

Parameterizing the Simplest Grassmann–Gaussian Relations for Pachner Move 3–3

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Abstract. We consider relations in Grassmann algebra corresponding to the four-dimensional Pachner move 3–3, assuming that there is just one Grassmann variable on each 3-face, and a 4-simplex weight is a Grassmann–Gaussian exponent depending on these variables on its five 3-faces. We show that there exists a large family of such relations; the problem is in finding their algebraic-topologically meaningful parameterization. We solve this problem in part, providing two nicely parameterized subfamilies of such relations. For the second of them, we further investigate the nature of some of its parameters: they turn out to correspond to an exotic analogue of middle homologies. In passing, we also provide the 2–4 Pachner move relation for this second case.

Key words: four-dimensional Pachner moves; Grassmann algebras; Clifford algebras; maximal isotropic Euclidean subspaces

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1 Introduction

Discrete topological field theories – specifically, field theories on piecewise linear (PL) manifolds – are definitely a challenging research subject. As there are now many interesting topological quantum field theories (TQFT’s) in three dimensions, it looks reasonable to concentrate on the four-dimensional case. Such a theory is expected to bring about interesting results both by itself and when compared with the existing theories on smooth manifolds.

As explained, for instance, in [11, Section 1], it makes sense first to construct algebraic relations corresponding to *Pachner moves*. And the simplest nontrivial relations of such kind arise, as we believe, in *Grassmann algebras*. In three dimensions, a relation corresponding to Pachner move 2–3 is often called *pentagon relation*, and there are some Grassmann-algebraic constructions for pentagon relation, presented, in particular, in paper [10]. As we hope to demonstrate here, the four-dimensional case has its own specific beauty; it is more complicated but also yields to systematic investigation.

If we consider an *ansatz* – a (tentative) specific form of quantities or expressions entering in our relations, and consider the relations as equations for the ansatz parameters, and if our ansatz is simple enough, then it may happen that the *existence* of many solutions for such equations follows already from parameter counting.

In this paper, we take the simplest possible form (6) of Grassmann-algebraic relation corresponding to Pachner move 3–3 – with just one Grassmann variable on each 3-face, and further assume that the Grassmann weight of a 4-simplex has the form of a Grassmann–Gaussian exponent, depending on the five variables on the 3-faces. A heuristic parameter count shows that there exists a large – and intriguing – family of relations of such form. We prefer to go further and prove the rigorous Theorem 4, formulated in terms of *isotropic linear spaces of Grassmann*

differential operators annihilating our Grassmann–Gaussian exponents. In doing so, we not only prove the existence of the 4-simplex weights satisfying the 3–3 relations, but discover some interesting operators (namely, (16) and (17)) that may deserve further investigation; at least, they have an elegant form (namely, (28) and (29)) in one specific case.

Having proved our Theorem 4, we are naturally led to the problem of finding an *algebraic-topologically meaningful parameterization* of our Grassmann weights, which would enable us to move further and construct topological field theories. In the present paper, we make two steps in this direction by presenting two explicitly – and nicely – parameterized *subfamilies* of such weights, found largely by guess-and-try method. The first subfamily resembles the (more cumbersome) constructions in [7, 8] – both are related to *exotic homologies*. The striking new fact is, however, that this is now only a subfamily of something mysterious, on whose nature only our parameterized second family sheds some additional light.

Some of the results of this paper have first appeared, in a preliminary form, in the preprint [9]. Below,

- in Section 2, we recall the basic definitions from the theory of Grassmann algebras and Berezin integral,
- in Section 3, we recall the four-dimensional Pachner moves, mainly moves 3–3 and 2–4 with which we will be dealing in this paper, and introduce some notational conventions,
- in Section 4, we introduce a 3–3 relation for Grassmann 4-simplex weights. First, we do it in a general form, then we specialize the weights to be Grassmann–Gaussian exponents and explain their connection with isotropic spaces of Grassmann differential operators,
- in Section 5, based on these isotropic spaces, we show the existence of a vast family of 4-simplex weights satisfying the 3–3 relation. The way we do it is constructive; what lacks in it is a parameterization for this whole family relevant for algebraic-topological applications,
- in Section 6, we present two subfamilies of Grassmann 4-simplex weights satisfying the 3–3 relation where such parameterization has been obtained,
- in Section 7, we do some preparational work in order to expose some exotic-homological structures lying behind the second of the mentioned subfamilies. Namely, we introduce, for a given triangulated four-manifold, a sequence of two linear mappings – supposedly a fragment of an exotic chain complex, prove their chain property (their composition vanishes), and present computational evidence showing that they provide an exotic analogue of usual middle (i.e., second) homologies, and
- in Section 8, guided by the fact that the mentioned exotic-homological structures manifest themselves more clearly for the Pachner move 2–4, we present the relations corresponding to this move, study a new factor – edge weight – appearing in these relations, and then formulate the relations for both moves 3–3 and 2–4 using these exotic-homological terms.

2 Grassmann algebras and Berezin integral

In this paper, a *Grassmann algebra* is an associative algebra over the field \mathbb{C} of complex numbers, with unity, generators x_i – also called Grassmann variables – and relations

$$x_i x_j = -x_j x_i.$$

This implies that, in particular, $x_i^2 = 0$, so each element of a Grassmann algebra is a polynomial of degree ≤ 1 in each x_i .

The *degree* of a Grassmann monomial is its total degree in all Grassmann variables. If an algebra element consists of monomials of only odd or only even degrees, it is called *odd* or, respectively, *even*. If all the monomials have degree 2, we call such element a Grassmannian *quadratic form*.

The *exponent* is defined by its usual Taylor series. We call the exponent of a quadratic form *Grassmann–Gaussian exponent*. Here is an example of it:

$$\exp(x_1x_2 + x_3x_4) = 1 + x_1x_2 + x_3x_4 + x_1x_2x_3x_4.$$

There are two kinds of derivations in a Grassmann algebra: left derivative $\frac{\partial}{\partial x_i}$ and right derivative $\overleftarrow{\frac{\partial}}{\partial x_i}$, with respect to a Grassmann variable x_i . These are \mathbb{C} -linear operations in Grassmann algebra defined as follows. Let f be an element not containing variable x_i , then

$$\frac{\partial}{\partial x_i} f = f \overleftarrow{\frac{\partial}}{\partial x_i} = 0, \quad (1)$$

and

$$\frac{\partial}{\partial x_i}(x_i f) = f, \quad (fx_i) \overleftarrow{\frac{\partial}}{\partial x_i} = f. \quad (2)$$

From (1) and (2), the following *Leibniz rules* follow: if f is either even or odd, then

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial}{\partial x_i} f \cdot g + \epsilon f \frac{\partial}{\partial x_i} g, \quad (gf) \overleftarrow{\frac{\partial}}{\partial x_i} = g \cdot f \overleftarrow{\frac{\partial}}{\partial x_i} + \epsilon g \overleftarrow{\frac{\partial}}{\partial x_i} f, \quad (3)$$

where $\epsilon = 1$ for an even f and $\epsilon = -1$ for an odd f .

The *Grassmann–Berezin calculus* of anticommuting variables is in many respects parallel to the usual calculus, see [1] and especially [2]. Still, there are some peculiarities, and one of them is that the integral in a Grassmann algebra is the same operation as derivative; more specifically, *Berezin integral* in a variable x_i is defined, traditionally, as the *right* derivative w.r.t. x_i . Independently, Berezin integral is defined as follows: it is a \mathbb{C} -linear operator in Grassmann algebra satisfying

$$\int dx_i = 0, \quad \int x_i dx_i = 1, \quad \int gh dx_i = g \int h dx_i,$$

where g does not contain x_i . Multiple integral is defined according to the following Fubini rule:

$$\int \cdots \int f dx_1 dx_2 \cdots dx_n = \int \left(\cdots \int \left(\int f dx_1 \right) dx_2 \cdots \right) dx_n. \quad (4)$$

In “differential” notations, integral (4) is

$$f \overleftarrow{\frac{\partial}}{\partial x_1} \overleftarrow{\frac{\partial}}{\partial x_2} \cdots \overleftarrow{\frac{\partial}}{\partial x_n}. \quad (5)$$

3 Pachner moves in four dimensions

Pachner moves [12] are elementary local rebuildings of a manifold triangulation. A triangulation of a PL manifold can be transformed into any other triangulation using a finite sequence of Pachner moves.

In four dimensions, each Pachner move replaces a cluster of 4-simplices with a cluster of some other 4-simplices, occupying the same place in the triangulation and having the same

boundary. There are five (types of) Pachner moves in four dimensions: $3 \rightarrow 3$, $2 \leftrightarrow 4$, and $1 \leftrightarrow 5$, where the numbers indicate how many 4-simplices have been withdrawn and how many have replaced them. As the withdrawn and the replacing clusters of 4-simplices have the same common boundary, we can glue them together in a natural way (forgetting for a moment about the rest of the manifold); then, for all Pachner moves, they must form together a sphere S^4 triangulated in six 4-simplices as the boundary of a 5-simplex, which we denote $\partial\Delta^5$. More details and a pedagogical introduction can be found in [11].

More traditional notations for the mentioned moves are 3-3, 2-4, 4-2, 1-5, and 5-1; we will be using these notations as well.

Move 3-3 is, in some informal sense, central: experience shows that if we have managed to find an algebraic formula whose structure can be regarded as reflecting the structure of the move, then we can also find (usually more complicated) formulas corresponding to the other Pachner moves. This may be compared to the three-dimensional case, where the popular “pentagon relation” often corresponds to the “central” three-dimensional Pachner move 2-3, while it is believed that, having done something interesting with this pentagon equation, one will be also able to work with the move 1-4.

We call the initial cluster of 4-simplices in a move the *left-hand side* (l.h.s.) of that move, and the final cluster – its *right-hand side* (r.h.s.). All moves in this paper will involve six vertices denoted $i = 1, \dots, 6$. Below are some more details.

3.1 Move $3 \rightarrow 3$

It transforms, in the notations used in this paper, the cluster of three 4-simplices 12345, 12346 and 12356 situated around the 2-face 123 into the cluster of three other 4-simplices, 12456, 13456 and 23456, situated around the 2-face 456. The inner 3-faces (tetrahedra) are 1234, 1235 and 1236 in the l.h.s., and 1456, 2456 and 3456 in the r.h.s. The boundary of both sides consists of nine tetrahedra listed below in table (15).

There are no inner edges (1-faces) or vertices (0-faces) in either side of this move.

3.2 Moves $2 \leftrightarrow 4$

We describe move $2 \rightarrow 4$; move $4 \rightarrow 2$ is its inverse. Move $2 \rightarrow 4$ replaces, in the notations of this paper, the cluster of two 4-simplices 12345 and 12346 with the cluster of four 4-simplices 12356, 12456, 13456 and 23456. The boundary of both sides consists of eight tetrahedra 1235, 1236, 1245, 1246, 1345, 1346, 2345 and 2346.

In the l.h.s., there is one inner tetrahedron 1234, no inner 2-faces and no inner edges.

In the r.h.s., there are six inner tetrahedra 1256, 1356, 1456, 2356, 2456 and 3456, and four inner 2-faces: 156, 256, 356 and 456. It turns out especially important – see Subsection 8.1 – that there *is* one inner edge, namely 56, in the r.h.s.

There are no inner vertices in either side of this move.

3.3 Moves $1 \leftrightarrow 5$

We don’t work with these moves in this paper, so we only indicate that the move $1 \rightarrow 5$ adds a new vertex 6 inside the 4-simplex 12345, thus dividing it into five new 4-simplices. Move $5 \rightarrow 1$ is, of course, its inverse.

3.4 A few conventions

Any side of a Pachner move, as well as a single 4-simplex, is a triangulated four-manifold with boundary. In Section 7, we also consider an arbitrary *orientable* triangulated four-manifold.

Here are some of our conventions concerning manifolds, their simplices, and also some complex parameters appearing in our theory, like *vertex coordinates* (see Section 6).

Convention 1. All manifolds in this paper are assumed to be *oriented*. In the case of Pachner moves, the orientation is defined so that, for the 4-simplex 12345, it is given by this order of its vertices.

Convention 2. We denote by N_k the number of k -simplices in a triangulation, and by N'_k the number of *inner* k -simplices. Vertices are numbered from 1 through N_0 (as we have already done for Pachner moves, where $N_0 = 6$).

Convention 3. When simplices are written in terms of their vertices, these go in the increasing order of their numbers (again, as we have already done).

Convention 4. If the order of vertices is unknown, we use the following notation. Let there be, for instance, a 4-simplex whose set of vertices is $\{i, j, k, l, m\}$, then we denote it, omitting the commas for brevity, as $\{ijklm\}$.

Convention 5. The complex parameters appearing in our theory – to be exact, the eighteen parameters in Section 5 and vertex coordinates introduced in Section 6 – lie in the *general position* with respect to any considered algebraic formula, unless the opposite is explicitly stated. For instance, there is no division by zero in formula (26). Moreover, concerning vertex coordinates, all functions of them in this paper are *rational*, so the reader can assume that the coordinates are *indeterminates* over \mathbb{C} (and we extended \mathbb{C} to the relevant field of rational functions).

4 Relation 3–3 with Grassmann–Gaussian exponents: generalities

4.1 The form of relation 3–3

The Grassmann-algebraic Pachner move relations for move 3–3, considered in this paper, have the following general form:

$$\begin{aligned} & \iiint \mathcal{W}_{12345} \mathcal{W}_{12346} \mathcal{W}_{12356} dx_{1234} dx_{1235} dx_{1236} \\ &= \text{const} \iiint \mathcal{W}_{12456} \mathcal{W}_{13456} \mathcal{W}_{23456} dx_{1456} dx_{2456} dx_{3456}. \end{aligned} \quad (6)$$

Here Grassmann variables x_{ijkl} are attached to all 3-faces, i.e., tetrahedra $t = ijkl$; the *Grassmann weight* \mathcal{W}_{ijklm} of a 4-simplex $u = ijklm$ depends on (i.e., contains) the variables on its 3-faces, e.g., \mathcal{W}_{12345} depends on x_{1234} , x_{1235} , x_{1245} , x_{1345} and x_{2345} . Also, \mathcal{W}_u may depend on parameters attached to the 4-simplex u or/and its subsimplices. The integration goes in variables on *inner* three-faces in the corresponding side of Pachner move, while the result depends on the variables on boundary faces. Finally, const in the right-hand side is a numeric factor.

Formula (6) appears to give the simplest possible form for a Grassmann-algebraic relation imitating the 3–3 move.

Remark 1. We expect formula (6) to be the first step in constructing a fermionic topological field theory, based on our previous experience. As yet, the most detailed construction of such kind can be found in paper [6], although for a three-dimensional case and a simpler theory. Note especially

- the analogy between either side of (6) and [6, formula (10)] describing what happens when we glue together manifolds with boundary,

- also the analogy between either side of (6) if the \mathcal{W}_u are as in formula (7) below, and [6, formula (8)] for a multi-component fermionic state-sum invariant of a manifold with boundary,
- and the discussion in [6, Section 5] of the modification of Atiyah's TQFT axioms for anti-commuting variables.

Remark 2. As an example of a more heavyweight relation corresponding to the same Pachner move 3–3, we can cite [7, formula (38)], where, in particular, *two* Grassmann variables live on each tetrahedron.

Further simplification is achieved by using *Grassmann–Gaussian exponents* (which corresponds to *free fermions* in physical language) and assuming that

$$\mathcal{W}_u = \exp\left(\frac{1}{2} \sum_{t_1, t_2 \subset u} \alpha_{t_1 t_2}^{(u)} x_{t_1} x_{t_2}\right), \quad (7)$$

where t_1 and t_2 are two 3-faces of u , and $\alpha_{t_1 t_2}^{(u)} \in \mathbb{C}$ are numeric coefficients, with the antisymmetry condition

$$\alpha_{t_1 t_2}^{(u)} = -\alpha_{t_2 t_1}^{(u)}. \quad (8)$$

We hope to demonstrate in this paper that the relation (6) is interesting already in the case of such exponents.

4.2 Isotropic subspaces of operators

The exponent (7) is characterized, up to a factor that does not depend on those x_t that enter in it, by the equations

$$\left(\partial_t - \sum_{t' \subset u} \alpha_{t t'}^{(u)} x_{t'}\right) \mathcal{W}_u = 0 \quad \text{for all } t \subset u, \quad (9)$$

where we denote $\partial_t = \partial/\partial x_t$. Generalizing the operators in the big parentheses in (9), we consider \mathbb{C} -linear combinations of operators of left differentiations and multiplying by Grassmann generators:

$$d = \sum_t (\beta_t \partial_t + \gamma_t x_t), \quad (10)$$

where t runs over all 3-faces in a given triangulated manifold.

Remark 3. We are going to apply differential equations like (9) to analysing the integral identity (6). Such kind of analysis is a well known and widely used tool, especially in the bosonic context. For example, holonomic and q -holonomic functions are studied via the difference/differential equations they satisfy, see [3, 4, 13].

We regard the anticommutator of two operators (10) (defined as $[A, B]_+ = AB + BA$ for operators A and B) as their *scalar product*:

$$\langle d^{(1)}, d^{(2)} \rangle \stackrel{\text{def}}{=} [d^{(1)}, d^{(2)}]_+ = \sum_t \left(\beta_t^{(1)} \gamma_t^{(2)} + \beta_t^{(2)} \gamma_t^{(1)} \right). \quad (11)$$

With this scalar product, operators (10) form a *complex Euclidean space*, while all polynomials of these operators form a *Clifford algebra*.

Recall that an *isotropic*, or *totally singular*, subspace of a complex Euclidean space \mathbb{C}^{2n} is such linear subspace where the scalar product identically vanishes. We will need some basic facts about isotropic subspaces; for the reader's convenience, we formulate them as the following Theorem 1 and give it a simple proof. Much more interesting facts about Clifford algebras and isotropic subspaces in Euclidean spaces can be found, e.g., in [5].

Theorem 1. *Maximal isotropic spaces in complex Euclidean space \mathbb{C}^{2n} have dimension n . The manifold of these maximal isotropic spaces – isotropic Grassmannian – splits up in two connected components.*

Proof. The first statement is an easy exercise. The second can be proved as follows. Let $V \subset \mathbb{C}^{2n}$ be a maximal isotropic subspace. For a *generic* orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_{2n}$, the orthogonal projection of V onto the space W spanned by the first half of basis vectors, i.e., $\mathbf{e}_1, \dots, \mathbf{e}_n$, coincides with the whole W . Also, considering the manifold B of orthonormal bases, it has two connected components (the determinant of transition matrix is 1 within a component, and -1 between the components), and the same components remain in $B \setminus S$ – the result of taking away the set S of non-generic (in the sense indicated above) bases: the components cannot split any further, because S has complex codimension ≥ 1 and thus real codimension ≥ 2 . Taking some liberty, we call, in this proof, the components of B *orientations* of \mathbb{C}^{2n} .

We prefer to arrange basis vectors in a column, and vector coordinates in a row; so, an arbitrary vector in \mathbb{C}^{2n} is written like $(\alpha_1 \ \dots \ \alpha_n) \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_{2n} \end{pmatrix}$, and vectors in \mathbb{C}^{2n} are identified with row vectors if a basis is given. For a space V and basis $\mathbf{e}_1, \dots, \mathbf{e}_{2n}$ such as in the previous paragraph, we can represent V as the linear span of the rows of the following matrix:

$$(\mathbf{1}_n \ iO), \tag{12}$$

where $\mathbf{1}_n$ is the identity matrix and O is an orthogonal matrix, both of sizes $n \times n$, and $i = \sqrt{-1}$. The determinant of O is either 1 or -1 , and its sign obviously cannot change within one component of $B \setminus S$. So, given a fixed orientation of \mathbb{C}^{2n} and a maximal isotropic space V , we get either 1 or -1 as $\det O$.

It remains to prove that there is no further splitting between maximal isotropic spaces. Consider two such spaces, V_1 and V_2 . There exists a generic, in the above sense, basis $\mathbf{e}_1, \dots, \mathbf{e}_{2n}$ for *both* of them. Using this basis, V_1 and V_2 can be written in terms of matrices (12), and the corresponding orthogonal matrices O_1 and O_2 belong to the same connected component in the space of orthogonal matrices. ■

Theorem 2.

- (i) *For a given weight \mathcal{W}_u of the form (7), the operators d satisfying equation*

$$d\mathcal{W}_u = 0 \tag{13}$$

form a five-dimensional isotropic linear space.

- (ii) *For the set of equations (13) corresponding to a five-dimensional isotropic space V of operators (10) with t running over the five 3-faces of a 4-simplex u , there exists a nonzero Grassmann algebra element \mathcal{W}_u , containing only the Grassmann generators x_t and satisfying all these equations. This \mathcal{W}_u is determined by these equations uniquely up to a numeric factor.*
- (iii) *The element \mathcal{W}_u from item (ii) is even for one connected component of the set of five-dimensional isotropic spaces V , and odd for the other. In the first case, it is, for a generic V , a Grassmann–Gaussian exponent (7).*

Proof. (i) Five such linearly independent equations are already written in (9). It follows from the antisymmetry (8) of coefficients $\alpha_{tt'}$ that any two operators d written in the big parentheses in (9) anticommute (including the case where they coincide). This means that the scalar product (11) vanishes.

(ii) We denote the ten-dimensional Euclidean space of all operators (10), where $t \subset u$, simply as \mathbb{C}^{10} . There exists an orthogonal transform O of \mathbb{C}^{10} sending V into the subspace generated by the five $\frac{\partial}{\partial x_t}$, and to O corresponds, according to the general theory, a \mathbb{C} -linear automorphism B of the Grassmann algebra such that

$$Oy = ByB^{-1} \quad \text{for } y \in \mathbb{C}^{10}.$$

As $\frac{\partial}{\partial x_t} 1 = 0$ for all five t , it follows that $\mathcal{W}_u = B^{-1}1$ is annihilated by all operators in V . On the other hand, if there were two linearly independent \mathcal{W}_u annihilated by all operators in V , it would follow that the two corresponding algebra elements $B\mathcal{W}_u$ would be annihilated by all five $\frac{\partial}{\partial x_t}$, but this holds only for the one-dimensional space of constants.

(iii) A Zariski open set of even elements \mathcal{W}_u is already provided, and it consists of Grassmann–Gaussian exponents (7). A similar Zariski open set of odd elements \mathcal{W}_u can be described as follows. Take any *odd* number of indices t and make the interchange $\partial_t \leftrightarrow x_t$ in the equations (9) for these t . An easy exercise shows that, for a given antisymmetric matrix $\alpha_{tt'}^{(u)}$, the operators in the left-hand sides of the resulting equations span an isotropic subspace, and annihilate a one-dimensional space of *odd* \mathcal{W}_u 's. Now take, for every $\alpha_{tt'}^{(u)}$, those \mathcal{W}_u that satisfy the resulting equations.

For instance, we can make the mentioned interchange for all $n = 5$ indices t , so that the \mathcal{W}_u 's in the resulting set will satisfy equations

$$\left(x_t - \sum_{t' \subset u} \alpha_{tt'}^{(u)} \partial_{t'} \right) \mathcal{W}_u = 0 \quad \text{for all } t \subset u. \quad \blacksquare$$

Theorem 3.

- (i) For weights \mathcal{W} of the form (7), both sides of (6) satisfy 9-dimensional spaces of isotropic equations, i.e., equations

$$d(\text{l.h.s.}) = 0 \quad \text{and} \quad d(\text{r.h.s.}) = 0$$

where in both cases the relevant operators d form a 9-dimensional isotropic space.

- (ii) Such 9-dimensional space of isotropic equations determines the l.h.s. or r.h.s. of (6) up to a numeric factor, if it is also assumed that this l.h.s. or r.h.s. depends only on Grassmann variables on the boundary 3-faces, as explained after formula (6).

Proof. (i) First, we consider the integrand $\mathfrak{W} = \mathcal{W}_{12345}\mathcal{W}_{12346}\mathcal{W}_{12356}$ for the l.h.s. or $\mathfrak{W} = \mathcal{W}_{12456}\mathcal{W}_{13456}\mathcal{W}_{23456}$ for the r.h.s. It satisfies a 12-dimensional isotropic space of equations of the form

$$\partial_t \mathfrak{W} = \sum_{t'} \gamma_{t'} x_{t'} \mathfrak{W}$$

for each boundary or inner tetrahedron t ; these equations follow from equations (9) for individual weights and the first Leibniz rule in (3). Next, if t is an inner tetrahedron and if some operator $\sum_{t'} (\beta_{t'} \partial_{t'} + \gamma_{t'} x_{t'})$ anticommutes with the differentiation ∂_t – that is,

$$\gamma_t = 0 \tag{14}$$

in the sum – and if also

$$\sum_{t'} (\beta_{t'} \partial_{t'} + \gamma_{t'} x_{t'}) \mathfrak{W} = 0,$$

then $\partial_t \mathfrak{W}$ satisfies a similar equation, from which ∂_t and x_t are absent, namely

$$\sum_{t' \neq t} (\beta_{t'} \partial_{t'} + \gamma_{t'} x_{t'}) (\partial_t \mathfrak{W}) = -\partial_t \sum_{t'} (\beta_{t'} \partial_{t'} + \gamma_{t'} x_{t'}) \mathfrak{W} = 0,$$

and it is not hard to see that $\int \mathfrak{W} dx_t$ – the right derivative – satisfies the same equation.

Due to condition (14), now there remains, at least, an 11-dimensional isotropic space of equations instead of the 12-dimensional. Proceeding this way further with the two remaining inner tetrahedra t , we get, at least, a 9-dimensional space of equations. As an isotropic subspace in a 18-dimensional complex Euclidean space (nine boundary tetrahedra t , operators ∂_t and x_t for each of them) cannot be more than 9-dimensional, it is exactly 9-dimensional.

(ii) This is proved in full analogy with similar statement in item (ii) in Theorem 2. ■

Remark 4. This time, each side of (6) is easily shown to be an *odd* element – namely, of Grassmann degree 3, and this determines the connected component in the manifold of maximal isotropic subspaces where our subspaces belong. This will be important for the construction in Section 5, see Remark 5.

5 A large family of Grassmann–Gaussian weights satisfying relation 3–3

In this section, we construct a 18-parameter family of Grassmann weights depending on the variables x_{ijkl} on the boundary tetrahedra of either l.h.s. or r.h.s. of Pachner move 3–3 and such that a weight in this family can be represented as *both* the l.h.s. and r.h.s. of (6), with all the 4-simplex weights \mathcal{W}_{ijklm} having the form (7). Although the search for an *algebraic-topologically meaningful* parameterization for these weights is still in progress, the very existence of such family is already of interest; moreover, some properties of these weights can be seen already from the parameterization given below.

5.1 Heuristic parameter count

Before presenting our construction below in Subsection 5.2, we would like to explain it heuristically, using parameter counting. For a single 4-simplex, the corresponding isotropic space of operators, spanned by the operators in big parentheses in (9), depends on 10 parameters. When we compose the l.h.s. or r.h.s. of (6) (not yet demanding that l.h.s. be equal to r.h.s.), there are thus 30 parameters. Three of them are, however, redundant, because of the possible scalings of variables x_t on three *inner* tetrahedra t – it is easily seen that such scalings may only multiply the considered integrals by a numeric factor. So, we have $3 \times 10 - 3 = 27$ essential parameters in each side of (6).

On the other hand, a 9-dimensional isotropic subspace in an 18-dimensional complex Euclidean space is determined by 36 parameters. So, requiring this equalness, we subtract 36 parameters and are left with $2 \times 27 - 36 = 18$ parameters.

5.2 Rigorous construction

There are nine boundary tetrahedra in the l.h.s. or r.h.s. of Pachner move 3–3. We see it convenient to arrange them in the following table, where also 4-simplices are indicated by small

numbers to which the tetrahedra belong:

	12456	13456	23456	
12345	1245	1345	2345	(15)
12346	1246	1346	2346	
12356	1256	1356	2356	

Thus, the tetrahedra in every row correspond to a 4-simplex in the l.h.s., and the tetrahedra in every column correspond to a 4-simplex in the r.h.s. of the move.

First, we introduce nine nonvanishing parameters $\varkappa_t \in \mathbb{C}$ for all tetrahedra $t = ijkl$ in the table, and then eighteen *orthonormal* vectors-operators – a pair

$$\mathbf{e}_t = \frac{1}{\varkappa_t} \frac{\partial}{\partial x_t} + \varkappa_t x_t, \quad \mathbf{f}_t = i \left(\frac{1}{\varkappa_t} \frac{\partial}{\partial x_t} - \varkappa_t x_t \right)$$

for each t ; here

$$i = \sqrt{-1}.$$

Then, we introduce six more parameters: λ_u for each table row, and μ_u for each table column, where $u = ijklm$ is the corresponding 4-simplex. With these parameters, we construct the following six *isotropic and mutually orthogonal* vectors:

$$\mathbf{g}_{ijklm} = \mathbf{e}_{ijlm} + i \cos \lambda_{ijklm} \mathbf{e}_{iklm} + i \sin \lambda_{ijklm} \mathbf{e}_{jklm} \quad (16)$$

for the table rows, and

$$\mathbf{h}_{ijklm} = \mathbf{f}_{ijkl} + i \cos \mu_{ijklm} \mathbf{f}_{ijkm} + i \sin \mu_{ijklm} \mathbf{f}_{ijlm} \quad (17)$$

for the table columns.

Next, we bring into consideration six more unit vectors, orthogonal to each other and to all \mathbf{g}_u and \mathbf{h}_u :

$$\mathbf{p}_{ijklm} = \sin \lambda_{ijklm} \mathbf{e}_{iklm} - \cos \lambda_{ijklm} \mathbf{e}_{jklm} \quad (18)$$

for each row and

$$\mathbf{q}_{ijklm} = \sin \mu_{ijklm} \mathbf{f}_{ijkm} - \cos \mu_{ijklm} \mathbf{f}_{ijlm} \quad (19)$$

for each column, and an orthogonal 3×3 matrix

$$A = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi' & \sin \psi' \\ 0 & -\sin \psi' & \cos \psi' \end{pmatrix} \begin{pmatrix} \cos \psi'' & \sin \psi'' & 0 \\ -\sin \psi'' & \cos \psi'' & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ψ , ψ' and ψ'' – Euler angles for A – are our three remaining parameters. With these vectors and matrix, we construct isotropic, and orthogonal to each other as well as to all \mathbf{g}_u and \mathbf{h}_u , vectors \mathbf{r} , \mathbf{s} and \mathbf{t} . It is convenient for us to arrange these vectors in a column, and we define them as follows:

$$\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{12345} \\ \mathbf{p}_{12346} \\ \mathbf{p}_{12356} \end{pmatrix} + iA \begin{pmatrix} \mathbf{q}_{12456} \\ \mathbf{q}_{13456} \\ \mathbf{q}_{23456} \end{pmatrix}. \quad (20)$$

Remark 5. The plus sign before the second term in (20) cannot be changed to minus without making change(s) elsewhere in our construction. As a direct calculation shows, this sign ensures that the isotropic space spanned by vectors \mathbf{g}_u , \mathbf{h}_u , \mathbf{r} , \mathbf{s} and \mathbf{t} (see item (i) below in Theorem 4) belongs to the desired connected component, according to Remark 4.

Theorem 4.

- (i) *The linear space \mathfrak{V} spanned by vectors \mathbf{g}_{12345} , \mathbf{g}_{12346} , \mathbf{g}_{12356} , \mathbf{h}_{12456} , \mathbf{h}_{13456} , \mathbf{h}_{23456} , \mathbf{r} , \mathbf{s} and \mathbf{t} is 9-dimensional isotropic – a maximal isotropic subspace in the 18-dimensional space of operators (10) for tetrahedra t in the table (15).*
- (ii) *The 18 parameters \varkappa_t , λ_u , μ_u , ψ , ψ' and ψ'' , used in our construction, are independent: the Jacobian matrix of the mapping from the space of these parameters to the Grassmannian (which consists of 9-dimensional subspaces in the mentioned 18-dimensional linear space) has rank 18 in a generic point.*
- (iii) *For generic parameters, \mathfrak{V} is such that there exist such weights \mathcal{W}_u of the form (7) for all 4-simplices in the l.h.s. and r.h.s. of (6) that both sides of (6) are turned into zero by all operators in \mathfrak{V} .*

Proof. (i) This follows directly from our construction.

(ii) This is shown by a direct calculation (enough to find that rank is 18 for some specific values of parameters).

(iii) We begin with considering the four vectors \mathbf{g}_{12346} , \mathbf{g}_{12356} , \mathbf{s} and \mathbf{t} , see (16) and (20). They are linearly independent, and their expansions in terms of the basis vectors \mathbf{e}_t and \mathbf{f}_t have zero coefficients if t belongs to the first row of table (15); we visualize this fact by saying that our four vectors have the form $\begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$. Moreover, \mathbf{g}_{12346} and \mathbf{g}_{12356} have the forms $\begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{pmatrix}$; in this proof, we call vectors of such forms *second-row* and *third-row* vectors, respectively.

Next, we consider the orthogonal projections of linear combinations $\sigma\mathbf{s} + \tau\mathbf{t}$ (where $\sigma, \tau \in \mathbb{C}$) onto the space of third-row vectors. There exist two such linear combinations

$$\sigma_d\mathbf{s} + \tau_d\mathbf{t} \quad \text{and} \quad \sigma_x\mathbf{s} + \tau_x\mathbf{t} \tag{21}$$

whose projections, called $\mathbf{d}_{1236}^{(12356)}$ and $\mathbf{x}_{1236}^{(12356)}$ (where the 4-simplex 12356 corresponds to the third row, and the *inner* tetrahedron 1236 is common for it and the “second-row” 4-simplex 12346), satisfy

$$\left\langle \mathbf{d}_{1236}^{(12356)}, \mathbf{d}_{1236}^{(12356)} \right\rangle = \left\langle \mathbf{x}_{1236}^{(12356)}, \mathbf{x}_{1236}^{(12356)} \right\rangle = 0, \quad \left\langle \mathbf{d}_{1236}^{(12356)}, \mathbf{x}_{1236}^{(12356)} \right\rangle = 1.$$

As \mathbf{s} and \mathbf{t} lie in an isotropic subspace, this means also that the projections $\mathbf{d}_{1236}^{(12346)}$ and $\mathbf{x}_{1236}^{(12346)}$ of the same vectors (21) onto the space of *second-row* vectors satisfy

$$\left\langle \mathbf{d}_{1236}^{(12346)}, \mathbf{d}_{1236}^{(12346)} \right\rangle = \left\langle \mathbf{x}_{1236}^{(12346)}, \mathbf{x}_{1236}^{(12346)} \right\rangle = 0, \quad \left\langle \mathbf{d}_{1236}^{(12346)}, \mathbf{x}_{1236}^{(12346)} \right\rangle = -1.$$

Now the three operators

$$\frac{\partial}{\partial x_{1236}} + \mathbf{d}_{1236}^{(12356)}, \quad x_{1236} - \mathbf{x}_{1236}^{(12356)} \quad \text{and} \quad \mathbf{g}_{12356} \tag{22}$$

span a three-dimensional isotropic subspace in the (10-dimensional) space of operators (10) for which $t \in \partial(12356)$ (boundary of the 4-simplex 12356), while the three operators

$$\frac{\partial}{\partial x_{1236}} + \mathbf{d}_{1236}^{(12346)}, \quad x_{1236} + \mathbf{x}_{1236}^{(12346)} \quad \text{and} \quad \mathbf{g}_{12346} \quad (23)$$

span a three-dimensional isotropic subspace in the space of operators (10) for which $t \in \partial(12346)$.

To move further, we note that our space \mathfrak{V} depends on the vectors \mathbf{p}_u and \mathbf{q}_u only modulo the six vectors \mathbf{g}_u and \mathbf{h}_u : the definitions (18) and (19) can be changed by adding any linear combinations of \mathbf{g}_u 's and \mathbf{h}_u 's to their right-hand sides, and this does not affect \mathfrak{V} . In particular, this means that each \mathbf{q}_u can be changed to a linear combination of itself and \mathbf{h}_u (17) (with the

same $u = ijklm$) in such two ways that all the new \mathbf{q}_u 's will fit into the pattern $\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}$

in the first case, and $\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ * & * & * \end{pmatrix}$ in the second case. Then, in full analogy with what we have

already done, we obtain the analogues of operators (22) and (23) for the two remaining inner tetrahedra (namely, 1234 and 1235, respectively, instead of 1236) in the l.h.s. of the move 3–3 and their adjoining 4-simplices.

As a result, we get a five-dimensional isotropic space of operators for a 4-simplex in the l.h.s. For instance, for 12356, it is spanned by the operators (22) and also

$$\frac{\partial}{\partial x_{1235}} + \mathbf{d}_{1235}^{(12356)} \quad \text{and} \quad x_{1235} + \mathbf{x}_{1235}^{(12356)}$$

(the signs before $\mathbf{x}_t^{(u)}$ must be different for the two 4-simplices u containing the tetrahedron t).

Of course, the five-dimensional isotropic space of operators can be found in a similar way also for the 4-simplices in the r.h.s. of our Pachner move. Then we define the weights \mathcal{W}_u for all the six 4-simplices as satisfying each the corresponding five equations, according to item (ii) in Theorem 2, and (see the proof of item (ii) in Theorem 3) the relation (6) does hold.

What remains is to show that the above construction can be performed in such way that we get *even* Grassmann elements – exponents (7) (see item (iii) in Theorem 2) as our weights \mathcal{W}_u . We think that the easiest way to do this is to refer to the nontrivial example given below in Subsection 6.2, where the weights have the form (7), and the construction of isotropic spaces mentioned in this proof works well; then it is extended to the general case by continuity. So, to within this example, Theorem 4 is proven. \blacksquare

6 Two explicit constructions of parameterized weights satisfying relation 3–3

The first construction, given in Subsection 6.2, is a particular case of the construction in the previous Section 5. It depends on five (six, one of which is redundant) parameters – so, the nature of remaining $18 - 5 = 13$ parameters is still mysterious. Note that the rank of matrix $(\alpha_{t_1 t_2}^{(u)})$ (compare formula (7) with (25), (26) and (27)) is 4 for this construction, as is the rank of a generic antisymmetric 5×5 matrix, hence this rank is also 4 in the general case of Section 5.

The second construction, given in Subsection 6.3, is probably a *limiting* case of the construction in Section 5, because here $\text{rank}(\alpha_{t_1 t_2}^{(u)}) = 2$. Despite this kind of degeneracy, the second construction exhibits extremely interesting relations to exotic homologies, studied in Sections 7 and 8.

6.1 Some formulas common for the two constructions

We are going to present, in Subsections 6.2 and 6.3, two explicit constructions of nicely parameterized Grassmann four-simplex weights \mathcal{W}_{ijklm} of the form (7), satisfying the 3–3 algebraic relation (6). In this subsection, we write out some formulas that belong to both of them.

First, in both cases a quantity φ_{ijk} is introduced for each 2-face ijk . These φ_{ijk} enter both in the expressions for weights and in the multiplier const in (6). To be more exact, our relations here look as follows:

$$\begin{aligned} & \frac{1}{\varphi_{123}} \iiint \mathcal{W}_{12345} \mathcal{W}_{12346} \mathcal{W}_{12356} \, dx_{1234} \, dx_{1235} \, dx_{1236} \\ &= -\frac{1}{\varphi_{456}} \iiint \mathcal{W}_{12456} \mathcal{W}_{13456} \mathcal{W}_{23456} \, dx_{1456} \, dx_{2456} \, dx_{3456}. \end{aligned} \quad (24)$$

Second, the following Grassmannian quadratic form is used in both cases. For a 4-simplex $u = ijklm$, let abc be its 2-face, and $d_1 < d_2$ – two remaining vertices. We put

$$\Phi_{ijklm} = \sum_{\substack{\text{over 2-faces } abc \\ \text{of } ijklm}} \epsilon_{d_1abcd_2} \varphi_{abc} x_{\{abcd_1\}} x_{\{abcd_2\}}, \quad (25)$$

where $\epsilon_{d_1abcd_2} = 1$ if the order d_1abcd_2 of vertices determines the orientation of $ijklm$ induced by the fixed orientation of the manifold – l.h.s. or r.h.s. of a Pachner move in our case – and $\epsilon_{d_1abcd_2} = -1$ otherwise. Recall also Convention 4 concerning the curly brackets in the subscripts in (25).

Remark 6. In practical calculations, we use formula

$$\epsilon_{d_1abcd_2} = p_{ijklm} \epsilon_{d_1abcd_2}^{ijklm},$$

where p_{ijklm} reflects the consistent orientation of 4-simplices. Namely, for the simplices in the l.h.s. of move 3–3, $p_{12345} = -p_{12346} = p_{23456} = 1$, and for the simplices in the r.h.s. $p_{12456} = -p_{13456} = p_{23456} = 1$. As for $\epsilon_{d_1abcd_2}^{ijklm}$, it is the sign of permutation between the sequences of its subscripts and superscripts.

6.2 First family of weights

Let a complex number ξ_i be put in correspondence to every vertex $i = 1, \dots, 6$. We call these numbers *vertex coordinates*. Then we define φ_{ijk} as follows:

$$\varphi_{ijk} = \frac{\xi_i - \xi_j}{1 + \xi_i \xi_j} \cdot \frac{\xi_j - \xi_k}{1 + \xi_j \xi_k} \cdot \frac{\xi_k - \xi_i}{1 + \xi_k \xi_i}, \quad (26)$$

then Φ_{ijklm} according to (25), and then the weight \mathcal{W}_{ijklm} as the following Grassmann–Gaussian exponent:

$$\mathcal{W}_{ijklm} = \exp \Phi_{ijklm}. \quad (27)$$

These formulas for weights first appeared in [9, Appendix].

Theorem 5. *The weights \mathcal{W}_{ijklm} with φ_{ijk} defined according to (26) satisfy the relation (24).*

Proof. Direct computer calculation. ■

Direct calculations show that the isotropic spaces of operators for the weights introduced in this Subsection fit well into the scheme of Section 5. We think that this subject of isotropic spaces for 4-simplex weights deserves a detailed study in further works; right here we write out, just for illustration, the elegant explicit formulas for one \mathfrak{g}_u and one \mathfrak{h}_u (and, looking at them, it will not be hard to guess the formulas for other \mathfrak{g}_u and \mathfrak{h}_u). First, introduce the following auxiliary quantities:

$$r_{ijkl} = \frac{(\xi_i \xi_j + 1)(\xi_k \xi_l + 1)(\xi_i \xi_j \xi_k \xi_l - \xi_k \xi_l + \xi_j \xi_l + \xi_i \xi_l + \xi_j \xi_k + \xi_i \xi_k - \xi_i \xi_j + 1)}{(\xi_k - \xi_i)(\xi_k - \xi_j)(\xi_l - \xi_i)(\xi_l - \xi_j)},$$

$$s_{ijkl} = -\frac{(\xi_j - \xi_i)(\xi_l - \xi_k)(\xi_j \xi_k \xi_l + \xi_i \xi_k \xi_l - \xi_i \xi_j \xi_l - \xi_i \xi_j \xi_k + \xi_l + \xi_k - \xi_j - \xi_i)}{(\xi_i \xi_k + 1)(\xi_j \xi_k + 1)(\xi_i \xi_l + 1)(\xi_j \xi_l + 1)}.$$

Then, \mathfrak{g}_{12345} is proportional to the following vector:

$$\begin{aligned} \mathfrak{g}_{12345} \propto & r_{1245} \frac{\partial}{\partial x_{1245}} + s_{1245} x_{1245} + r_{1345} \frac{\partial}{\partial x_{1345}} - s_{1345} x_{1345} \\ & + r_{2345} \frac{\partial}{\partial x_{2345}} + s_{2345} x_{2345}, \end{aligned} \quad (28)$$

and \mathfrak{h}_{12456} is proportional to the following vector:

$$\begin{aligned} \mathfrak{h}_{12456} \propto & r_{1245} \frac{\partial}{\partial x_{1245}} - s_{1245} x_{1245} + r_{1246} \frac{\partial}{\partial x_{1246}} + s_{1246} x_{1246} \\ & + r_{1256} \frac{\partial}{\partial x_{1256}} - s_{1256} x_{1256}. \end{aligned} \quad (29)$$

6.3 Second family of weights

This time, let each vertex i have *three* complex coordinates ξ_i, η_i, ζ_i over field \mathbb{C} . We define φ_{ijk} as the following determinant:

$$\varphi_{ijk} = \begin{vmatrix} \xi_i & \xi_j & \xi_k \\ \eta_i & \eta_j & \eta_k \\ \zeta_i & \zeta_j & \zeta_k \end{vmatrix}. \quad (30)$$

Then we define the quantity

$$h_{ijklm} = \alpha \xi_n + \beta \eta_n + \gamma \zeta_n, \quad (31)$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, and $n \in \{1, \dots, 6\}$ is the number *missing* in the set $\{i, j, k, l, m\}$.

Finally, we define the 4-simplex weight \mathcal{W}_{ijklm} as follows:

$$\mathcal{W}_{ijklm} = h_{ijklm} + \Phi_{ijklm}. \quad (32)$$

Theorem 6. *The weight (32) is a Grassmann–Gaussian exponent:*

$$\mathcal{W}_{ijklm} = h_{ijklm} \exp(\Phi_{ijklm}/h_{ijklm}). \quad (33)$$

Proof. A direct calculation shows that the form Φ_{ijklm} has now rank 2. So, the Grassmann exponent in (33) cannot include terms of degree > 2 , and the terms of degree ≤ 2 are exactly as in (32). \blacksquare

Theorem 7. *The weights \mathcal{W}_{ijklm} defined in this Subsection satisfy the 3–3 relation (24).*

Proof. Direct calculation. We used our package PL¹ for manipulations in Grassmann algebra. ■

Remark 7. Formula (31) is simple and works well, but conceals the real nature of quantities h_{ijklm} . This will be explained below in Sections 7 and 8, and we will rephrase the statement of Theorem 7 in new terms, as part of Theorem 11.

7 An exotic analogue of middle homologies

The terms h_{ijklm} of zero Grassmann degree in weights (32) have actually an exotic homological nature. This becomes especially clear if we consider not only move 3–3, but also move 2–4, and this we are going to do in Section 8. Right here, we are presenting the sequence (34) of two linear mappings and some related notions and statements. Then, in the end of Subsection 8.2, we will explain that what is essential in a set of terms h_{ijklm} corresponds to an element in $\text{Ker } g_4 / \text{Im } g_3$, at least in the case of the mentioned Pachner moves. To be exact, the permitted h_{ijklm} 's will correspond to $\text{Ker } g_4$, while the result for the Grassmann weight of either side of the move will not change under a change of the h_{ijklm} 's corresponding to an element in $\text{Im } g_3$.

Sequence (34) is expected to be part of a longer chain complex, but we don't need here that complex in full.

Let there be an oriented triangulated PL manifold M with boundary. We introduce \mathbb{C} -linear spaces $\mathbb{C}^{N'_1}$ and \mathbb{C}^{N_4} whose bases are *inner* edges and (all) 4-simplices of M , respectively (notations in accordance with Convention 2), and two \mathbb{C} -linear mappings between them as follows:

$$\mathbb{C}^{N'_1} \xrightarrow{g_3} \mathbb{C}^{N_4} \xrightarrow{g_4} \mathbb{C}^{N'_1}. \quad (34)$$

We use notations g_3 and g_4 because these mappings have a clear analogy with mappings g_3 and g_4 in [8, formula(13)]; we leave the definition and discussion of other g_i (namely, g_1, g_2, g_5 and g_6) for further papers.

A set of admissible values for h_{ijklm} will correspond to an element of $\text{Ker } g_4$, while such Grassmann weights as the right-hand side of formula (37) below do not change when an element of $\text{Im } g_3$ is added to it.

By definition, the matrix element of mapping g_3 between an edge $b = ij$ and a 4-simplex u vanishes unless $b \subset u$. Assuming $b \subset u$, we can write $u = \{ijklm\}$, which means, according to Convention 4, that u has vertices i, j, k, l and m but they all don't necessarily go in the increasing order. In this case, the matrix element of g_3 is

$$\frac{1}{\varphi_{ijk}\varphi_{ijl}\varphi_{ijm}}. \quad (35)$$

Recall that φ_{ijk} is defined in (30).

Similarly, by definition, the matrix element of mapping g_4 between a 4-simplex u and an edge b is nonzero only if $b \subset u$. We write again $u = \{ijklm\}$ and $b = ij$, and define this matrix element as

$$\epsilon_{ijklm}\varphi_{klm}, \quad (36)$$

where $\epsilon_{ijklm} = 1$ if the sequence $ijklm$, in this order, gives the consistent (with the fixed orientation of manifold M) orientation of u , and $\epsilon_{ijklm} = -1$ otherwise.

¹Korepanov A.I., Korepanov I.G., Sadykov N.M., PL: Piecewise-linear topology using GAP, <http://sf.net/projects/plgap/>.

Theorem 8. *Mappings g_3 and g_4 defined according to (35) and (36) form a chain, in the sense that*

$$g_4 \circ g_3 = 0.$$

Proof. We first show this for $M = \partial\Delta^5$ – the sphere S^4 triangulated into five 4-simplices as the boundary of 5-simplex 123456. Consider an edge $a \subset S^4$ as a basis vector in the leftmost space in (34), an edge $b \subset S^4$ as a basis vector in the rightmost space in (34), and the matrix element of mapping $g_4 \circ g_3$ between a and b . This matrix element is the sum of products of expressions (35) and (36) over those 4-simplices u that contain both a and b ; we call such a product *contribution* of u .

There are three cases; in all three $ijklmn$ is a permutation of the set of vertices $1, \dots, 6$.

Case 1. Edges a and b coincide, $a = b = ij$. The matrix element is

$$\begin{aligned} & \frac{\epsilon_{ijklm}\varphi_{klm}}{\varphi_{ijk}\varphi_{ijl}\varphi_{ijm}} + \frac{\epsilon_{ijkln}\varphi_{kln}}{\varphi_{ijk}\varphi_{ijl}\varphi_{ijn}} + \frac{\epsilon_{ijkmn}\varphi_{kmn}}{\varphi_{ijk}\varphi_{ijm}\varphi_{ijn}} + \frac{\epsilon_{ijlmn}\varphi_{lmn}}{\varphi_{ijl}\varphi_{ijm}\varphi_{ijn}} \\ &= \epsilon_{ijklm} \frac{\varphi_{klm}\varphi_{ijn} - \varphi_{kln}\varphi_{ijm} + \varphi_{kmn}\varphi_{ijl} - \varphi_{lmn}\varphi_{ijk}}{\varphi_{ijk}\varphi_{ijl}\varphi_{ijm}\varphi_{ijn}} = 0. \end{aligned}$$

The numerator, after the reducing to a common denominator, vanishes: this is a *Plücker relation*.

Case 2. Edges a and b have one vertex in common, $a = ij$, $b = ik$. The matrix element is

$$\frac{\epsilon_{ikjlm}\varphi_{jlm}}{\varphi_{ijk}\varphi_{ijl}\varphi_{ijm}} + \frac{\epsilon_{ikjln}\varphi_{jln}}{\varphi_{ijk}\varphi_{ijl}\varphi_{ijn}} + \frac{\epsilon_{ikjmn}\varphi_{jmn}}{\varphi_{ijk}\varphi_{ijm}\varphi_{ijn}} = \epsilon_{ikjlm} \frac{\varphi_{jlm}\varphi_{ijn} - \varphi_{jln}\varphi_{ijm} + \varphi_{jmn}\varphi_{ijl}}{\varphi_{ijk}\varphi_{ijl}\varphi_{ijm}\varphi_{ijn}} = 0.$$

Here the numerator is obtained from the numerator in our Case 1 by letting $k = j$.

Case 3. Edges a and b do not intersect, $a = ij$, $b = kl$. The matrix element is

$$\frac{\epsilon_{klijm}\varphi_{ijm}}{\varphi_{ijk}\varphi_{ijl}\varphi_{ijm}} + \frac{\epsilon_{klijn}\varphi_{ijn}}{\varphi_{ijk}\varphi_{ijl}\varphi_{ijn}} = 0.$$

Next, we prove the theorem again for $M = S^4$, but triangulated in an *arbitrary* way. This arbitrary triangulation can be achieved by a sequence of Pachner moves performed on the initial triangulation considered above. Each Pachner move replaces some 4-simplices with some other ones, in such way that the withdrawn and the replacing 4-simplices will form together a sphere $\partial\Delta^5$ if we change the orientation of, say, the withdrawn 4-simplices. It follows then from what we have proved for $\partial\Delta^5$ that the contribution of all the replacing 4-simplices into any matrix element of $g_4 \circ g_3$ is the same as the contribution of all the withdrawn 4-simplices, including the cases where an edge appears or disappears during the move.

Finally, it remains to say that any manifold M has locally the same structure as S^4 , and any matrix element of $g_4 \circ g_3$ consists only of obviously local contributions. \blacksquare

Experimental result. *For a closed oriented 4-dimensional PL manifold M , the vector space $\text{Ker } g_4 / \text{Im } g_3$ is six times (i.e., isomorphic to the direct sum of six copies of) usual second homologies $H_2(M; \mathbb{C})$.*

This has been checked (in particular) for $M = T^4, T^2 \times S^2, S^1 \times S^3, S^2 \times S^2, S^4$, and the Kummer surface.

8 Relation 2–4 and the exotic homological nature of h_{ijklm}

As we have already said, it is the relation 2–4 that makes especially clear the exotic homological nature of free terms h_{ijklm} . Here it is:

$$\begin{aligned} \int \mathcal{W}_{12345} \mathcal{W}_{12346} dx_{1234} &= -\frac{1}{\varphi_{156}\varphi_{256}\varphi_{356}\varphi_{456}} \int \dots \int \mathcal{W}_{12356} \mathcal{W}_{12456} \mathcal{W}_{13456} \mathcal{W}_{23456} \\ &\quad \times w_{56} dx_{1256} dx_{1356} dx_{1456} dx_{2356} dx_{2456} dx_{3456}. \end{aligned} \quad (37)$$

It involves a new factor w_{56} in the integrand in its r.h.s. – the *edge weight* of the inner edge 56. So, we first define this weight in Subsection 8.1, and then return to our quantities h_{ijklm} in Subsection 8.2.

8.1 The edge weight

For any edge $a = ij$ in a triangulation of a 4-manifold with boundary, we define a Grassmann differential operator $\partial_a = \partial_{ij}$ as the following sum over all tetrahedra $t = \{ijkl\}$ containing this edge:

$$\partial_{ij} = \sum_{t=\{ijkl\}} \frac{1}{\varphi_{ijk}\varphi_{ijl}} \partial_t. \quad (38)$$

Lemma. *The weight \mathcal{W}_u (32) of a 4-simplex u satisfies “edge equations”*

$$\partial_a \mathcal{W}_u = 0$$

for any edge a .

Proof. Direct calculation (which is, of course, nontrivial only if $a \subset u$). ■

Specifically, here is how the operator (38) looks for the inner edge 56 of the cluster of four 4-simplices corresponding to the r.h.s. of relation (37):

$$\begin{aligned} \partial_{56} = & \frac{1}{\varphi_{156}\varphi_{256}} \partial_{1256} + \frac{1}{\varphi_{156}\varphi_{356}} \partial_{1356} + \frac{1}{\varphi_{156}\varphi_{456}} \partial_{1456} \\ & + \frac{1}{\varphi_{256}\varphi_{356}} \partial_{2356} + \frac{1}{\varphi_{256}\varphi_{456}} \partial_{2456} + \frac{1}{\varphi_{356}\varphi_{456}} \partial_{3456}. \end{aligned} \quad (39)$$

Theorem 9. *The right-hand side of (37) does not change if an expression is added to w_{56} whose “edge 56 derivative” is zero, as follows:*

$$w_{56} \mapsto w_{56} + \tilde{w}_{56}, \quad \partial_{56} \tilde{w}_{56} = 0.$$

Proof. We have to prove that

$$\int \cdots \int \mathcal{W}_{12356} \mathcal{W}_{12456} \mathcal{W}_{13456} \mathcal{W}_{23456} \tilde{w}_{56} dx_{1256} dx_{1356} dx_{1456} dx_{2356} dx_{2456} dx_{3456} = 0. \quad (40)$$

As the “edge 56 derivative” of every factor in the integrand

$$\mathcal{X} = \mathcal{W}_{12356} \mathcal{W}_{12456} \mathcal{W}_{13456} \mathcal{W}_{23456} \tilde{w}_{56}.$$

vanishes, it vanishes for all the product:

$$\partial_{56} \mathcal{X} = 0. \quad (41)$$

It is an easy exercise (just break \mathcal{X} into the even and odd parts) to see that the right analogue of (41) also holds:

$$\mathcal{X} \overleftarrow{\partial}_{56} = 0,$$

where $\overleftarrow{\partial}_{56}$ is the same linear combination (39), but with left derivatives replaced with the right ones. Finally, recall that the integration means, according to Section 2, the right differentiation, and specifically in (40) this can be represented as follows (compare (5)):

$$\mathcal{X} \overleftarrow{\partial}_{1256} \overleftarrow{\partial}_{1356} \cdots \overleftarrow{\partial}_{3456} = \varphi_{156} \varphi_{256} \mathcal{X} \overleftarrow{\partial}_{56} \overleftarrow{\partial}_{1356} \cdots \overleftarrow{\partial}_{3456} = 0. \quad \blacksquare$$

Due to Theorem 9, the following definition of w_{56} is not surprising:

$$w_{56} = \partial_{56}^{-1} 1, \quad (42)$$

by which we understand *any* Grassmann algebra element w_{56} such that $\partial_{56} w_{56} = 1$. For instance, we can take $w_{56} = \varphi_{156} \varphi_{256} x_{1256}$.

8.2 The quantities h_{ijklm}

The quantities h_{ijklm} for *both* moves 3–3 and 2–4 considered in this paper can be obtained as follows. First, glue together both sides of a Pachner move in the natural way – identifying like-named boundary simplices. This gives $S^4 = \partial\Delta^5$ (recall that this means a 4-sphere triangulated as the boundary of a 5-simplex).

Remark 8. This gluing implies that we have changed the orientation in one of the sides. Nevertheless, our mapping g_3 is defined in such way that the orientation issues do not affect the definition of allowable set of values for h_{ijklm} given below.

Theorem 10. *The sequence (34) for $S^4 = \partial\Delta^5$ is exact: $\text{Ker } g_4 = \text{Im } g_3$.*

Proof. Taking into account Theorem 8, it is enough to show that

$$\text{rank } g_3 + \text{rank } g_4 = 6,$$

where 6 is the number of 4-simplices in the triangulation and thus the dimension of the middle space in (34). This is done by a direct calculation; both ranks prove to be 3. ■

Remark 9. Compare this also with the Experimental result on page 16.

We now take an arbitrary chain c_{edges} on edges of our S^4 – element of the first linear space in (34), and then its image under g_3 – a chain on 4-simplices. We call the resulting coefficients at 4-simplices $u = ijklm$, both in the l.h.s. and r.h.s. of our Pachner move, an *allowable set* of values for h_{ijklm} .

Theorem 11. *Let there be an allowable set of values $h_u = h_{ijklm}$ for the six 4-simplices in both sides of either Pachner move 3–3 or 2–4, and, in case of move 2–4, let there be chosen any edge weight w_{56} according to (42). Let also the 4-simplex weights be defined according to (32), with the quadratic forms Φ_u defined according to (25) and (30). Then, the relation (24) holds for move 3–3 or, respectively, (37) holds for move 2–4.*

Proof. Direct calculation. ■

Remark 10. It is an easy exercise to show that our allowable sets of values $h_u = h_{ijklm}$ for Pachner moves can be represented in the form (31) (although this form may disguise their nature). Hence, the part of Theorem 11 dealing with move 3–3 says the same as Theorem 7, as was promised in Remark 7.

We explain now why we said in the first paragraph of Section 7 that what is essential in a set of terms h_{ijklm} corresponds to an element in $\text{Ker } g_4 / \text{Im } g_3$. First, we note that in both sides of the move 3–3, as well as in the l.h.s. of move 2–4, there are *no* inner edges. So, the sequence (34) assumes the form

$$0 \xrightarrow{g_3} \mathbb{C}^{N_4} \xrightarrow{g_4} 0,$$

i.e., the factor $\text{Ker } g_4 / \text{Im } g_3$ obviously gives *all* possible values for h_{ijklm} , which agrees with Theorem 11.

Remark 11. Of course, Theorem 11 tells us more, namely, also how to make the h_{ijklm} 's agree for the two sides, so that (24) or (37) holds.

Now consider the r.h.s. of the 2–4 relation (37). As it is equal to the l.h.s., it does not depend on the coefficient in c_{edges} at the edge 56, the latter being absent from the l.h.s. of the Pachner move, while being the only inner edge for its r.h.s. Consider the sequence (34) for the r.h.s. of move 2–4. The first and third spaces in this sequence are one-dimensional, the only basis vector being the edge 56. Any allowable set of h_u , with the u 's in this r.h.s., obviously makes a *cycle* in the sense that it is annihilated by the mapping g_4 . So, the r.h.s. of (37) is determined by this cycle modulo a *boundary* – image of g_3 .

Especially interesting question for future research is to uncover the analogue(s) of such exotic-homological structures for more general 4-simplex weights described in Section 5.

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²GAP – Groups, Algorithms, Programming – a System for Computational Discrete Algebra, <http://gap-system.org/>.

³Maxima, a computer algebra system, <http://maxima.sourceforge.net>.