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Semistability of Principal Bundles on a Kähler Manifold with a Non-Connected Structure Group

Indranil BISWAS † *and Tomás L. GÓMEZ* ‡

† *School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India*
E-mail: *indranil@math.tifr.res.in*

‡ *Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Nicolás Cabrera 15, Campus Cantoblanco UAM, 28049 Madrid, Spain*
E-mail: *tomas.gomez@icmat.es*

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Abstract. We investigate principal G-bundles on a compact Kähler manifold, where G is a complex algebraic group such that the connected component of it containing the identity element is reductive. Defining (semi)stability of such bundles, it is shown that a principal G-bundle E_G admits an Einstein–Hermitian connection if and only if E_G is polystable. We give an equivalent formulation of the (semi)stability condition. A question is to compare this definition with that of [Gómez T.L., Langer A., Schmitt A.H.W., Sols I., *Ramanujan Math. Soc. Lect. Notes Ser.*, Vol. 10, Ramanujan Math. Soc., Mysore, 2010, 281–371].

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1 Introduction

Let X be a compact connected Kähler manifold equipped with a Kähler form ω. Let G be a connected complex reductive group. The connected component, containing the identity element, of the center of G will be denoted by $Z_0(G)$. In the study of principal G-bundles on X, it is usually assumed that the group G is connected.

In [8], Ramanathan gave the following definition. A principal G-bundle E is stable (respectively, semistable) if for all proper parabolic subgroups P and all reductions of structure group $E_P \subset E|_U$ on a big open subset $U \subset X$ (complement of a Zariski closed subset of codimension at least two), and for all strictly dominant characters $\chi : P \longrightarrow \mathbb{C}^*$, the associated line bundle $E_P(\chi)$ over U satisfies the inequality

$$\deg(E_P(\chi)) < 0 \qquad (\text{respectively, } \deg(E_P(\chi)) \leq 0),$$

where the degree is calculated with respect to the Kähler form ω. We recall that a strictly dominant character of P is a character of P trivial on $Z_0(G)$ such that the dual of the line bundle on G/P associated to the character is ample.

A reduction $E_P \subset E|_U$ on a big open set $U \subset X$ to a parabolic subgroup P is called admissible if for every character χ of P trivial on $Z_0(G)$, we have $\deg(E_P(\chi)) = 0$. A semistable principal G-bundle on X is called polystable if there exists a reduction $E_{L(P)} \subset E_G$ to a Levi factor $L(P)$ of a parabolic subgroup $P \subset G$ such that the following two conditions hold:

- the principal $L(P)$-bundle $E_{L(P)}$ is stable, and
- the reduction of E_G to P obtained by extending the structure group of $E_{L(P)}$ to P using the inclusion $L(P) \hookrightarrow P$ is admissible.

Ramanathan used this notion in [9, 10] to construct the moduli space of semistable principal bundles when $\dim X = 1$. In [11] Ramanathan and Subramanian proved that a principal G-bundle admits an Einstein–Hermitian connection if and only if it is polystable.

Behrend defined stability for group schemes in [3]. A reductive group scheme $\mathcal{G}/X$ on X is stable (respectively, semistable) if for all parabolic subgroup schemes $\mathcal{P}/X$, we have

$$\deg(\mathcal{P}/X) := \deg(\mathrm{ad}(\mathcal{P})) < 0 \qquad (\text{respectively, } \deg(\mathrm{ad}(\mathcal{P})) \leq 0),$$

where $\mathrm{ad}(\mathcal{P})$ is the associated Lie algebra bundle on X. Then he defines a principal G-bundle on X to be stable (respectively, semistable) if the associated group scheme $\mathrm{Ad}(E)$ is semistable (respectively, semistable).

If G is connected, then it is easy to see that Behrend's definition of stability coincides with Ramanathan's, because the normalizer $N_G(P)$ of a parabolic subgroup P in G is equal to P when G is connected.

If G is reductive but not connected, the moduli space of principal G-bundles has been constructed in [5] for projective varieties. Given a 1-parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow [G, G] \subset G$, a parabolic subgroup $P(\lambda)$ is defined as follows [12]

$$P(\lambda) = \left\{ g \in G : \lim_{t \to \infty} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G \right\} \tag{1}$$

and the condition of stability is checked only with these parabolic subgroups. The construction in [5] is done using Geometric Invariant Theory, and one obtains a stability condition involving Hilbert polynomials. Looking only at the leading coefficients one obtains the associated "slope" stability, as usual, which only involves degrees. This is the stability we consider, since this is the one which is expected to correspond to the existence of Einstein–Hermitian connections.

Here we address the problem of finding a condition for the existence of an Einstein–Hermitian connection on a principal G-bundle on X when G is not connected.

If G_0 is the connected component of identity of G, then $E \longrightarrow E/G_0$ is a principal G_0-bundle on $Y := E/G_0$, and Y is a finite étale cover of X. In Section 2 we define the principal G-bundle E to be polystable when the principal G_0-bundle $E \longrightarrow Y$ is polystable (cf. Definition 1), and prove that this is a necessary and sufficient condition for the existence of an Einstein–Hermitian connection on E. In Section 3 we show that this definition of polystability is equivalent to checking the usual condition for those parabolic subgroups of G of the form $N_G(\mathfrak{p})$, where $\mathfrak{p}$ is a parabolic subalgebra of the Lie algebra $\mathfrak{g}$ of G. Note that $N_G(\mathrm{Lie}(P)) = N_G(P^0)$ for any parabolic subgroup P, where $\mathrm{Lie}(P)$ is the Lie algebra and P^0 is the connected component of P containing the identity element.

We remark that, if G is connected, all parabolic subgroups are of this form because for a connected reductive group, $N_G(\mathfrak{p}) = N_G(P) = P$.

It is natural to ask whether our condition is equivalent to the condition in [5] in terms of 1-parameter subgroups. For any parabolic subgroup P, it is $P \subset N_G(\mathrm{Lie}(P))$. Hence, a reduction of structure group to P gives reduction to $N_G(\mathrm{Lie}(P))$, and therefore if a principal G-bundle is (semi)stable in the sense of [5], then it is also (semi)stable in our sense.

The implication in the other direction is not clear, since there exist examples of non-connected groups G and 1-parameter subgroups λ such that $N_G(P(\lambda)^0)$ is strictly larger than $P(\lambda)$. It would be interesting to be able to compare the two definitions.

2 Connections on principal bundles

2.1 Semistable and polystable bundles

Let G be a reductive linear algebraic group defined over $\mathbb{C}$. We do not assume that G is connected. Let $G_0 \subset G$ be the connected component containing the identity element. We note that G_0 is a normal subgroup of G. The quotient group

$$\Gamma := G/G_0 \tag{2}$$

parametrizes the connected components of G.

Let X be a compact connected Kähler manifold equipped with a Kähler form ω. Let

$$E_G \longrightarrow X$$

be a holomorphic principal G-bundle on X. Consider the quotient map

$$E_G \xrightarrow{\phi} E_G/G_0 =: Y. \tag{3}$$

The natural projection

$$f : Y \longrightarrow X \tag{4}$$

is a unramified Galois covering map with Galois group Γ (defined in (2)). The pulled back form $f^*\omega$ is a Kähler form on Y. It should be clarified that Y need not be connected.

The projection ϕ in (3) makes E_G a holomorphic principal G_0-bundle on Y.

Definition 1. The principal G-bundle E_G on X is called *semistable* (respectively, *stable*) if for each connected component Y' of Y, the principal G_0-bundle

$$\phi^{-1}(Y') \longrightarrow Y'$$

is semistable (respectively, stable). Similarly, E_G on X is called *polystable* if the principal G_0-bundle $\phi^{-1}(Y') \longrightarrow Y'$ is polystable for every connected component Y' of Y.

Lemma 1. *A principal G-bundle E_G on X is semistable* (*respectively, polystable*) *if for some connected component Y' of Y, the principal G_0-bundle*

$$\phi^{-1}(Y') \longrightarrow Y'$$

is semistable (*respectively, polystable*). *The same criterion holds for stability.*

Proof. Take two connected components Y_1 and Y_2 of Y. Since the covering map f in (4) is Galois, there is an element g of the Galois group such that the automorphism g of Y takes Y_1 to Y_2. Let

$$\widetilde{g} := g|_{Y_1} : Y_1 \longrightarrow Y_2$$

be this isomorphism. Let E_1 (respectively, E_2) be the restriction of the principal G_0-bundle $E_G \longrightarrow Y$ to Y_1 (respectively, Y_2). Since $f \circ \widetilde{g} = f$, and $\mathrm{Lie}(G) = \mathrm{Lie}(G_0)$, we have

$$\widetilde{g}^*\mathrm{ad}(E_2) = \mathrm{ad}(E_1). \tag{5}$$

On the other hand, a principal G_0-bundle is semistable (respectively, polystable) if and only if the corresponding adjoint vector bundle is semistable (respectively, polystable); see [1, p. 214, Proposition 2.10] and [1, p. 224, Corollary 3.8]. Therefore, from (5) we conclude that E_1 is semistable (respectively, polystable) if and only if E_2 is semistable (respectively, polystable).

This isomorphism in (5) is compatible with the Lie algebra structure of the fibers of the two adjoint bundles. We recall that a principal G_0-bundle F_{G_0} is stable if for every parabolic subalgebra bundle $\widetilde{\mathfrak{p}} \subset \mathrm{ad}(F_{G_0})$, we have

$$\mathrm{degree}(\widetilde{\mathfrak{p}}) < 0$$

(see [3]). Therefore, if E_1 is stable, then E_2 is also stable. ∎

2.2 Einstein–Hermitian connections

Any two maximal compact subgroups of G differ by an inner automorphism of G. Fix a maximal compact subgroup $K \subset G$. The quotient G/K is a contractible manifold, in particular, G/K is connected.

Take a holomorphic principal G-bundle E_G over X. A *Hermitian structure* on E_G is a C^∞ reduction of structure group of E_G to K. Since G/K is contractible, and any C^∞ fiber bundle with a contractible fiber is trivial, it follows immediately that E_G admits Hermitian structures.

Any two connections on the principal G-bundle E_G differ by a smooth 1-form with values in $\mathrm{ad}(E_G)$. Two C^∞ connections ∇_1 and ∇_2 on the principal G-bundle E_G are called *equivalent* if $\nabla_1 - \nabla_2$ is of type $(1,0)$ [7, p. 87]. The complex structure on the total space of E_G defines an equivalence class of connections on E_G [7, p. 87, Proposition 2].

Let $E_K \subset E_G$ be a Hermitian structure. Then there is a unique connection ∇ on the principal K-bundle E_K such that the connection $\widetilde{\nabla}$ on E_G induced by ∇ lies in the equivalence class of connections defined by the complex structure on E_G [2, pp. 191–192, Proposition 5]. This $\widetilde{\nabla}$ is called the *Chern connection* corresponding to E_K.

Let $\mathrm{ad}(E_K) \longrightarrow X$ be the adjoint vector bundle for the principal K-bundle E_K. Let

$$\mathcal{K}(\nabla) \in \Omega^{1,1}(\mathrm{ad}(E_K))$$

be the curvature of the above connection ∇ on E_K. The Hermitian structure E_K is called *Einstein–Hermitian* if the section

$$\Lambda_\omega \mathcal{K}(\nabla) \in \Omega^0(\mathrm{ad}(E_K))$$

is given by some element of the center of the Lie algebra of K; here Λ_ω is the adjoint of multiplication of differential forms by the Kähler form ω.

Theorem 1. *A holomorphic principal G-bundle E_G on X admits an Einstein–Hermitian structure if and only if E_G is polystable.*

Given a polystable principal G-bundle E_G over X, the Chern connection on E_G corresponding to a Einstein–Hermitian structure on E_G is independent of the choice of Einstein–Hermitian structure.

Proof. First assume that $E_G \longrightarrow X$ admits an Einstein–Hermitian structure. The connection on the adjoint vector bundle $\mathrm{ad}(E_G)$ induced by an Einstein–Hermitian connection on E_G is also Einstein–Hermitian. Therefore, $\mathrm{ad}(E_G)$ is polystable. Take any connected component Y' of Y. Let $\overline{f}$ be the restriction of f to Y'. Let $E' \longrightarrow Y'$ be the principal G_0-bundle obtained by restricting $E_G \longrightarrow Y$ to Y'.

Since $\overline{f}^* \mathrm{ad}(E_G) = \mathrm{ad}(E')$, and $\mathrm{ad}(E_G)$ is polystable, we conclude that $\mathrm{ad}(E')$ is polystable [4, p. 439, Proposition 2.3]. Hence E' is polystable [1, p. 224, Corollary 3.8].

To prove the converse, assume that the principal G_0-bundle $E_G \longrightarrow X$ is polystable. Take a connected component Y' of Y. As before, $\overline{f}$ is the restriction of f to Y', and $E' \longrightarrow Y'$ is the principal G_0-bundle obtained by restricting $E_G \longrightarrow Y$ to Y'. The adjoint vector bundle $\mathrm{ad}(E')$ is polystable because E' is polystable. Since $\overline{f}$ is an étale covering map, an Einstein–Hermitian connection on $\mathrm{ad}(E')$ produces an Einstein–Hermitian connection on the direct image $\overline{f}_* \mathrm{ad}(E')$. We note that this uses the fact that the Kähler form on Y' is the pullback of the Kähler form on X.

Since $\overline{f}_* \mathrm{ad}(E')$ admits an Einstein–Hermitian connection, it follows that $\overline{f}_* \mathrm{ad}(E')$ is polystable. Since $\mathrm{ad}(E_G)$ is a direct summand of $\overline{f}_* \mathrm{ad}(E')$, we conclude that $\mathrm{ad}(E_G)$ is polystable.

The Einstein–Hermitian connection on $\overline{f}_* \mathrm{ad}(E')$ preserves $\mathrm{ad}(E_G)$, because $\mathrm{ad}(E_G)$ is a direct summand of $\overline{f}_* \mathrm{ad}(E')$. The connection ∇ on $\mathrm{ad}(E_G)$ obtained this way is Einstein–Hermitian.

Take the Einstein–Hermitian connection on $\mathrm{ad}(E')$ to be one given by an Einstein–Hermitian connection on E'. Therefore, the Einstein–Hermitian connection on $\mathrm{ad}(E')$ is compatible with the Lie algebra structure of the fibers of $\mathrm{ad}(E')$. This implies that the above connection ∇ on $\mathrm{ad}(E_G)$ is compatible with the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$. Therefore, ∇ defines a connection on the principal $\mathrm{Aut}(\mathrm{Lie}(G))$-bundle

$$E_{\mathrm{Aut}(\mathrm{Lie}(G))} = E_G \times^G \mathrm{Aut}(\mathrm{Lie}(G))$$

associated to E_G for the homomorphism $G \longrightarrow \mathrm{Aut}(\mathrm{Lie}(G))$ given by the adjoint action of G on $\mathrm{Lie}(G)$. This connection on $E_{\mathrm{Aut}(\mathrm{Lie}(G))}$ given by ∇ will be denoted by ∇_0. This connection ∇_0 is Einstein–Hermitian because ∇ is so.

Define $G^{ab} := G/[G,G]$. The quotient homomorphism $G \longrightarrow G^{ab}$ will be denoted by q. Let $G_0^{ab} \subset G^{ab}$ be the connected component containing the identity element. Let

$$\beta : \ G^{ab} \ \longrightarrow \ G^{ab}$$

be the homomorphism defined by $z \longmapsto z^n$, where n is the order of the quotient group G^{ab}/G_0^{ab}. Note that $\beta(G_0^{ab}) = G_0^{ab}$, and the homomorphism

$$G^{ab}/G_0^{ab} \ \longrightarrow \ G^{ab}/G_0^{ab}$$

given by β is the trivial homomorphism. Hence $\beta(G^{ab}) = G_0^{ab}$. Define

$$\gamma := \beta \circ q : \ G \ \longrightarrow \ G_0^{ab}.$$

Since G_0^{ab} is a torus, the principal G_0^{ab}-bundle $E_{G_0^{ab}}$ on X obtained by extending the structure group of E_G using γ has a unique Einstein–Hermitian connection. We will denote this Einstein–Hermitian connection on $E_{G_0^{ab}}$ by ∇'.

The connection ∇_0 (respectively, ∇') is a 1-form on the total space $E_{\mathrm{Aut}(\mathrm{Lie}(G))}$ (respectively, $E_{G_0^{ab}}$) with values in the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathrm{Lie}(G)))$ (respectively, $\mathrm{Lie}(G_0^{ab})$). Using the natural map $E_G \longrightarrow E_{\mathrm{Aut}(\mathrm{Lie}(G))}$ (respectively, $E_G \longrightarrow E_{G_0^{ab}}$), the 1-form ∇_0 (respectively, ∇') pulls back to a 1-form on E_G with values in $\mathrm{Lie}(\mathrm{Aut}(\mathrm{Lie}(G)))$ (respectively, $\mathrm{Lie}(G_0^{ab})$); this 1-form on E_G will be denoted by $\widehat{\nabla}$ (respectively, $\widehat{\nabla'}$). Note that

$$\mathrm{Lie}(G) = \mathrm{Lie}(\mathrm{Aut}(\mathrm{Lie}(G))) \oplus \mathrm{Lie}\left(G_0^{ab}\right).$$

Therefore, $\widehat{\nabla} + \widehat{\nabla'}$ is a 1-form on E_G with values in the Lie algebra $\mathrm{Lie}(G)$. It is straightforward to check that this $\mathrm{Lie}(G)$-valued 1-form $\widehat{\nabla} + \widehat{\nabla'}$ defines a connection on E_G. This connection on E_G will be denoted by $\widetilde{\nabla}$.

Fix a point $x_0 \in X$. Let $\mathrm{Aut}((E_G)_{x_0})$ denote the group of automorphisms of $(E_G)_{x_0}$ that commute with the action of G on $(E_G)_{x_0}$. Note that $\mathrm{Aut}((E_G)_{x_0})$ is identified with the fiber of the adjoint bundle $\mathrm{Ad}(E_G)_{x_0}$, and it is isomorphic to G. Consider parallel translations of the fiber $(E_G)_{x_0}$, along loops based at x_0, with respect to the above connection $\widetilde{\nabla}$. These together produce a subgroup of $\mathrm{Aut}((E_G)_{x_0})$. It can be shown that this subgroup is contained in a compact subgroup of $\mathrm{Aut}((E_G)_{x_0})$. Indeed, this follows from the fact that the holonomies of both ∇_0 and ∇' are compact. Therefore, possibly taking an extension of structure group, we get a Hermitian structure on E_G. (The above subgroup of $\mathrm{Aut}((E_G)_{x_0})$ is a conjugate of a subgroup of K.) This Hermitian structure is Einstein–Hermitian because both ∇_0 and ∇' are so.

An Einstein–Hermitian connection on the principal G-bundle $E_G \longrightarrow X$ pulls back to an Einstein–Hermitian connection on the principal G_0-bundle $E' \longrightarrow Y'$. Therefore, the uniqueness of the Einstein–Hermitian connection on E_G follows from the uniqueness of the Einstein–Hermitian connection on E'. To explain this, from [11, p. 24, Theorem 1] we know that a stable bundle has a unique Einstein–Hermitian connection, and from [6, p. 111, Theorem 3.27] we know that for any decomposition of a polystable vector bundle F into a direct sum of stable vector bundles, each direct summand is preserved by any Einstein–Hermitian connection on F. Now apply this to the adjoint vector bundle $\mathrm{ad}(E')$ and the principal $G_0/[G_0, G_0]$-bundle associated to E'. ∎

3 Equivalence of stability conditions

For a parabolic subalgebra $\mathfrak{p}$ of $\mathrm{Lie}(G)$, by $N_G(\mathfrak{p})$ we will denote the subgroup of G that preserves $\mathfrak{p}$ by the adjoint action. In this section, by a parabolic subgroup of G we will mean a group of the form $N_G(\mathfrak{p})$ for some parabolic subalgebra $\mathfrak{p}$. As before, by $Z_0(G)$ we will denote the connected component of the center of G containing the identity element.

Definition 2. A principal G-bundle is called *adjoint semistable* (respectively, *adjoint stable*) if for all reductions to a proper parabolic subgroup P, and all reductions of structure group $E_P \subset E|_U$ on a big open subset $U \subset X$, and for all strictly dominant characters $\chi : P \longrightarrow \mathbb{C}^*$, the associated line bundle $E_P(\chi)$ satisfies

$$\deg(E_P(\chi)) < 0 \qquad (\text{respectively, } \deg(E_P(\chi)) \leq 0),$$

where the degree is calculated with respect to the Kähler form ω.

A character of P is called dominant if the restriction to $P_0 := P \bigcap G_0$ is dominant.

Recall that a reduction $E_P \subset E|_U$ on a big open set $U \subset X$ to a parabolic subgroup P is called admissible if for every nontrivial character χ of P trivial on $Z_0(G)$, we have $\deg(E_P(\chi)) = 0$.

A semistable principal G-bundle on X is called *adjoint polystable* if there exists a reduction $E_{L(P)} \subset E_G$ to a Levi factor $L(P)$ of a parabolic subgroup $P \subset G$ such that the following two conditions hold:

- the principal $L(P)$-bundle $E_{L(P)}$ is stable, and
- the reduction of E_G to P obtained by extending the structure group of $E_{L(P)}$ to P is admissible.

Lemma 2. *A principal G-bundle E on X is adjoint semistable if and only if it is semistable in the sense of Definition 1.*

Proof. Recall that a principal G-bundle E on X induces a principal G_0-bundle E_0 on $Y{=}E/G_0$. By Definition 1, a principal G-bundle E is semistable if and only if the restriction of E_0 to a connected component Y' of Y is semistable. This is equivalent to $\mathrm{ad}(E_0)$ being semistable (cf. [1, p. 214, Proposition 2.10]).

Note that $\mathrm{ad}(E_0)$ is isomorphic to $f^*\mathrm{ad}(E)$, which is Γ-equivariant. If $\mathrm{ad}(E_0)$ is semistable, the it is also equivariantly semistable. On the other hand, suppose that it is unstable. Its Harder–Narasimhan filtration is unique, so it will be equivariant, and hence $\mathrm{ad}(E_0)$ will be equivariantly unstable. Therefore, $\mathrm{ad}(E_0)$ is semistable if and only if it is equivariantly semistable.

Taking the quotient by Γ, this is equivalent to the vector bundle $\mathrm{ad}(E)$ on X being semistable. We remark that the proof of [1, p. 214, Proposition 2.10] also works for disconnected groups if we use Definition 2. So $\mathrm{ad}(E)$ is semistable if and only if E is adjoint semistable. ∎

Lemma 3. *A principal G-bundle E on X is adjoint polystable if and only if it is polystable in the sense of Definition 1.*

Proof. If E is polystable in the sense of Definition 1, then from Theorem 1 it follows that the adjoint vector bundle $\mathrm{ad}(E)$ is polystable. Conversely, if $\mathrm{ad}(E)$ is polystable, then $f^*\mathrm{ad}(E)$ is polystable because an Einstein–Hermitian connection on $\mathrm{ad}(E)$ pulls back to an Einstein–Hermitian connection on $f^*\mathrm{ad}(E)$. If $f^*\mathrm{ad}(E)$ is polystable, from [1, p. 224, Corollary 3.8] we know that E is polystable in the sense of Definition 1.

On the other hand, $\mathrm{ad}(E)$ is polystable if and only if E is adjoint polystable; its proof is identical to the proof of [1, p. 224, Corollary 3.8]. ■

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