simple calculation with $I \in L^{\mathcal{H}}(c_L, c_{L,I})$ and $W \in L^{W(2,2)}(c_L, c_W)$ shows that

$$\langle I(n)m', m\rangle = \langle m', -I(-n)m + \delta_{n,0}2c_{L,I}\rangle, \qquad \langle W(n)m', m\rangle = \langle m', W(-n)m\rangle.$$

Therefore we get the following result (the first and third relations were given in [4]):

Lemma 3.5.

$$L^{\mathcal{H}}(h,h_I)^* \cong L^{\mathcal{H}}(h,-h_I+2c_{L,I}), \qquad L^{W(2,2)}(h,h_W)^* \cong L^{W(2,2)}(h,h_W).$$

In particular,

$$L^{\mathcal{H}}(h,(1\pm p)c_{L,I})^* \cong L^{\mathcal{H}}(h,(1\mp p)c_{L,I}).$$

Directly from Theorem 3.3 and Lemma 3.5 follows

Corollary 3.6. *Let* $p \in \mathbb{Z}_{>0}$.

(i) *If* $(h,(1-p)c_{L,I})$ *is typical for* $\mathcal{H}$ *(equivalently if* $\left(h,\frac{1-p^2}{24}c_W\right)$ *is typical for* $W(2,2)$*) then*

$$L^{\mathcal{H}}(h,(1-p)c_{L,I}) \cong_{W(2,2)} L^{W(2,2)}\left(h,\frac{1-p^2}{24}c_W\right).$$

(ii) *If* $(h_{p,r},(1-p)c_{L,I}) \in \mathcal{AT}_{\mathcal{H}}(c_L,c_{L,I})$ *(equivalently if* $\left(h_{p,r},\frac{1-p^2}{24}c_W\right) \in \mathcal{AT}_{W(2,2)}(c_L,c_W)$*) then*

$$L^{\mathcal{H}}(h_{p,r},(1-p)c_{L,I}) \cong_{W(2,2)} \widetilde{L}^{W(2,2)}\left(h_{p,r},\frac{1-p^2}{24}c_W\right)^*$$

and the short sequence of $W(2,2)$*-modules*

$$0 \to L^{W(2,2)}\left(h_{p,r},\frac{1-p^2}{24}c_W\right) \to L^{\mathcal{H}}\left(h_{p,r},(1-p)c_{L,I}\right) \to L^{W(2,2)}\left(h_{p,r}+rp,\frac{1-p^2}{24}c_W\right) \to 0$$

is exact.

From Lemma 3.1, Theorem 3.3 and Corollary 3.6 follow assertions of Theorem 1.1.

Finally, we show that Verma module over $\mathcal{H}$ is an infinite direct sum of irreducible $W(2,2)$-modules. Recall that $V^{\mathcal{H}}(h,(1-p)c_{L,I})$ has a series of singular vectors v^-_{ip}, $i \in \mathbb{Z}_{\geq 0}$ (for $i=0$, we set $v^-_0 = v_{h,h_I}$) which generate a descending chain of Verma submodules over $\mathcal{H}$:

$$\begin{array}{rl}
\langle v_{h,h_I}\rangle_{\mathcal{H}} & = V^{\mathcal{H}}(h,h_I) \\
\cup\!\mid & \\
\langle v^-_p\rangle_{\mathcal{H}} & \cong V^{\mathcal{H}}(h+p,h_I) \\
\cup\!\mid & \\
\vdots & \\
\cup\!\mid & \\
\langle v^-_{ip}\rangle_{\mathcal{H}} & \cong V^{\mathcal{H}}(h+ip,h_I) \\
\cup\!\mid & \\
\langle v^-_{(i+1)p}\rangle_{\mathcal{H}} & \cong V^{\mathcal{H}}(h+(i+1)p,h_I) \\
\cup\!\mid & \\
\vdots &
\end{array}$$

Therefore one may identify $V^{\mathcal{H}}(h+ip,h_I)$ with a submodule of $V^{\mathcal{H}}(h,h_I)$ and a singular vector $v^-_{ip} \in V^{\mathcal{H}}(h,h_I)$ with the highest weight vector $v_{h+ip,h_I} \in V^{\mathcal{H}}(h+ip,h_I)$. We will prove that in a typical case each of those vectors generates an irreducible $W(2,2)$-submodule.

Theorem 3.7. *Let $p \in \mathbb{Z}_{>0}$. Suppose that $(h, (1-p)c_{L,I}) \notin \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$. Then we have the following isomorphism of $W(2,2)$-modules*

$$V^{\mathcal{H}}(h, (1-p)c_{L,I}) \cong_{W(2,2)} \bigoplus_{i \geq 0} L^{W(2,2)}\left(h + ip, \frac{1-p^2}{24}c_W\right).$$

Proof. First we notice that the vertex algebra homorphism $\Psi\colon L^{W(2,2)}(c_W, c_L) \to L^{\mathcal{H}}(c_W, c_L)$, for every $i \in \mathbb{Z}_{\geq 0}$ induces the following non-trivial homomorphism of $W(2,2)$-modules:

$$\Psi^{(i)}\colon\ V^{W(2,2)}\left(h + ip, \frac{1-p^2}{24}c_W\right) \to \langle v_{ip}^- \rangle_{W(2,2)} \subset V^{\mathcal{H}}(h + ip, (1-p)c_{L,I}),$$

which maps the highest weight vector of $V^{W(2,2)}\big(h + ip, \frac{1-p^2}{24}c_W\big)$ to v_{ip}^-. Since $(h, \frac{1-p^2}{24}c_W)$ is typical it follows from Proposition 2.3(i) that $(h + ip, \frac{1-p^2}{24}c_W)$ are typical for all $i \in \mathbb{Z}_{>0}$ as well.

Let $h_W = \frac{1-p^2}{24}c_W$. Consider the homomorphism $\Psi^{(i)}\colon V^{W(2,2)}(h+ip, h_W) \to V^{\mathcal{H}}(h+ip, h_I)$ above. Applying (3.2), we get

$$\Psi^{(i)}(W(-p)v_{h+ip,h_W}) = \sum_{i=1}^{p-1} I(-i)I(i-p)v_{h+ip,h_I},$$

so $I(-p)v_{h+ip,h_I} \notin \operatorname{Im}\Psi^{(i)}$. Since the Verma modules $V^{W(2,2)}(h + ip, h_W)$ and $V^{\mathcal{H}}(h + ip, h_I)$ have equal characters, it follows that $\operatorname{Ker}\Psi^{(i)}$ contains a singular vector in $V^{W(2,2)}(h + ip, h_W)$ of conformal weight $h+(i+1)p$. Since the weight $(h+ip, h_W)$ is typical, the maximal submodule in $V^{W(2,2)}(h + ip, h_W)$ is generated by this singular vector so we conclude that $\operatorname{Ker}\Psi^{(i)}$ is the maximal submodule in $V^{W(2,2)}(h + ip, h_W)$. Therefore

$$\operatorname{Im}\Psi^{(i)} = \langle v_{h+ip,h_I} \rangle_{W(2,2)} \cong L^{W(2,2)}(h + ip, h_W).$$

In this way we get a series of $W(2,2)$-monomorphisms

$$L^{W(2,2)}(h + ip, h_W) \hookrightarrow V^{\mathcal{H}}(h, (1-p)c_{L,I}), \qquad i \in \mathbb{Z}_{\geq 0} \tag{3.6}$$

mapping v_{h+ip,h_W} to a singular vector v_{ip}^-. Let v_{jp}^- be an $\mathcal{H}$-singular vector in $V^{\mathcal{H}}(h + ip, (1-p)c_{L,I})$ of weight $h + jp$, for $j > i$. By Lemma 3.2, v_{jp}^- is singular for $W(2,2)$ and therefore $v_{jp}^- \notin \langle v_{h+ip,h_I} \rangle_{W(2,2)}$ for $j > i$. We conclude that the images of morphisms (3.6) have trivial pairwise intersections (since these images are non-isomorphic irreducible modules), so their sum is direct. The assertion follows from the observation that the character of this sum is

$$\sum_{i=0}^{\infty} q^{h+ip}(1-q^p)\prod_{k\geq 1}\left(1-q^k\right)^{-2} = q^h \prod_{k \geq 1}\left(1-q^k\right)^{-2} = \operatorname{char} V^{\mathcal{H}}(h, (1-p)c_{L.I}). \qquad \blacksquare$$

Remark 3.8. It is interesting to notice that our Theorem 3.7 shows that $V^{\mathcal{H}}(h, h_I)$ can be considered as a $W(2,2)$-analogue of certain Feigin–Fuchs modules for the Virasoro algebra which are also direct sums of infinitely many irreducible modules (cf. [11], [2, Theorem 5.1]).

From the previous theorem follows

$$\begin{array}{c} V^{\mathcal{H}}(h, h_I) \\ \| \\ \langle v_{h,h_I} \rangle_{W(2,2)} = L^{W(2,2)}(h, h_W) \\ \oplus \end{array}$$

$$\begin{aligned}\langle v_p^-\rangle_{W(2,2)} &\cong L^{W(2,2)}(h+p,h_W)\\ &\oplus\\ &\vdots\\ &\oplus\\ \langle v_{ip}^-\rangle_{W(2,2)} &\cong L^{W(2,2)}(h+ip,h_W)\\ &\oplus\\ &\vdots\end{aligned}$$

In atypical case however, the $W(2,2)$-submodules generated by $\mathcal{H}$–singular vectors are nested as follows. Consider $V^{\mathcal{H}}(h_{p,r},h_I)$ where $(h_{p,r},h_I)\in\mathcal{AT}_{\mathcal{H}}(c_L,c_{L,I})$. Then Ψ^0 maps a cosingular vector $u_{rp}\in V^{W(2,2)}(h_{p,r},h_W)$ to a singular vector v_{rp}^-. In other words we have

$$\langle v_{rp}^-\rangle_{W(2,2)}\subseteq\langle v_{h_{p,r},h_I}\rangle\cong_{W(2,2)}\widetilde{L}^{W(2,2)}(h_{p,r},h_W).$$

Using the same argument in view of Proposition 2.3 we see that

$$\langle v_{(r-i)p}^-\rangle_{W(2,2)}\subseteq\langle v_{ip}^-\rangle_{W(2,2)}\cong\widetilde{L}^{W(2,2)}(h_{p,r}+ip,h_W),\qquad i=1,\dots,\lfloor\frac{r-1}{2}\rfloor.$$

Therefore,

$$\begin{aligned}&\langle v_{h_{p,r},h_I}\rangle_{W(2,2)}/\langle v_{rp}^-\rangle_{W(2,2)}\cong L^{W(2,2)}(h_{p,r},h_W),\\ &\langle v_{ip}^-\rangle_{W(2,2)}/\langle v_{(r-i)p}^-\rangle_{W(2,2)}\cong L^{W(2,2)}(h_{p,r}+ip,h_W),\qquad i<\frac{r-1}{2},\\ &\langle v_{ip}^-\rangle_{W(2,2)}\cong L^{W(2,2)}(h_{p,r}+ip,h_W),\qquad i\geq\frac{r-1}{2}.\end{aligned}$$

In this case, $I(-p)^{r-i}v_{h_{p,r},h_I}$ are $W(2,2)$-cosingular vectors in $V^{\mathcal{H}}(h_{p,r},h_I)$.

Example 3.9. Consider $p=1$ case. Singular vector in $V^{\mathcal{H}}(h,0)$ is $u_1'=\big(L(-1)+\frac{h}{c_{L,I}}I(-1)\big)v_{h,0}$, and u_1' generates a copy of $V^{\mathcal{H}}(h+1,0)$.

$r=1$: $\Psi\colon V^{W(2,2)}(0,0)\to V^{\mathcal{H}}(0,0)$ maps a singular vector $u_1'=W(-1)v_{0,0}$ to 0 and a cosingular vector $u_1=L(-1)v_{0,0}$ to $\mathcal{H}$-singular vector $v_1^-=L(-1)v_{0,0}$. We get the short exact sequence of $W(2,2)$-modules

$$0\to L^{W(2,2)}(0,0)\to L^{\mathcal{H}}(0,0)\to L^{W(2,2)}(1,0)\to 0,$$

which is an expansion of (3.1) considered as a homomorphism of $W(2,2)$-modules. The rightmost module is generated by a projective image of $I(-1)v_{0,0}$. Therefore, $L^{\mathcal{H}}(c_L,c_{L,I})$ is generated over $W(2,2)$ by $v_{0,0}$ and $I(-1)v_{0,0}$.

$r\in\mathbb{Z}_{>0}$: In general, a cosingular vector $u_{rp}\in V^{W(2,2)}\big(\frac{1-r}{2},0\big)$ maps to a singular vector $v_r^-\in V^{\mathcal{H}}\big(\frac{1-r}{2},0\big)$ of weight $\frac{1+r}{2}$.

$$v_r^-=\prod_{i=0}^{r-1}\left(L(-1)+\frac{1-r+2i}{2c_{L,I}}I(-1)\right)v_{\frac{1-r}{2},0}.$$

4 Screening operators and $W(2,2)$-algebra

We think that the vertex algebra $L^{W(2,2)}(c_L,c_W)$ is a very interesting example of non-rational vertex algebra, which admits similar fusion ring of representations as some $\mathcal{W}$-algebras appearing in LCFT (cf. [1, 2, 10, 13]). Since $\mathcal{W}$-algebras appearing in LCFT are realized as kernels of screening operators acting on certain modules for Heisenberg vertex algebras, it is natural to ask if $L^{W(2,2)}(c_L, c_W)$ admits similar realization. In [4] we embedded the $W(2,2)$-algebra as a subalgebra of the Heisenberg–Virasoro vertex algebra. In this section we shall construct a screening operator S_1 such that the kernel of this operator is exactly $L^{W(2,2)}(c_L, c_W)$.

Let us first construct a non-semisimple extension of the vertex algebra $L^{\mathcal{H}}(c_L, c_{L,I})$. Recall that the Lie algebra $\mathcal{H}$ admits the triangular decomposition (2.2). Let $E = \mathrm{span}_{\mathbb{C}}\{v^0, v^1\}$ be 2-dimensional $\mathcal{H}^{\geq 0} = \mathcal{H}^0 \oplus \mathcal{H}^+$-module such that $\mathcal{H}^+$ acts trivially on E and

$$L(0)v^i = v^i, \qquad i = 0,1, \qquad I(0)v^1 = v^0, \qquad I(0)v^0 = 0,$$
$$C_L v^i = c_L v^i, \qquad C_{L,I} v^i = c_{L,I} v^i, \qquad C_I v^i = 0, \qquad i = 1,2.$$

Consider now induced $\mathcal{H}$-module

$$\widetilde{E} = U(\mathcal{H}) \otimes_{U(\mathcal{H}^{\geq 0})} E.$$

By construction, $\widetilde{E}$ is a non-split self-extension of the Verma module $V^{\mathcal{H}}(1,0)$:

$$0 \to V^{\mathcal{H}}(1,0) \to \widetilde{E} \to V^{\mathcal{H}}(1,0) \to 0.$$

Moreover, $\widetilde{E}$ is a restricted module for $\mathcal{H}$ and therefore it is a module over vertex operator algebra $L^{\mathcal{H}}(c_L, c_{L,I})$. Since

$$\widetilde{E} \cong E \otimes U(\mathcal{H}^-)$$

as a vector space, the operator $L(0)$ defines a $\mathbb{Z}_{\geq 0}$-gradation on $\widetilde{E}$.

Note that $(L(-1) + I(-1)/c_{L,I})v_0$ is a singular vector in $\widetilde{E}$ and it generates the proper submodule. Finally we define the quotient module

$$\mathcal{U} = \frac{\widetilde{E}}{U(\mathcal{H}).(L(-1) + I(-1)/c_{L,I})v_0}.$$

Proposition 4.1. *$\mathcal{U}$ is a $\mathbb{Z}_{\geq 0}$-graded module for the vertex operator algebra $L^{\mathcal{H}}(c_L, c_{L,I})$:*

$$\mathcal{U} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathcal{U}(m), \qquad L(0)|\mathcal{U}(m) \equiv (m+1)\,\mathrm{Id}\,.$$

The lowest component $\mathcal{U}(0) \cong E$. Moreover, $\mathcal{U}$ is a non-split extension of the Verma module $V^{\mathcal{H}}(1,0)$ by the simple highest weight module $L^{\mathcal{H}}(1,0)$:

$$0 \to L^{\mathcal{H}}(1,0) \to \mathcal{U} \to V^{\mathcal{H}}(1,0) \to 0.$$

Proof. By construction $\mathcal{U}$ is a graded quotient of a $\mathbb{Z}_{\geq 0}$-graded $L^{\mathcal{H}}(c_L, c_{L,I})$-module $\widetilde{E}$. The lowest component is $\mathcal{U}(0) \cong E$. Submodule $U(\mathcal{H}).v^0$ is isomorphic to $L^{\mathcal{H}}(1,0)$, and the projective image of v^1 generates the Verma module $V^{\mathcal{H}}(1,0)$ since $I(0)v^1 = v^0$. For the same reason, this exact sequence does not split. ■

Now we consider $L^{\mathcal{H}}(c_L, c_{L,I})$-module

$$\mathcal{V}^{\mathrm{ext}} := L^{\mathcal{H}}(c_L, c_{L,I}) \oplus \mathcal{U}.$$

By using [16, Theorem 4.8.1] (see also [3, 17]) we have that $\mathcal{V}^{\mathrm{ext}}$ has the structure of a vertex operator algebra with vertex operator map Y_{ext} defined as follows:

$$Y_{\mathrm{ext}}(a_1 + w_1, z)(a_2 + w_2) = Y(a_1, z)(a_2 + w_2) + e^{zL(-1)} Y(a_2, -z) w_1,$$

where $a_1, a_2 \in L^{\mathcal{H}}(c_L, c_{L,I})$, $w_1, w_2 \in \mathcal{U}$.

Take now $v^i \in E \subset \mathcal{U}$, $i = 0, 1$ as above and define

$$S_i(z) = Y_{\text{ext}}\big(v^i, z\big) = \sum_{n\in\mathbb{Z}} S_i(n) z^{-n-1}.$$

By construction

$$S_1(z) \in \operatorname{End}\big(L^{\mathcal{H}}(c_L, c_{L,I}), L^{\mathcal{H}}(1,0)\big)((z)).$$

Proposition 4.2. *For all $n, m \in \mathbb{Z}$ we have:*

$$[L(n), S_i(m)] = -mS_i(n+m), \qquad i = 0, 1,$$
$$[W(n), S_0(m)] = 0, \qquad [W(n), S_1(m)] = 2mc_{L,I} S_0(n+m).$$

In particular, $S_0(0)$ and $S_1(0)$ are screening operators. Moreover,

$$S_1 = S_1(0) \colon \ L^{\mathcal{H}}(c_L, c_{L,I}) \to L^{\mathcal{H}}(1,0)$$

is nontrivial and $S_1(0)I(-1)\mathbf{1} = -v_0$.

Proof. Since $L(k)v^i = \delta_{k,0}v^i$ for $k \geq 0$, commutator formula gives that

$$[L(n), S_i(m)] = -mS_i(n+m).$$

Next we calculate $[W(n), S_1(m)]$. We have

$$W(-1)v^1 = 2I(-1)v^0 = -2c_{L,I}L(-1)v^0,$$
$$W(0)v^1 = -2c_{L,I}v^0, \qquad W(n)v^1 = 0, \qquad n \geq 0.$$

This implies that

$$[W(n), S_1(m)] = 2c_{L,I} m S_0(n+m).$$

Since $W(n)v^0 = 0$ for $n \geq -1$ we get

$$[W(n), S_0(m)] = 0.$$

Therefore we have proved that $S_i(0)$, $i = 0, 1$ are screening operators. Next we have

$$S_1(0)I(-1)\mathbf{1} = \operatorname{Res}_z Y_{\text{ext}}\big(v^1, z\big)I(-1) = \operatorname{Res}_z e^{zL(-1)}Y(I(-1)\mathbf{1}, -z)v^1 = -v_0.$$

The proof follows. ■

Theorem 4.3. *S_1 is a derivation of the vertex algebra $\mathcal{V}^{\text{ext}}$ and we have*

$$\operatorname{Ker}_{L^{\mathcal{H}}(c_L, c_{L,I})} S_1 \cong L^{W(2,2)}(c_L, c_W).$$

Proof. By construction $S_1 = \operatorname{Res}_z Y_{\text{ext}}(v^1, z)$, so S_1 is a derivation so $\overline{W} = \operatorname{Ker}_{L^{\mathcal{H}}(c_L, c_{L,I})} S_1$ is a vertex subalgebra of $L^{\mathcal{H}}(c_L, c_{L,I})$. Since

$$S_1 L(-2)\mathbf{1} = S_1 W(-2)\mathbf{1} = 0$$

we have that $L^{W(2,2)}(c_L, c_W) \subset \overline{W}$. Since $S_1 I(-1)\mathbf{1} \neq 0$, we have that $I(-1)\mathbf{1}$ does not belong to $\overline{W}$. By using the fact that $L^{\mathcal{H}}(c_L, c_{L,I})$ is as $W(2,2)$-module generated by singular vector $\mathbf{1}$ and cosingular vector $I(-1)\mathbf{1}$ (see Example 3.9) we get that $\overline{W} = L^{W(2,2)}(c_L, c_W)$. The proof follows. ■

Remark 4.4. Of course, every $\mathcal{V}_{\text{ext}}$-module becomes a $W(2,2)$-module with screening operator S_1. Similar statement holds for intertwining operators. Constructions of such modules and intertwining operators require different techniques which we will present in our forthcoming paper [5].

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