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The Algebra of a q-Analogue of Multiple Harmonic Series*

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Abstract. We introduce an algebra which describes the multiplication structure of a family of q-series containing a q-analogue of multiple zeta values. The double shuffle relations are formulated in our framework. They contain a q-analogue of Hoffman's identity for multiple zeta values. We also discuss the dimension of the space spanned by the linear relations realized in our algebra.

Key words: multiple harmonic series; q-analogue

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1 Introduction

In this article we introduce an algebra to formalize the multiplication structure of a q-analogue of multiple zeta values.

An *admissible index* is an ordered set of positive integers $(k_1, \dots, k_r)$ with $k_1 \geq 2$. For an admissible index $\mathbf{k} = (k_1, \dots, k_r)$, the *multiple zeta value* (MZV) $\zeta(\mathbf{k})$ is defined by

$$\zeta(\mathbf{k}) := \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

The vector space spanned by MZVs over $\mathbb{Q}$ is closed under multiplication. There are two ways to calculate the product of MZVs. One way is to calculate the product directly from the above definition of MZVs shuffling the indices n_i. Another way is to use an iterated integral representation, called the Drinfel'd integral [2, 10]. By calculating the product of MZVs in two ways above, we obtain different expressions. As a result we get linear relations among MZVs, which are called the *double shuffle relations*.

In [4] Hoffman gives an algebraic formulation to describe the multiplication structure of MZVs. The two ways to calculate the product are realized as two different operations of multiplication on a non-commutative polynomial ring, which we call in this paper the harmonic product and the integral shuffle product, respectively. Using the algebraic setup, an extension of the double shuffle relations is given in [6], and it is conjectured that it contains all linear relations among MZVs.

In this paper we consider the multiplication structure of a q-analogue of MZVs. Fix a complex parameter q such that $0 < |q| < 1$. For an admissible index $\mathbf{k} = (k_1, \dots, k_r)$, a *$q$-analogue of multiple zeta values* [7, 11] is defined by

$$\zeta_q(\mathbf{k}) := \sum_{n_1 > \cdots > n_r > 0} \frac{q^{(k_1-1)n_1 + \cdots + (k_r-1)n_r}}{[n_1]^{k_1} \cdots [n_r]^{k_r}}, \tag{1}$$

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$$[n] := \frac{1-q^n}{1-q}$$

is the q-integer. In the limit $q \to 1$, we restore the MZV $\zeta(\mathbf{k})$.

The harmonic product of qMZVs can be defined naturally, and we also have an iterated integral representation of qMZV [11]. However the vector space spanned by qMZVs over $\mathbb{Q}$ is not presumably closed under the multiplication arising from the integral representation. To overcome the difficulty we consider a larger class of q-series allowing the factor $q^n/[n]$ in the sum (1). Such extension is proposed also in [8]. Then the enlarged vector space of q-series is closed under the harmonic product and the integral shuffle product. The main result of this paper is to formulize the multiplication structure by extending Hoffman's algebra. Thus we can consider the double shuffle relations for qMZVs.

There are many linear relations over $\mathbb{Q}$ among qMZVs. An important feature in the q-analogue case is that there are inhomogeneous linear relations in the following sense. For an admissible index $\mathbf{k} = (k_1, \ldots, k_r)$, the *modified qMZV* $\bar{\zeta}_q(\mathbf{k})$ is defined by

$$\bar{\zeta}_q(\mathbf{k}) := (1-q)^{-|\mathbf{k}|}\zeta_q(\mathbf{k}) = \sum_{n_1 > \cdots > n_r > 0} \frac{q^{(k_1-1)n_1 + \cdots + (k_r-1)n_r}}{(1-q^{n_1})^{k_1} \cdots (1-q^{n_r})^{k_r}}, \tag{2}$$

where $|\mathbf{k}|$ is the *weight* of $\mathbf{k}$ defined by $|\mathbf{k}| := \sum_{i=1}^{r} k_i$. In [9] it is observed that there are linear relations among the modified qMZVs with different weight. Taking the limit $q \to 1$ in such relations, the highest weight terms only survive and we obtain linear relations for MZVs. It suggests that we should consider the vector space spanned by the modified qMZVs rather than the original qMZVs.

Our double shuffle relations contain linear relations for the modified qMZVs. However they do not suffice to get all linear relations. In this article we also give some relations among q-series containing the factor $q^n/[n]$, which we call the resummation duality, as a supply of linear relations (see Theorem 4 below). By computer experiment it is checked that the double shuffle relations and the resummation duality give all linear relations among the modified qMZVs up to weight 7.

The paper is organized as follows. In Section 2 we give the algebraic setup to formalize the multiplication structure of qMZVs. To define the integral shuffle product we make use of an extended version of a q-analogue of multiple polylogarithms (of one variable). In Section 3 we discuss the double shuffle relations. As an example we prove Hoffman's identity for qMZV (see Proposition 7 below) in our algebraic framework. Note that it is derived from Ohno's relation and the duality for qMZV [1]. At last we prove the resummation duality and show some computer experiment about the dimension of the $\mathbb{Q}$-linear space spanned by the relations among the modified qMZVs obtained in this paper.

2 Shuffle products

2.1 Algebraic setup

Let $\hbar$ be a formal variable and $\mathcal{C} := \mathbb{Q}[\hbar]$ the coefficient ring. Denote by $\mathfrak{H}$ the non-commutative polynomial algebra over $\mathcal{C}$ freely generated by alphabet $\{x, y, \rho\}$. Set

$$\xi := y - \rho, \qquad z_k := x^{k-1}y, \quad k \geq 1.$$

Let $\mathfrak{H}^1$ be the subalgebra of $\mathfrak{H}$ freely generated by the set $A := \{\xi\} \cup \{z_k\}_{k \geq 1}$. Note that any element of A is homogeneous and the degree of ξ and z_k ($k \geq 1$) is 1 and k, respectively.

Define the $\mathcal{C}$-submodule $\widehat{\mathfrak{H}}^0$ of $\mathfrak{H}^1$ by

$$\widehat{\mathfrak{H}}^0 := \mathcal{C} + \xi\mathfrak{H}^1 + \sum_{k\geq 2} z_k\mathfrak{H}^1.$$

We denote by $\mathfrak{H}^0$ the $\mathcal{C}$-submodule of $\widehat{\mathfrak{H}}^0$ generated by 1 and the words $z_{k_1}\dots z_{k_r}$ with $k_1\geq 2$ and $k_2,\dots,k_r\geq 1$.

Hereafter we fix a complex parameter q such that $0<|q|<1$. We endow $\mathbb{C}$ with $\mathcal{C}$-module structure such that $\hbar$ acts as multiplication by $1-q$. Denote by $\mathfrak{z}$ the $\mathcal{C}$-submodule of $\mathfrak{H}$ generated by A. For a positive integer n we define the $\mathcal{C}$-linear map $I_{\cdot}(n):\mathfrak{z}\to\mathbb{C}$ by

$$I_\xi(n) := \frac{q^n}{[n]}, \qquad I_{z_k}(n) := \frac{q^{(k-1)n}}{[n]^k}.$$

Note that

$$I_\rho(n) = I_{z_1-\xi}(n) = 1-q. \tag{3}$$

Now we define the $\mathcal{C}$-linear map $Z_q:\widehat{\mathfrak{H}}^0\to\mathbb{C}$ by $Z_q(1)=1$ and

$$Z_q(u_1\dots u_r) := \sum_{n_1>\cdots>n_r>0}\prod_{i=1}^r I_{u_i}(n_i),$$

where $r\geq 1$ and $u_i\in A$. The infinite sum in the right hand side absolutely converges because there exists a positive constant M such that $|1/[n]|\leq M$ for all $n\geq 1$. If $\mathbf{k}=(k_1,\dots,k_r)$ is an admissible index, the value $Z_q(z_{k_1}\dots z_{k_r})$ is equal to the qMZV (1).

2.2 Harmonic product

We define the harmonic product on $\mathfrak{H}^1$ generalizing the algebraic formulation given in [6]. Consider the commutative product $\circ$ on $\mathfrak{z}$ by setting

$$z_k\circ z_l = z_{k+l}+\hbar z_{k+l-1}, \qquad \xi\circ z_k = z_{k+1}, \qquad \xi\circ\xi = z_2-\hbar\xi$$

for $k,l\geq 1$ and extending by $\mathcal{C}$-linearity. Define the $\mathcal{C}$-bilinear product $*$ on $\mathfrak{H}^1$ inductively by setting

$$\begin{aligned}&1*w=w, \qquad w*1=w,\\&(u_1w)*(u_2w') = u_1(w*u_2w')+u_2(u_1w*w')+(u_1\circ u_2)(w*w')\end{aligned}$$

for $w,w'\in\mathfrak{H}^1$ and $u_1,u_2\in A$. It is commutative and associative because the product $\circ$ is commutative and associative. Let us call $*$ the *harmonic product* on $\mathfrak{H}^1$. Then the $\mathcal{C}$-submodule $\widehat{\mathfrak{H}}^0$ is a subalgebra of $\mathfrak{H}^1$ with respect to the harmonic product.

Theorem 1. *For any $w,w'\in\widehat{\mathfrak{H}}^0$ we have $Z_q(w*w')=Z_q(w)Z_q(w')$.*

Proof. For a positive integer N we define the $\mathcal{C}$-linear map $F_{\cdot}(N):\mathfrak{H}^1\to\mathbb{C}$ by $F_1(N)=1$ and

$$F_{u_1\dots u_r}(N) = \sum_{N>n_1>\cdots>n_r>0}\prod_{i=1}^r I_{u_i}(n_i), \tag{4}$$

where $u_i\in A$. Note that $F_{uw}(N)=\sum_{N>m>0} I_u(m)F_w(m)$ for $u\in A$ and $w\in\mathfrak{H}^1$. We have $Z_q(w)=\lim_{N\to\infty}F_w(N)$ for any $w\in\widehat{\mathfrak{H}}^0$, and hence it suffices to prove that $F_{w*w'}(N)=$

$F_w(N)F_{w'}(N)$ for words w, w' in A starting with ξ or z_k $(k \ge 2)$. Let us prove it by induction on the sum of the degrees of w and w'. Note that if $w = 1$ or $w' = 1$ the equality is trivial. Let $w, w' \in \widehat{\mathfrak{H}}^0$ be words and $u_1, u_2 \in A$. Then

$$F_{u_1 w}(N)F_{u_2 w'}(N) = \sum_{N>m>0} I_{u_1}(m)F_w(m)F_{u_2 w'}(m) + \sum_{N>m>0} I_{u_2}(m)F_{w'}(m)F_{u_1 w}(m) + \sum_{N>m>0} I_{u_1}(m)I_{u_2}(m)F_w(m)F_{w'}(m).$$

Now the desired equality follows from the induction hypothesis and

$$I_{z_k}(m)I_{z_l}(m) = I_{z_{k+l}+\hbar z_{k+l-1}}(m), \qquad I_\xi(m)^2 = I_{z_2-\hbar\xi}(m), \qquad I_\xi(m)I_{z_k}(m) = I_{z_{k+1}}(m)$$

for $k, l \ge 2$ and $m \ge 1$. ■

2.3 Integral shuffle product

Let us define the $\mathcal{C}$-bilinear product $ш$ on $\mathfrak{H}$ inductively as follows. We set $1 \,ш\, w = w \,ш\, 1 = w$ for any $w \in \mathfrak{H}$. For $u, v \in \{x, y, \rho\}$ and $w, w' \in \mathfrak{H}$, we set

$$uw \,ш\, vw' = u(w \,ш\, vw') + v(uw \,ш\, w') + \alpha(u,v)(w \,ш\, w'),$$

where $\alpha(u,v)$ is determined by

$$\alpha(x,x) = \hbar x, \qquad \alpha(x,y) = \alpha(y,x) = 0, \qquad \alpha(y,y) = -y\rho$$

and

$$\alpha(u,\rho) = \alpha(\rho,u) = -u\rho \tag{5}$$

for $u \in \{x, y, \rho\}$. Then the product $ш$ is commutative because of the symmetry of α.

Lemma 1. *For any $w, w' \in \mathfrak{H}$, we have $\rho w \,ш\, w' = \rho(w \,ш\, w')$.*

Proof. From the property (5), we see that

$$\rho w \,ш\, uw' - \rho(w \,ш\, uw') = u\left(\rho w \,ш\, w' - \rho(w \,ш\, w')\right)$$

for $u \in \{x, y, \rho\}$ and $w, w' \in \mathfrak{H}$. Using this formula repeatedly we find that $\rho w \,ш\, w' - \rho(w \,ш\, w') = w'\left(\rho w \,ш\, 1 - \rho(w \,ш\, 1)\right) = 0$ for $w, w' \in \mathfrak{H}$. ■

Proposition 1. *The product $ш$ is associative.*

Proof. We prove $(w_1 \,ш\, w_2) \,ш\, w_3 = w_1 \,ш\, (w_2 \,ш\, w_3)$ for $w_i \in \mathfrak{H}$ $(i = 1, 2, 3)$ by induction on the sum of the degrees of w_1, w_2 and w_3. If $w_i = 1$ for some i, it is trivial. Suppose that $w_i = u_i w'_i$ $(i = 1, 2, 3)$ for $u_i \in \{x, y, \rho\}$ and $w'_i \in \mathfrak{H}$. Lemma 1 implies that if $u_i = \rho$ for some i, the desired equality follows from the induction hypothesis. Thus we should check the associativity in the case where any u_i is x or y. Here let us consider the case where $(u_1, u_2, u_3) = (x, y, y)$. We have

$$\begin{aligned}
(xw'_1 \,ш\, yw'_2) \,ш\, yw'_3 &= \big(x(w'_1 \,ш\, yw'_2) + y(xw'_1 \,ш\, w'_2)\big) \,ш\, yw'_3 \\
&= x\big((w'_1 \,ш\, yw'_2) \,ш\, yw'_3\big) + y\big(x(w'_1 \,ш\, yw'_2) \,ш\, w'_3\big) \\
&\quad + y\big((xw'_1 \,ш\, w'_2) \,ш\, yw'_3 + y(xw'_1 \,ш\, w'_2) \,ш\, w'_3 - \rho((xw'_1 \,ш\, w'_2) \,ш\, w'_3)\big) \\
&= x\big((w'_1 \,ш\, yw'_2) \,ш\, yw'_3\big) \\
&\quad + y\left((xw'_1 \,ш\, w'_2) \,ш\, yw'_3 + (xw'_1 \,ш\, yw'_2) \,ш\, w'_3 - \rho((xw'_1 \,ш\, w'_2) \,ш\, w'_3)\right).
\end{aligned}$$

Now apply the induction hypothesis and use the equality

$$\rho\big(xw_1' ш (w_2' ш w_3')\big) = xw_1' ш \big(\rho(w_2' ш w_3')\big),$$

which follows from Lemma 1. Then we obtain

$$x\big(w_1' ш (yw_2' ш yw_3')\big) + y\big(xw_1' ш \big(w_2' ш yw_3' + yw_2' ш w_3' - \rho(w_2' ш w_3')\big)\big).$$

It is equal to $xw_1' ш (yw_2' ш yw_3')$. The proof for the other cases is similar. ■

Thus an associative commutative product ш is defined on $\mathfrak{H}$. We call it the *integral shuffle product*.

Proposition 2. *The $\mathcal{C}$-submodule $\widehat{\mathfrak{H}}^0$ of $\mathfrak{H}$ is closed under the integral shuffle product.*

Proof. First let us prove that

$$z_k w ш z_l w' \in \sum_{j \geq \min(k,l)} z_j \mathfrak{H}^1 \tag{6}$$

for $k, l \geq 1$ and $w, w' \in \mathfrak{H}^1$. If $k = l = 1$ it follows from $\alpha(y,y) = -y\rho$. If $k = 1$ and $l \geq 2$, we find the above property by induction on l using

$$yw ш z_l w' = y(w ш z_l w') + x(yw ш z_{l-1} w')$$

and $y = z_1$, $xz_j = z_{j+1}$ $(j \geq 1)$. For $k, l \geq 2$ we obtain it by induction again from

$$z_k w ш z_l w' = x(z_{k-1} w ш z_l w' + z_k w ш z_{l-1} w' + \hbar z_{k-1} w ш z_{l-1} w').$$

It remains to prove that $\xi w ш z_k w'$ and $\xi w ш \xi w'$ belong to $\widehat{\mathfrak{H}}^0$ for $w, w' \in \mathfrak{H}^1$ and $k \geq 2$. It follows from the property (6) and

$$\begin{aligned} &\xi w ш z_k w' = \xi(w ш z_k w') + x(yw ш z_{k-1} w'), &(7)\\ &\xi w ш \xi w' = \xi\big(w ш \xi w' + \xi w ш w' - \rho(w ш w')\big). \end{aligned}$$

■

In the rest of this section we prove the following theorem.

Theorem 2. *For any $w, w' \in \widehat{\mathfrak{H}}^0$ we have $Z_q(w ш w') = Z_q(w) Z_q(w')$.*

Thus we define the two operations of multiplication, the harmonic product and the integral shuffle product, on $\widehat{\mathfrak{H}}^0$. They describe the multiplication structure of a family of q-series $Z_q(w)$ containing qMZVs. Note that we can formally restore Hoffman's algebra for MZVs [4] by setting $\hbar = 0$ and $\rho = 0$.

2.4 A q-analogue of multiple polylogarithms

To prove Theorem 2 we introduce an extended version of a q-analogue of multiple polylogarithms (of one variable). Denote by $\mathcal{F}$ the ring of holomorphic functions on the unit disk $|t| < 1$. We consider $\mathcal{F}$ as a $\mathcal{C}$-module by $(\hbar f)(t) := (1-q) f(t)$ for $f \in \mathcal{F}$. Define the $\mathcal{C}$-linear map $\widehat{\mathfrak{H}}^0 \ni w \mapsto L_w \in \mathcal{F}$ by $L_1(t) = 1$ and

$$L_{\xi w}(t) := \sum_{n=1}^{\infty} \frac{t^n}{[n]} F_w(n), \qquad L_{z_k w}(t) := \sum_{n=1}^{\infty} \frac{t^n}{[n]^k} F_w(n)$$

for $w \in \mathfrak{H}^1$ and $k \geq 2$, where $F_w(n)$ is defined by (4).

Consider the q-difference operator $\mathcal{D}_q$ defined by

$$(\mathcal{D}_q f)(t) := \frac{f(t) - f(qt)}{(1-q)t}.$$

To describe the function $\mathcal{D}_q L_w$ ($w \in \widehat{\mathfrak{H}}^0$) we introduce the two maps Δ_j ($j = 0, 1$) as follows. Set

$$\widetilde{\mathfrak{H}}^0 := \xi \mathfrak{H}^1 + \sum_{k \geq 2} z_k \mathfrak{H}^1, \qquad \mathfrak{h}^{\geq a} := \mathcal{C} + \sum_{k \geq a} z_k \mathfrak{H}^1, \quad a = 1, 2.$$

Let $\Delta_0 : \mathfrak{h}^{\geq 2} \to \widetilde{\mathfrak{H}}^0$ be the $\mathcal{C}$-linear map defined by

$$\Delta_0(1) = 0, \qquad \Delta_0(z_k w) = \begin{cases} \xi w, & k = 2, \\ z_{k-1} w, & k \geq 3 \end{cases}$$

for $w \in \mathfrak{H}^1$. Next we define the $\mathcal{C}$-linear map $\Delta_1 : \mathfrak{h}^{\geq 1} \to \widehat{\mathfrak{H}}^0$ by $\Delta_1(1) = 1$ and

$$\Delta_1(z_k w) = \left(\sum_{a=2}^{k} \binom{k-1}{a-1} (-\hbar)^{k-a} z_a + (-\hbar)^{k-1} \xi \right) w$$

for $k \geq 1$ and $w \in \mathfrak{H}^1$.

Now note that $\widehat{\mathfrak{H}}^0$ is decomposed into the $\mathcal{C}$-submodules

$$\widehat{\mathfrak{H}}^0 = \mathfrak{h}^{\geq 2} \oplus \left(\bigoplus_{r \geq 0} \xi \rho^r \mathfrak{h}^{\geq 1} \right).$$

Proposition 3. *For $w \in \mathfrak{h}^{\geq 2}$ we have*

$$(\mathcal{D}_q L_w)(t) = \frac{1}{t} L_{\Delta_0(w)}(t). \tag{8}$$

For $w \in \mathfrak{h}^{\geq 1}$ and $r \geq 0$ it holds that

$$(\mathcal{D}_q L_{\xi \rho^r w})(t) = \frac{((1-q)t)^r}{(1-t)^{r+1}} L_{\Delta_1(w)}(t). \tag{9}$$

Proof. The equality (8) follows from $\mathcal{D}_q(t^n) = [n] t^{n-1}$ for $n \geq 0$. Let us prove (9). If $w = u_1 \ldots u_s \in \mathfrak{h}^{\geq 1}$ ($u_i \in A$) is a word we have

$$L_{\xi \rho^r w}(t) = (1-q)^r \sum_{n > n_1 > \cdots > n_s > 0} \binom{n - n_1 - r}{r} \frac{t^n}{[n]} \prod_{i=1}^{s} I_{u_i}(n_i)$$

because of (3). Therefore

$$\begin{aligned} (\mathcal{D}_q L_{\xi \rho^r w})(t) &= (1-q)^r \sum_{n_1 > \cdots > n_s > 0} \left(\sum_{n = n_1 + 1}^{\infty} \binom{n - n_1 - r}{r} t^{n-1} \right) \prod_{i=1}^{s} I_{u_i}(n_i) \\ &= \frac{((1-q)t)^r}{(1-t)^{r+1}} \sum_{n_1 > \cdots > n_s > 0} t^{n_1} \prod_{i=1}^{s} I_{u_i}(n_i). \end{aligned}$$

Here we used the equality

$$\frac{1}{(1-x)^{k+1}}=\sum_{j=0}^{\infty}\binom{k+j}{j}x^{j}, \qquad |x|<1, \tag{10}$$

for any non-negative integer k. Now the equality (9) follows from

$$t^{n}I_{z_k}(n)=\sum_{a=1}^{k}\binom{k-1}{a-1}\bigl(-(1-q)\bigr)^{k-a}\frac{t^{n}}{[n]^{a}}$$

for $k\geq 1$. ■

2.5 Multiplication structure of the q-analogue of multiple polylogarithms

Let us prove that the set of functions $\{L_w\}_{w\in\widehat{\mathfrak{H}}^0}$ is closed under multiplication. Define the $\mathcal{C}$-linear map $I_0:\widetilde{\mathfrak{H}}^0\to\mathfrak{h}^{\geq 2}\cap\widetilde{\mathfrak{H}}^0$ by

$$I_0(\xi w)=z_2 w, \qquad I_0(z_k w)=z_{k+1}w, \qquad w\in\mathfrak{H}^1, \quad k\geq 2,$$

and the $\mathcal{C}$-linear map $I_1:\widehat{\mathfrak{H}}^0\to\mathfrak{h}^{\geq 1}$ by

$$I_1(1)=1, \qquad I_1(\xi w)=z_1 w, \qquad I_1(z_k w)=\sum_{a=1}^{k}\binom{k-1}{a-1}\hbar^{k-a}z_a$$

for $w\in\mathfrak{H}^1$ and $k\geq 2$. We have the following property.

Lemma 2. *The maps $\Delta_0 I_0$ and $\Delta_1 I_1$ are identities on $\widetilde{\mathfrak{H}}^0$ and $\widehat{\mathfrak{H}}^0$, respectively.*

Proposition 4.

(1) *Let $w\in\widetilde{\mathfrak{H}}^0$. Suppose that $f\in\mathcal{F}$ satisfies $f(0)=0$ and $(\mathcal{D}_q f)(t)=L_w(t)/t$. Then $f=L_{I_0(w)}$.*

(2) *Let $w\in\widehat{\mathfrak{H}}^0$ and $r\geq 0$. Suppose that $f\in\mathcal{F}$ satisfies $f(0)=0$ and*

$$(\mathcal{D}_q f)(t)=\frac{((1-q)t)^{r}}{(1-t)^{r+1}}L_w(t).$$

Then $f=L_{\xi\rho^{r}I_1(w)}$.

Proof. Note that the initial value problem $\mathcal{D}_q f=g$ and $f(0)=0$ for a given $g\in\mathcal{F}$ has a unique solution in $\mathcal{F}$ if it exists. Therefore it suffices to check that the function f given above is a solution to the initial value problems in (1) or (2). We have $f(0)=0$ because the image of I_0 or $\xi\rho^{r}I_1$ is contained in $\widetilde{\mathfrak{H}}^0$. Proposition 3 and Lemma 2 imply that the function f is a solution. ■

To write down the structure of multiplication of the functions L_w ($w\in\widehat{\mathfrak{H}}^0$), let us define the commutative $\mathcal{C}$-bilinear product $\star$ on $\widehat{\mathfrak{H}}^0$. Set $1\star w=w\star 1=w$ for $w\in\widehat{\mathfrak{H}}^0$. In general we define the product $\star$ inductively as follows. For $w,w'\in\mathfrak{h}^{\geq 2}$ we set

$$w\star w'=I_0\left(\Delta_0(w)\star w'+w\star\Delta_0(w')-\hbar\Delta_0(w)\star\Delta_0(w')\right).$$

For $w\in\mathfrak{h}^{\geq 2}$, $w'\in\mathfrak{h}^{\geq 1}$ and $r\geq 0$, set

$$w\star\xi\rho^{r}w'=I_0(\Delta_0(w)\star\xi\rho^{r}w')+\xi\rho^{r}I_1\bigl(\bigl(w-\hbar\Delta_0(w)\bigr)\star\Delta_1(w')\bigr).$$

For $w, w' \in \mathfrak{h}^{\geq 1}$ and $r, s \geq 0$,

$$\xi\rho^r w \star \xi\rho^s w' = \xi\rho^r I_1(\Delta_1(w) \star \xi\rho^s w') + \xi\rho^s I_1\big(\xi\rho^r w \star \Delta_1(w')\big) - \xi\rho^{r+s+1} I_1\big(\Delta_1(w) \star \Delta_1(w')\big).$$

Since the image of I_0 is contained in $\widetilde{\mathfrak{H}}^0$, the product $\star$ is well-defined.

Proposition 5. *For any $w, w' \in \widehat{\mathfrak{H}}^0$ we have $L_{w \star w'} = L_w L_{w'}$.*

Proof. It suffices to prove in the case where w and w' are homogeneous. Let us prove it by induction on the sum of the degrees of w and w'. Note that the desired equality is trivial if w or w' belongs to $\mathcal{C}$. Otherwise the function $L_w L_{w'}$ has a zero at $t = 0$. Now calculate $\mathcal{D}_q(L_w L_{w'})$ by using the formula

$$\big(\mathcal{D}_q(fg)\big)(t) = (\mathcal{D}_q f)(t)g(t) + f(t)(\mathcal{D}_q g)(t) - (1-q)t(\mathcal{D}_q f)(t)(\mathcal{D}_q g)(t)$$

for $f, g \in \mathcal{F}$. The q-difference of L_w and $L_{w'}$ is written in terms of the maps Δ_0 and Δ_1 as described in Proposition 3. Here note that if w is homogeneous the degree of $\Delta_0(w)$ is less than that of w. Now the induction hypothesis implies that $\mathcal{D}_q(L_w L_{w'})$ is given in terms of the product $\star$. Use Proposition 4 to restore the original function $L_w L_{w'}$, and we get the desired equality from the definition of the product $\star$. ■

2.6 Proof of Theorem 2

Let us prove Theorem 2. First we describe a relation between the qMZV and the function L_w.

Lemma 3. *Define the $\mathcal{C}$-linear map $e : \widehat{\mathfrak{H}}^0 \to \widehat{\mathfrak{H}}^0$ by setting $e(1) = 1$ and*

$$e(\xi w) = \xi w, \qquad e(z_k w) = \left(\sum_{a=2}^{k} \binom{k-2}{a-2} \hbar^{k-a} z_a \right) w$$

for $w \in \mathfrak{H}^1$ and $k \geq 2$. Then we have $L_w(q) = Z_q(e(w))$ for any $w \in \widehat{\mathfrak{H}}^0$.

Proof. It follows from $q^n/[n] = I_\xi(n)$ and $q^n/[n]^k = I_{e(z_k)}(n)$ for $k \geq 2$ and $n \geq 1$. ■

Note that the map e given in Lemma 3 is an isomorphism on the $\mathcal{C}$-module $\widehat{\mathfrak{H}}^0$. Its inverse is given by $e^{-1}(1) = 1$, $e^{-1}(\xi w) = \xi w$ and

$$e^{-1}(z_k w) = \left(\sum_{a=2}^{k} \binom{k-2}{a-2} (-\hbar)^{k-a} z_a \right) w$$

for $w \in \mathfrak{H}^1$ and $k \geq 2$.

Now Theorem 2 is reduced to the following proposition because of Proposition 5 and Lemma 3.

Proposition 6. *It holds that $e \star (e^{-1} \times e^{-1}) = ш$ on $\widehat{\mathfrak{H}}^0 \times \widehat{\mathfrak{H}}^0$.*

In the proof of Proposition 6 we use the properties below.

Lemma 4.

(1) $e(z_k w) = (x + \hbar) e(z_{k-1} w)$ *for $k \geq 3$ and $w \in \mathfrak{H}^1$.*

(2) $(1 - \hbar\Delta_0) e^{-1} = \Delta_1$ *on $\mathfrak{h}^{\geq 2}$.*

(3) $\Delta_1(z_k w) = -\hbar\Delta_1(z_{k-1} w) + e^{-1}(z_k w)$ *for $k \geq 2$ and $w \in \mathfrak{H}^1$.*

(4) $\Delta_0 e^{-1}(z_k w) = \Delta_1(z_{k-1}w)$ *for* $k \geq 2$ *and* $w \in \mathfrak{H}^1$.

(5) $eI_0 e^{-1}(\xi w) = xyw$ *for* $w \in \mathfrak{H}^1$, *and* $eI_0 e^{-1}(w) = (x+\hbar)w$ *for* $w \in \sum_{a\geq 2} z_a \mathfrak{H}^1$.

(6) $I_1 I_0(w) = (x+\hbar)I_1(w)$ *for* $w \in \widehat{\mathfrak{H}}^0$.

The proof is straightforward.

Lemma 5. *For any* $w, w' \in \mathfrak{h}^{\geq 1}$ *it holds that* $I_1(\Delta_1(w) \star \Delta_1(w')) = w ш w'$.

Proof. We can assume without loss of generality that w and w' are homogeneous. If $w = 1$ or $w' = 1$, it is trivial since $I_1\Delta_1$ is the identity on $\mathfrak{h}^{\geq 1}$ (Lemma 2). Let us prove the desired equality by induction on the sum of the degrees of w and w'.

First consider the case where $w = z_1\rho^r w_1$ and $w' = z_1\rho^s w_2$ for $r, s \geq 0$ and $w_1, w_2 \in \mathfrak{H}^1$. From the definition of Δ_1 and $\star$ we have

$$\begin{aligned} I_1\big(\Delta_1(w) \star \Delta_1(w')\big) &= I\big(\xi\rho^r w_1 \star \xi\rho^s w_2\big) \\ &= I_1\big(\xi\rho^r I_1(\Delta_1(w_1) \star \xi\rho^s w_2) + \xi\rho^s I_1\big(\xi\rho^s w_1 \star \Delta_1(w_2)\big) \\ &\quad - \xi\rho^{r+s+1} I_1\big(\Delta_1(w_1) \star \Delta_1(w_2)\big)\big) \\ &= y\rho^r I_1\big(\Delta_1(w_1) \star \Delta_1(y\rho^s w_2)\big) + y\rho^s I_1\big(\Delta_1(y\rho^s w_1) \star \Delta_1(w_2)\big) \\ &\quad - y\rho^{r+s+1} I_1\big(\Delta_1(w_1) \star \Delta_1(w_2)\big). \end{aligned}$$

Apply the induction hypothesis and we get

$$y\rho^r(w_1 ш y\rho^s w_2) + y\rho^s(y\rho^s w_1 ш w_2) - y\rho^{r+s+1}(w_1 ш w_2).$$

It is equal to $z_1\rho^r w_1 ш z_1\rho^s w_2$ because of Lemma 1.

Next let us consider the case where $w = z_1\rho^r w_1$ and $w' = z_k w_2$ for $r \geq 0$, $k \geq 2$ and $w_1, w_2 \in \mathfrak{H}^1$. Using Lemma 4 (3) and the induction hypothesis, we find that

$$I_1\big(\Delta_1(w) \star \Delta_1(w')\big) = -\hbar\big(y\rho^r w_1 ш z_{k-1}w_2\big) + I_1\big(\xi\rho^r w_1 \star e^{-1}(z_k w_2)\big).$$

From Lemma 4 (2), (4) and $e^{-1}(z_k w) \in \mathfrak{h}^{\geq 2}$, we have

$$\xi\rho^r w_1 \star e^{-1}(z_k w_2) = I_0\big(\Delta_1(y\rho^r w_1) \star \Delta_1(z_{k-1}w_2)\big) + \xi\rho^r I_1\big(\Delta_1(w_1) \star \Delta_1(z_k w_2)\big).$$

Use Lemma 4 (6) to calculate the image of the first term by I_1. Now we can apply the induction hypothesis and see that

$$I_1\big(\xi\rho^r w_1 \star e^{-1}(z_k w_2)\big) = (x+\hbar)(y\rho^r w_1 ш z_{k-1}w_2) + y\rho^r(w_1 ш z_k w_2).$$

Therefore

$$I_1\big(\Delta_1(w) \star \Delta_1(w')\big) = x\big(y\rho^r w_1 ш z_{k-1}w_2\big) + y\rho^r(w_1 ш z_k w_2).$$

It is equal to $z_1\rho^r w_1 ш z_k w_2$.

Finally suppose that $w = z_k w_1$ and $w' = z_l w_2$ for $k, l \geq 2$ and $w_1, w_2 \in \mathfrak{H}^1$. From Lemma 4 (3) we get

$$\begin{aligned} \Delta_1(w) \star \Delta_1(w') &= e^{-1}(z_k w_1) \star e^{-1}(z_l w_2) - \hbar\Delta_1(z_k w_1) \star \Delta_1(z_{l-1}w_2) \\ &\quad - \hbar\Delta_1(z_{k-1}w_1) \star \Delta_1(z_l w_2) - \hbar^2\Delta_1(z_{k-1}w_1) \star \Delta_1(z_{l-1}w_2). \end{aligned}$$

Using the induction hypothesis we have

$$\begin{aligned} I_1\big(\Delta_1(w) \star \Delta_1(w')\big) &= I_1\big(e^{-1}(z_k w_1) \star e^{-1}(z_l w_2)\big) - \hbar z_k w_1 ш z_{l-1}w_2 \\ &\quad - \hbar z_{k-1}w_1 ш z_l w_2 - \hbar^2 z_{k-1}w_1 ш z_{l-1}w_2. \end{aligned}$$

Since $e^{-1}(z_k w_1)$ and $e^{-1}(z_l w_2)$ belong to $\mathfrak{h}^{\geq 2}$, we see that $I_1(e^{-1}(z_k w_1) \star e^{-1}(z_l w_2))$ is equal to

$$(x+\hbar) I_1\big(\Delta_1(z_{k-1}w_1) \star \Delta_1(z_l w_2) + \Delta_1(z_k w_1) \star \Delta_1(z_{l-1}w_2) + \hbar \Delta_1(z_{k-1}w_1) \star \Delta_1(z_{l-1}w_2)\big).$$

using Lemma 4 (3), (4) and (6). Now apply the induction hypothesis again. As a result we find that $I_1(\Delta_1(w) \star \Delta_1(w'))$ is equal to

$$x(z_{k-1}w_1 \mathbin{ш} z_l w_2 + z_k w_1 \mathbin{ш} z_{l-1} w_2 + \hbar z_{k-1} w_1 \mathbin{ш} z_{l-1} w_2) = z_k w_1 \mathbin{ш} z_l w_2.$$

This completes the proof. ■

Proof of Proposition 6. It suffices to prove that $e(w \star w') = e(w) \mathbin{ш} e(w')$ for homogeneous elements $w, w' \in \widehat{\mathfrak{H}}^0$. If $w = 1$ or $w' = 1$, then it is trivial. Now we divide into four cases:

(*i*) $w = \xi \rho^r w_1$ and $w' = \xi \rho^s w_2$ for $r, s \geq 0$ and $w_1, w_2 \in \mathfrak{h}^{\geq 1}$,

(*ii*) $w = z_k w_1$ and $w' = \xi \rho^r w_2$ for $k \geq 2$, $w_1 \in \mathfrak{H}^1$, $r \geq 0$ and $w_2 \in \mathfrak{h}^{\geq 1}$,

(*iii*) $w = z_2 w_1$ and $w' = z_l w_2$ for $l \geq 2$ and $w_1, w_2 \in \mathfrak{H}^1$,

(*iv*) $w = z_k w_1$ and $w' = z_l w_2$ for $k, l \geq 3$ and $w_1, w_2 \in \mathfrak{H}^1$.

Case (*i*) It holds that

$$\begin{aligned} w \star w' = {} & \xi \rho^r I_1\big(\Delta_1(w_1) \star \Delta_1(y \rho^s w_2)\big) + \xi \rho^s I_1\big(\Delta_1(y \rho^r w_1) \star \Delta_1(w_2)\big) \\ & - \xi \rho^{r+s+1} I_1\big(\Delta_1(w_1) \star \Delta_1(w_2)\big). \end{aligned}$$

Using Lemma 5, we see that

$$e(w \star w') = \xi \rho^r (w_1 \mathbin{ш} y \rho^s w_2) + \xi \rho^s (y \rho^r w_1 \mathbin{ш} w_2) + \xi \rho^{r+s+1} (w_1 \mathbin{ш} w_2) = e(w) \mathbin{ш} e(w').$$

Case (*ii*) We proceed by induction on k. Let $k = 2$. Using the result in the Case (*i*) and Lemma 5 we see that

$$e(w \star w') = e I_0 e^{-1} \left(e(\xi w_1) \mathbin{ш} e(\xi \rho^r w_2) \right) + \xi \rho^r (z_2 w_1 \mathbin{ш} w_2).$$

Because of the equality (7), it holds that

$$e(\xi w_1) \mathbin{ш} e(\xi \rho^r w_2) = \xi \big(w_1 \mathbin{ш} y \rho^r w_2 + y w_1 \mathbin{ш} \rho^r w_2 - \rho(w_1 \mathbin{ш} \rho^r w_2) \big).$$

Using Lemma 4 (5) we get

$$\begin{aligned} e I_0 e^{-1} \left(e(\xi w_1) \mathbin{ш} e(\xi \rho^r w_2) \right) &= xy \big(w_1 \mathbin{ш} y \rho^r w_2 + y w_1 \mathbin{ш} \rho^r w_2 - \rho(w_1 \mathbin{ш} \rho^r w_2) \big) \\ &= x \big(y w_1 \mathbin{ш} y \rho^r w_2 \big). \end{aligned}$$

Hence

$$e(w \star w') = x \big(y w_1 \mathbin{ш} y \rho^r w_2 \big) + (y - \rho) \rho^r (x y w_1 \mathbin{ш} w_2) = e(w) \mathbin{ш} e(w').$$

This completes the proof for the case $k = 2$.

Suppose that $k \geq 3$. Using Lemma 4 (2) we have

$$e(w \star w') = e I_0 \big(z_{k-1} w_1 \star \xi \rho^r w_2 \big) + \xi \rho^r I_1 \big(\Delta_1 e(z_k w_1) \star \Delta_1(w_2) \big).$$

Note that $e(z_k w_1) \in \mathfrak{h}^{\geq 1}$. From the induction hypothesis and Lemma 5, we get

$$e(w \star w') = e I_0 e^{-1} (e(z_{k-1} w_1) \mathbin{ш} \xi \rho^r w_2) + \xi \rho^r (e(z_k w_1) \mathbin{ш} w_2).$$

Because of Lemma 5 (1), the second term in the right hand side is equal to

$$\xi\rho^r((x+\hbar)e(z_{k-1}w_1) ш w_2).$$

Let us calculate the first term. Set

$$\theta_k = \sum_{a=2}^{k} \binom{k-2}{a-2} \hbar^{k-a} z_{a-1} \in \sum_{a\geq 1} z_a \mathfrak{H}^1.$$

Then $e(z_{k-1}w_1) = x\theta_{k-1}w_1$. Hence

$$e(z_{k-1}w_1) ш \xi\rho^r w_2 = x(\theta_{k-1}w_1 ш y\rho^r w_2) + \xi(x\theta_{k-1}w_1 ш \rho^r w_2).$$

Note that the first term in the right hand side belongs to $\sum_{a\geq 2} z_a \mathfrak{H}^1$. Using Lemma 5 (5) we find that

$$\begin{aligned} eI_0e^{-1}(e(z_{k-1}w_1) ш \xi\rho^r w_2) &= (x+\hbar)x(\theta_{k-1}w_1 ш y\rho^r w_2) + xy(x\theta_{k-1}w_1 ш \rho^r w_2) \\ &= x\big((x+\hbar)\theta_{k-1}w_1 ш y\rho^r w_2\big). \end{aligned}$$

Thus we obtain

$$\begin{aligned} e(w \star w') &= x\big((x+\hbar)\theta_{k-1}w_1 ш y\rho^r w_2\big) + \xi\rho^r((x+\hbar)x\theta_{k-1}w_1 ш w_2) \\ &= x(x+\hbar)\theta_{k-1}w_1 ш \xi\rho^r w_2 = e(w) ш e(w'). \end{aligned}$$

Case (iii) We proceed by induction on l. Let $l=2$. From the result in the Case (ii) we have

$$e(w \star w') = eI_0e^{-1}\left(e(\xi w_1) ш e(z_2 w_2) + e(z_2 w_1) ш e(\xi w_2) - \hbar e(\xi w_1) ш e(\xi w_2)\right).$$

It holds that

$$\begin{aligned} &e(\xi w_1) ш e(z_2 w_2) + e(z_2 w_1) ш e(\xi w_2) - \hbar e(\xi w_1) ш e(\xi w_2) \\ &\quad = \xi\left(w_1 ш (x-\hbar)y w_2 + (x-\hbar)y w_1 ш w_2 + \hbar\rho(w_1 ш w_2)\right) \\ &\qquad + 2xy(w_1 ш y w_2 + y w_1 ш w_2 - \rho(w_1 ш w_2)). \end{aligned}$$

Using Lemma 5 (5) we get

$$\begin{aligned} e(w \star w') &= xy\left(w_1 ш (x-\hbar)y w_2 + (x-\hbar)y w_1 ш w_2 + \hbar\rho(w_1 ш w_2)\right) \\ &\quad + 2(x+\hbar)xy(w_1 ш y w_2 + y w_1 ш w_2 - \rho(w_1 ш w_2)) \\ &= x\left(y(w_1 ш xy w_2 + xy w_1 ш w_2) + 2x(y w_1 ш y w_2)\right) + \hbar x(y w_1 ш y w_2) \\ &= x\left(y w_1 ш xy w_2 + xy w_1 ш y w_2 + \hbar y w_1 ш y w_2\right) = e(w) ш e(w'). \end{aligned}$$

Next consider the case where $l \geq 3$. From the result in the case (ii) and Lemma 5 (1), we get

$$e(w \star w') = eI_0e^{-1}(\xi w_1 ш x e(z_{l-1}w_2) + z_2 w_1 ш e(z_{l-1}w_2)).$$

Note that

$$\begin{aligned} &\xi w_1 ш x e(z_{l-1}w_2) + z_2 w_1 ш e(z_{l-1}w_2) \\ &\quad = \xi\big(w_1 ш x e(z_{l-1}w_2)\big) + x\big(y w_1 ш e(z_{l-1}w_2)\big) + xy w_1 ш e(z_{l-1}w_2). \end{aligned}$$

The second and third terms in the right hand side belong to $\sum_{a\geq 2} z_a\mathfrak{H}^1$. Therefore we obtain

$$\begin{aligned} e(w\star w') &= xy\big(w_1 ш xe(z_{l-1}w_2)\big) + (x+\hbar)\left\{x\big(yw_1 ш e(z_{l-1}w_2)\big) + xyw_1 ш e(z_{l-1}w_2)\right\} \\ &= x\big(yw_1 ш xe(z_{l-1}w_2)\big) + \hbar x\big(yw_1 ш e(z_{l-1}w_2)\big) + (x+\hbar)\big(xyw_1 ш e(z_{l-1}w_2)\big) \\ &= xyw_1 ш xe(z_{l-1}w_2) + \hbar xyw_1 ш e(z_{l-1}w_2) \\ &= xyw_1 ш (x+\hbar)e(z_{l-1}w_2) = e(w) ш e(w') \end{aligned}$$

using Lemma 5 (1) again.

Case (iv) We proceed by induction on $k+l$. From the result in the Case (iii) and the induction hypothesis we find that

$$e(w\star w') = eI_0e^{-1}\big(e(z_{k-1}w_1) ш e(z_lw_2) + e(z_kw_1) ш e(z_{l-1}w_2) - \hbar e(z_{k-1}w_1) ш e(z_{l-1}w_2)\big).$$

Using Lemma 5 (1) we have

$$\begin{aligned} & e(z_{k-1}w_1) ш e(z_lw_2) + e(z_kw_1) ш e(z_{l-1}w_2) - \hbar e(z_{k-1}w_1) ш e(z_{l-1}w_2) \\ &\quad = e(z_{k-1}w_1) ш xe(z_{l-1}w_2) + xe(z_{k-1}w_1) ш e(z_{l-1}w_2) + \hbar e(z_{k-1}w_1) ш e(z_{l-1}w_2). \end{aligned}$$

It belongs to $\sum_{a\geq 2} z_a\mathfrak{H}^1$. Hence Lemma 5 (5) implies that

$$\begin{aligned} e(w\star w') &= (x+\hbar)\{e(z_{k-1}w_1) ш xe(z_{l-1}w_2) + xe(z_{k-1}w_1) ш e(z_{l-1}w_2) \\ &\qquad + \hbar e(z_{k-1}w_1) ш e(z_{l-1}w_2)\} \\ &= (x+\hbar)e(z_{k-1}w_1) ш (x+\hbar)e(z_{l-1}w_2) = e(w) ш e(w'). \end{aligned}$$

This completes the proof. ■

3 Linear relations among the modified qMZVs

3.1 Double shuffle relation

We regard $\widehat{\mathfrak{H}}^0$ as a graded $\mathbb{Q}$-module by setting the degree of x, y, ρ and $\hbar$ to be one, and call the degree the *weight* on $\widehat{\mathfrak{H}}^0$. Denote the homogeneous component of weight d by $\widehat{\mathfrak{H}}^0_d$. Now we define the $\mathbb{Q}$-linear map $\bar{Z}_q : \widehat{\mathfrak{H}}^0 \to \mathbb{C}$ by $\bar{Z}_q(w) := (1-q)^{-d}Z_q(w)$ for $w \in \widehat{\mathfrak{H}}^0_d$. If $\mathbf{k} = (k_1,\ldots,k_r)$ is an admissible index, $\bar{Z}_q(z_{k_1}\ldots z_{k_r})$ is equal to the modified qMZV $\bar{\zeta}_q(\mathbf{k})$ defined by (2). Set $\mathfrak{H}^0_d := \mathfrak{H}^0 \cap \widehat{\mathfrak{H}}^0_d$. Then we have

$$\bar{Z}_q\big(\mathfrak{H}^0_d\big) = \sum_{|\mathbf{k}|\leq d} \mathbb{Q}\bar{\zeta}_q(\mathbf{k}).$$

From the definition of the harmonic product $*$ and the integral shuffle product ш, we see that $\widehat{\mathfrak{H}}^0 = \oplus_{d\geq 0}\widehat{\mathfrak{H}}^0_d$ is a commutative graded $\mathbb{Q}$-algebra with respect to either $*$ or ш. Now we obtain the following theorem from Theorems 1 and 2.

Theorem 3. *Denote by S_d $(d\geq 0)$ the $\mathbb{Q}$-subspace of $\widehat{\mathfrak{H}}^0_d$ spanned by the elements $w*w' - w ш w'$ where w and w' are homogeneous elements of $\widehat{\mathfrak{H}}^0$ such that the sum of the weights of w and w' is equal to d. Then $S_d \subset \ker \bar{Z}_q$.*

Thus we obtain linear relations among the modified qMZVs as the image of $S_d \cap \mathfrak{H}^0$. Let us call such relations the *double shuffle relations*.

As an example of the double shuffle relations we prove a q-analogue of Hoffman's identity for MZVs [3]:

Proposition 7. *Let $(k_1, \dots, k_r)$ be an admissible index. Then we have*

$$\sum_{1 \le i \le r} \zeta_q(k_1, \dots, k_i + 1, \dots, k_r) = \sum_{\substack{1 \le i \le r \\ k_i \ge 2}} \sum_{a=0}^{k_i - 2} \zeta_q(k_1, \dots, k_{i-1}, k_i - a, a + 1, k_{i+1}, \dots, k_r).$$

Proof. The proof is similar to that for MZVs given in [5]. From the definition of the harmonic product we have

$$\xi * z_{k_1} \cdots z_{k_r} = \sum_{i=1}^{r+1} z_{k_1} \cdots z_{k_{i-1}} \xi z_{k_i} \cdots z_{k_r} + \sum_{i=1}^{r} z_{k_1} \cdots z_{k_i + 1} \cdots z_{k_r}.$$

For $\alpha \ge 1$ and $w \in \mathfrak{H}^1$, it holds that

$$y \,\text{ш}\, x^\alpha w = \sum_{j=0}^{\alpha-1} x^j y x^{\alpha-j} w + x^\alpha (y \,\text{ш}\, w),$$
$$y \,\text{ш}\, y^\alpha w = \sum_{j=1}^{\alpha} y^j \xi y^{\alpha-j} w + y^\alpha (y \,\text{ш}\, w).$$

Using these formulas we obtain

$$\xi \,\text{ш}\, z_{k_1} \cdots z_{k_r} = \sum_{i=1}^{r+1} z_{k_1} \cdots z_{k_{i-1}} \xi z_{k_i} \cdots z_{k_r} + \sum_{\substack{1 \le i \le r \\ k_i \ge 2}} \sum_{a=0}^{k_i - 2} z_{k_1} \cdots z_{k_{i-1}} z_{k_i - a} z_{a+1} z_{k_{i+1}} \cdots z_{k_r}.$$

Hence we get the desired equality from Theorem 3. ■

3.2 Resummation duality

The double shuffle relations do not contain all linear relations among the modified qMZVs. We give another family of linear relations to make up for this lack.

Theorem 4. *For a positive integer k, set*

$$\varphi_k := \sum_{a=2}^{k} (-\hbar)^{k-a} z_a + (-\hbar)^{k-1} \xi.$$

Let r be a positive integer and α_i, β_i $(1 \le i \le r)$ non-negative integers. Then we have

$$\bar{Z}_q\big(\varphi_{\alpha_1+1} \rho^{\beta_1} \cdots \varphi_{\alpha_r+1} \rho^{\beta_r}\big) = \bar{Z}_q(\varphi_{\beta_r+1} \rho^{\alpha_r} \cdots \varphi_{\beta_1+1} \rho^{\alpha_1}). \tag{11}$$

Proof. Note that

$$I_{\varphi_k}(n) = (1-q)^k \frac{q^{kn}}{(1-q^n)^k}$$

for $k \ge 1$. Hence we have

$$\begin{aligned} &\bar{Z}_q\big(\varphi_{\alpha_1+1} \rho^{\beta_1} \cdots \varphi_{\alpha_r+1} \rho^{\beta_r}\big) \\ &\quad = (1-q)^{\sum_{i=1}^{r} (\alpha_i + \beta_i + 1)} \sum_{n_1 > \cdots > n_r > 0} \prod_{i=1}^{r} \binom{n_i - n_{i+1} - 1}{\beta_i} \frac{q^{(\alpha_i+1) n_i}}{(1-q^{n_i})^{\alpha_i+1}}, \end{aligned}$$

where $n_{r+1}=0$. Expand $1/(1-q^{n_i})^{\alpha_i+1}$ by using (10). Then we get

$$(1-q)^{\sum_{i=1}^{r}(\alpha_i+\beta_i+1)}\sum_{n_1>\cdots>n_r>0}\sum_{s_1,\ldots,s_r=0}^{\infty}\prod_{i=1}^{r}\binom{n_i-n_{i+1}-1}{\beta_i}\binom{\alpha_i+s_i}{\alpha_i}q^{(\alpha_i+s_i+1)n_i}.$$

Now take the sum over $n_1,\ldots,n_r$ successively using (10) again. As a result we obtain

$$(1-q)^{\sum_{i=1}^{r}(\alpha_i+\beta_i+1)}\sum_{s_1,\ldots,s_r=0}^{\infty}\prod_{i=1}^{r}\binom{\alpha_i+s_i}{\alpha_i}\left(\frac{q^{\sum_{j=1}^{i}(\alpha_j+s_j+1)}}{1-q^{\sum_{j=1}^{i}(\alpha_j+s_j+1)}}\right)^{\beta_i+1}.$$

Setting $m_i=\sum_{j=1}^{i}(\alpha_i+s_i+1)$ $(1\le i\le r)$ we see that it is equal to

$$(1-q)^{\sum_{i=1}^{r}\alpha_i}\sum_{m_r>\cdots>m_1>0}\prod_{i=1}^{r}\binom{m_i-m_{i-1}-1}{\alpha_i}\frac{q^{(\beta_i+1)m_i}}{[m_i]^{\beta_i+1}},$$

where $m_0=0$. This is the right hand side of (11). ■

Let us call the property (11) the *resummation duality*. Denote by R_d $(d\ge 0)$ the $\mathbb{Q}$-subspace of $\widehat{\mathfrak{H}}^0_d$ spanned by the elements

$$\varphi_{\alpha_1+1}\rho^{\beta_1}\cdots\varphi_{\alpha_r+1}\rho^{\beta_r}-\varphi_{\beta_r+1}\rho^{\alpha_r}\cdots\varphi_{\beta_1+1}\rho^{\alpha_1}$$

with $r>0$, $\alpha_i,\beta_i\ge 0$ $(1\le i\le r)$ and $\sum_{i=1}^{r}(\alpha_i+\beta_i+1)=d$. The resummation duality implies that $R_d\subset\ker\bar{Z}_q$.

Recall that the $\mathbb{Q}$-vector space spanned by the modified qMZVs

$$Z_{\le d}:=\sum_{|\mathbf{k}|\le d}\mathbb{Q}\bar{\zeta}_q(\mathbf{k})$$

is realized as $\bar{Z}_q(\mathfrak{H}^0_d)$ in our framework. The $\mathbb{Q}$-subspaces S_d, defined in Theorem 3, and R_d are contained in $\ker\bar{Z}_q$. Therefore the subspace

$$N_{\le d}:=\mathfrak{H}^0\cap(S_d+R_d)$$

describes some linear relations among the modified qMZVs.

By computer experiment we can find a lower bound of the dimension of $Z_{\le d}$ [9], and calculate the dimension of $N_{\le d}$. The result up to weight 7 is given as follows:

d	2	3	4	5	6	7
# of admissible indices	1	3	7	15	31	63
lower bound of dim $Z_{\le d}$	1	2	4	7	11	18
dim $N_{\le d}$	0	1	3	8	20	45

The second line above gives the number of admissible indices whose weight is less than or equal to d. We see that the sum of the values in the third line and the fourth one is equal to the number of admissible indices. Therefore the third line gives the dimension of $Z_{\le d}$ exactly and the space $N_{\le d}$ describes all linear relations among the modified qMZVs up to weight 7.

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