The Master T-Operator for Inhomogeneous XXX Spin Chain and mKP Hierarchy[⋆]

Anton ZABRODIN [†1][†2][†3][†4]

[†1] *Institute of Biochemical Physics, 4 Kosygina, 119334, Moscow, Russia*

[†2] *ITEP, 25 B. Cheremushkinskaya, 117218, Moscow, Russia*
   E-mail: *zabrodin@itep.ru*

[†3] *National Research University Higher School of Economics,*
   20 Myasnitskaya Ulitsa, Moscow 101000, Russia

[†4] *MIPT, Institutskii per. 9, 141700, Dolgoprudny, Moscow region, Russia*

Abstract. Following the approach of [Alexandrov A., Kazakov V., Leurent S., Tsuboi Z., Zabrodin A., *J. High Energy Phys.* **2013** (2013), no. 9, 064, 65 pages, arXiv:1112.3310], we show how to construct the master T-operator for the quantum inhomogeneous GL(N) XXX spin chain with twisted boundary conditions. It satisfies the bilinear identity and Hirota equations for the classical mKP hierarchy. We also characterize the class of solutions to the mKP hierarchy that correspond to eigenvalues of the master T-operator and study dynamics of their zeros as functions of the spectral parameter. This implies a remarkable connection between the quantum spin chain and the classical Ruijsenaars–Schneider system of particles.

Key words: quantum integrable spin chains; classical many-body systems; quantum-classical correspondence; master T-operator; tau-function

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1   Introduction

The master T-operator was introduced in [2] (in a preliminary form, it was previously discussed in [14]). It is a generating function for commuting conserved quantities of quantum spin chains and associated integrable vertex models which unifies the transfer matrices on all levels of the nested Bethe ansatz and Baxter's Q-operators in one commuting family.

It was also proven in [2] that the master T-operator, as a function of infinitely many auxiliary parameters ("times"), one of which being the quantum spectral parameter, satisfies the same hierarchy of bilinear Hirota equations as the classical tau-function does. This means that any eigenvalue of the master T-operator is a tau-function of a classical integrable hierarchy. For finite spin chains with GL(N)-invariant R-matrices this tau-function is a polynomial in the quantum spectral parameter. The close connection of the spin chain spectral problem with integrable many-body systems of classical mechanics comes from the dynamics of zeros of the polynomial tau-functions. This is a further development of earlier studies [15, 19, 38, 39, 40] clarifying the role of the Hirota bilinear difference equation [11, 24] in quantum integrable models.

In this paper we review the results of [2] and make the connection with classical many-body systems more precise. The presentation here is deliberately made as close as possible to that of [3], where a similar correspondence between the quantum Gaudin model and the classical

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Calogero–Moser many-body system was established using the connection of the former model with the Kadomtsev–Petviashvili (KP) hierarchy. Similarly to that paper, here we discuss the correspondence between integrable systems of different kinds:

  (i) Quantum integrable magnets (spin chains) of XXX-type,

 (ii) The classical modified Kadomtsev–Petviashvili (mKP) hierarchy,

(iii) The classical Ruijsenaars–Schneider (RS) system of particles.

The link (i)-(ii) is the correspondence between quantum spin chains with the $GL(N)$-invariant rational R-matrices and the classical mKP hierarchy based on the construction of the master T-operator [2, 41, 42]. The link (ii)-(iii) is a well-known story about dynamics of poles of rational solutions to soliton equations, see [1, 12, 17, 18, 20, 34, 37]. The composition of (i)-(ii) and (ii)-(iii) implies the connection between the quantum XXX-model and the classical rational RS model [32] which was first mentioned in [2]. The link (i)-(iii) also extends the correspondence between the quantum Gaudin model and the classical Calogero–Moser system earlier established in [26, 27] using different arguments.

Following [2], we show how to construct the master T-operator for the $GL(N)$-based inhomogeneous spin chain with twisted boundary conditions using the co-derivative operation [16]. We call such models "spin chains" in a rather broad sense, not implying the existence of any local Hamiltonian of Heisenberg type. (Integrable local interactions in general do not exist for inhomogeneous spin chains.) However, even in the general case of arbitrary inhomogeneity parameters u_i the model still makes sense as a generalized spin chain with non-local interactions. The "spin variables" are vectors from the spaces $\mathbb{C}^N$ at each site. In fact one may prefer to keep in mind integrable lattice models of statistical mechanics rather than spin chains as such. In either case the final goal of the theory is the diagonalization of the transfer matrices which is usually achieved by the nested Bethe ansatz method in one form or another.

The master T-operator depends on an infinite number of auxiliary "time variables" $\mathbf{t} = \{t_0, t_1, t_2, \ldots\}$ (where t_0 can be identified with the spectral parameter u) and satisfies the bilinear identity for the classical mKP hierarchy. Hence any of its eigenvalues is a mKP tau-function. Here is a short dictionary of the XXX-mKP correspondence:

XXX **chain**		**mKP hierarchy**
master T-operator	$\longleftrightarrow$	τ*-function*
spectral parameter	$\longleftrightarrow$	*the t_0-variable*
higher transfer matrices	$\longleftrightarrow$	*Plücker coordinates*

Moreover, from the explicit form of the R-matrix and the Yang–Baxter equation it follows that this tau-function is a polynomial in $u = t_0$. Therefore, according to [20, 12], the dynamics of its roots in t_i with $i \geq 1$ is given by equations of motion of the rational RS system of particles. It should be mentioned that the method for deriving the dynamics of roots is similar to that used in deriving the Bethe equations in Sklyanin's separation of variables method [35, 36]. This is not particularly surprising because, according to [30] (see also [19, 38]), the nested Bethe ansatz equations themselves can be understood as an integrable many-body system of RS type in discrete time.

The XXX-RS correspondence implies that the "inhomogeneities at sites" u_i in the XXX-chain should be identified with initial coordinates of the RS particles while eigenvalues of the spin chain Hamiltonians are their initial velocities. Eigenvalues of the Lax matrix for the rational RS model coincide with eigenvalues of the twist matrix (with certain multiplicities). Therefore, with fixed integrals of motion in the RS model determined by invariants of the twist matrix, there are a finite number of solutions for their values which correspond to different eigenstates of

the spin chain. In other words, the eigenstates of the spin chain Hamiltonians are in one-to-one correspondence with (a finite number of) intersection points of two Lagrangian submanifolds in the phase space of the classical RS model. One of them is the hyperplane of fixed u_i's and another is the submanifold of constant levels of the RS Hamiltonians in involution. In short, the dictionary of the XXX-RS correspondence is as follows:

$\boldsymbol{XXX}$ **chain**		**Ruijsenaars–Schneider**
inhomogeneities at the sites	$\longleftrightarrow$	*initial coordinates*
eigenvalues of Hamiltonians	$\longleftrightarrow$	*initial momenta*
twist parameters	$\longleftrightarrow$	*integrals of motion*

This "quantum-classical correspondence" was also discussed [7, 8, 29] in the context of super-symmetric gauge theories and branes.

2   The quantum spin chain

Consider generalized quantum integrable spin chains with $\mathrm{GL}(N)$-invariant R-matrix

$$R(u) = I \otimes I + \frac{1}{u} \sum_{a,b=1}^{N} e_{ab} \otimes e_{ba}.$$

Here u is the spectral parameter and I is the unity matrix. By e_{ab} we denote the basis in the space of $N \times N$ matrices such that e_{ab} has only one non-zero element (equal to 1) at the place ab: $(e_{ab})_{cd} = \delta_{ac}\delta_{bd}$. Note that $P = \sum_{ab} e_{ab} \otimes e_{ba}$ is the permutation matrix in the space $\mathbb{C}^N \otimes \mathbb{C}^N$ with the defining property $P(v \otimes w) = w \otimes v$ for any vectors $v, w \in \mathbb{C}^N$, so the R-matrix can be written as $R(u) = I \otimes I + \frac{1}{u}P$. The $\mathrm{GL}(N)$-invariance of this R-matrix means that $g \otimes g\, R(u) = R(u)g \otimes g$ for any $g \in \mathrm{GL}(N)$.

A more general $\mathrm{GL}(N)$-invariant R-matrix is

$$R_\lambda(u) = I \otimes I + \frac{1}{u} \sum_{a,b=1}^{N} e_{ab} \otimes \pi_\lambda(\mathsf{e}_{ba}), \qquad (2.1)$$

which acts in the tensor product of the vector representation space $\mathbb{C}^N$ and an arbitrary finite-dimensional irreducible representation π_λ of the algebra $U(gl(N))$ with highest weight λ. We identify λ with the Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\ell = \ell(\lambda)$ non-zero rows, where $\lambda_i \in \mathbb{Z}_+$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. By e_{ab} we denote the generators of the algebra $U(gl(N))$ with the commutation relations $\mathsf{e}_{ab}\mathsf{e}_{a'b'} - \mathsf{e}_{a'b'}\mathsf{e}_{ab} = \delta_{a'b}\mathsf{e}_{ab'} - \delta_{ab'}\mathsf{e}_{a'b}$. In this notation we have $e_{ab} = \pi_{(1)}(\mathsf{e}_{ab})$, where $\pi_{(1)}$ is the N-dimensional vector representation corresponding to the 1-box diagram $\lambda = (1)$.

For working in multiple tensor product spaces like $(\mathbb{C}^N)^{\otimes n} = \underbrace{\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}_{n}$ the following notation is convenient. For any $g \in \mathrm{End}\,(\mathbb{C}^N)$ we write $g^{(i)} = I^{\otimes(i-1)} \otimes g \otimes I^{\otimes(n-i)} \in \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big)$. In particular, the generators of $\mathrm{GL}(N)$ can be realized as $e_{ab}^{(i)} := I^{\otimes(i-1)} \otimes e_{ab} \otimes I^{\otimes(n-i)}$. They commute for any $i \neq j$ because they act in different spaces. In this notation, $P_{ij} = P_{ji} = \sum_{a,b} e_{ab}^{(i)} e_{ba}^{(j)}$ $(i \neq j)$ is the operator acting by permutation of the i-th and j-th tensor factors in the space $(\mathbb{C}^N)^{\otimes n}$. We have: $P_{ij}g^{(j)} = g^{(i)}P_{ij}$, and $P_{ij}g^{(k)} = g^{(k)}P_{ij}$ for $k \neq i, j$.

Fix a matrix $g \in \mathrm{GL}(N)$ called the twist matrix which is assumed to be diagonalizable. A family of commuting operators acting in the space $\mathcal{V} = (\mathbb{C}^N)^{\otimes n}$ (quantum transfer matrices

or T-operators) can be constructed as

$$\mathsf{T}_\lambda(u) = \mathrm{tr}_{\pi_\lambda}\Big(R_\lambda^{10}(u - u_1) R_\lambda^{20}(u - u_2) \cdots R_\lambda^{n0}(u - u_n)(I^{\otimes n} \otimes \pi_\lambda(g)) \Big), \qquad (2.2)$$

where u_i are arbitrary complex parameters which are assumed to be all distinct. The trace is taken in the auxiliary space V_λ where the representation π_λ is realized. By $R_\lambda^{j0}(u)$ we denote the R-matrix (2.1) acting in the tensor product of the j-th local space $\mathbb{C}^N$ of the chain and the space V_λ labeled by 0. More precisely, let us denote (2.1) symbolically as $R_\lambda(u) = \sum_i a_i \otimes b_i$. Then $R_\lambda^{j0}(u)$ is realized as $R_\lambda^{j0}(u) = \sum_i I^{\otimes(j-1)} \otimes a_i \otimes I^{\otimes(n-j)} \otimes b_i$, where $j = 1, 2, \ldots, n$. Here the operator b_i acts in the auxiliary space V_λ. It follows from the Yang–Baxter equation that the T-operators with the same g, u_i commute for all u, λ and can be simultaneously diagonalized. The normalization used above is such that $\mathsf{T}_\varnothing(u) = I^{\otimes n}$. Another useful normalization is

$$T_\lambda(u) = \prod_{j=1}^{n}(u - u_j) \cdot \mathsf{T}_\lambda(u).$$

In this normalization all $T_\lambda(u)$ and all their eigenvalues are polynomials in u of degree n.

For $n = 0$ the transfer matrix (2.2) is the character of g in the representation π_λ: $\mathsf{T}_\lambda(u) = \mathrm{tr}_{\pi_\lambda} g := \chi_\lambda(g)$, It is given by the Schur polynomial $s_\lambda(\mathbf{y})$ of the variables $\mathbf{y} = \{y_1, y_2, \ldots\}$, where $y_k = \frac{1}{k}\,\mathrm{tr}\,g^k$:

$$\chi_\lambda(g) = s_\lambda(\mathbf{y}) = \det_{i,j=1,\ldots,\ell(\lambda)} h_{\lambda_i - i + j}(\mathbf{y}),$$

(the Jacobi–Trudi formula). Here the complete symmetric polynomials $h_k(\mathbf{y}) = s_{(k)}(\mathbf{y})$ are defined by

$$\exp\big(\xi(\mathbf{y}, z)\big) = \sum_{k=0}^{\infty} h_k(\mathbf{y}) z^k, \qquad \xi(\mathbf{y}, z) := \sum_{k \geq 1} y_k z^k.$$

It is convenient to set $h_k = 0$ for $k < 0$. Let $w_1, \ldots, w_N$ be the eigenvalues of $g \in \mathrm{GL}(N)$ realized as an element of $\mathrm{End}\,(\mathbb{C}^N)$. Then $y_k = \frac{1}{k}\,(w_1^k + \cdots + w_N^k)$ and

$$\chi_\lambda(g) = \frac{\det_{1 \leq i,j \leq N}\big(w_j^{\lambda_i + N - i}\big)}{\det_{1 \leq i,j \leq N}\big(w_j^{N-i}\big)}$$

(see [23]). This formula implies that $\chi_\varnothing(g) = s_\varnothing(\mathbf{y}) = 1$.

A more explicit construction of the quantum transfer matrices $T_\lambda(u)$ was suggested in [16]. It uses the special derivative operator on the group $\mathrm{GL}(N)$ called there the co-derivative operator. In fact it is a sort of "matrix logarithmic derivative". The precise definition is as follows. Let g be an element of the group $\mathrm{GL}(N)$ and f be any function of g with values in $\mathrm{End}\,(L)$, where L is the space of any $\mathrm{GL}(N)$-representation. The (left) co-derivative is defined as

$$\mathsf{D} f(g) = \frac{\partial}{\partial \varepsilon} \sum_{ab} e_{ab} \otimes f\big(e^{\varepsilon \mathbf{e}_{ba}} g\big)\Big|_{\varepsilon = 0}.$$

The right hand side belongs to $\mathrm{End}\,(\mathbb{C}^N \otimes L)$. In particular, the result of the action of D on a scalar function is a linear operator in $\mathbb{C}^N$, acting by D twice we get an operator in $\mathbb{C}^N \otimes \mathbb{C}^N$ and so on. For example:

$$\mathsf{D} \det g = \det g \cdot I, \qquad \mathsf{D}\,\mathrm{tr}\,g^m = m g^m, \qquad \mathsf{D} g^m = \sum_{k=0}^{m-1} P\big(g^k \otimes g^{m-k}\big) \quad \text{for } k \geq 1.$$

When the number of tensor factors is more than two another notation is more convenient. Let $V_i \cong \mathbb{C}^N$ be copies of $\mathbb{C}^N$ and $\mathcal{V} = V_1 \otimes \cdots \otimes V_n$ as before. Then, applying the matrix derivatives to a scalar function f several times, we can embed the result into $\mathrm{End}(\mathcal{V})$ according to the formulas

$$\mathsf{D}_{i_1} f(g) = \sum_{a,b} e_{ab}^{(i_1)} f\left(e^{\varepsilon \mathbf{e}_{ba}} g\right)\Big|_{\varepsilon=0},$$

$$\mathsf{D}_{i_2} \mathsf{D}_{i_1} f(g) = \frac{\partial}{\partial \varepsilon_2} \frac{\partial}{\partial \varepsilon_1} \sum_{a_2 b_2} \sum_{a_1 b_1} e_{a_2 b_2}^{(i_2)} e_{a_1 b_1}^{(i_1)} f\left(e^{\varepsilon_1 \mathbf{e}_{b_1 a_1}^{(i_1)}} e^{\varepsilon_2 \mathbf{e}_{b_2 a_2}^{(i_2)}} g\right)\Big|_{\varepsilon_1 = \varepsilon_2 = 0},$$

and so on. The lower indices of D show in which tensor factors the resulting operator acts non-trivially. In this notation, the examples given above read: $\mathsf{D}_i \mathrm{tr}\, g = g^{(i)}$, $\mathsf{D}_i \mathrm{tr}\, g^{(j)} = P_{ij} g^{(j)}$ $(i \neq j)$. For many other formulas of this type see [2, Appendix D].

According to [16] the transfer matrix (2.2) can be represented as a chain of operators of the form $1 + \frac{\mathsf{D}_i}{u - u_i}$ acting on the character:

$$\mathsf{T}_\lambda(u) = \left(1 + \frac{\mathsf{D}_n}{u - u_n}\right) \cdots \left(1 + \frac{\mathsf{D}_1}{u - u_1}\right) \chi_\lambda(g).$$

For simplicity we assume that the twist matrix g is diagonal: $g = \mathrm{diag}\,(w_1, w_2, \ldots, w_N)$. By analogy with the Gaudin model, one may introduce (non-local) spin chain "Hamiltonians" as residues of the $\mathsf{T}_{(1)}(u)$ at $u = u_i$:

$$\mathsf{T}_{(1)}(u) = \mathrm{tr}\, g + \sum_{i=1}^{n} \frac{H_i}{u - u_i}. \tag{2.3}$$

Explicitly, they have the form:

$$H_i = \overrightarrow{\prod_{j=i+1}^{n}} \left(1 + \frac{P_{ij}}{u_i - u_j}\right) g^{(i)} \overrightarrow{\prod_{j=1}^{i-1}} \left(1 + \frac{P_{ij}}{u_i - u_j}\right).$$

(Here and below we write $\overrightarrow{\prod_{j=1}^{m}} A_j = A_1 A_2 \cdots A_m$ and $\overleftarrow{\prod_{j=1}^{m}} A_j = A_m \cdots A_2 A_1$ for ordered products.)

For example, for $n = 3$ we have:

$$H_1 = \left(1 + \frac{P_{12}}{u_1 - u_2}\right)\left(1 + \frac{P_{13}}{u_1 - u_3}\right) g^{(1)},$$

$$H_2 = \left(1 + \frac{P_{23}}{u_2 - u_3}\right) g^{(2)} \left(1 + \frac{P_{21}}{u_2 - u_1}\right),$$

$$H_3 = g^{(3)} \left(1 + \frac{P_{31}}{u_3 - u_1}\right)\left(1 + \frac{P_{32}}{u_3 - u_2}\right).$$

Similar non-local operators were discussed in [10].

It is easy to check that the operators

$$M_a = \sum_{l=1}^{n} e_{aa}^{(l)} \tag{2.4}$$

commute with the Hamiltonians H_i: $[H_i, M_a] = 0$ (for diagonal g). Therefore, common eigenstates of the Hamiltonians can be classified according to eigenvalues of the operators M_a. Let

$$\mathcal{V} = \bigotimes_{i=1}^{n} V_i = \bigoplus_{m_1, \ldots, m_N} \mathcal{V}(\{m_a\})$$

be the decomposition of the Hilbert space of the spin chain $\mathcal{V}$ into the direct sum of eigenspaces of the operators M_a with eigenvalues $m_a \in \mathbb{Z}_{\geq 0}$, $a = 1, \ldots, N$. Then eigenstates of the H_i's belong to the spaces $\mathcal{V}(\{m_a\})$. Since $\sum_a e_{aa} = I$ is the unit matrix, $\sum_a M_a = nI^{\otimes n}$, and hence

$$\sum_{a=1}^{N} m_a = n.$$

Note also that

$$\sum_{i=1}^{n} H_i = (\mathsf{D}_1 + \cdots + \mathsf{D}_n)\operatorname{tr} g = \sum_{i=1}^{n} g^{(i)} = \sum_{i=1}^{n}\sum_{a=1}^{N} e_{aa}^{(i)} w_a = \sum_{a=1}^{N} w_a M_a.$$

3 The master T-operator and the mKP hierarchy

3.1 The master T-operator

The master T-operator for the spin chain can be defined as

$$T(u, \mathbf{t}) = (u - u_n + \mathsf{D}_n) \cdots (u - u_1 + \mathsf{D}_1) \exp\left(\sum_{k \geq 1} t_k \operatorname{tr} g^k\right), \tag{3.1}$$

where $\mathbf{t} = \{t_1, t_2, \ldots\}$ is an infinite set of "time parameters". These operators commute for all values of the parameters: $[T(u, \mathbf{t}), T(u', \mathbf{t}')] = 0$.

The Cauchy–Littlewood identity

$$\sum_{\lambda} \chi_\lambda(g) s_\lambda(\mathbf{t}) = \exp\left(\sum_{k \geq 1} t_k \operatorname{tr} g^k\right)$$

implies that the expansion of $T(u, \mathbf{t})$ in the Schur functions is

$$T(u, \mathbf{t}) = \sum_{\lambda} T_\lambda(u) s_\lambda(\mathbf{t}). \tag{3.2}$$

The sum is taken over all Young diagrams including the empty one. Therefore, the T-operators $T_\lambda(u)$ can be restored from the master T-operator according to the formula

$$T_\lambda(u) = s_\lambda(\tilde{\partial}) T(u, \mathbf{t})\Big|_{\mathbf{t}=0}, \tag{3.3}$$

where $\tilde{\partial} = \{\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \ldots\}$. In particular,

$$T_{(1)}(u) = \partial_{t_1} T(u, \mathbf{t})\Big|_{\mathbf{t}=0}, \qquad T_{(1^2)}(u) = \frac{1}{2}\big(\partial_{t_1}^2 - \partial_{t_2}\big) T(u, \mathbf{t})\Big|_{\mathbf{t}=0}.$$

Given $z \in \mathbb{C}$, we will use the standard notation $\mathbf{t} \pm [z^{-1}]$ for the following special shift of the time variables:

$$\mathbf{t} \pm [z^{-1}] := \left\{t_1 \pm z^{-1}, t_2 \pm \frac{1}{2}z^{-2}, t_3 \pm \frac{1}{3}z^{-3}, \ldots\right\}.$$

As we shall see below, $T(u, \mathbf{t} \pm [z^{-1}])$ regarded as functions of z with fixed $\mathbf{t}$ plays an important role. Here we only note that equation (3.3) implies that $T(u, 0 \pm [z^{-1}])$ is the generating series for T-operators corresponding to the one-row and one-column diagrams respectively:

$$T\big(u, [z^{-1}]\big) = \sum_{s \geq 0} z^{-s} T_{(s)}(u), \qquad T\big(u, -[z^{-1}]\big) = \sum_{a=0}^{N} (-z)^{-a} T_{(1^a)}(u). \tag{3.4}$$

3.2 The bilinear identity and Hirota equations

The following statement was proved in [2]:

Theorem 3.1. *The master T-operator* (3.1) *satisfies the bilinear identity for the mKP hierarchy* [6, 13]:

$$\oint_{\mathcal{C}_{[0,\infty]}} z^{u-u'} e^{\xi(\mathbf{t}-\mathbf{t}',z)} T\left(u,\mathbf{t}-[z^{-1}]\right) T\left(u,\mathbf{t}'+[z^{-1}]\right) \mathrm{d}z = 0 \tag{3.5}$$

for all $\mathbf{t}$, $\mathbf{t}'$ *and* u, u', *where the integration contour* $\mathcal{C}_{[0,\infty]}$ *encircles the cut* $[0,\infty]$ *between* 0 *and* ∞ *(including the points* 0 *and* ∞*) and does not enclose any singularities coming from the* T-*factors.*

This means that each eigenvalue of the master T-operator is a tau-function of the mKP hierarchy. Equation (3.2) is the expansion of the tau-function in Schur polynomials [9, 31, 33]. The functional relations for quantum transfer matrices [4, 5, 21, 22] can be then interpreted as Plücker relations for coefficients of the expansion.

Setting $u' = u$ and $t'_k = t_k - \frac{1}{k}\left(z_1^{-k} + z_2^{-k} + z_3^{-k}\right)$ in (3.5) and taking the residues we arrive at the 3-term Hirota equation

$$(z_2 - z_3)T\left(u,\mathbf{t}-[z_1^{-1}]\right) T\left(u,\mathbf{t}-[z_2^{-1}]-[z_3^{-1}]\right)$$
$$+ (z_3 - z_1)T\left(u,\mathbf{t}-[z_2^{-1}]\right) T\left(u,\mathbf{t}-[z_3^{-1}]-[z_1^{-1}]\right)$$
$$+ (z_1 - z_2)T\left(u,\mathbf{t}-[z_3^{-1}]\right) T\left(u,\mathbf{t}-[z_1^{-1}]-[z_2^{-1}]\right) = 0. \tag{3.6}$$

Setting $u' = u - 1$, $t'_k = t_k - \frac{1}{k}\left(z_1^{-k} + z_2^{-k}\right)$, we obtain another 3-term Hirota equation

$$z_2 T\left(u+1,\mathbf{t}-[z_2^{-1}]\right) T\left(u,\mathbf{t}-[z_1^{-1}]\right) - z_1 T\left(u+1,\mathbf{t}-[z_1^{-1}]\right) T\left(u,\mathbf{t}-[z_2^{-1}]\right)$$
$$+ (z_1 - z_2)T(u+1,\mathbf{t})T\left(u,\mathbf{t}-[z_1^{-1}]-[z_2^{-1}]\right) = 0.$$

Due to (3.9) (see below), it can be formally regarded as a particular case of (3.6) in the limit $z_3 \to 0$.

3.3 The Baker–Akhiezer functions

According to the general scheme, the Baker–Akhiezer (BA) function and its adjoint corresponding to the tau-function (3.1) are given by the formulas [6, 13]

$$\psi_u(\mathbf{t};z) = z^u e^{\xi(\mathbf{t},z)} T^{-1}(u,\mathbf{t})T\left(u,\mathbf{t}-[z^{-1}]\right), \tag{3.7}$$
$$\psi_u^*(\mathbf{t};z) = z^{-u} e^{-\xi(\mathbf{t},z)} T^{-1}(u,\mathbf{t})T\left(u,\mathbf{t}+[z^{-1}]\right). \tag{3.8}$$

For brevity, we will refer to both ψ and ψ^* as BA functions. In terms of the BA functions, the bilinear identity (3.5) can be written as

$$\oint_{\mathcal{C}_{[0,\infty]}} \psi_u(\mathbf{t};z)\psi_{u'}^*(\mathbf{t}';z)\mathrm{d}z = 0.$$

Using the definition (3.1), we have:

$$T\left(u,\mathbf{t}-[z^{-1}]\right) = z^{-N}\left(u-u_n+\mathsf{D}_n\right)\cdots\left(u-u_1+\mathsf{D}_1\right)\left[\det(zI-g)e^{\operatorname{tr}\xi(\mathbf{t},g)}\right],$$

$$T\left(u,\mathbf{t}+[z^{-1}]\right) = z^{N}\left(u-u_n+\mathsf{D}_n\right)\cdots\left(u-u_1+\mathsf{D}_1\right)\left[\frac{e^{\operatorname{tr}\xi(\mathbf{t},g)}}{\det(zI-g)}\right].$$

Note that because $(\det g)^{-1}\mathsf{D}\det g = \mathsf{D} + 1$, we have

$$\lim_{z \to 0}\left(z^{\pm N}T\left(u, \mathbf{t} \mp \left[z^{-1}\right]\right)\right) = (\det g)^{\pm 1}T(u \pm 1, \mathbf{t}). \tag{3.9}$$

For the BA functions we can thus write:

$$\psi_u(\mathbf{t}; z) = z^{u-N}e^{\xi(\mathbf{t},z)}T^{-1}(u, \mathbf{t})\overset{\overleftarrow{n}}{\underset{i=1}{\prod}}(u - u_i + \mathsf{D}_i)\left[\det\left(zI - g\right)e^{\operatorname{tr}\xi(\mathbf{t},g)}\right], \tag{3.10}$$

$$\psi_u^*(\mathbf{t}; z) = z^{N-u}e^{-\xi(\mathbf{t},z)}T^{-1}(u, \mathbf{t})\overset{\overleftarrow{n}}{\underset{i=1}{\prod}}(u - u_i + \mathsf{D}_i)\left[\frac{e^{\operatorname{tr}\xi(\mathbf{t},g)}}{\det\left(zI - g\right)}\right]. \tag{3.11}$$

From these formulas we see that $z^{-u}e^{-\xi(\mathbf{t},z)}\psi_u(\mathbf{t}; z)$ is a polynomial in z^{-1} of degree N while $z^u e^{\xi(\mathbf{t},z)}\psi_u^*(\mathbf{t}; z)$ is a rational function of z with poles at the points $z = w_i$ (eigenvalues of the matrix g) of at least first order because of $\det(zI - g)$ in the denominator. Moreover, since each co-derivative raises the order of the poles, these poles may be actually of a higher order, up to $n + 1$. Also, as is seen from the second formula, this function has a zero of order N at $z = 0$. (We assume that $w_a \neq 0$.)

Regarded as functions of u, both $z^{-u}\psi_u$ and $z^u\psi_u^*$ are rational functions of u with n zeros and n poles which are simple in general position. From (3.1) and (3.10), (3.11) it follows that

$$\lim_{u \to \infty}\left(z^{-u}e^{-\xi(\mathbf{t},z)}\psi_u(\mathbf{t}; z)\right) = z^{-N}\det(zI - g), \tag{3.12}$$

$$\lim_{u \to \infty}\left(z^u e^{\xi(\mathbf{t},z)}\psi_u^*(\mathbf{t}; z)\right) = z^N(\det(zI - g))^{-1}.$$

The BA functions satisfy the following differential-difference equations:

$$\partial_{t_1}\psi_u(\mathbf{t}; z) = \psi_{u+1}(\mathbf{t}; z) + V(u, \mathbf{t})\psi_u(\mathbf{t}; z), \tag{3.13}$$

$$-\partial_{t_1}\psi_u^*(\mathbf{t}; z) = \psi_{u-1}^*(\mathbf{t}; z) + V(u - 1, \mathbf{t})\psi_u^*(\mathbf{t}; z), \tag{3.14}$$

where

$$V(u, \mathbf{t}) = \partial_{t_1}\log\frac{T(u + 1, \mathbf{t})}{T(u, \mathbf{t})}. \tag{3.15}$$

We also note the formulas for the *stationary* BA functions $\psi_u(z) := \psi_u(0; z)$, $\psi_u^*(z) := \psi_u^*(0; z)$ which directly follow from (3.10), (3.11):

$$\psi_u(z) = z^{u-N}\overset{\overleftarrow{n}}{\underset{i=1}{\prod}}\left(1 + \frac{\mathsf{D}_i}{u - u_i}\right)\det(zI - g), \tag{3.16}$$

$$\psi_u^*(z) = z^{N-u}\overset{\overleftarrow{n}}{\underset{i=1}{\prod}}\left(1 + \frac{\mathsf{D}_i}{u - u_i}\right)\frac{1}{\det(zI - g)}.$$

Below we will also need the relation

$$\partial_{t_m}\log\frac{T(u + 1, \mathbf{t})}{T(u, \mathbf{t})} = \operatorname{res}_\infty\left(\psi_u(\mathbf{t}; z)\psi_{u+1}^*(\mathbf{t}; z)z^m\,\mathrm{d}z\right). \tag{3.17}$$

(Here $\operatorname{res}_\infty(\ldots) \equiv \frac{1}{2\pi i}\oint_\infty(\ldots)$ and $\frac{1}{2\pi i}\oint_\infty z^{-1}\mathrm{d}z = 1$.) This relation can be derived from the bilinear identity (3.5) in the following way. Applying $\partial_{t_m'}$ and putting $u' = u + 1$, $t_k' = t_k$ afterwards, we get:

$$\oint_{\mathcal{C}_{[0,\infty]}}\left(-z^{m-1}T\left(u, \mathbf{t} - \left[z^{-1}\right]\right)T\left(u + 1, \mathbf{t} + \left[z^{-1}\right]\right)\right.$$

$$\left. + z^{-1}T\left(u, \mathbf{t} - \left[z^{-1}\right]\right)\partial_{t_m}T\left(u + 1, \mathbf{t} + \left[z^{-1}\right]\right)\right)\mathrm{d}z = 0.$$

The first term is regular at $z = 0$ and thus contributes to the integral by the residue at ∞ while the second term has residues at the points 0, ∞ and the both contribute to the integral. Using (3.9), we can find the residues:

$$\mathrm{res}_\infty \left(z^{m-1} T \left(u, \mathbf{t} - \left[z^{-1} \right] \right) T \left(u + 1, \mathbf{t} + \left[z^{-1} \right] \right) \right)$$
$$= T(u, \mathbf{t}) \partial_{t_m} T(u + 1, \mathbf{t}) - T(u + 1, \mathbf{t}) \partial_{t_m} T(u, \mathbf{t})$$

Dividing both sides by $T(u, \mathbf{t}) T(u + 1, \mathbf{t})$, we obtain (3.17).

4 Zeros of the master T-operator as Ruijsenaars–Schneider particles

The eigenvalues of the master T-operator are polynomials in the spectral parameter u:

$$T(u, \mathbf{t}) = e^{t_1 \,\mathrm{tr}\, g + t_2 \,\mathrm{tr}\, g^2 + \cdots} \prod_{k=1}^{n} (u - u_k(t_1, t_2, \ldots)).$$

The roots of each eigenvalue have their own dynamics in the times t_k. These dynamics are known to be given by the rational RS model [32] (see [12, 20], which extend the methods developed by Krichever [18] and Shiota [34] for dymamics of poles of solutions to the KP hierarchy). The inhomogeneity parameters of the spin chain play the role of coordinates of the RS particles at $t_i = 0$: $u_j = u_j(0)$. In particular, we have $T(u, 0) = T_\varnothing(u) = \prod_{k=1}^{n} (u - u_k)$.

From (3.4) we see that $T(u) := T_{(1)}(u) = \partial_{t_1} T(u, \mathbf{t}) \big|_{\mathbf{t}=0}$. Therefore,

$$\mathsf{T}_{(1)}(u) = \frac{T_{(1)}(u)}{T_\varnothing(u)} = \partial_{t_1} \log T(u, \mathbf{t}) \big|_{\mathbf{t}=0} = \mathrm{tr}\, g - \sum_{k=1}^{n} \frac{\dot{u}_k(0)}{u - u_k}.$$

Comparing with (2.3), we conclude that the initial velocities are equal (up to sign) to the eigenvalues of the spin chain Hamiltonians:

$$\dot{u}_i = -H_i. \tag{4.1}$$

This unexpected connection between the quantum spin chain and the classical RS model was mentioned in [2]. A similar relation between quantum Hamiltonians in Gaudin model and velocities of particles in the classical Calogero–Moser model was found in [26, 27] within a different framework, see also [25, 28] for further developments.

4.1 Lax pair for the RS model from dynamics of poles

Following Krichever's method [18], let us derive equations of motion for the t_1-dynamics of the u_i's. Essentially, the derivation below is not specific to the master T-operator case but only depends on the polynomiality of the tau-function. The specific part is the particular normalization of the BA functions.

It is convenient to denote $t_1 = t$ and put all other times to zero since they are irrelevant for this derivation. Correspondingly, we will write $T(u, t)$ instead of $T(u, \mathbf{t})$ and $\partial_t u_k = \dot{u}_k$, etc. From (3.15) we see that

$$V(u, t) = \partial_t \log \frac{T(u + 1, t)}{T(u, t)} = \sum_{k=1}^{n} \left(\frac{\dot{u}_k}{u - u_k} - \frac{\dot{u}_k}{u - u_k + 1} \right).$$

The method of [18] is to perform the pole expansion of the linear problem (3.13) for the BA function ψ. From (3.10) we have the pole expansion of the BA function

$$\psi = z^u e^{tz} \left(c_0(z) + \sum_{i=1}^{n} \frac{c_i(z,t)}{u - u_i(t)} \right),$$

where $c_0(z) = \det(I - z^{-1}g)$ (see (3.12)). Substituting this into (3.13), we obtain

$$\sum_{i=1}^{n} \left(\frac{zc_i + \dot{c}_i}{u - u_i} + \frac{c_i \dot{u}_i}{(u - u_i)^2} \right) - \sum_{i=1}^{n} \frac{zc_i - c_0 \dot{u}_i}{u - u_i + 1}$$
$$- \sum_{i=1}^{n} \frac{c_0 \dot{u}_i}{u - u_i} - \sum_{i=1}^{n} \frac{c_i}{u - u_i} \sum_{k=1}^{n} \left(\frac{\dot{u}_k}{u - u_k} - \frac{\dot{u}_k}{u - u_k + 1} \right) = 0.$$

The l.h.s. is a rational function of u with first order poles at $u = u_i$ and $u = u_i - 1$ (possible poles of the second order cancel automatically) vanishing at infinity. Therefore, to solve the linear problem it is enough to cancel all the poles. Representing the l.h.s. as a sum of simple pole terms and equating the coefficients in front of each pole to zero, we get the following system of equations for $i = 1, \ldots, n$:

$$zc_i - c_0 \dot{u}_i - \dot{u}_i \sum_{k=1}^{n} \frac{c_k}{u_i - u_k - 1} = 0,$$

$$\dot{c}_i + zc_i - c_0 \dot{u}_i - c_i \sum_{k \neq i} \frac{\dot{u}_k}{u_i - u_k} - \dot{u}_i \sum_{k \neq i} \frac{c_k}{u_i - u_k} + c_i \sum_{k=1}^{n} \frac{\dot{u}_k}{u_i - u_k + 1} = 0.$$

These equations can be rewritten in the matrix form:

$$(zI - Y)\mathsf{c} = c_0(z)\dot{U}\mathbf{1},$$
$$\dot{\mathsf{c}} = T\mathsf{c}, \tag{4.2}$$

where $\mathsf{c} = (c_1, c_2, \ldots, c_n)^{\mathsf{t}}$, $\mathbf{1} = (1, 1, \ldots, 1)^{\mathsf{t}}$ are n-component vectors and the $n \times n$ matrices $U = U(t)$, $Y = Y(t)$, $T = T(t)$ are given by

$$U_{ij} = u_i \delta_{ij}, \qquad Y_{ij} = \frac{\dot{u}_i}{u_i - u_j - 1}, \tag{4.3}$$

$$T_{ij} = \left(\sum_{k \neq i} \frac{\dot{u}_k}{u_i - u_k} - \sum_{k \neq i} \frac{\dot{u}_k}{u_i - u_k + 1} \right) \delta_{ij} + \left(\frac{\dot{u}_i}{u_i - u_j} - \frac{\dot{u}_i}{u_i - u_j - 1} \right) (1 - \delta_{ij}).$$

Note that $Y = \dot{U}Q$, $T = \tilde{T} - Y$, where

$$Q_{ij} = \frac{1}{u_i - u_j - 1}, \tag{4.4}$$

$$\tilde{T}_{ij} = \left(\sum_{k \neq i} \frac{\dot{u}_k}{u_i - u_k} - \sum_{k} \frac{\dot{u}_k}{u_i - u_k + 1} \right) \delta_{ij} + \frac{\dot{u}_i}{u_i - u_j} (1 - \delta_{ij}). \tag{4.5}$$

The compatibility condition of the system (4.2) is $([T, Y] - \dot{Y})\mathsf{c} = c_0(\ddot{U} - T\dot{U})\mathbf{1}$ or

$$-(\ddot{U}Q + M)\mathsf{c} = c_0(\ddot{U} - T\dot{U})\mathbf{1},$$

where $M := \dot{U}\big(\dot{Q}+QT-\dot{U}^{-1}T\dot{U}Q\big)$. A straightforward calculation shows that $M = WQ$, where W is the diagonal matrix $W = \mathrm{diag}(W_1,\ldots,W_n)$ with elements

$$W_i = \sum_{k\neq i} \frac{2\dot{u}_i\dot{u}_k}{(u_i - u_k)((u_i - u_k)^2 - 1)}.$$

Therefore, $[T,Y] - \dot{Y} = -(\ddot{U}Q + M) = -(\ddot{U} + W)Q$. Since Q is a non-degenerate matrix, the matrix equation $[T,Y] - \dot{Y} = 0$ is equivalent to $\ddot{U} + W = 0$. At the same time one can easily check that

$$(T\dot{U}\mathbf{1})_i = \sum_k T_{ik}\dot{u}_k = -W_i$$

and so the compatibility condition for the linear system (4.2) is $\ddot{U} + W = 0$ which yields the equations of motion for the RS model with n particles

$$\ddot{u}_i = -\sum_{k\neq i} \frac{2\dot{u}_i\dot{u}_k}{(u_i - u_k)((u_i - u_k)^2 - 1)}, \qquad i = 1,\ldots,n. \tag{4.6}$$

Their derivation implies that they can be represented in the Lax form

$$\dot{Y} = [T,Y]. \tag{4.7}$$

and the matrices Y, T form the Lax pair for the model. The matrix Y is the Lax matrix for the RS model. As is seen from (4.7), the time evolution preserves its spectrum, i.e., the coefficients $\mathcal{J}_k$ of the characteristic polynomial

$$\det(zI - Y(t)) = \sum_{k=0}^{n} \mathcal{J}_k z^{n-k}$$

are integrals of motion.

In a similar way, substituting the adjoint BA function

$$\psi^* = z^{-u}e^{-tz}\left(c_0^{-1}(z) + \sum_{i=1}^{n} \frac{c_i^*(z,t)}{u - u_i(t)}\right)$$

into the adjoint linear problem (3.14), we get

$$\mathsf{c}^{*\mathsf{t}}\dot{U}^{-1}(zI - Y) = -c_0^{-1}(z)\mathbf{1}^{\mathsf{t}},$$
$$\partial_t(\mathsf{c}^{*\mathsf{t}}\dot{U}^{-1}) = -\mathsf{c}^{*\mathsf{t}}\dot{U}^{-1}T, \tag{4.8}$$

where $\mathsf{c}^{*\mathsf{t}} = (c_1^*, c_2^*, \ldots, c_n^*)$ and $\mathbf{1}^{\mathsf{t}} = (1,1,\ldots,1)$. Note that $\mathbf{1}^{\mathsf{t}}T = 0$.

Using (4.2), (4.8), we find the solutions for the vectors c, c^*:

$$\mathsf{c}(z,t) = c_0(z)(zI - Y(t))^{-1}\dot{U}\mathbf{1},$$
$$\mathsf{c}^{*\mathsf{t}}(z,t) = -c_0^{-1}(z)\mathbf{1}^{\mathsf{t}}(zI - Y(t))^{-1}\dot{U}. \tag{4.9}$$

For the functions ψ, ψ^* themselves we then have:

$$\psi = c_0(z)z^u e^{tz}\big(1 + \mathbf{1}^{\mathsf{t}}(uI - U(t))^{-1}(zI - Y(t))^{-1}\dot{U}\mathbf{1}\big),$$
$$\psi^* = c_0^{-1}(z)z^{-u}e^{-tz}\big(1 - \mathbf{1}^{\mathsf{t}}(zI - Y(t))^{-1}(uI - U(t))^{-1}\dot{U}\mathbf{1}\big). \tag{4.10}$$

Let us mention some properties of the matrices U, Y to be used in the calculations below. As is well known (and easy to check), the matrix $[U,Y] - Y$ has rank 1. More precisely, the matrices U, Y satisfy the commutation relation

$$[U,Y] = Y + U\mathbf{1}\otimes\mathbf{1}^{\mathsf{t}} \tag{4.11}$$

(here $\mathbf{1}\otimes\mathbf{1}^{\mathsf{t}}$ is the $n\times n$ matrix of rank 1 with all entries equal to 1).

Lemma 4.1. *For any $k \geq 0$ the following equality holds:*

$$1^{\mathsf{t}} Y^k \dot{U} 1 = -\operatorname{tr} Y^{k+1}.$$

Indeed, we have: $1^{\mathsf{t}} Y^k \dot{U} 1 = \operatorname{tr}\left(1 \otimes 1^{\mathsf{t}} Y^k \dot{U}\right) = \operatorname{tr}\left((\dot{U}1 \otimes 1^{\mathsf{t}})Y^k\right) = \operatorname{tr}\left(([U, Y] - Y)Y^k\right) = -\operatorname{tr} Y^{k+1} + \operatorname{tr}[U, Y^{k+1}]$ but the last trace is 0 as trace of a commutator.

4.2 Eigenvalues of the Lax matrix

Here we prove that the eigenvalues of the Lax matrix Y are the same as the eigenvalues of the twist matrix g with appropriate multiplicities.

Theorem 4.2. *The Lax matrix Y has eigenvalues w_a with multiplicities $m_a \geq 0$ such that $m_1 + \cdots + m_N = n$.*

Indeed, let us compare the large $|u|$ expansions of (3.16) and (4.10). From (3.16) we have:

$$\psi_u(z) = \det\left(I - z^{-1}g\right)z^u\left(1 - \frac{1}{u}\sum_i \sum_a \frac{e_{aa}^{(i)} w_a}{z - w_a} + O\!\left(u^{-2}\right)\right).$$

The expansion of (4.10) at $t = 0$ gives (using Lemma 4.1):

$$\psi_u(z) = \det\left(I - z^{-1}g\right)z^u\left(1 - \frac{1}{u}\operatorname{tr}\frac{Y_0}{zI - Y_0} + O\!\left(u^{-2}\right)\right),$$

where we set $Y_0 := Y(0)$. Therefore, we conclude that

$$\operatorname{tr}\frac{Y_0}{zI - Y_0} = \sum_i \sum_a \frac{e_{aa}^{(i)} w_a}{z - w_a}$$

and, since $\operatorname{tr}\left(zI - Y_0\right)^{-1} = \partial_z \log \det(zI - Y_0)$, we have

$$\det(zI - Y_0) = \prod_{a=1}^{N}(z - w_a)^{\sum\limits_{i=1}^{n} e_{aa}^{(i)}} = \prod_{a=1}^{N}(z - w_a)^{M_a},$$

where M_a is the operator (2.4). Hence we see that the M_a is the "operator multiplicity" of the eigenvalue w_a. In the sector $\mathcal{V}(\{m_a\})$ the multiplicity becomes equal to m_a.

Less formal arguments are as follows. The singularities of the vectors $\mathsf{c}(z, t)$, $\mathsf{c}^*(z, t)$ as functions of z are the same as the singularities of the functions ψ, ψ^* in the finite part of the complex plane. From (3.10) we see that $\mathsf{c}(z, t)$ has a pole of order N at $z = 0$ and no other poles. At the same time the first equation in (4.9) states that there are possible poles at eigenvalues of the matrix $Y(t)$ (which do not depend on time). Therefore, they must be canceled by zeros of $c_0(z) = z^{-N}\det(zI - g)$ which are at $z = w_a$ and are assumed to be simple. If all eigenvalues of Y are distinct, such a cancellation is only possible if $n \leq N$. However, the most interesting setting for the quantum spin chains is quite opposite: $n > N$ or even $n \gg N$ (large chain length at a fixed rank of the symmetry algebra). We conclude that in this case the Lax matrix has to have multiple eigenvalues. At first glance, a multiple eigenvalue w_a with multiplicity $m_a \geq 2$ might lead to an unwanted pole of ψ at $z = w_a$ coming from the higher order pole of the matrix $(zI - Y)^{-1}$ which now can not be cancelled by the *simple* zero of $\det(zI - g)$. In fact higher order poles do not appear in the vector $(zI - Y)^{-1}1$ because $\dot{U}1$ is a special vector for the matrix Y which can be decomposed into N Jordan blocks of sizes $m_a \times m_a$. However, they do appear in the co-vector $1^{\mathsf{t}}(zI - Y)^{-1}$ and the function ψ^* has multiple poles at $z = w_a$ (with multiplicities $m_a + 1$).

4.3   Equations of motion in Hamiltonian form

The momenta v_i canonically conjugate to the coordinates of the RS particles u_i can be introduced by the formula

$$\dot{u}_i = -e^{-v_i} \prod_{k \neq i} \frac{u_i - u_k + 1}{u_i - u_k} \qquad \text{or} \qquad v_i = -\log(-\dot{u}_i) + \sum_{k \neq i} \log \frac{u_i - u_k + 1}{u_i - u_k}.$$

Then the Hamiltonian form of the RS equations of motion (4.6) is

$$\begin{pmatrix} \dot{u}_i \\ \dot{v}_i \end{pmatrix} = \begin{pmatrix} \partial_{v_i} \mathcal{H}_1 \\ -\partial_{u_i} \mathcal{H}_1 \end{pmatrix}$$

with the Hamiltonian

$$\mathcal{H}_1 = \operatorname{tr} Y = \sum_{i=1}^{n} e^{-v_i} \prod_{k \neq i} \frac{u_i - u_k + 1}{u_i - u_k}.$$

This result was generalized to the difference KP hierarchy (which is essentially equivalent to the mKP hierarchy) in [12]:

$$\begin{pmatrix} \partial_{t_m} u_i \\ \partial_{t_m} v_i \end{pmatrix} = \begin{pmatrix} \partial_{v_i} \mathcal{H}_m \\ -\partial_{u_i} \mathcal{H}_m \end{pmatrix}, \qquad \mathcal{H}_m = \operatorname{tr} Y^m. \tag{4.12}$$

The $\mathcal{H}_m$'s are higher integrals of motion (Hamiltonians) for the RS model. They are known to be in involution [32]. This agrees with the commutativity of the mKP flows.

For completeness, we give a derivation of (4.12) which is a version of the arguments from [12, 34]. The main technical tool is equation (3.17) which states that

$$\sum_k \left(\frac{\partial_{t_m} u_k}{u - u_k} - \frac{\partial_{t_m} u_k}{u - u_k + 1} \right) = \operatorname{res}_\infty \left[\left(c_0 + \sum_i \frac{c_i}{u - u_i} \right) \left(c_0^{-1} + \frac{c_i^*}{u - u_i + 1} \right) z^{m-1} \mathrm{d}z \right].$$

Matching coefficients in front of the poles, we get

$$\partial_{t_m} u_i = -(\dot{u}_i)^{-1} \operatorname{res}_\infty \left(c_i c_i^* z^m \mathrm{d}z \right).$$

Inserting here (4.9), we continue the chain of equalities:

$$\begin{aligned}
\partial_{t_m} u_i &= \operatorname{res}_\infty \left[\left(1^{\mathrm{t}} (zI - Y)^{-1} \dot{U} \right)_i (\dot{u}_i)^{-1} \left((zI - Y)^{-1} \dot{U} 1 \right)_i z^m \mathrm{d}z \right] \\
&= \operatorname{res}_\infty \left[\left(1^{\mathrm{t}} (zI - Y)^{-1} \right)_i \left((zI - Y)^{-1} \dot{U} 1 \right)_i z^m \mathrm{d}z \right] \\
&= \operatorname{res}_\infty \left[1^{\mathrm{t}} (zI - Y)^{-1} E_{ii} (zI - Y)^{-1} \dot{U} 1 z^m \mathrm{d}z \right] \\
&= \operatorname{res}_\infty \left[\operatorname{tr} \left((\dot{U} 1 \otimes 1^{\mathrm{t}}) (zI - Y)^{-1} E_{ii} (zI - Y)^{-1} \right) z^m \mathrm{d}z \right].
\end{aligned}$$

The next steps are to use the commutation relation (4.11) and notice that $E_{ii} Y = \dot{u}_i \frac{\partial Y}{\partial \dot{u}_i}$:

$$\begin{aligned}
\partial_{t_m} u_i &= \operatorname{res}_\infty \left[\operatorname{tr} \left((-Y + UY - YU)(zI - Y)^{-1} E_{ii} (zI - Y)^{-1} \right) z^m \mathrm{d}z \right] \\
&= -\operatorname{res}_\infty \left[\operatorname{tr} \left((zI - Y)^{-1} \frac{\partial Y}{\partial \log \dot{u}_i} (zI - Y)^{-1} \right) z^m \mathrm{d}z \right] \\
&\quad + \operatorname{res}_\infty \left[\operatorname{tr} \left(E_{ii} (zI - Y)^{-1} (UY - YU)(zI - Y)^{-1} \right) z^m \mathrm{d}z \right].
\end{aligned}$$

The trace in the last term is

$$\operatorname{tr}\big(E_{ii}(zI-Y)^{-1}(UY-YU)(zI-Y)^{-1}\big) = \operatorname{tr}\big(E_{ii}[U,(zI-Y)^{-1}]\big) = \big([U,(zI-Y)^{-1}]\big)_{ii},$$

which is equal to zero because the matrix U is diagonal. We are left with

$$\partial_{t_m}u_i = -\operatorname{res}_\infty\left[\operatorname{tr}\left((zI-Y)^{-1}\frac{\partial Y}{\partial \log \dot u_i}(zI-Y)^{-1}\right)z^m dz\right]$$

$$= -\operatorname{res}_\infty\left[\frac{\partial}{\partial \log \dot u_i}\operatorname{tr}\frac{1}{zI-Y}z^m dz\right] = -\frac{\partial}{\partial \log \dot u_i}\operatorname{tr} Y^m = \partial_{v_i}\operatorname{tr} Y^m.$$

This proves the first equality in (4.12). Note that another form of the equation $\partial_{t_m}u_i = \partial\operatorname{tr} Y^m/\partial v_i$ is

$$\partial_{t_m}u_i = -m\operatorname{tr}\big(E_{ii}Y^m\big) = -m\big(Y^m\big)_{ii}. \tag{4.13}$$

The proof of the second equality in (4.12) is more involved. Here we will closely follow [12]. First, using the Lax equation $\dot Y = [\tilde T, Y]$ and cyclicity of the trace, we take the t_1-derivative of (4.13) to get:

$$\partial_{t_m}\dot u_i = -m\operatorname{tr}\big(Y^m[E_{ii},\tilde T]\big)$$

(recall that $\tilde T = T + Y$, see (4.5)). With the help of this formula we can find $\partial_{t_m}v_i$:

$$\partial_{t_m}v_i = -\dot u_i^{-1}\partial_{t_m}\dot u_i + \sum_{j=1}^n\sum_{l\neq i}\left(\partial_{u_j}\log\frac{u_i-u_l+1}{u_i-u_l}\right)\partial_{t_m}u_j$$

$$= m\dot u_i^{-1}\operatorname{tr}\left(Y^m[E_{ii},\tilde T]\right) - m\sum_{j=1}^n\sum_{l\neq i}\left(\partial_{u_j}\log\frac{u_i-u_l+1}{u_i-u_l}\right)\operatorname{tr}\big(Y^m E_{jj}\big)$$

$$= m\operatorname{tr}\big(A^{(i)}Y^{m-1}\big),$$

where

$$A^{(i)} = \dot u_i^{-1}\big(Y E_{ii}\tilde T' - \tilde T' E_{ii}Y\big) - \sum_{j=1}^n\sum_{l\neq i}\left(\partial_{u_j}\log\frac{u_i-u_l+1}{u_i-u_l}\right)E_{jj}Y.$$

Here $\tilde T'$ is the matrix $\tilde T$ (see (4.4)) with zeros on the main diagonal, $\tilde T'_{ij} = \tilde T_{ij} - \delta_{ij}\tilde T_{ii}$. One can show that

$$-A^{(i)} = \partial_{u_i}Y + [C^{(i)}, Y] \tag{4.14}$$

where $C^{(i)}$ is the matrix

$$C^{(i)} = \sum_{l=1}^n\frac{E_{ll}}{u_{li}+1} - \sum_{l\neq i}\frac{E_{ll}}{u_{li}}$$

(here and below $u_{ij}\equiv u_i - u_j$). From this it immediately follows that $\partial_{t_m}v_i = \partial_{u_i}\operatorname{tr}(Y^m)$, which is the second equality in (4.12). The most direct way to prove (4.14) is to calculate matrix elements of both sides. For example, matrix elements of the matrix $A^{(i)}$ are as follows:

$$A_{jk}^{(i)} = -Y_{jk}\left(\frac{1-\delta_{ij}}{u_{ik}-1} - \frac{1-\delta_{ik}}{u_{ik}} - \frac{\delta_{ik}}{u_{ij}+1} + 1 + \delta_{ij}\sum_{l\neq i}\left(\frac{1}{u_{il}+1} - \frac{1}{u_{il}}\right)\right).$$

4.4   Determinant formula for the master T-operator

There is an explicit determinant representation of the master T-operator. Let $U_0 = U(0)$ be the diagonal matrix $U_0 = \mathrm{diag}(u_1, u_2, \ldots, u_n)$, where $u_i = u_i(0)$ and Y_0 be the Lax matrix (4.3) at $\mathbf{t} = 0$, with $\dot{u}_i(0) = -H_i$ (see (4.1)). Then

$$T(u, \mathbf{t}) = e^{\mathrm{tr}\,\xi(\mathbf{t}, g)} \det\left(uI - U_0 + \sum_{k \geq 1} k t_k Y_0^k\right). \tag{4.15}$$

Substituting this into (3.7), (3.8) we find formulas for the stationary BA functions:

$$\psi_u(z) = \det(zI - g) z^{u-N} \frac{\det\big((uI - U_0)(zI - Y_0) - Y\big)}{\det(uI - U_0) \det(zI - Y_0)}, \tag{4.16}$$

$$\psi_u^*(z) = \frac{z^{N-u}}{\det(zI - g)} \frac{\det\big((zI - Y_0)(uI - U_0) + Y\big)}{\det(uI - U_0) \det(zI - Y_0)}. \tag{4.17}$$

Let us show that these formulas are equivalent to the stationary versions of (4.10). Using commutation relation (4.11), we have:

$$
\begin{aligned}
\det\big((uI - U_0)(zI - Y_0) - Y\big) &= \det\big((zI - Y_0)(uI - U_0) + [U_0, Y_0] - Y\big) \\
&= \det\Big((zI - Y_0)(uI - U_0) + \dot{U}(\mathbf{1} \otimes \mathbf{1}^{\mathbf{t}})\Big) \\
&= \det(uI - U_0) \det(zI - Y_0) \det\Big(I + (uI - U_0)^{-1}(zI - Y_0)^{-1}\dot{U}(\mathbf{1} \otimes \mathbf{1}^{\mathbf{t}})\Big) \\
&= \det(uI - U_0) \det(zI - Y_0)\big(1 + \mathrm{tr}\big((uI - U_0)^{-1}(zI - Y_0)^{-1}\dot{U}(\mathbf{1} \otimes \mathbf{1}^{\mathbf{t}})\big)\big) \\
&= \det(uI - U_0) \det(zI - Y_0)\big(1 + \mathbf{1}^{\mathbf{t}}(uI - U_0)^{-1}(zI - Y_0)^{-1}\dot{U}\mathbf{1}\big)
\end{aligned}
$$

and similarly for (4.17). These formulas show that (4.10) and (4.16), (4.17) are indeed equivalent.

Let us stress that determinant formulas of the type (4.15), (4.16) and (4.17) are not new in the context of polynomial tau-functions of classical integrable hierarchies (see, e.g., [12, 34, 37]). The new observation is that the master T-operator for quantum XXX spin chains has exactly this form.

4.5   Spectrum of the spin chain Hamiltonians from the classical RS model

It follows from the above arguments that the eigenvalues of the (non-local) spin chain Hamiltonians H_i, $i = 1, \ldots, n$ (2.3), can be found in the framework of the classical RS system with n particles as follows. Consider the matrix

$$Y_0 = \begin{pmatrix} H_1 & \dfrac{H_1}{u_2 - u_1 + 1} & \dfrac{H_1}{u_3 - u_1 + 1} & \cdots & \dfrac{H_1}{u_n - u_1 + 1} \\ \dfrac{H_2}{u_1 - u_2 + 1} & H_2 & \dfrac{H_2}{u_3 - u_2 + 1} & \cdots & \dfrac{H_2}{u_n - u_2 + 1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{H_n}{u_1 - u_n + 1} & \dfrac{H_n}{u_2 - u_n + 1} & \dfrac{H_n}{u_3 - u_n + 1} & \cdots & H_n \end{pmatrix}. \tag{4.18}$$

The spectrum of the H_i's in the space $\mathcal{V}(\{m_a\})$ is determined by the conditions

$$\mathrm{tr}\,Y_0^j = \sum_{a=1}^N m_a w_a^j \qquad \text{for all} \quad j \geq 1,$$

i.e., given the initial coordinates u_i and the action variables $\mathcal{H}_j = \operatorname{tr} Y_0^j$ one has to find possible values of the initial velocities $\dot{u}_i = -H_i$. This is equivalent to n algebraic equations for n quantities $H_1, \ldots, H_n$.

In other words, the eigenstates of the quantum Hamiltonians correspond to the intersection points of two Lagrangian manifolds in the phase space of the RS model. One of them is the Lagrangian hyperplane defined by fixing the u_i's and the other one is the Lagrangian submanifold obtained by fixing values of the involutive integrals of motion $\mathcal{H}_i$'s, with the latter being determined by eigenvalues of the spin chain twist matrix. This purely classical prescription appears to be equivalent to the Bethe ansatz solution and solves the spectral problem for the quantum spin chain.

Example 4.3. Consider the vector $\mathsf{v}_a \in \mathbb{C}^N$ with components $(\mathsf{v}_a)_b = \delta_{ab}$. Since $P_{ij}(\mathsf{v}_a)^{\otimes n} = (\mathsf{v}_a)^{\otimes n}$, the vector $(\mathsf{v}_a)^{\otimes n}$ is an eigenstate for the Hamiltonians H_i with the eigenvalues

$$
w_a \prod_{j=1, j\neq i}^{n} \frac{u_i - u_j + 1}{u_i - u_j}.
$$

It is also an eigenvector for the operators M_b with eigenvalues $m_b = n\delta_{ab}$. The matrix (4.18) in this case is the $n \times n$ Jordan block with the only eigenvector $\mathbf{1}$ with eigenvalue w_a and $\operatorname{tr} Y_0^j = nw_a^j$.

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