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Tableau Formulas for One-Row Macdonald Polynomials of Types C_n and D_n*

Boris FEIGIN †1, *Ayumu HOSHINO* †2, *Masatoshi NOUMI* †3, *Jun SHIBAHARA* †4 *and Jun'ichi SHIRAISHI* †5

†1 *National Research University Higher School of Economics, International Laboratory of Representation Theory and Mathematical Physics, Moscow, Russia*
E-mail: *borfeigin@gmail.com*

†2 *Kagawa National College of Technology, 355 Chokushi-cho, Takamatsu, Kagawa 761-8058, Japan*
E-mail: *hoshino@t.kagawa-nct.ac.jp*

†3 *Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan*
E-mail: *noumi@math.kobe-u.ac.jp*

†4 *Hamamatsu University School of Medicine, 1-20-1 Handayama, Higashi-ku, Hamamatsu city, Shizuoka 431-3192, Japan*
E-mail: *s070521math@yahoo.co.jp*

†5 *Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan*
E-mail: *shiraish@ms.u-tokyo.ac.jp*

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Abstract. We present explicit formulas for the Macdonald polynomials of types C_n and D_n in the one-row case. In view of the combinatorial structure, we call them "tableau formulas". For the construction of the tableau formulas, we apply some transformation formulas for the basic hypergeometric series involving very well-poised balanced ${}_{12}W_{11}$ series. We remark that the correlation functions of the deformed $\mathcal{W}$ algebra generators automatically give rise to the tableau formulas when we principally specialize the coordinate variables.

Key words: Macdonald polynomials; deformed $\mathcal{W}$ algebras

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1 Introduction

I.G. Macdonald introduced the symmetric polynomials $P_\lambda(x;q,t)$ as a (q,t)-deformation of the Schur polynomials $s_\lambda(x)$. Then he extended this construction to the cases of the symmetric Laurent polynomials invariant under the actions of the Weyl groups of simple root systems [10]. For type A_n, he gave an explicit combinatorial formula for $P_\lambda(x;q,t)$, usually called the "tableau formula". In [1] it was shown that the tableau formula for $P_\lambda(x;q,t)$ of type A_n can be interpreted as certain specialization of the correlation functions of the deformed $\mathcal{W}$ algebras of type A_n. One of our motivations is to explore a little further the correspondence between the Macdonald polynomials and the deformed $\mathcal{W}$ algebras associated with simple root systems. More precisely, we calculate the correlation functions of $\mathcal{W}$ algebras of types C_n and D_n with

*This paper is a contribution to the Special Issue on Orthogonal Polynomials, Special Functions and Applications. The full collection is available at http://www.emis.de/journals/SIGMA/OPSFA2015.html principal specializations in coordinate variables (Definition 7.8). As a result, we obtain certain combinatorial expressions, which we regard as tableau formulas. Then, on the basis of Lassalle's explicit formulas [8] (see [4] for a proof and their generalization), we prove that they are actually the Macdonald polynomials of types C_n and D_n in the one-row case (Theorem 7.10).

We need to recall briefly the Kashiwara–Nakashima tableaux. In the study of quantum algebras [5], they gave combinatorial descriptions of the crystal bases for the integrable highest weight representations by using the "semi-standard tableaux" of respective types. For the crystal bases of the symmetric tensor representations $V(r\Lambda_1)$ of types C_n and D_n, the semi-standard tableau is defined to be a one-row diagram (r) of size r filled with entries in the ordered set $I=\{1,2,\dots,n,\overline{n},\overline{n-1},\dots,\overline{1}\}$ being arranged in the weakly increasing manner. The orderings of I are defined by (6.1) for C_n and by (4.1) for D_n respectively. For a semi-standard tableau of shape (r), denote by θ_i the number of the letter $i\in I$ in the tableau. Then we have $\theta_1+\theta_2+\cdots+\theta_n+\theta_{\overline{n}}+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{1}}=r$. For type D_n, we have an aditional condition $\theta_n\theta_{\overline{n}}=0$ due to the structure of the ordering.

We now present the main results of this paper (Theorems 4.3, 6.2 and 6.5). Let $P^{(C_n)}_{(r)}(x;q,t,T)$ and $P^{(D_n)}_{(r)}(x;q,t)$ be the Macdonald polynomials of types C_n and D_n respectively attached to a single row (r). As for the notation, see Appendix A.2.

Theorem 1.1. *We have the following tableau formulas in the one-row cases:*

$$\begin{aligned}
P^{(C_n)}_{(r)}(x;q,t,t^2/q) &= \frac{(q;q)_r}{(t;q)_r}\sum_{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r}\prod_{k\in I}\frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}}\\
&\quad\times\prod_{1\le l\le n}\frac{(t^{n-l+1}q^{\theta_l+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}(t^{n-l+2}q^{\theta_{l+1}+\cdots+\theta_{\overline{l+1}}-1};q)_{\theta_{\overline{l}}}}{(t^{n-l+2}q^{\theta_l+\cdots+\theta_{\overline{l+1}}-1};q)_{\theta_{\overline{l}}}(t^{n-l+1}q^{\theta_{l+1}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}}\\
&\quad\times x_1^{\theta_1-\theta_{\overline{1}}}x_2^{\theta_2-\theta_{\overline{2}}}\cdots x_n^{\theta_n-\theta_{\overline{n}}},
\end{aligned}\tag{1.1}$$

$$\begin{aligned}
P^{(D_n)}_{(r)}(x;q,t) &= \frac{(q;q)_r}{(t;q)_r}\sum_{\substack{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r\\ \theta_n\theta_{\overline{n}}=0}}\prod_{k\in I}\frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}}\\
&\quad\times\prod_{1\le l\le n-1}\frac{(t^{n-l-1}q^{\theta_l+\theta_{l+1}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}(t^{n-l}q^{\theta_{l+1}+\theta_{l+2}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}}{(t^{n-l}q^{\theta_l+\theta_{l+1}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}(t^{n-l-1}q^{\theta_{l+1}+\theta_{l+2}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}}\\
&\quad\times x_1^{\theta_1-\theta_{\overline{1}}}x_2^{\theta_2-\theta_{\overline{2}}}\cdots x_n^{\theta_n-\theta_{\overline{n}}}.
\end{aligned}\tag{1.2}$$

Here and hereafter, we use the standard notation of q-shifted factorials

$$(z;q)_\infty=\prod_{k=0}^{\infty}(1-q^kz),\qquad (z;q)_k=\frac{(z;q)_\infty}{(q^kz;q)_\infty},\qquad k\in\mathbb{Z},$$
$$(a_1,a_2,\dots,a_r;q)_k=(a_1;q)_k(a_2;q)_k\cdots(a_r;q)_k,\qquad k\in\mathbb{Z}.$$

Remark 1.2. These tableau formulas for $P^{(C_n)}_{(r)}(x;q,t,t^2/q)$ and $P^{(D_n)}_{(r)}(x;q,t)$ are obtained by principally specializing the correlation functions of the deformed $\mathcal{W}$ algebras of types C_n and D_n respectively. See Theorem 7.10.

We can extend the tableau formula of type C_n in (1.1) as follows to general q, t and T.

Theorem 1.3. *Set $\theta:=\min(\theta_n,\theta_{\overline{n}})$ for simplicity of display. We have*

$$P^{(C_n)}_{(r)}(x;q,t,T)=\frac{(q;q)_r}{(t;q)_r}\sum_{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r}\prod_{k\in I\setminus\{n,\overline{n}\}}\frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}}\frac{(t;q)_{|\theta_n-\theta_{\overline{n}}|}}{(q;q)_{|\theta_n-\theta_{\overline{n}}|}}$$

$$\times \prod_{1\le l\le n-1}\left(\frac{(t^{n-l-1}q^{\theta_l+\cdots+\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}}{(t^{n-l}q^{\theta_l+\cdots+\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}}\right.$$
$$\left.\times \frac{(t^{n-l}q^{\theta_{l+1}+\cdots\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}}{(t^{n-l-1}q^{\theta_{l+1}+\cdots+\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}}\right)$$
$$\times \frac{(T;q)_\theta (t^n q^{r-2\theta};q)_{2\theta}}{(q;q)_\theta (Tt^{n-1}q^{r-\theta};q)_\theta (t^{n-1}q^{r-2\theta+1};q)_\theta} x_1^{\theta_1-\theta_{\overline{1}}} x_2^{\theta_2-\theta_{\overline{2}}}\cdots x_n^{\theta_n-\theta_{\overline{n}}}. \tag{1.3}$$

Remark 1.4. At present, we do not know any $\mathcal{W}$ algebra which explains the formula (1.3).

There are several combinatorial expressions for the Macdonald polynomials studied from some different points of view. See [9, 13, 14] for example. It would be an intriguing problem to find possible connections between those formulas and ours obtained in this paper.

This paper is organized as follows. In Sections 2, 3 and 5 we construct the transformation formulas for the basic hypergeometric series for proving our tableau formulas. In Sections 4 and 6, we prove the tableau formulas for the one-row Macdonald polynomials of types C_n and D_n respectively. In Section 7, we recall the deformed $\mathcal{W}$ algebras of types C_n and D_n, and then prove that the correlation functions with principal specialization give us the tableau formulas for the Macdonald polynomials of types C_n and D_n in the one-row case respectively. In Appendix A, we recall briefly the Koornwinder polynomials, and the Macdonald polynomials of types C_n and D_n as degenerations of the Koornwinder polynomials.

Throughout this paper, we use the standard notation for the basic hypergeometric series as

$$ {}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,\dots,b_r\end{matrix};q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_n}{(q,b_1,b_2,\dots,b_r;q)_n}z^n,$$
$$ {}_{r+1}W_r(a_1;a_4,a_5,\dots,a_{r+1};q,z)={}_{r+1}\phi_r\left[\begin{matrix}a_1,qa_1^{1/2},-qa_1^{1/2},a_4,\dots,a_{r+1}\\ a_1^{1/2},-a_1^{1/2},qa_1/a_4,\dots,qa_1/a_{r+1}\end{matrix};q,z\right].$$

We call the ${}_{r+1}W_r$ series *very well-poised* basic hypergeometric series. Moreover, we call a ${}_{r+1}W_r$ series *very well-poised balanced* when it satisfies the balancing condition $(a_4a_5\cdots a_{r+1})z = \big(\pm(a_1q)^{\frac12}\big)^{r-3}$.

2 Transformation formula I

In this section, we give a transformation formula of basic hypergeometric series. We show that a very well-poised balanced ${}_{12}W_{11}$ series is transformed to a ${}_4\phi_3$ series which is neither balanced nor well-poised. Recall the following proposition:

Proposition 2.1 ([12, Proposition 7.3])**.** *We have for $r,\theta\in\mathbb{Z}_{\ge 0}$*

$$ {}_{r+9}W_{r+8}\left(a;q^{-\theta},q^\theta af,a_1,\dots,a_r,\left(\frac{aq}{f}\right)^{\frac12},-\left(\frac{aq}{f}\right)^{\frac12},\left(\frac{aq^2}{f}\right)^{\frac12},-\left(\frac{aq^2}{f}\right)^{\frac12};q,z\right) \tag{2.1}$$
$$=\frac{(aq,f^2/q;q)_\theta}{(af,f;q)_\theta}\sum_{m\ge0}\frac{(q/f,q^{-\theta},aq/f;q)_m}{(q,q^{-\theta}q^2/f^2,aq;q)_m}q^m\,{}_{r+5}W_{r+4}\big(a;q^{-m},q^m aq/f,a_1,\dots,a_r;q,z\big).$$

The main result in this section is as follows:

Theorem 2.2. *Assume $af = a_2a_3$, then*

$$
\begin{aligned}
&{}_{12}W_{11}\left(a; q^{-\theta}, q^\theta af, f, a_2, a_3, \left(\frac{aq}{f}\right)^{\frac12}, -\left(\frac{aq}{f}\right)^{\frac12}, \left(\frac{aq^2}{f}\right)^{\frac12}, -\left(\frac{aq^2}{f}\right)^{\frac12}; q, q/f\right)\\
&\quad = \frac{(aq, af/a_2;q)_\theta}{(af, aq/a_2;q)_\theta} {}_4\phi_3\left[\begin{matrix} q^{-\theta}, q^{-\theta}a_2/a, f, a_2 \\ q^{-\theta+1}a_2/af, q^{-\theta+1}/f, aq/a_3\end{matrix}; q, q^2/f^2\right].
\end{aligned} \tag{2.2}
$$

Remark 2.3. By one of the anonymous referees, it was pointed out that (2.2) is a special case of the transformation formula obtained by Langer, Schlosser and Warnaar [7, equation (4.2)]. Namely, we have (2.2) by letting $d \to 0$ (or $d \to \infty$) in the $p = 0$ case of [7, equation (4.2)]. The authors thank the referee for informing them of this fact.

Remark 2.4. The ${}_4\phi_3$ series in the right hand side of (2.2) is neither balanced nor well-poised. However it has the following structure: set for simplicity $u_1 := q^{-\theta}$, $u_2 := q^{-\theta}a_2/a$, $u_3 := f$, $u_4 := a_2$, $v_1 := q$, $v_2 := q^{-\theta+1}a_2/af$, $v_3 := q^{-\theta+1}/f$ and $v_4 := aq/a_3$. Then we have

(i) $u_1v_1 = u_3v_3 = q^{-\theta+1}$ and $u_2v_4 = u_4v_2 = q^{-\theta+1}a_2/a_3$,

(ii) $u_1v_4 = u_4v_3 = aq^{-\theta+1}/a_3$ and $u_2v_1 = u_3v_2 = q^{-\theta+1}a_2/a$.

Proof of Theorem 2.2. Applying (2.1) to the left hand side of (2.2), we have

$$
\begin{aligned}
\text{l.h.s. of (2.2)} &= \frac{(aq, f^2/q;q)_\theta}{(af, f;q)_\theta}\\
&\quad\times \sum_{m\ge 0} \frac{(q/f, q^{-\theta}, aq/f;q)_m}{(q, q^{-\theta+2}/f^2, aq;q)_m} q^m\, {}_8W_7\left(a; q^{-m}, aq^{m+1}/f, f, a_2, a_3; q, q/f\right).
\end{aligned} \tag{2.3}
$$

We apply Watson's formula [3, p. 35, equation (2.5.1)]

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, q^{-n} \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}\end{matrix}; q, \frac{a^2q^{2+n}}{bcde}\right]\\
&\quad = \frac{(aq, aq/de;q)_n}{(aq/d, aq/e;q)_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, d, e, aq/bc \\ aq/b, aq/c, deq^{-n}/a\end{matrix}; q, q\right],
\end{aligned} \tag{2.4}
$$

to the right hand side of (2.3) with the substitutions $b = aq^{m+1}/f$, $c = a_3$, $d = a_2$ and $e = f$. Then we have

$$
\begin{aligned}
\text{r.h.s. of (2.3)} &= \frac{(aq, f^2/q;q)_\theta}{(af, f;q)_\theta} \sum_{m\ge 0} \frac{(q^{-\theta}, aq/a_2f, q/f;q)_m}{(q^{-\theta+2}/f^2, aq/a_2, q;q)_m} q^m\\
&\quad\times \sum_{j=0}^{m} \frac{(f, a_2, q^{-m}f/a_3, q^{-m};q)_j}{(aq/a_3, q^{-m}a_2f/a, q^{-m}f, q;q)_j} q^j.
\end{aligned} \tag{2.5}
$$

Now we need to change the order of the summation. Setting $s := m - j$ and using the condition $af = a_2a_3$, we have

$$
\begin{aligned}
\text{r.h.s. of (2.5)} &= \frac{(aq, f^2/q;q)_\theta}{(af, f;q)_\theta} \sum_{j=0}^{\theta} \frac{(q^{-\theta}, f, a_2;q)_j}{(q^{-\theta+2}/f^2, aq/a_3, q;q)_j} (q/f)^{2j}\\
&\quad\times \sum_{s=0}^{\theta-j} \frac{(aq/a_2f, q/f, q^{-\theta+j};q)_s}{(aq/a_2, q, q^{-\theta+j+2}/f^2;q)_s} q^s.
\end{aligned} \tag{2.6}
$$

As a final step, we apply the q-Saalschütz transformation formula [3, p. 13, equation (1.7.2)] to the summation with respect to s of (2.6). Then we have

$$\text{r.h.s. of (2.6)} = \frac{(aq,q;q)_\theta}{(af,f;q)_\theta}\sum_{j=0}^{\theta}\frac{(af/a_2,f;q)_{\theta-j}}{(aq/a_2,q;q)_{\theta-j}}\frac{(f,a_2;q)_j}{(q,aq/a_3;q)_j}$$
$$= \frac{(aq,af/a_2;q)_\theta}{(af,aq/a_2;q)_\theta}{}_4\phi_3\left[\begin{matrix} q^{-\theta},q^{-\theta}a_2/a,f,a_2 \\ q^{-\theta+1}a_2/af,q^{-\theta+1}/f,aq/a_3\end{matrix};q,q^2/f^2\right].$$

This completes the proof of Theorem 2.2. ■

In what follows, we use Theorem 2.2 in the form

$${}_{12}W_{11}\left(a;q^{-\theta},q^\theta af,f,a_2,a_3,\left(\frac{aq}{f}\right)^{\frac12},-\left(\frac{aq}{f}\right)^{\frac12},\left(\frac{aq^2}{f}\right)^{\frac12},-\left(\frac{aq^2}{f}\right)^{\frac12};q,q/f\right)$$
$$= \frac{(aq,q;q)_\theta}{(af,f;q)_\theta}\sum_{j=0}^{\theta}\frac{(af/a_2,f;q)_{\theta-j}}{(aq/a_2,q;q)_{\theta-j}}\frac{(f,a_2;q)_j}{(q,aq/a_3;q)_j}. \tag{2.7}$$

3 Transformation formula II

In this section, we present a transformation formula which will be used to describe the Macdonald polynomials of types D_n and C_n.

Theorem 3.1. *Let $n\in\mathbb{Z}_{\geq 2}$. Fix $K,m_1,m_2,\dots,m_n\in\mathbb{Z}_{\geq 0}$ arbitrarily. Set $m_{l,n}:=\sum_{k=l}^{n}m_k$, $\phi_{l,n}:=\sum_{k=l}^{n}\phi_k$ for simplicity of display. We have*

$$\sum_{\substack{\phi_1,\phi_2,\dots,\phi_{n-1},i\geq 0\\ \phi_1+\phi_2+\cdots+\phi_{n-1}+i=K}}\left(\prod_{1\leq l\leq n-1}\frac{(t;q)_{\phi_l}(t;q)_{\phi_l+m_l}}{(q;q)_{\phi_l}(q;q)_{\phi_l+m_l}}\right.$$
$$\left.\times\frac{(t^{n-l-1}q^{\phi_l+2\phi_{l+1,n-1}+m_{l,n}+1};q)_{\phi_l}(t^{n-l}q^{2\phi_{l+1,n-1}+m_{l+1,n}};q)_{\phi_l}}{(t^{n-l}q^{\phi_l+2\phi_{l+1,n-1}+m_{l,n}};q)_{\phi_l}(t^{n-l-1}q^{2\phi_{l+1,n-1}+m_{l+1,n}+1};q)_{\phi_l}}\right)$$
$$\times\frac{(t;q)_{m_n}}{(q;q)_{m_n}}\frac{(t;q)_i(t^nq^{2K+m_{1,n}-2i};q)_i}{(q;q)_i(t^{n-1}q^{2K+m_{1,n}-2i+1};q)_i}$$
$$=\sum_{\substack{\phi_1,\phi_2,\dots,\phi_{n-1},\phi_n\geq 0\\ \phi_1+\phi_2+\cdots+\phi_n=K}}\prod_{1\leq j\leq n}\frac{(t;q)_{\phi_j}(t;q)_{\phi_j+m_j}}{(q;q)_{\phi_j}(q;q)_{\phi_j+m_j}}. \tag{3.1}$$

We prove Theorem 3.1 by induction on n. In order to clarify the structure of our proof, we first confirm the case $n=2$ in Section 3.1, and then treat the general case in Section 3.2.

3.1 The case $n=2$

Proposition 3.2. *Fix $K,m_1,m_2\in\mathbb{Z}_{\geq 0}$ arbitrarily. We have*

$$\sum_{\substack{\phi_1,i\geq 0\\ \phi_1+i=K}}\frac{(t;q)_{m_1+\phi_1}(t;q)_{m_2}(t,q^{m_1+m_2+\phi_1+1},tq^{m_2};q)_{\phi_1}(t,t^2q^{2K+m_1+m_2-2i};q)_i}{(q;q)_{m_1+\phi_1}(q;q)_{m_2}(q,tq^{m_1+m_2+\phi_1},q^{m_2+1};q)_{\phi_1}(q,tq^{2K+m_1+m_2-2i+1};q)_i}$$
$$=\sum_{\substack{\phi_1,\phi_2\geq 0\\ \phi_1+\phi_2=K}}\frac{(t;q)_{m_1+\phi_1}(t;q)_{m_2+\phi_2}(t;q)_{\phi_1}(t;q)_{\phi_2}}{(q;q)_{m_1+\phi_1}(q;q)_{m_2+\phi_2}(q;q)_{\phi_1}(q;q)_{\phi_2}}. \tag{3.2}$$

Proof. One finds that the summation in l.h.s. of (3.2) with respect to ϕ_1 is given by the following ${}_{12}W_{11}$ series

$$\text{l.h.s. of (3.2)} = \frac{(t,t^2q^{m_1+m_2};q)_K(t;q)_{m_1}(t;q)_{m_2}}{(q,tq^{m_1+m_2+1};q)_K(q;q)_{m_1}(q;q)_{m_2}} \times {}_{12}W_{11}\left(a;q^{-K},q^Kaf,f,b,c,\left(\frac{aq}{f}\right)^{\frac{1}{2}},-\left(\frac{aq}{f}\right)^{\frac{1}{2}},\left(\frac{aq^2}{f}\right)^{\frac{1}{2}},-\left(\frac{aq^2}{f}\right)^{\frac{1}{2}};q,q/f\right), \quad (3.3)$$

where $a=tq^{m_1+m_2}$, $b=tq^{m_1}$, $c=tq^{m_2}$, $f=t$. Then applying formula (2.7), the r.h.s. of (3.3) is rewritten as

$$\frac{(t;q)_{m_1}(t;q)_{m_2}}{(q;q)_{m_1}(q;q)_{m_2}}\sum_{\phi_1=0}^{K}\frac{(tq^{m_2},t;q)_{K-\phi_1}}{(q^{m_2+1},q;q)_{K-\phi_1}}\frac{(t,tq^{m_1};q)_{\phi_1}}{(q,q^{m_1+1};q)_{\phi_1}} = \sum_{\phi_1=0}^{K}\frac{(t;q)_{m_1+\phi_1}(t;q)_{m_2+K-\phi_1}(t;q)_{\phi_1}(t;q)_{K-\phi_1}}{(q;q)_{m_1+\phi_1}(q;q)_{m_2+K-\phi_1}(q;q)_{\phi_1}(q;q)_{K-\phi_1}}.$$

This completes the proof of Proposition 3.2. ∎

3.2 The general case

Assume the validity of the transformation formula (3.1) for $n-1$. We have

$$\begin{aligned}\text{l.h.s. of (3.1)} &= \sum_{\phi_{n-1}=0}^{K}\sum_{\phi_{n-2}=0}^{K-\phi_{n-1}}\cdots\sum_{\phi_2=0}^{K-\phi_{3,n-1}}\prod_{2\le l\le n-1}\frac{(t;q)_{\phi_l}(t;q)_{\phi_l+m_l}}{(q;q)_{\phi_l}(q;q)_{\phi_l+m_l}}\\ &\times\frac{(t^{n-l-1}q^{\phi_l+2\phi_{l+1,n-1}+m_{l,n}+1};q)_{\phi_l}(t^{n-l}q^{2\phi_{l+1,n-1}+m_{l+1,n}};q)_{\phi_l}}{(t^{n-l}q^{\phi_l+2\phi_{l+1,n-1}+m_{l,n}};q)_{\phi_l}(t^{n-l-1}q^{2\phi_{l+1,n-1}+m_{l+1,n}+1};q)_{\phi_l}}\\ &\times\sum_{\phi_1=0}^{K-\phi_{2,n-1}}\frac{(t;q)_{\phi_1}(t;q)_{\phi_1+m_1}(t;q)_{m_n}(t,t^nq^{2\phi_{1,n-1}+m_{1,n}};q)_{K-\phi_{1,n-1}}}{(q;q)_{\phi_1}(q;q)_{\phi_1+m_1}(q;q)_{m_n}(q,t^{n-1}q^{2\phi_{1,n-1}+m_{1,n}+1};q)_{K-\phi_{1,n-1}}}\\ &\times\frac{(t^{n-2}q^{\phi_1+2\phi_{2,n-1}+m_{1,n}+1};q)_{\phi_1}(t^{n-1}q^{2\phi_{2,n-1}+m_{2,n}};q)_{\phi_1}}{(t^{n-1}q^{\phi_1+2\phi_{2,n-1}+m_{1,n}};q)_{\phi_1}(t^{n-2}q^{2\phi_{2,n-1}+m_{2,n}+1};q)_{\phi_1}}.\end{aligned} \quad (3.4)$$

The summation with respect to ϕ_1 in (3.4) can be written as follows

$$\frac{(t;q)_{m_1}(t;q)_{m_n}(t,t^nq^{2\phi_{2,n-1}+m_{1,n}};q)_{K-\phi_{2,n-1}}}{(q;q)_{m_1}(q;q)_{m_n}(q,t^{n-1}q^{2\phi_{2,n-1}+m_{1,n}+1};q)_{K-\phi_{2,n-1}}} \times {}_{12}W_{11}\left(a;q^{-\theta},q^{\theta}af,f,b,c,\left(\frac{aq}{f}\right)^{\frac{1}{2}},-\left(\frac{aq}{f}\right)^{\frac{1}{2}},\left(\frac{aq^2}{f}\right)^{\frac{1}{2}},-\left(\frac{aq^2}{f}\right)^{\frac{1}{2}};q,q/f\right), \quad (3.5)$$

where $a=t^{n-1}q^{2\phi_{2,n-1}+m_{1,n}}$, $b=tq^{m_1}, c=t^{n-1}q^{2\phi_{2,n-1}+m_{2,n}}$, $f=t$, $\theta=K-\phi_{2,n-1}$. Applying the formula (2.7), (3.5) is transformed into

$$\frac{(t;q)_{m_1}(t;q)_{m_n}}{(q;q)_{m_1}(q;q)_{m_n}} \times\sum_{j=0}^{K-\phi_{2,n-1}}\frac{(t^{n-1}q^{2\phi_{2,n-1}+m_{2,n}};q)_{K-j-\phi_{2,n-1}}(t;q)_{K-j-\phi_{2,n-1}}(t;q)_j(tq^{m_1};q)_j}{(t^{n-2}q^{2\phi_{2,n-1}+m_{2,n}+1};q)_{K-j-\phi_{2,n-1}}(q;q)_{K-j-\phi_{2,n-1}}(q;q)_j(q^{m_1+1};q)_j}. \quad (3.6)$$

Using (3.6) and changing the order of the summations, we can express the r.h.s. of (3.4) as follows

$$\sum_{j=0}^{K}\frac{(t;q)_j(t;q)_{j+m_1}}{(q;q)_j(q;q)_{j+m_1}}\sum_{\phi_{n-1}=0}^{K-j}\sum_{\phi_{n-2}=0}^{K-j-\phi_{n-1}}\cdots\sum_{\phi_3=0}^{K-j-\phi_{4,n-1}}\prod_{3\le l\le n-1}\frac{(t;q)_{\phi_l}(t;q)_{\phi_l+m_l}}{(q;q)_{\phi_l}(q;q)_{\phi_l+m_l}}$$
$$\times\frac{(t^{n-l-1}q^{\phi_l+2\phi_{l+1,n-1}+m_{l,n}+1};q)_{\phi_l}(t^{n-l}q^{2\phi_{l+1,n-1}+m_{l+1,n}};q)_{\phi_l}}{(t^{n-l}q^{\phi_l+2\phi_{l+1,n-1}+\phi_{l,n}};q)_{\phi_l}(t^{n-l-1}q^{2\phi_{l+1,n-1}+m_{l+1,n}+1};q)_{\phi_l}}$$
$$\times\sum_{\phi_2=0}^{K-j-\phi_{3,n-1}}\frac{(t;q)_{\phi_2+m_2}(t;q)_{m_n}(t;q)_{\phi_2}(t^{n-1}q^{2\phi_{2,n-1}+m_{2,n-1}};q)_{K-j-\phi_{2,n-1}}(t;q)_{K-j-\phi_{2,n-1}}}{(q;q)_{\phi_2+m_2}(q;q)_{m_n}(q;q)_{\phi_2}(t^{n-2}q^{2\phi_{2,n-1}+m_{2,n}+1};q)_{K-j-\phi_{2,n-1}}(q;q)_{K-j-\phi_{2,n-1}}}$$
$$\times\frac{(t^{n-2}q^{2\phi_{3,n-1}+m_{3,n}};q)_{\phi_2}(t^{n-3}q^{\phi_2+2\phi_{3,n-1}+m_{2,n}+1};q)_{\phi_2}}{(t^{n-3}q^{2\phi_{3,n-1}+m_{3,n}+1};q)_{\phi_2}(t^{n-2}q^{\phi_2+2\phi_{3,n-1}+m_{2,n}};q)_{\phi_2}}. \tag{3.7}$$

By the induction hypothesis, (3.7) can be rewritten as

$$\sum_{j=0}^{K}\frac{(t;q)_j(t;q)_{j+m_1}}{(q;q)_j(q;q)_{j+m_1}}\sum_{\phi_2+\phi_3+\cdots+\phi_{n-1}+\phi=K-j}\frac{(t;q)_{\phi}(t;q)_{\phi+m_n}}{(q;q)_{\phi}(q;q)_{\phi+m_n}}\prod_{2\le l\le n-1}\frac{(t;q)_{\phi_l}(t;q)_{\phi_l+m_l}}{(q;q)_{\phi_l}(q;q)_{\phi_l+m_l}}$$
$$=\sum_{j+\phi_2+\phi_3+\cdots+\phi_{n-1}+\phi=K}\frac{(t;q)_j(t;q)_{j+m_1}}{(q;q)_j(q;q)_{j+m_1}}\frac{(t;q)_{\phi}(t;q)_{\phi+m_n}}{(q;q)_{\phi}(q;q)_{\phi+m_n}}\prod_{2\le l\le n-1}\frac{(t;q)_{\phi_l}(t;q)_{\phi_l+m_l}}{(q;q)_{\phi_l}(q;q)_{\phi_l+m_l}}$$
$$=\text{r.h.s. of (3.1)}.$$

Hence we have completed the proof of Theorem 3.1.

4 Tableau formulas for Macdonald polynomials of type D_n

In this section, we investigate the tableau formula for the one-row Macdonald polynomials of type D_n. Let $I := \{1,2,\ldots,n-1,n,\overline{n},\overline{n-1},\ldots,\overline{1}\}$ be the index set with the ordering

$$1\prec 2\prec\cdots\prec n-1 \begin{smallmatrix}\curlywedge & n & \curlyvee\\ \curlyvee & \overline{n} & \curlywedge\end{smallmatrix} \overline{n-1}\prec\cdots\prec\overline{1}. \tag{4.1}$$

Denoting by Λ_1 the first fundamental weight of type D_n, let $P_{(r)}^{(D_n)}(x;q,t)$ be the Macdonald polynomials of type D_n associated with the weights $r\Lambda_1$ for $r\in\mathbb{Z}_{\ge 0}$.

We recall Lassalle's formula for $P_{(r)}^{(D_n)}(x;q,t)$. Lassalle introduced $G_r(x;q,t)$ defined by the generating function

$$\prod_{i=1}^{n}\frac{(tux_i;q)_\infty}{(ux_i;q)_\infty}\frac{(tu/x_i;q)_\infty}{(u/x_i;q)_\infty}=\sum_{r\ge 0}G_r(x;q,t)u^r. \tag{4.2}$$

Comparing the coefficient of u^r of the equation (4.2), we obtain

$$G_r(x;q,t)=\sum_{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r}\prod_{i\in I}\frac{(t;q)_{\theta_i}}{(q;q)_{\theta_i}}x_1^{\theta_1-\theta_{\overline{1}}}x_2^{\theta_2-\theta_{\overline{2}}}\cdots x_n^{\theta_n-\theta_{\overline{n}}}, \tag{4.3}$$

where θ_i, $\theta_{\overline{i}}\in\mathbb{Z}_{\ge 0}$, $i=1,2,\ldots,n$.

The following theorem [4, Theorem 5.2] was conjectured by Lassalle [8].

Theorem 4.1 ([4, 8])**.** *For any positive integer r we have*

$$G_r(x;q,t)=\sum_{i=0}^{[r/2]}\frac{(t;q)_{r-2i}}{(q;q)_{r-2i}}P^{(D_n)}_{(r-2i)}(x;q,t)\frac{(t;q)_i(t^nq^{r-2i};q)_i}{(q;q)_i(t^{n-1}q^{r-2i+1};q)_i}. \tag{4.4}$$

Conversely

$$P^{(D_n)}_{(r)}(x;q,t)=\frac{(q;q)_r}{(t;q)_r}\sum_{i=0}^{[r/2]}G_{r-2i}(x;q,t)t^i\frac{(1/t;q)_i(t^nq^{r-i};q)_i}{(q;q)_i(t^{n-1}q^{r-i};q)_i}\frac{1-t^nq^{r-2i}}{1-t^nq^{r-i}}. \tag{4.5}$$

Remark 4.2. If we insert (4.3) in (4.5), we have an explicit combinatorial formula for $P^{(D_n)}_{(r)}(x;q,t)$. However, it is not clear how we can extract the combinatorics of the Kashiwara–Nakashima tableaux of type D from this Lassalle's version.

Here, we establish the tableau formula for $P^{(D_n)}_{(r)}(x;q,t)$.

Theorem 4.3. *We have*

$$P^{(D_n)}_{(r)}(x;q,t)=\frac{(q;q)_r}{(t;q)_r}\sum_{\substack{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r\\ \theta_n\theta_{\overline{n}}=0}}\prod_{k\in I}\frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \tag{4.6}$$
$$\times\prod_{1\le l\le n-1}\frac{(t^{n-l}q^{\theta_{l+1}+\theta_{l+2}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}(t^{n-l-1}q^{\theta_l+\theta_{l+1}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}}{(t^{n-l-1}q^{\theta_{l+1}+\theta_{l+2}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}(t^{n-l}q^{\theta_l+\theta_{l+1}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}}x_1^{\theta_1-\theta_{\overline{1}}}x_2^{\theta_2-\theta_{\overline{2}}}\cdots x_n^{\theta_n-\theta_{\overline{n}}}.$$

Remark 4.4. Set $X:=t^{n-l}q^{\theta_{l+1}+\theta_{l+2}+\cdots+\theta_{\overline{l+1}}}$ and $Y:=t^{n-l-1}q^{\theta_{l+1}+\theta_{l+2}+\cdots+\theta_{\overline{l+1}}}$ for simplicity. The last product in (4.6) can be rewritten as

$$\prod_{1\le l\le n-1}\frac{(X;q)_{\theta_{\overline{l}}}(q^{\theta_l}Y;q)_{\theta_{\overline{l}}}}{(Y;q)_{\theta_{\overline{l}}}(q^{\theta_l}X;q)_{\theta_{\overline{l}}}}=\prod_{1\le l\le n-1}\frac{(X;q)_{\theta_l}(X;q)_{\theta_{\overline{l}}}(Y;q)_{\theta_l+\theta_{\overline{l}}}}{(Y;q)_{\theta_l}(Y;q)_{\theta_{\overline{l}}}(X;q)_{\theta_l+\theta_{\overline{l}}}}.$$

This implies that r.h.s. of (4.6) has the symmetry $(\mathbb{Z}/2\mathbb{Z})^n$. Namely, it is invariant under the exchange $x_l\leftrightarrow\frac{1}{x_l}$, $1\le l\le n-1$.

Remark 4.5. It would be an intriguing problem to show the factorization of the Macdonald polynomial $P^{(D_n)}_{(r)}(x;q,t)$ from our formula (4.6) when we make the principal specialization:

$$P^{(D_n)}_{(r)}\big(t^{n-1},\dots,t,1;q,t\big)=t^{-r(n-1)}\frac{(t^n;q)_r(t^{2(n-1)};q)_r}{(t;q)_r(t^{(n-1)};q)_r}.$$

Remark 4.6. Setting $n=1$ in (4.6), we have

$$P^{(D_1)}_{(r)}(x;q,t)=x^r+x^{-r}.$$

Setting $n=2$ in (4.6), we have

$$P^{(D_2)}_{(r)}(x;q,t)=\frac{(q;q)_r}{(t;q)_r}\sum_{\substack{\theta_1+\theta_2+\theta_{\overline{2}}+\theta_{\overline{1}}=r\\ \theta_2\theta_{\overline{2}}=0}}\frac{(t;q)_{\theta_1}}{(q;q)_{\theta_1}}\frac{(t;q)_{\theta_2}}{(q;q)_{\theta_2}}\frac{(t;q)_{\theta_{\overline{2}}}}{(q;q)_{\theta_{\overline{2}}}}\frac{(t;q)_{\theta_{\overline{1}}}}{(q;q)_{\theta_{\overline{1}}}}$$
$$\times\frac{(tq^{\theta_2+\theta_{\overline{2}}};q)_{\theta_{\overline{1}}}}{(qq^{\theta_2+\theta_{\overline{2}}};q)_{\theta_{\overline{1}}}}\frac{(qq^{\theta_1+\theta_2+\theta_{\overline{2}}};q)_{\theta_{\overline{1}}}}{(tq^{\theta_1+\theta_2+\theta_{\overline{2}}};q)_{\theta_{\overline{1}}}}x_1^{\theta_1-\theta_{\overline{1}}}x_2^{\theta_2-\theta_{\overline{2}}}$$

$$= \left(\frac{(q;q)_r}{(t;q)_r} \sum_{\mu_1+\mu_2=r} \frac{(t;q)_{\mu_1}}{(q;q)_{\mu_1}} \frac{(t;q)_{\mu_2}}{(q;q)_{\mu_2}} x_1^{(\mu_1-\mu_2)/2} x_2^{-(\mu_1-\mu_2)/2} \right) \times \left(\frac{(q;q)_r}{(t;q)_r} \sum_{\nu_1+\nu_2=r} \frac{(t;q)_{\nu_1}}{(q;q)_{\nu_1}} \frac{(t;q)_{\nu_2}}{(q;q)_{\nu_2}} x_1^{(\nu_1-\nu_2)/2} x_2^{(\nu_1-\nu_2)/2} \right),$$

which shows the symmetry $D_2 = A_1 \times A_1$.

Proof of Theorem 4.3. Let $\{\Psi^{(D_n)}_{(r)}(x;q,t)\}_{r\in\mathbb{Z}_{\geq 0}}$ be a certain collection of Laurent polynomials. By using Lassalle's formula (4.4), it is easily proved by induction that the infinite system of equalities for $\Psi^{(D_n)}_{(r)}(x;q,t)$

$$G_r(x;q,t) = \sum_{i=0}^{[r/2]} \frac{(t;q)_{r-2i}}{(q;q)_{r-2i}} \Psi^{(D_n)}_{(r-2i)}(x;q,t) \frac{(t;q)_i (t^n q^{r-2i};q)_i}{(q;q)_i (t^{n-1}q^{r-2i+1};q)_i}, \qquad r \in \mathbb{Z}_{\geq 0}, \tag{4.7}$$

gives us $\Psi^{(D_n)}_{(r)}(x;q,t) = P^{(D_n)}_{(r)}(x;q,t)$, $r \in \mathbb{Z}_{\geq 0}$.

Set

$$\Psi^{(D_n)}_{(r)}(x;q,t) = \frac{(q;q)_r}{(t;q)_r} \sum_{\substack{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r \\ \theta_n\theta_{\overline{n}}=0}} \prod_{k\in I} \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \times \prod_{1\leq l\leq n-1} \frac{(t^{n-l}q^{\theta_{l+1}+\theta_{l+2}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}} (t^{n-l-1}q^{\theta_l+\theta_{l+1}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}}{(t^{n-l-1}q^{\theta_{l+1}+\theta_{l+2}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}} (t^{n-l}q^{\theta_l+\theta_{l+1}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}} \times x_1^{\theta_1-\theta_{\overline{1}}} x_2^{\theta_2-\theta_{\overline{2}}} \cdots x_n^{\theta_n-\theta_{\overline{n}}}.$$

We prove this family of Laurent polynomials satisfies (4.7). In view of the $(\mathbb{Z}/2\mathbb{Z})^n$ symmetry of $\Psi^{(D_n)}_{(r)}(x;q,t)$, it is sufficient to consider in (4.7) the coefficients of the monomials $x_1^{m_1}\cdots x_n^{m_n}$ with nonnegative powers $m_1,\dots,m_n \in \mathbb{Z}_{\geq 0}$ only. Let $r \in \mathbb{Z}_{\geq 0}$, and fix $m_1,\dots,m_n \in \mathbb{Z}_{\geq 0}$ arbitrarily. Set $K := \frac{1}{2}(r - m_1 - m_2 - \cdots - m_n)$ for simplicity. Setting

$$\theta_k = m_k + \phi_k, \qquad \theta_{\overline{k}} = \phi_k, \qquad 1 \leq k \leq n-1, \qquad \theta_n = m_n, \qquad \theta_{\overline{n}} = 0,$$

one finds that the coefficients of the monomials $x_1^{m_1}\cdots x_n^{m_n}$ in (4.7) is exactly given by l.h.s. of (3.1). On the other hand, the coefficients of the monomials $x_1^{m_1}\cdots x_n^{m_n}$ in $G_r(x;q,t)$ is clearly r.h.s. of (3.1). Hence we have proved (4.7), which establishes the tableau formula $P^{(D_n)}_{(r)}(x;q,t) = \Psi^{(D_n)}_{(r)}(x;q,t)$. ■

5 Transformation formula III

In this section, we present a transformation formula to describe the Macdonald polynomials of type C_n.

Theorem 5.1. *Let $n \in \mathbb{Z}_{\geq 2}$. Fix $K, m_1, m_2, \dots, m_n \in \mathbb{Z}_{\geq 0}$ arbitrarily. Set $m_{l,n} := \sum_{k=l}^{n} m_k$, $\phi_{l,n} := \sum_{k=l}^{n} \phi_k$ for simplicity of display. We have*

$$\sum_{\substack{\phi_1,\phi_2,\dots,\phi_n,i\geq 0 \\ \phi_1+\phi_2+\cdots+\phi_n+i=K}} \prod_{1\leq k\leq n} \frac{(t;q)_{\phi_k}(t;q)_{\phi_k+m_k}}{(q;q)_{\phi_k}(q;q)_{\phi_k+m_k}} \cdot (t^2/q)^i \frac{(t^{-1}q;q)_i (t^n q^{2K+m_{1,n}-2i};q)_i}{(q;q)_i (t^{n+1}q^{2K+m_{1,n}-2i};q)_i}$$

$$\times \prod_{1\le l\le n} \frac{(t^{n-l+1}q^{\phi_l+\phi_{l+1,n}+m_{l,n}};q)_{\phi_l}(t^{n-l+2}q^{2\phi_{l+1,n}+m_{l+1,n}-1};q)_{\phi_l}}{(t^{n-l+2}q^{\phi_l+\phi_{l+1,n}+m_{l,n}-1};q)_{\phi_l}(t^{n-l+1}q^{2\phi_{l+1,n}+m_{l+1,n}};q)_{\phi_l}}$$
$$= \sum_{\substack{\phi_1,\phi_2,\dots,\phi_n\ge 0\\ \phi_1+\phi_2+\cdots+\phi_n=K}} \prod_{1\le j\le n} \frac{(t;q)_{\phi_j}(t;q)_{\phi_j+m_j}}{(q;q)_{\phi_j}(q;q)_{\phi_j+m_j}}. \tag{5.1}$$

We prove Theorem 5.1 by induction on n. In Section 5.1 we show Theorem 5.1 in the case of $n=2$ and in Section 5.2 we treat the general case.

5.1 The case $n = 2$

Setting $n=2$, we have

$$\text{r.h.s. of (5.1)} = \sum_{\phi_1=0}^{K} \frac{(t;q)_{m_1+\phi_1}(t;q)_{\phi_1}(t;q)_{m_2+K-\phi_1}(t;q)_{K-\phi_1}}{(q;q)_{m_1+\phi_1}(q;q)_{\phi_1}(q;q)_{m_2+K-\phi_1}(q;q)_{K-\phi_1}}$$
$$= \sum_{\phi_1=0}^{K} \frac{(t;q)_{m_1}(t;q)_{m_2}(tq^{m_1};q)_{\phi_1}(t;q)_{\phi_1}(tq^{m_2};q)_{K-\phi_1}(t;q)_{K-\phi_1}}{(q;q)_{m_1}(q;q)_{m_2}(q^{m_1+1};q)_{\phi_1}(q;q)_{\phi_1}(q^{m_2+1};q)_{K-\phi_1}(q;q)_{K-\phi_1}}.$$

Then we have

$$\text{l.h.s. of (5.1)} = \sum_{\phi_2=0}^{K}\sum_{\phi_1=0}^{K-\phi_2} \frac{(t;q)_{\phi_2}(t;q)_{\phi_2+m_2}(t;q)_{\phi_1}(t;q)_{\phi_1+m_1}}{(q;q)_{\phi_2}(q;q)_{\phi_2+m_2}(q;q)_{\phi_1}(q;q)_{\phi_1+m_1}}$$
$$\times \frac{(t^2q^{m_1+m_2+2\phi_2+\phi_1};q)_{\phi_1}(t^3q^{m_2+2\phi_2-1};q)_{\phi_1}}{(t^3q^{m_1+m_2+2\phi_2+\phi_1-1};q)_{\phi_1}(t^2q^{m_2+2\phi_2};q)_{\phi_1}}$$
$$\times \frac{(tq^{m_2+\phi_2};q)_{\phi_2}(t^2q^{-1};q)_{\phi_2}}{(t^2q^{m_2-1+\phi_2};q)_{\phi_2}(t;q)_{\phi_2}}(t^2/q)^{K-\phi_2-\phi_1}$$
$$\times \frac{(q/t;q)_{K-\phi_2-\phi_1}(t^2q^{m_1+m_2+2\phi_2+2\phi_1};q)_{K-\phi_2-\phi_1}}{(q;q)_{K-\phi_2-\phi_1}(t^3q^{m_1+m_2+2\phi_2+2\phi_1};q)_{K-\phi_2-\phi_1}}$$
$$= \sum_{\phi_2=0}^{K} \frac{(t;q)_{m_2+\phi_2}(t;q)_{\phi_2}(tq^{m_2+\phi_2};q)_{\phi_2}(t^2q^{-1};q)_{\phi_2}(q/t;q)_{K-\phi_2}(t^2q^{m_1+m_2+2\phi_2};q)_{K-\phi_2}}{(q;q)_{m_2+\phi_2}(q;q)_{\phi_2}(tq^{m_2-1+\phi_2};q)_{\phi_2}(t;q)_{\phi_2}(q;q)_{K-\phi_2}(t^3q^{m_1+m_2+2\phi_2};q)_{K-\phi_2}}$$
$$\times\, {}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, q^{\phi_2-K} \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq^{K-\phi_2+1}\end{matrix}; q, \frac{a^2q^{K-\phi_2+2}}{bcde}\right], \tag{5.2}$$

where $a=t^3q^{m_1+m_2-1+\phi_2}$, $b=tq^{m_1}$, $c=t^2q^{K+m_1+m_2+\phi_2}$, $d=t$, $e=t^3q^{m_2-1+2\phi_2}$, $\frac{a^2q^{K-\phi_2+2}}{bcde}=q/t$. Applying Watson's transformation formula (2.4) to the ${}_8\phi_7$ series of (5.2), we have

$${}_8\phi_7 \text{ series of (5.2)} = \frac{(t^3q^{m_1+m_2+2\phi_2}, t^{-1}q^{m_1+1};q)_{K-\phi_2}}{(t^2q^{m_1+m_2+2\phi_2}, q^{m_1+1};q)_{K-\phi_2}}$$
$$\times\, {}_4\phi_3\left[\begin{matrix} q^{\phi_2-K-m_2}, t, t^3q^{m_2-1+2\phi_2}, q^{\phi_2-K} \\ t^2q^{m_2+2\phi_2}, tq^{\phi_2-K}, tq^{\phi_2-K-m_1}\end{matrix}; q, q\right]. \tag{5.3}$$

Using Sears' transformation formulas [3, p. 242, Appendix III, equations (III.15) and (III.16)], we rewrite the ${}_4\phi_3$ series in (5.3) as follows

$${}_4\phi_3\left[\begin{matrix} q^{\phi_2-K-m_2}, t, t^3q^{m_2-1+2\phi_2}, q^{\phi_2-K} \\ t^2q^{m_2+2\phi_2}, tq^{\phi_2-K}, tq^{\phi_2-K-m_1}\end{matrix}; q, q\right] = \frac{(q^{m_1-K+\phi_2}, t^{-1}q^{1-m_2-K-\phi_2};q)_{K-\phi_2}}{(tq^{-m_1-K-\phi_2}, t^2q^{1-m_2-K-\phi_2};q)_{K-\phi_2}}$$

$$\times\ {}_4\phi_3\left[\begin{matrix} tq^{m_1}, t, t^{-2}q^{1-m_2-K-\phi_2}, q^{\phi_2-K} \\ tq^{\phi_2-K}, t^{-1}q^{1-m_2-K-\phi_2}, q^{m_1+1} \end{matrix}; q, q\right]. \tag{5.4}$$

Sears' transformation gives us

$$\begin{aligned} {}_4\phi_3 \text{ series in } (5.4) &= \frac{(t^{-2}q^{1-m_2-K-\phi_2}, q; q)_{K-\phi_2}}{(t^{-1}q^{1-m_2-K-\phi_2}, t^{-1}q; q)_{K-\phi_2}} \\ &\quad \times\ {}_4\phi_3\left[\begin{matrix} t, t^2q^{m_1+m_2+k+\phi_2}, t^{-1}q, q^{\phi_2-K} \\ q^{m_1+1}, q, t^2q^{m_2} \end{matrix}; q, q\right], \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} {}_4\phi_3 \text{ series in } (5.5) &= \frac{(t^{-1}q, tq^{m_2+2\phi_2}; q)_{K-\phi_2}}{(q, t^2q^{m_2+2\phi_2}; q)_{K-\phi_2}} t^{K-\phi_2} \\ &\quad \times\ {}_4\phi_3\left[\begin{matrix} t, t^{-2}q^{1-m_2-K-\phi_2}, tq^{m_1}, q^{\phi_2-K} \\ q^{m_1+1}, tq^{\phi_2-K}, t^{-1}q^{1-m_2-K-\phi_2} \end{matrix}; q, q\right]. \end{aligned}$$

Then we have

$$\begin{aligned} &\text{l.h.s. of } (5.2) \\ &= \sum_{\phi_2=0}^{K} \frac{(t;q)_{m_1}(t)_{m_2}(t^2q^{m_2+\phi_2};q)_{\phi_2}(t^2q^{-1};q)_{\phi_2}(t^{-1}q;q)_{K-\phi_2}(tq^{m_2+2\phi_2};q)_{K-\phi_2}}{(q;q)_{m_1}(q;q)_{m_2}(q;q)_{\phi_2}(q^{m_2+1};q)_{\phi_2}(t^2q^{m_2-1+\phi_2};q)_{\phi_2}(q;q)_{K-\phi_2}(t^2q^{m_2+2\phi_2};q)_{K-\phi_2}} \\ &\quad \times\ {}_4\phi_3\left[\begin{matrix} t, t^{-2}q^{1-m_2-K-\phi_2}, tq^{m_1}, q^{\phi_2-K} \\ q^{m_1+1}, tq^{\phi_2-K}, t^{-1}q^{1-m_2-K-\phi_2} \end{matrix}; q, q\right] \\ &= \sum_{\phi_1=0}^{K} \frac{(t;q)_{m_1}(t;q)_{m_2}(t^{-1}q;q)_K(tq^{m_2};q)_K(q^{-K}, t^{-2}q^{1-m_2-K}, t, tq^{m_1};q)_{\phi_1}}{(q;q)_{m_1}(q;q)_{m_2}(q;q)_K(t^2q^{m_2};q)_K(tq^{-K}, t^{-1}q^{1-m_2-K}, q, q^{m_1+1};q)_{\phi_1}} \\ &\quad \times \sum_{\phi_2=0}^{K-\phi_1} \frac{(t^2q^{m_2-1}, tq^{\frac{m_2+1}{2}}, -tq^{\frac{m_2+1}{2}}, t^2q^{-1}, tq^{m_2+K-\phi_1}, q^{\phi_1-K};q)_{\phi_2}}{(q, tq^{\frac{m_2-1}{2}}, -tq^{\frac{m_2-1}{2}}, q^{m_2+1}, tq^{\phi_1-K}, t^2q^{m_2+K-\phi_1};q)_{\phi_2}}(t^2/q)^K(q/t)^{\phi_2}q^{\phi_1} \\ &= \sum_{\phi_1=0}^{K} \frac{(t;q)_{m_1}(t;q)_{m_2}(t^{-1}q;q)_K(tq^{m_2};q)_K(q^{-K}, t^{-2}q^{1-m_2-K}, t, tq^{m_1};q)_{\phi_1}}{(q;q)_{m_1}(q;q)_{m_2}(q;q)_K(t^2q^{m_2};q)_K(tq^{-K}, t^{-1}q^{1-m_2-K}, q, q^{m_1+1};q)_{\phi_1}}(t^2/q)^Kq^{\phi_1} \\ &\quad \times\ {}_6\phi_5\left[\begin{matrix} t^2q^{m_2-1}, tq^{\frac{m_2+1}{2}}, -tq^{\frac{m_2+1}{2}}, t^2q^{-1}, tq^{m_2+K-\phi_1}, q^{\phi_1-K} \\ q, tq^{\frac{m_2-1}{2}}, -tq^{\frac{m_2-1}{2}}, q^{m_2+1}, tq^{\phi_1-K}, t^2q^{m_2+K-\phi_1} \end{matrix}; q, q/t\right] \\ &= \sum_{\phi_1=0}^{k} \frac{(t;q)_{m_1}(t;q)_{m_2}(t^{-1}q;q)_K(tq^{m_2};q)_K(q^{-K}, t^{-2}q^{1-m_2-K}, t, tq^{m_1};q)_{\phi_1}}{(q;q)_{m_1}(q;q)_{m_2}(q;q)_K(t^2q^{m_2};q)_K(tq^{-K}, t^{-1}q^{1-m_2-K}, q, q^{m_1+1};q)_{\phi_1}} \\ &\quad \times \frac{(t^2q^{m_2};q)_{K-\phi_1}(t^{-1}q^{1-K+\phi_1};q)_{K-\phi_1}}{(q^{m_2+1};q)_{K-\phi_1}(tq^{\phi_1-K};q)_{K-\phi_1}}(t^2/q)^Kq^{\phi_1} = \text{r.h.s. of } (5.1). \end{aligned}$$

5.2 The general case

We have

$$\begin{aligned} \text{l.h.s. of } (5.1) &= \sum_{\phi_{2,n}=0}^{K} \prod_{2\le k\le n} \frac{(t;q)_{\phi_k}(t;q)_{\phi_k+m_k}}{(q;q)_{\phi_k}(q;q)_{\phi_k+m_k}} \\ &\quad \times \prod_{2\le l\le n} \frac{(t^{n-l+1}q^{\phi_l+\phi_{l+1,n}+m_{l,n}};q)_{\phi_l}(t^{n-l+2}q^{2\phi_{l+1,n}+m_{l+1,n}-1};q)_{\phi_l}}{(t^{n-l+2}q^{\phi_l+\phi_{l+1,n}+m_{l,n}-1};q)_{\phi_l}(t^{n-l+1}q^{2\phi_{l+1,n}+m_{l+1,n}};q)_{\phi_l}} \end{aligned}$$

$$\times \sum_{\phi_1=0}^{K-\phi_{2,n}} \frac{(t^n q^{\phi_1+\phi_{2,n}+m_{1,n}};q)_{\phi_1}(t^{n+1}q^{2\phi_{2,n}+m_{2,n}-1};q)_{\phi_1}}{(t^{n+1}q^{\phi_1+\phi_{2,n}+m_{1,n}-1};q)_{\phi_1}(t^n q^{2\phi_{2,n}+m_{2,n}};q)_{\phi_1}}$$
$$\times (t^2/q)^{K-\phi_{2,n}-\phi_1} \frac{(t^{-1}q;q)_{K-\phi_{2,n}-\phi_1}(t^n q^{2\phi_{2,n}+m_{1,n}+2\phi_1};q)_{K-\phi_{2,n}-\phi_1}}{(q;q)_{K-\phi_{2,n}-\phi_1}(t^{n+1}q^{2\phi_{2,n}+m_{1,n}+2\phi_1};q)_{K-\phi_{2,n}-\phi_1}}. \tag{5.6}$$

Here, we can describe the summation with respect to ϕ_1 in (5.6) as follows:

$$\frac{(t;q)_{m_1}(t^n q^{m_{1,n}+2\phi_{2,n}},t^{-1}q;q)_{K-\phi_{2,n}}}{(q;q)_{m_1}(t^{n+1}q^{m_{1,n}+2\phi_{2,n}},q;q)_{K-\phi_{2,n}}}(t^2/q)^{K-\phi_{2,n}}$$
$$\times {}_8\phi_7\left[\begin{matrix} t^{n+1}q^{m_{1,n}+2\phi_{2,n}-1}, t^{\frac{n+1}{2}}q^{\frac{m_{1,n}+2\phi_{2,n}+1}{2}}, \\ t^{\frac{n+1}{2}}q^{\frac{m_{1,n}+2\phi_{2,n}-1}{2}}, -t^{\frac{n+1}{2}}q^{\frac{m_{1,n}+2\phi_{2,n}-1}{2}}, \end{matrix}\right.$$
$$\left.\begin{matrix} -t^{\frac{n+1}{2}}q^{\frac{m_{1,n}+2\phi_{2,n}+1}{2}}, t, tq^{m_1}, t^{n+1}q^{m_{2,n}+2\phi_{2,n}-1}, t^n q^{m_{1,n}+2\phi_{2,n}+K}, q^{\phi_{2,n}-K} \\ t^n q^{m_{1,n}+2\phi_{2,n}}, t^n q^{m_{2,n}+2\phi_{2,n}}, q^{m_1+1}, tq^{\phi_{2,n}-K}, t^{n+1}q^{m_{1,n}+2\phi_{2,n}+K} \end{matrix}; q, q/t\right]$$
$$= \frac{(t;q)_{m_1}(t^n q^{m_{1,n}+2\phi_{2,n}}, t^{-1}q, t^{-n}q^{1-K-m_{2,n}-\phi_{2,n}};q)_{K-\phi_{2,n}}}{(q;q)_{m_1}(q^{m_1+1},q,tq^{\phi_{2,n}-K};q)_{K-\phi_{2,n}}}(t^2/q)^{K-\phi_{2,n}}$$
$$\times {}_4\phi_3\left[\begin{matrix} t^{n-1}q^{m_{2,n}+2\phi_{2,n}}, t^{n+1}q^{m_{2,n}+2\phi_{2,n}-1}, t^n q^{m_{1,n}+\phi_{2,n}+K}, q^{\phi_{2,n}-K} \\ t^n q^{m_{1,n}+2\phi_{2,n}}, t^n q^{m_{2,n}+2\phi_{2,n}}, t^n q^{m_{2,n}+2\phi_{2,n}} \end{matrix}; q, q\right]. \tag{5.7}$$

Applying Sears' ${}_4\phi_3$ transformation formula [3, p. 41, equation (2.10.4)] to the r.h.s. of (5.7), we have

$$\text{r.h.s. of (5.7)} = (t^2/q)^{K-\phi_{2,n}} \frac{(t;q)_{m_1}(t^{-1}q;q)_{K-\phi_{2,n}}(t^{n-1}q^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_{2,n}}}{(q;q)_{m_1}(q;q)_{K-\phi_{2,n}}(t^n q^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_{2,n}}}$$
$$\times {}_4\phi_3\left[\begin{matrix} tq^{m_1}, t, t^{-n}q^{1-K-m_{2,n}-\phi_{2,n}}, q^{\phi_{2,n}-K} \\ q^{m_1+1}, tq^{\phi_{2,n}-K}, t^{-n+1}q^{1-K-m_{2,n}-\phi_{2,n}} \end{matrix}; q, q\right]. \tag{5.8}$$

Using the following two formulas

$$\frac{(t^{-1}q;q)_{K-\phi_{2,n}}(q^{\phi_{2,n}-K};q)_{\phi_1}}{(q;q)_{K-\phi_{2,n}}(tq^{\phi_{2,n}-K};q)_{\phi_1}} = \frac{(t^{-1}q;q)_{K-\phi_{2,n}-\phi_1}}{(q;q)_{K-\phi_{2,n}-\phi_1}}t^{-\phi_1},$$
$$\frac{(t^{n-1}q^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_{2,n}}(t^{-n}q^{1-K-m_{2,n}-\phi_{2,n}};q)_{\phi_1}}{(t^n q^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_{2,n}}(t^{-n+1}q^{1-K-m_{2,n}-\phi_{2,n}};q)_{\phi_1}} = \frac{(t^{n-1}q^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_{2,n}-\phi_1}}{(t^n q^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_{2,n}-\phi_1}}t^{-\phi_1},$$

we have

$$\text{r.h.s. of (5.8)}$$
$$= \sum_{\phi_1=0}^{K-\phi_{2,n}} \frac{(t;q)_{m_1}(t;q)_{\phi_1}(tq^{m_1};q)_{\phi_1}(t^{-1}q;q)_{K-\phi_{2,n}-\phi_1}(t^{n-1}q^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_{2,n}-\phi_1}}{(q;q)_{m_1}(q;q)_{\phi_1}(q^{m_1+1};q)_{\phi_1}(q;q)_{K-\phi_{2,n}-\phi_1}(t^n q^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_{2,n}-\phi_1}}$$
$$\times (t^2/q)^{K-\phi_{2,n}-\phi_1}.$$

Then we have

$$\text{l.h.s. of (5.6)} = \sum_{\phi_{2,n}=0}^{K} \prod_{2\le k\le n} \frac{(t;q)_{\phi_k}(t;q)_{\phi_k+m_k}}{(q;q)_{\phi_k}(q;q)_{\phi_k+m_k}}$$
$$\times \prod_{2\le l\le n} \frac{(t^{n-l+1}q^{\phi_l+\phi_{l+1,n}+m_{l,n}};q)_{\phi_l}(t^{n-l+2}q^{2\phi_{l+1,n}+m_{l+1,n}-1};q)_{\phi_l}}{(t^{n-l+2}q^{\phi_l+\phi_{l+1,n}+m_{l,n}-1};q)_{\phi_l}(t^{n-l+1}q^{2\phi_{l+1,n}+m_{l+1,n}};q)_{\phi_l}}$$

$$\times \sum_{\phi_1=0}^{K-\phi_{2,n}} \frac{(t;q)_{m_1}(t;q)_{\phi_1}(tq^{m_1};q)_{\phi_1}(t^{-1}q;q)_{K-\phi_{2,n}-\phi_1}(t^{n-1}q^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_{2,n}-\phi_1}}{(q;q)_{m_1}(q;q)_{\phi_1}(q^{m_1+1};q)_{\phi_1}(q;q)_{K-\phi_{2,n}-\phi_1}(t^nq^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_{2,n}-\phi_1}}$$
$$\times (t^2/q)^{K-\phi_{2,n}-\phi_1}$$
$$= \sum_{\phi_1=0}^{K} \frac{(t;q)_{\phi_1}(t;q)_{\phi_1+m_1}}{(q;q)_{\phi_1}(q;q)_{\phi_1+m_1}} \sum_{\phi_{2,n}=0}^{K-\phi_1} \prod_{2\le k\le n} \frac{(t;q)_{\phi_k}(t;q)_{\phi_k+m_k}}{(q;q)_{\phi_k}(q;q)_{\phi_k+m_k}}$$
$$\times \prod_{2\le l\le n} \frac{(t^{n-l+1}q^{\phi_l+\phi_{l+1,n}+m_{l,n}};q)_{\phi_l}(t^{n-l+2}q^{2\phi_{l+1,n}+m_{l+1,n}-1};q)_{\phi_l}}{(t^{n-l+2}q^{\phi_l+\phi_{l+1,n}+m_{l,n}-1};q)_{\phi_l}(t^{n-l+1}q^{2\phi_{l+1,n}+m_{l+1,n}};q)_{\phi_l}}$$
$$\times \frac{(t^{-1}q;q)_{K-\phi_1-\phi_{2,n}}(t^{n-1}q^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_1-\phi_{2,n}}}{(q;q)_{K-\phi_1-\phi_{2,n}}(t^nq^{m_{2,n}+2\phi_{2,n}};q)_{K-\phi_1-\phi_{2,n}}}(t^2/q)^{K-\phi_1-\phi_{2,n}}$$
$$= \sum_{\phi_1=0}^{K} \frac{(t;q)_{\phi_1}(t;q)_{\phi_1+m_1}}{(q;q)_{\phi_1}(q;q)_{\phi_1+m_1}} \sum_{\phi_2+\cdots+\phi_n+i=K-\phi_1} \prod_{2\le k\le n} \frac{(t;q)_{\phi_k}(t;q)_{\phi_k+m_k}}{(q;q)_{\phi_k}(q;q)_{\phi_k+m_k}}$$
$$\times \frac{(t^{-1}q;q)_i(t^{n-1}q^{m_{2,n}+2\phi_{2,n}};q)_i}{(q;q)_i(t^nq^{m_{2,n}+2\phi_{2,n}};q)_i}(t^2/q)^i$$
$$\times \prod_{2\le l\le n} \frac{(t^{n-l+1}q^{\phi_l+\phi_{l+1,n}+m_{l,n}};q)_{\phi_l}(t^{n-l+2}q^{2\phi_{l+1,n}+m_{l+1,n}-1};q)_{\phi_l}}{(t^{n-l+2}q^{\phi_l+\phi_{l+1,n}+m_{l,n}-1};q)_{\phi_l}(t^{n-l+1}q^{2\phi_{l+1,n}+m_{l+1,n}};q)_{\phi_l}}. \tag{5.9}$$

By the induction hypothesis, we obtain

$$\text{r.h.s. of (5.9)} = \sum_{\phi_1=0}^{K} \frac{(t;q)_{\phi_1}(t;q)_{\phi_1+m_1}}{(q;q)_{\phi_1}(q;q)_{\phi_1+m_1}} \sum_{\phi_2+\cdots+\phi_n=K-\phi_1} \prod_{2\le k\le n} \frac{(t;q)_{\phi_k}(t;q)_{\phi_k+m_k}}{(q;q)_{\phi_k}(q;q)_{\phi_k+m_k}}$$
$$= \sum_{\phi_1+\phi_2+\cdots+\phi_n=K} \prod_{1\le j\le n} \frac{(t;q)_{\phi_j}(t;q)_{\phi_j+m_j}}{(q;q)_{\phi_j}(q;q)_{\phi_j+m_j}} = \text{r.h.s. of (5.1)}.$$

6 Tableau formulas for Macdonald polynomials of type C_n

In this section, we establish the tableau formulas for the Macdonald polynomials of type C_n. Let $I := \{1, 2, \dots, n-1, n, \overline{n}, \overline{n-1}, \dots, \overline{1}\}$ be the index set with the ordering

$$1 \prec 2 \prec \cdots \prec n-1 \prec n \prec \overline{n} \prec \overline{n-1} \prec \cdots \prec \overline{1}. \tag{6.1}$$

Denoting by Λ_1 the first fundamental weight of type C_n, let $P^{(C_n)}_{(r)}(x;q,t,T)$ be the Macdonald polynomials of type C_n associated with the weights $r\Lambda_1$ for $r \in \mathbb{Z}_{\ge 0}$.

The following theorem [4, Theorem 5.1] was conjectured by Lassalle [8].

Theorem 6.1 ([4, 8])**.** *For any positive integer r we have*

$$G_r(x;q,t) = \sum_{i=0}^{[r/2]} \frac{(t;q)_{r-2i}}{(q;q)_{r-2i}} P^{(C_n)}_{(r-2i)}(x;q,t,T)T^i \frac{(t/T;q)_i(t^nq^{r-2i};q)_i}{(q;q)_i(Tt^{n-1}q^{r-2i+1};q)_i}. \tag{6.2}$$

Conversely

$$P^{(C_n)}_{(r)}(x;q,t,T) = \frac{(q;q)_r}{(t;q)_r} \sum_{i=0}^{[r/2]} G_{r-2i}(x;q,t)t^i \frac{(T/t;q)_i(t^nq^{r-i};q)_i}{(q;q)_i(Tt^{n-1}q^{r-i};q)_i} \frac{1-t^nq^{r-2i}}{1-t^nq^{r-i}}.$$

First we prove the tableau formula for $P^{(C_n)}_{(r)}(x;q,t,t^2/q)$.

Theorem 6.2. *We have*

$$P^{(C_n)}_{(r)}(x;q,t,t^2/q) = \frac{(q;q)_r}{(t;q)_r} \sum_{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r} \prod_{k\in I} \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \tag{6.3}$$
$$\times \prod_{1\le l\le n} \frac{(t^{n-l+1}q^{\theta_l+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}(t^{n-l+2}q^{\theta_{l+1}+\cdots+\theta_{\overline{l+1}}-1};q)_{\theta_{\overline{l}}}}{(t^{n-l+2}q^{\theta_l+\cdots+\theta_{\overline{l+1}}-1};q)_{\theta_{\overline{l}}}(t^{n-l+1}q^{\theta_{l+1}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}} x_1^{\theta_1-\theta_{\overline{1}}} x_2^{\theta_2-\theta_{\overline{2}}} \cdots x_n^{\theta_n-\theta_{\overline{n}}}.$$

Remark 6.3. It would be an intriguing problem to show the factorization of the Macdonald polynomial $P^{(C_n)}_{(r)}(x;q,t,t^2/q)$ from our formula (6.3) when we make the principal specialization

$$P^{(C_n)}_{(r)}\big((t^2/q)^{1/2}t^{n-1},\dots,(t^2/q)^{1/2}t,(t^2/q)^{1/2};q,t,t^2/q\big) = q^{r/2}t^{-rn}\frac{(t^n;q)_r(t^{2(n+1)}/q^2;q)_r}{(t;q)_r(t^{n+1}/q;q)_r}.$$

Remark 6.4. Setting $n=1$ in (6.3) we have,

$$P^{(C_1)}_{(r)}(x;q,t,t^2/q) = \sum_{\theta_1=0}^{r} \frac{(t^2/q,q^{-r};q)_{\theta_1}}{(q,t^{-2}q^{2-r};q)_{\theta_1}}(q/t)^{2\theta_1}x^{-r+2\theta_1} = x^{-r}\,{}_2\phi_1\left[\begin{matrix} t^2/q,q^{-r} \\ t^{-2}q^{2-r}\end{matrix};q,(qx/t)^2\right].$$

Setting $n=2$ in (6.3) we have

$$P^{(C_2)}_{(r)}(x;q,t,t^2/q) = \frac{(q;q)_r}{(t;q)_r}\sum_{\theta_1+\theta_2+\theta_{\overline{2}}+\theta_{\overline{1}}=r} \frac{(t;q)_{\theta_1}(t;q)_{\theta_{\overline{1}}}(t;q)_{\theta_2}(t;q)_{\theta_{\overline{2}}}}{(q;q)_{\theta_1}(q;q)_{\theta_{\overline{1}}}(q;q)_{\theta_2}(q;q)_{\theta_{\overline{2}}}}$$
$$\times \frac{(t^2q^{\theta_1+\theta_2+\theta_{\overline{2}}},t^3q^{\theta_2+\theta_{\overline{2}}-1};q)_{\theta_{\overline{1}}}}{(t^3q^{\theta_1+\theta_2+\theta_{\overline{2}}-1},t^2q^{\theta_2+\theta_{\overline{2}}};q)_{\theta_{\overline{1}}}} \frac{(tq^{\theta_2},t^2/q;q)_{\theta_{\overline{2}}}}{(t^2q^{\theta_2-1},t;q)_{\theta_{\overline{2}}}} x_1^{\theta_1-\theta_{\overline{1}}}x_2^{\theta_2-\theta_{\overline{2}}}.$$

Proof of Theorem 6.2. Let $\{\Psi^{(C_n)}_{(r)}(x;q,t,T)\}_{r\in\mathbb{Z}_{\ge0}}$ be a certain collection of Laurent polynomials. By using Lassalle's formula (6.2), it is proved by induction that the infinite system of equalities for $\Psi^{(C_n)}_{(r)}(x;q,t,T)$

$$G_r(x;q,t) = \sum_{i=0}^{[r/2]} \frac{(t;q)_{r-2i}}{(q;q)_{r-2i}}\Psi^{(C_n)}_{(r-2i)}(x;q,t,T)T^i\frac{(t/T;q)_i(t^nq^{r-2i};q)_i}{(q;q)_i(Tt^{n-1}q^{r-2i+1};q)_i} \tag{6.4}$$

gives us $\Psi^{(C_n)}_{(r)}(x;q,t,T) = P^{(C_n)}_{(r)}(x;q,t,T)$, $r\in\mathbb{Z}_{\ge0}$. We use this argument with the specialization of the parameter $T=t^2/q$.

Set

$$\Psi^{(C_n)}_{(r)}(x;q,t,t^2/q) = \frac{(q;q)_r}{(t;q)_r} \sum_{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r} \prod_{k\in I} \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}}$$
$$\times \prod_{1\le l\le n} \frac{(t^{n-l+1}q^{\theta_l+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}(t^{n-l+2}q^{\theta_{l+1}+\cdots+\theta_{\overline{l+1}}-1};q)_{\theta_{\overline{l}}}}{(t^{n-l+2}q^{\theta_l+\cdots+\theta_{\overline{l+1}}-1};q)_{\theta_{\overline{l}}}(t^{n-l+1}q^{\theta_{l+1}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}} x_1^{\theta_1-\theta_{\overline{1}}} x_2^{\theta_2-\theta_{\overline{2}}} \cdots x_n^{\theta_n-\theta_{\overline{n}}}.$$

We prove this family of Laurent polynomials satisfies (6.4) with the specialization $T=t^2/q$. In view of the $(\mathbb{Z}/2\mathbb{Z})^n$ symmetry of $\Psi^{(C_n)}_{(r)}(x;q,t,t^2/q)$, it is sufficient to consider in (6.4) the coefficients of the monomials $x_1^{m_1}\cdots x_n^{m_n}$ with nonnegative powers $m_1,\dots,m_n\in\mathbb{Z}_{\ge0}$ only. Let $r\in\mathbb{Z}_{\ge0}$, and fix $m_1,\dots,m_n\in\mathbb{Z}_{\ge0}$ arbitrarily. Set $K:=\frac{1}{2}(r-m_1-m_2-\cdots-m_n)$ for simplicity. Setting

$$\theta_k = m_k+\phi_k, \qquad \theta_{\overline{k}} = \phi_k, \qquad 1\le k\le n,$$

one finds that the coefficients of the monomials $x_1^{m_1}\cdots x_n^{m_n}$ in (6.4) is exactly given by l.h.s. of (5.1). On the other hand, the coefficients of the monomials $x_1^{m_1}\cdots x_n^{m_n}$ in $G_r(x;q,t)$ is r.h.s. of (5.1). Hence we have proved (6.4) with $T=t^2/q$, which establishes the tableau formula $P^{(C_n)}_{(r)}(x;q,t,t^2/q)=\Psi^{(C_n)}_{(r)}(x;q,t,t^2/q)$. ■

Finally, we present a tableau formula for the one-row Macdonald polynomials $P^{(C_n)}_{(r)}(x;q,t,T)$ with general parameters (q,t,T).

Theorem 6.5. *Set $\theta:=\min(\theta_n,\theta_{\overline{n}})$. We have*

$$\begin{aligned} P^{(C_n)}_{(r)}(x;q,t,T) &= \frac{(q;q)_r}{(t;q)_r}\sum_{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r}\prod_{k\in I\backslash\{n,\overline{n}\}}\frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}}\frac{(t;q)_{|\theta_n-\theta_{\overline{n}}|}}{(q;q)_{|\theta_n-\theta_{\overline{n}}|}}\\ &\quad\times\prod_{1\le l\le n-1}\left(\frac{(t^{n-l-1}q^{\theta_l+\cdots+\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}}{(t^{n-l}q^{\theta_l+\cdots+\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}}\right.\\ &\qquad\left.\times\frac{(t^{n-l}q^{\theta_{l+1}+\cdots\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}}{(t^{n-l-1}q^{\theta_{l+1}+\cdots+\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}}\right)\\ &\quad\times\frac{(T;q)_\theta(t^nq^{r-2\theta};q)_{2\theta}}{(q;q)_\theta(Tt^{n-1}q^{r-\theta};q)_\theta(t^{n-1}q^{r-2\theta+1};q)_\theta}x_1^{\theta_1-\theta_{\overline{1}}}x_2^{\theta_2-\theta_{\overline{2}}}\cdots x_n^{\theta_n-\theta_{\overline{n}}}. \end{aligned}\tag{6.5}$$

Remark 6.6. It would be an intriguing problem to show the factorization of the Macdonald polynomial $P^{(C_n)}_{(r)}(x;q,t,T)$ from our formula (6.5) when we make the principal specialization:

$$P^{(C_n)}_{(r)}\big(T^{1/2}t^{n-1},\dots,T^{1/2}t,T^{1/2};q,t,T\big)=T^{-r/2}t^{-r(n-1)}\frac{(t^n;q)_r(t^{2(n-1)}T;q)_r}{(t;q)_r(t^nT;q)_r}.$$

Proof of Theorem 6.5. We prove that the system of equalities (6.4) is satisfied by the following Laurent polynomials

$$\begin{aligned} \Psi^{(C_n)}_{(r)}(x;q,t,T) &= \frac{(q;q)_r}{(t;q)_r}\sum_{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r}\prod_{k\in I\backslash\{n,\overline{n}\}}\frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}}\frac{(t;q)_{|\theta_n-\theta_{\overline{n}}|}}{(q;q)_{|\theta_n-\theta_{\overline{n}}|}}\\ &\quad\times\prod_{1\le l\le n-1}\left(\frac{(t^{n-l-1}q^{\theta_l+\cdots+\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}}{(t^{n-l}q^{\theta_l+\cdots+\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}}\right.\\ &\qquad\left.\times\frac{(t^{n-l}q^{\theta_{l+1}+\cdots\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\overline{l}}}}{(t^{n-l-1}q^{\theta_{l+1}+\cdots+\theta_{n-1}+|\theta_n-\theta_{\overline{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\overline{l}}}}\right)\\ &\quad\times\frac{(T;q)_\theta(t^nq^{r-2\theta};q)_{2\theta}}{(q;q)_\theta(Tt^{n-1}q^{r-\theta};q)_\theta(t^{n-1}q^{r-2\theta+1};q)_\theta}x_1^{\theta_1-\theta_{\overline{1}}}x_2^{\theta_2-\theta_{\overline{2}}}\cdots x_n^{\theta_n-\theta_{\overline{n}}}. \end{aligned}$$

Note that these $\Psi^{(C_n)}_{(r)}(x;q,t,T)$ have the $(\mathbb{Z}/2\mathbb{Z})^n$ symmetry.

Let $r\in\mathbb{Z}_{\ge0}$, and fix $m_1,\dots,m_n\in\mathbb{Z}_{\ge0}$ arbitrarily. We study the coefficient of the monomial $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$ in r.h.s. of (6.4), namely we consider the case $\theta_k-\theta_{\overline{k}}=m_k\ge0$ $(1\le k\le n)$. Set $K:=\frac{1}{2}(r-m_1-m_2-\cdots-m_n)$, $\theta_{l,m}:=\sum_{k=l}^{m}\theta_{\overline{k}}$, $\check{I}:=I\backslash\{n,\overline{n}\}$ for simplicity. Then the coefficient of the monomial $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$ in r.h.s. of (6.4) is expressed as follows

$$\sum_{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}+2i=r}\prod_{k\in\check{I}}\frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}}\frac{(t;q)_{m_n}}{(q;q)_{m_n}}$$

$$
\begin{aligned}
&\times \prod_{1\le l\le n-1} \frac{(t^{n-l-1}q^{\theta_{\bar l}+2\theta_{l,n-1}+m_{l,n}+1};q)_{\theta_{\bar l}}(t^{n-l}q^{2\theta_{l+1,n-1}+m_{l+1,n}};q)_{\theta_{\bar l}}}{(t^{n-l}q^{\theta_{\bar l}+2\theta_{l+1,n-1}+m_{l,n}};q)_{\theta_{\bar l}}(t^{n-l-1}q^{2\theta_{l+1,n-1}+m_{l+1,n}+1};q)_{\theta_{\bar l}}}\\
&\times \frac{(T;q)_\theta (t^n q^{r-2\theta};q)_{2\theta}}{(q;q)_\theta (Tt^{n-1}q^{r-\theta};q)_\theta (t^{n-1}q^{r-2\theta+1};q)_\theta} \frac{(t/T;q)_i (t^n q^{r-2i};q)_i}{(q;q)_i (Tt^{n-1}q^{r-2i+1};q)_i} T^i\\
&= \sum_{\theta_{1,n-1}=0}^{K} \sum_{\theta_{\bar n}=0}^{K-\theta_{1,n-1}} \prod_{k\in \check I} \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(t;q)_{m_n}}{(q;q)_{m_n}}\\
&\times \prod_{1\le l\le n-1} \frac{(t^{n-l-1}q^{\theta_{\bar l}+2\theta_{l,n-1}+m_{l,n}+1};q)_{\theta_{\bar l}}(t^{n-l}q^{2\theta_{l+1,n-1}+m_{l+1,n}};q)_{\theta_{\bar l}}}{(t^{n-l}q^{\theta_{\bar l}+2\theta_{l+1,n-1}+m_{l,n}};q)_{\theta_{\bar l}}(t^{n-l-1}q^{2\theta_{l+1,n-1}+m_{l+1,n}+1};q)_{\theta_{\bar l}}}\\
&\times \frac{(T;q)_{\theta_{\bar n}} (t^n q^{2\theta_{1,n-1}+m_{1,n}};q)_{2\theta_{\bar n}}}{(q;q)_{\theta_{\bar n}} (Tt^{n-1}q^{2\theta_{1,n-1}+m_{1,n}+\theta_{\bar n}};q)_{\theta_{\bar n}} (t^{n-1}q^{2\theta_{1,n-1}+m_{1,n}+1};q)_{\theta_{\bar n}}}\\
&\times \frac{(t/T;q)_{K-\theta_{1,n-1}-\theta_{\bar n}} (t^n q^{2\theta_{1,n-1}+m_{1,n}+2\theta_{\bar n}};q)_{K-\theta_{1,n-1}-\theta_{\bar n}}}{(q;q)_{K-\theta_{1,n-1}-\theta_{\bar n}} (Tt^{n-1}q^{2\theta_{1,n-1}+m_{1,n}+2\theta_{\bar n}+1};q)_{K-\theta_{1,n-1}-\theta_{\bar n}}} T^{K-\theta_{1,n-1}-\theta_{\bar n}}\\
&= \sum_{\theta_{1,n-1}=0}^{K} \prod_{k\in \check I} \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(t;q)_{m_n}}{(q;q)_{m_n}}\\
&\times \prod_{1\le l\le n-1} \frac{(t^{n-l-1}q^{\theta_{\bar l}+2\theta_{l,n-1}+m_{l,n}+1};q)_{\theta_{\bar l}}(t^{n-l}q^{2\theta_{l+1,n-1}+m_{l+1,n}};q)_{\theta_{\bar l}}}{(t^{n-l}q^{\theta_{\bar l}+2\theta_{l+1,n-1}+m_{l,n}};q)_{\theta_{\bar l}}(t^{n-l-1}q^{2\theta_{l+1,n-1}+m_{l+1,n}+1};q)_{\theta_{\bar l}}}\\
&\times \frac{(t/T;q)_{K-\theta_{1,n-1}} (t^n q^{2\theta_{1,n-1}+m_{1,n}};q)_{K-\theta_{1,n-1}}}{(q;q)_{K-\theta_{1,n-1}} (Tt^{n-1}q^{2\theta_{1,n-1}+m_{1,n}+1};q)_{K-\theta_{1,n-1}}} T^{K-\theta_{1,n-1}}\\
&\times {}_6W_5\big(Tt^{n-1}q^{2\theta_{1,n-1}+m_{1,n}}, T, t^n q^{\theta_{1,n-1}+m_{1,n}+K}, q^{-K+\theta_{1,n-1}}; q, q/t\big).
\end{aligned}\tag{6.6}
$$

By the summation formula for ${}_6\phi_5$ series [3, p. 34, equation (2.4.2)], we have

$$
{}_6W_5 \text{ series in (6.6)} = \frac{(Tt^{n-1}q^{2\theta_{1,n-1}+m_{1,n}+1};q)_{K-\theta_{1,n-1}} (t^{-1}q^{-K+\theta_{1,n}+1};q)_{K-\theta_{1,n-1}}}{(t^{n-1}q^{2\theta_{1,n-1}+m_{1,n}+1};q)_{K-\theta_{1,n-1}} (Tt^{-1}q^{-K+\theta_{1,n-1}-1};q)_{K-\theta_{1,n-1}}}.
$$

Note that the dependence on the parameter T in (6.6) disappears by the cancelation as

$$
\frac{(t/T;q)_{K-\theta_{1,n-1}} (t^{-1}q^{-K+\theta_{1,n-1}+1};q)_{K-\theta_{1,n-1}}}{(Tt^{-1}q^{-K+\theta_{1,n-1}-1};q)_{K-\theta_{1,n-1}}} T^{K-\theta_{1,n-1}} = (t;q)_{K-\theta_{1,n-1}}.
$$

Hence we have recast the coefficient of the monomial $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$ in r.h.s. of (6.4) as

$$
\begin{aligned}
&\sum_{\theta_{1,n-1}=0}^{K} \prod_{k\in \check I} \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(t;q)_{m_n}}{(q;q)_{m_n}}\\
&\quad\times \prod_{1\le l\le n-1} \frac{(t^{n-l-1}q^{\theta_{\bar l}+2\theta_{l,n-1}+m_{l,n}+1};q)_{\theta_{\bar l}}(t^{n-l}q^{2\theta_{l+1,n-1}+m_{l+1,n}};q)_{\theta_{\bar l}}}{(t^{n-l}q^{\theta_{\bar l}+2\theta_{l+1,n-1}+m_{l,n}};q)_{\theta_{\bar l}}(t^{n-l-1}q^{2\theta_{l+1,n-1}+m_{l+1,n}+1};q)_{\theta_{\bar l}}}\\
&\quad\times \frac{(t;q)_{K-\theta_{1,n-1}} (t^n q^{2\theta_{1,n-1}+m_{1,n}};q)_{K-\theta_{1,n-1}}}{(q;q)_{K-\theta_{1,n-1}} (t^{n-1}q^{2\theta_{1,n-1}+m_{1,n}+1};q)_{K-\theta_{1,n-1}}}.
\end{aligned}
$$

Changing the running indices as $\phi_k = \theta_{\bar k}$, one finds that this is nothing but l.h.s. of (3.1). Then Theorem 3.1 means that this is the coefficient of the monomial $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$ in $G_r(x;q,t)$. ■

7 Deformed $\mathcal{W}$ algebras of types C_l and D_l

In this section, we study a relation between the tableau formulas for Macdonald polynomials of types C_l and D_l and the deformed $\mathcal{W}$ algebras of types C_l and D_l. We briefly recall the definition of the deformed $\mathcal{W}$ algebras of types C_l and D_l [2].

Let $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ and $\{\omega_1, \omega_2, \dots, \omega_l\}$ be the sets of simple roots and of fundamental weights of a simple Lie algebra $\mathfrak{g}$ of rank l. Let $(\cdot, \cdot)$ be the invariant inner product on $\mathfrak{g}$ and $C = (C_{i,j})_{1\le i,j\le l}$ the *Cartan matrix* where $C_{i,j} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. Let $r^\vee$ be the maximal number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$ and set $D = \operatorname{diag}(r_1, r_2, \dots, r_l)$ where $r_i = r^\vee(\alpha_i, \alpha_i)/2$. Denote by $I = (I_{i,j})_{1\le i,j\le l}$ the incidence matrix where $I_{i,j} = 2\delta_{i,j} - C_{i,j}$. Let $B = (B_{i,j})_{1\le i,j\le l} = DC$ (i.e. $B_{i,j} = r^\vee(\alpha_i, \alpha_i)$). We define $l \times l$ matrices $C(q,t)$, $D(q,t)$, $B(q,t)$ and $M(q,t)$ as follows

$$\begin{aligned}
&C_{i,j}(q,t) = \big(q^{r_i}t^{-1} + q^{-r_i}t\big)\delta_{i,j} - [I_{i,j}]_q,\\
&D(q,t) = \operatorname{diag}([r_1]_q, [r_2]_q, \dots, [r_l]_q),\\
&B(q,t) = D(q,t)C(q,t),\\
&M(q,t) = D(q,t)C(q,t)^{-1} = D(q,t)B(q,t)^{-1}D(q,t),
\end{aligned}\tag{7.1}$$

where we use the standard notation $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$.

Let $\mathcal{H}_{q,t}$ be the Heisenberg algebra with generators $a_i[n]$ and $y_i[n]$ ($i = 1, 2, \dots, l$; $n \in \mathbb{Z}$) with the following relations

$$\begin{aligned}
&[a_i[n], a_j[m]] = \frac{1}{n}\big(q^{r_i n} - q^{-r_i n}\big)\big(t^n - t^{-n}\big)B_{i,j}\big(q^n, t^n\big)\delta_{n,-m},\\
&[a_i[n], y_j[m]] = \frac{1}{n}\big(q^{r_i n} - q^{-r_i n}\big)\big(t^n - t^{-n}\big)\delta_{i,j}\delta_{n,-m},\\
&[y_i[n], y_j[m]] = \frac{1}{n}\big(q^{r_i n} - q^{-r_i n}\big)\big(t^n - t^{-n}\big)M_{i,j}\big(q^n, t^n\big)\delta_{n,-m}.
\end{aligned}\tag{7.2}$$

Note that we have

$$a_j[n] = \sum_{i=1}^{l} C_{i,j}\big(q^n, t^n\big)y_i[n].$$

Introduce the generating series:

$$Y_i(z) := :\!\exp\Bigg(\sum_{m\neq 0} y_i[m]z^{-m}\Bigg)\!:.$$

Here we have used the standard notation for the normal ordering $:\cdots:$ for the Heisenberg generators defined as follows. We call the negative modes $a_i[-n]$, $y_i[-n]$ ($n > 0$) creation operators and positive modes $a_i[n]$, $y_i[n]$ ($n > 0$) annihilation operators. Then the normal ordered product $:\mathcal{O}:$ of an operator $\mathcal{O}$ is obtained by moving all the creation operators to the left and the annihilation operators to the right. For example we have

$$Y_i(z) = \exp\Bigg(\sum_{m>0} y_i[-m]z^m\Bigg)\exp\Bigg(\sum_{m>0} y_i[m]z^{-m}\Bigg).$$

Remark 7.1. In [2], suitable zero mode factors are included in the generating series $Y_i(z)$ to ensure reasonable commutation relations with the screening operators. In this paper, however, we omit writing them since we do not need any arguments based on the screening operators.

We define a set J and fields $\Lambda_i(z) = \Lambda_i^{(X)}(z)$, $i \in J$, for $X = C_l$ and D_l.

(i) The C_l series. $J := \{1, 2, \dots, l, \bar{l}, \overline{l-1}, \dots, \bar{1}\}$,

$$\Lambda_i(z) := {:}Y_i\big(zq^{-i+1}t^{i-1}\big)Y_{i-1}\big(zq^{-i}t^{i}\big)^{-1}{:}, \qquad i = 1, 2, \dots, l,$$
$$\Lambda_{\bar{i}}(z) := {:}Y_{i-1}\big(zq^{-2l+i-2}t^{2l-i}\big)Y_i\big(zq^{-2l+i-3}t^{2l-i+1}\big)^{-1}{:}, \qquad i = 1, 2, \dots, l. \tag{7.3}$$

(ii) The D_l series. $J := \{1, 2, \dots, l, \bar{l}, \overline{l-1}, \dots, \bar{1}\}$,

$$\Lambda_i(z) := {:}Y_i\big(zq^{-i+1}t^{i-1}\big)Y_{i-1}\big(zq^{-i}t^{i}\big)^{-1}{:}, \qquad i = 1, 2, \dots, l-2,$$
$$\Lambda_{l-1}(z) := {:}Y_l\big(zq^{-l+2}t^{l-2}\big)Y_{l-1}\big(zq^{-l+2}t^{l-2}\big)Y_{l-2}\big(zq^{-l+1}t^{l-1}\big){:},$$
$$\Lambda_l(z) := {:}Y_l\big(zq^{-l+2}t^{l-2}\big)Y_{l-1}\big(zq^{-l}t^{l}\big)^{-1}{:},$$
$$\Lambda_{\bar{l}}(z) := {:}Y_{l-1}\big(zq^{-l+2}t^{l-2}\big)Y_l\big(zq^{-l}t^{l}\big)^{-1}{:},$$
$$\Lambda_{\overline{l-1}}(z) := {:}Y_{l-2}\big(zq^{-l+1}t^{l-1}\big)Y_{l-1}\big(zq^{-l}t^{l}\big)^{-1}Y_l\big(zq^{-l}t^{l}\big){:},$$
$$\Lambda_{\bar{i}}(z) := {:}Y_{i-1}\big(zq^{-2l+i+2}t^{2l-i-2}\big)Y_i\big(zq^{-2l+i+1}t^{2l-i-1}\big)^{-1}{:}, \qquad i = 1, 2, \dots, l-2. \tag{7.4}$$

Definition 7.2 ([2]). Define the first generating fields $T^{(X)}(x, z)$ of the deformed $\mathcal{W}$ algebras of type $X = C_l$ or D_l with the independent indeterminates $x = (x_1, x_2, \dots, x_l)$:

$$T^{(X)}(x, z) := x_1\Lambda_1(z) + \cdots + x_l\Lambda_l(z) + \frac{1}{x_l}\Lambda_{\bar{l}}(z) + \cdots + \frac{1}{x_1}\Lambda_{\bar{1}}(z). \tag{7.5}$$

Remark 7.3. It is not an easy task to define the deformed $\mathcal{W}$ algebras purely in terms of the generators and relations, except for some simple cases such as the deformed Virasoro algebra. One of the simplest bypass ways is to regard the deformed $\mathcal{W}$ as the algebra generated by the $T^{(X)}(x, z)$ given in terms of the Heisenberg generators.

Remark 7.4. The x_i's ($i = 1, 2, \dots, l$) correspond to the zero mode factors, and they parametrize the highest weight condition for the representation of the $\mathcal{W}$ algebras.

Lemma 7.5. *For C_l and D_l series we obtain the following operator product expansions*

$$f(w/z)\Lambda_i(z)\Lambda_j(w) = \gamma_{i,j}(z, w)\,{:}\Lambda_i(z)\Lambda_j(w){:},$$

where

$$f(z) = f^{(X)}(z) := \exp\left(-\sum_{n=1}^{\infty}(q^n - q^{-n})(t^n - t^{-n})M_{1,1}(q^n, t^n)z^n\right),$$

and $\gamma_{i,j}(z, w) = \gamma_{i,j}^{(X)}(z, w)$ for $X = C_l$ or D_l given by

$$\gamma_{i,j}^{(C_l)}(z, w) = \begin{cases} 1, & i = j, \\ \gamma(w/z), & i \prec j,\ j \neq \bar{i}, \\ \gamma(z/w), & i \succ j,\ \bar{j} \neq i, \\ \gamma(w/z)\gamma\big(q^{2i-2l-2}t^{-2i+2l}w/z\big), & i \prec j,\ j = \bar{i}, \\ \gamma(z/w)\gamma\big(q^{-2i+2l+2}t^{2i-2l}z/w\big), & i \succ j,\ \bar{j} = i, \end{cases}$$

$$\gamma_{i,j}^{(D_l)}(z, w) = \begin{cases} 1 & i = j, \\ \gamma(w/z), & i \prec j,\ j \neq \bar{i}, \\ \gamma(z/w), & i \succ j,\ \bar{j} \neq i, \\ \gamma(w/z)\gamma\big(q^{2i-2l+2}t^{-2i+2l-2}w/z\big), & i \prec j,\ j = \bar{i}, \\ \gamma(z/w)\gamma\big(q^{-2i+2l-2}t^{2i-2l+2}z/w\big), & i \succ j,\ \bar{j} = i. \end{cases}$$

Here we have use the notation $\gamma(z) = \frac{(1-t^2z)(1-z/q^2)}{(1-z)(1-zt^2/q^2)}$.

A proof of Lemma 7.5 can be obtained by straightforward but pretty lengthy calculations using (7.1), (7.2), (7.3), (7.4). Therefore we safely can omit the detail.

Let $|0\rangle$ be the vacuum vector satisfying the annihilation conditions $a_i[n]|0\rangle = 0$ for all i and $n > 0$. Let $\mathcal{F}$ be the Fock module obtained by inducing up the one dimensional representation $\mathbb{C}|0\rangle$ of the algebra of annihilation operators to the whole Heisenberg algebra. Then one can check that the Fourier modes of the generator $T^{(X)}(x, z)$ acting on $\mathcal{F}$ are well defined. Let $\langle 0|$ to be the dual vacuum satisfying $\langle 0|a_i[-n] = 0$ for all i and $n > 0$.

Now we consider the correlation functions $\langle 0|T(x, z_1)T(x, z_2)\cdots T(x, z_r)|0\rangle$ of types C_l and D_l with the normalization $\langle 0|\Lambda_i(z)|0\rangle = 1$.

Proposition 7.6. *Let $X = C_l$ or D_l. Set $I_l := \{1, 2, \dots, l, \bar{l}, \overline{l-1}, \dots, \bar{1}\}$. Let $x_1, \dots, x_l$ be indeterminates and set $x_{\bar{i}} = 1/x_i$, $1 \le i \le l$. We have*

$$\prod_{i<j} f^{(X)}(z_j/z_i) \cdot \langle 0|T^{(X)}(x, z_1)T^{(X)}(x, z_2)\cdots T^{(X)}(x, z_r)|0\rangle$$
$$= F^{(X)}(x_1, \dots, x_l|z_i, \dots, z_r|q, t),$$

where

$$F^{(X)}(x_1, \dots, x_l|z_i, \dots, z_r|q, t) = \sum_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r \in I_l} x_{\varepsilon_1} x_{\varepsilon_2} \cdots x_{\varepsilon_r} \prod_{1 \le i < j \le r} \gamma_{\varepsilon_i, \varepsilon_j}(z_i, z_j).$$

Proof of Proposition 7.6. In view of the expression (7.5) we only need to know the matrix elements

$$\langle 0|\Lambda^{(X)}_{\varepsilon_1}(z_1)\Lambda^{(X)}_{\varepsilon_2}(z_2)\cdots\Lambda^{(X)}_{\varepsilon_r}(z_r)|0\rangle, \tag{7.6}$$

for any fixed $\varepsilon_1, \dots, \varepsilon_r \in I_l$. Then Proposition 7.5 and the normal ordering rule imply that

$$(7.6) = \prod_{i<j} \left(f^{(X)}(z_j/z_i)\right)^{-1} \cdot \prod_{1 \le i < j \le r} \gamma_{\varepsilon_i, \varepsilon_j}(z_i, z_j). \qquad \blacksquare$$

Remark 7.7. It is clearly seen from the definition that $F^{(X)}(x_1, \dots, x_l|z_i, \dots, z_r|q, t)$ is a symmetric rational function in z_i's. We conjecture that the $F^{(X)}(x_1, \dots, x_l|z_i, \dots, z_r|q, t)$ is a symmetric Laurent polynomial in x_i's associated with the Weyl group of corresponding type X (C_l or D_l). For the case of type A_l, we better understand the situation due to the theory of the shuffle algebra (we refer the reader to [1]).

Definition 7.8. By principally specializing the z_i's, set

$$\Phi^{(X)}_r(x|q, t) = F^{(X)}\big(x_1, \dots, x_l|q^{r-1}, q^{r-2}, \dots, 1|q^{1/2}, q^{1/2}t^{-1/2}\big).$$

Remark 7.9. Remark 7.7 implies that $\Phi^{(X)}_r(x|q, t)$ is a symmetric Laurent polynomial in x_i's associated with the Weyl group of corresponding type X (C_l or D_l). It is easy to check that the terms which do not vanish under the principal specialization correspond exactly to the set of semi-standard tableaux of type X.

Theorem 7.10. *We have*

(*i*) $\Phi^{(C_l)}_r(x|q, t) = P^{(C_l)}_{(r)}(x; q, t, t^2/q)$,

(*ii*) $\Phi^{(D_l)}_r(x|q, t) = P^{(D_l)}_{(r)}(x; q, t)$.

Proof. Straightforward calculation of $\Phi^{(X)}_r(x|q, t)$ gives us (1.1) for $X = C_l$ and (1.2) for $X = D_l$. $\blacksquare$

A Macdonald polynomials of types C_n and D_n

We recall briefly the Koornwinder polynomials, and the definitions of the Macdonald polynomials of types C_n and D_n as degenerations of the Koornwinder polynomials.

A.1 Koornwinder polynomials

Let (a,b,c,d,q,t) be a set of complex parameters with $|q|<1$. Set $\alpha=(abcdq^{-1})^{1/2}$ for simplicity. Let $x=(x_1,\dots,x_n)$ be a set of independent indeterminates. Koornwinder's q-difference operator $\mathcal{D}_x=\mathcal{D}_x(a,b,c,d|q,t)$ is defined by [6]

$$\begin{aligned}\mathcal{D}_x&=\sum_{i=1}^n\frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{\alpha t^{n-1}(1-x_i^2)(1-qx_i^2)}\prod_{j\neq i}\frac{(1-tx_ix_j)(1-tx_i/x_j)}{(1-x_ix_j)(1-x_i/x_j)}(T_{q,x_i}-1)\\&+\sum_{i=1}^n\frac{(1-a/x_i)(1-b/x_i)(1-c/x_i)(1-d/x_i)}{\alpha t^{n-1}(1-1/x_i^2)(1-q/x_i^2)}\prod_{j\neq i}\frac{(1-tx_j/x_i)(1-t/x_ix_j)}{(1-x_j/x_i)(1-1/x_ix_j)}(T_{q^{-1},x_i}-1),\end{aligned}$$

where $T_{q^{\pm1},x_i}f(x_1,\dots,x_i,\dots,x_n)=f(x_1,\dots,q^{\pm1}x_i,\dots,x_n)$. The Koornwinder polynomial $P_\lambda(x)=P_\lambda(x;a,b,c,d,q,t)$ with partition $\lambda=(\lambda_1,\dots,\lambda_n)$ (i.e., $\lambda_i\in\mathbb{Z}_{\geq0},\lambda_1\geq\cdots\geq\lambda_n$) is uniquely characterized by the two conditions (a) $P_\lambda(x)$ is a $S_n\ltimes(\mathbb{Z}/2\mathbb{Z})^n$ invariant Laurent polynomial having the triangular expansion in terms of the monomial basis $(m_\lambda(x))$ as $P_\lambda(x)=m_\lambda(x)+$lower terms, (b) $P_\lambda(x)$ satisfies $\mathcal{D}_xP_\lambda(x)=d_\lambda P_\lambda(x)$. The eigenvalue is given by

$$d_\lambda=\sum_{j=1}^n\langle abcdq^{-1}t^{2n-2j}q^{\lambda_j}\rangle\langle q^{\lambda_j}\rangle=\sum_{j=1}^n\langle\alpha t^{n-j}q^{\lambda_j};\alpha t^{n-j}\rangle,$$

where we used the notation $\langle x\rangle=x^{1/2}-x^{-1/2}$ and $\langle x;y\rangle=\langle xy\rangle\langle x/y\rangle=x+x^{-1}-y-y^{-1}$ for simplicity of display.

A.2 Macdonald polynomials of types C_n and D_n

We consider some degeneration of the Koornwinder polynomials to the Macdonald polynomials. As for the details, we refer the readers to [6] and [11]. Specializing the parameters in the Koornwinder polynomial $P_\lambda(x;a,b,c,d,q,t)$ as

$$(a,b,c,d,q,t)\to\big(-b^{1/2},ab^{1/2},-q^{1/2}b^{1/2},q^{1/2}ab^{1/2},q,t\big),$$

we obtain the Macdonald polynomial of type (BC_n,C_n) [6]

$$P_\lambda^{(BC_n,C_n)}(x;a,b,q,t)=P_\lambda\big(x;-b^{1/2},ab^{1/2},-q^{1/2}b^{1/2},q^{1/2}ab^{1/2},q,t\big).$$

Namely, setting

$$D_x^{(BC_n,C_n)}=\sum_{\sigma_1,\dots,\sigma_n=\pm1}\prod_{i=1}^n\frac{(1-ab^{1/2}x_i^{\sigma_i})(1+b^{1/2}x_i^{\sigma_i})}{1-x_i^{2\sigma_i}}\cdot\prod_{1\leq i<j\leq n}\frac{1-tx_i^{\sigma_i}x_i^{\sigma_j}}{1-x_i^{\sigma_i}x_i^{\sigma_j}}\cdot\prod_{i=1}^nT_{q^{\sigma_i/2},x_i},$$

we have

$$\begin{aligned}&P_\lambda^{(BC_n,C_n)}(x)=m_\lambda(x)+\text{lower terms},\\&D_x^{(BC_n,C_n)}P_\lambda^{(BC_n,C_n)}(x)=(ab)^{n/2}t^{n(n-1)/4}\left(\sum_{\sigma_1,\dots,\sigma_n=\pm1}s_1^{\sigma_1/2}\cdots s_n^{\sigma_n/2}\right)P_\lambda^{(BC_n,C_n)}(x),\end{aligned}$$

where $s_i=abt^{n-i}q^{\lambda_i}$.

Macdonald polynomials of type C_n and type D_n are obtained as follows

$$P_\lambda^{(C_n)}(x;b,q,t) = P_\lambda^{(BC_n,C_n)}(x;1,b,q,t),$$
$$P_\lambda^{(D_n)}(x;q,t) = P_\lambda^{(BC_n,C_n)}(x;1,1,q,t).$$

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