

Formal Integrability for the Nonautonomous Case of the Inverse Problem of the Calculus of Variations

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Abstract. We address the integrability conditions of the inverse problem of the calculus of variations for time-dependent SODE using the Spencer version of the Cartan–Kähler theorem. We consider a linear partial differential operator P given by the two Helmholtz conditions expressed in terms of semi-basic 1-forms and study its formal integrability. We prove that P is involutive and there is only one obstruction for the formal integrability of this operator. The obstruction is expressed in terms of the curvature tensor R of the induced nonlinear connection. We recover some of the classes of Lagrangian semisprays: flat semisprays, isotropic semisprays and arbitrary semisprays on 2-dimensional manifolds.

Key words: formal integrability; partial differential operators; Lagrangian semisprays; Helmholtz conditions

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1 Introduction

One of the most interesting problems of geometric mechanics is related to the integrability conditions of the inverse problem of the calculus of variations for time-dependent second-order ordinary differential equations (SODE). The inverse problem can be formulated as follows. Given a time-dependent system of SODE

$$\frac{d^2x^i}{dt^2} + 2G^i\left(t, x, \frac{dx}{dt}\right) = 0, \quad i \in \{1, \dots, n\},$$

under what conditions this system can be made equivalent, using a multiplier matrix g_{ij} , with the system of Euler–Lagrange equations of a regular Lagrangian

$$g_{ij}\left(t, x, \frac{dx}{dt}\right) \left(\frac{d^2x^i}{dt^2} + 2G^i\left(t, x, \frac{dx}{dt}\right)\right) = \frac{d}{dt} \left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} ?$$

In this case such a system is called variational. The necessary and sufficient conditions under which such a system is variational are known as the *Helmholtz conditions*.

This inverse problem was solved for the case $n = 1$ by Darboux [17], and for $n = 2$ by Douglas [20]. Douglas’s approach consists in an application of the Riquier theory of systems of partial differential equations [32], to a certain associated linear differential system. The generalization of its results in the higher dimensional case is a very difficult problem because the system provided by the Helmholtz conditions is extremely over-determined. Some of the first studies of the inverse problem in spaces of arbitrary dimension are those of Davis [18] and Kosambi [26].

There are different attempts to solve this problem. First, there are some reformulations of the Helmholtz conditions in better geometric forms, which are close enough to the first analytical formulations [20, 33, 34, 35], but undercover more of the geometry behind them [11, 13, 15, 16, 19, 28, 29]. The system of SODE is identified with a semispray on the first jet bundle of a fibred manifold over \mathbb{R} . The most important geometric tools induced by a semispray are nonlinear connection, Jacobi endomorphism, dynamical covariant derivative, linear connections and their curvatures. Some reformulations of the Helmholtz conditions are using either the special derivations along the tangent bundle projection introduced in [38], or the semi-basic 1-forms [5] and the Frölicher–Nijenhuis theory of derivations on the algebra of vector-valued forms [21].

Anderson and Thompson [2] analyzed the inverse problem based on the exterior differential system approach [4]. Using the variational bicomplex associated to a system of arbitrary order ordinary differential equations, they derived the fundamental system of equations for the variational multiplier and proved their sufficiency. They made a detailed study of two dimensional sprays and they proved, for general degrees of freedom, that all isotropic semisprays are variational. It means semisprays that have the associated Jacobi endomorphism a multiple of the identity. This correspond to the *Case I* of Douglas’s classification. This approach is continued in [1], where the case of Φ diagonalizable, with distinct eigenfunctions, is exposed in detail. The same Case I was proved to be variational also in [36]. This paper uses Riquier theory, but in a more geometric way. The process of repeated differentiations of equations and searching for new nontrivial relations is realized by intrinsic operations.

Another subcase of Douglas’s case II is discussed in [14]: separable systems of SODE. Any systems of SODE from this subcase is variational. They showed that any system of SODE in *Case III* with n degrees of freedom can be separated into n separate systems of two first-order equations. They also proved that there are systems separable in the above sense but not separable into single independent second-order equations. This case was treated in [9]. In [37] the authors reinvestigated the case $n = 2$ with their more intrinsic version of the Riquier algorithm. Their approach is based on the same underlying methodology as the analytical work of Douglas.

Another method of studying the integrability conditions of the inverse problem of the calculus of variation is the Spencer–Goldsmchmidt theory of formal integrability of partial differential operators, using two sufficient conditions provided by Cartan–Kähler theorem [12, 22, 40]. This method was applied for autonomous SODE in [23], using the Frölicher–Nijenhuis theory of derivations of vector-valued differential forms. Grifone and Muzsnay gave the first obstructions so that a *spray* (homogeneous semispray) is variational, for general degrees of freedom. In order to obtain a complete classification of variational sprays, they restricted their work to some particular cases. The Spencer theory is fully applied to the two dimensional case, corresponding to Douglas’s paper. For the general n -dimensional case, it is proved only that isotropic sprays are variational. It is important to notice that Grifone and Muzsnay’s analysis starts from the Euler–Lagrange partial differential operator, and not from the Helmholtz conditions.

For time independent, homogeneous SODE, the inverse problem is known as the projective metrizable problem. This problem and its formal integrability is studied in [7] using Spencer theory. It was shown that there exists only one first obstruction for the formal integrability of the projective metrizable operator, expressed in terms of the curvature tensor of the nonlinear connection induced by the spray. This obstruction correspond to second obstruction for the formal integrability of the Euler–Lagrange operator.

An interesting and new approach regarding variational PDE’s is the one of A. Prástaro [30, 31]. Using suitable cohomologies and integral bordism groups, the author characterizes variational systems constrained by means of PDE’s of submanifolds of fiber bundles. He presents a new algebraic topological characterization of global solutions of variational problems.

In this paper we address the integrability conditions of the inverse problem of the calculus of variations for time-dependent SODE using also the Spencer version of the Cartan–Kähler theorem. The proper setting is the first jet bundle $J^1\pi$ of an $(n+1)$ manifold M fibred over \mathbb{R} . In [5] it is proved that a time-dependent semispray is Lagrangian if and only if there exists a semi-basic 1-form θ on $J^1\pi$, that satisfies a differential system. This gives rise to a linear partial differential operator P . We study the formal integrability of P using two sufficient conditions provided by Cartan–Kähler theorem. We prove that the symbol $\sigma^1(P)$ is involutive (Theorem 3) and hence there is only one obstruction for the formal integrability of the operator P , which is due to curvature tensor R (Theorem 4). Based on this result, we recover some of the classes of Lagrangian semisprays: flat semisprays, isotropic semisprays and arbitrary semisprays on 2-dimensional jet spaces ($n = 1$).

The motivation for this article is double-folded. So far all the results about the inverse problem of the calculus of variations were obtained separately, in the autonomous and nonautonomous settings. This is due to the different frameworks involved: the tangent bundle TM (a vector bundle) and respectively the first jet bundle $J^1\pi$ (an affine bundle). The geometric tools are usually constructed in different ways, and special attention was given to the time-depending situation. This paper follows the line of [7] but naturally the proofs of the main theorems have some particularities due to the different setting.

Secondly, there are similarities between the formulation of the Helmholtz conditions for sprays in the autonomous setting and respectively for semisprays in the nonautonomous one [5, 6]. This is natural because $J^1\pi$ can be embedded in \widetilde{TM} (the tangent bundle with the zero section removed). Due to this embedding one can associate to any regular Lagrangian on the velocity-phase space $J^1\pi$ a homogeneous degenerate Lagrangian on the extended phase space \widetilde{TM} , such that the action defined by a curve in the jet formalism coincides with the action defined by the corresponding curve in the extended formalism. There are correspondences between the main geometric objects associated to these Lagrangians: Poincaré 1- and 2-forms, energies, canonical semisprays-sprays [3, 8, 10, 24]. Therefore, due to this homogeneous formalism, it is natural to expect such kind of similarities between the results corresponding to homogeneous structures on \widetilde{TM} and nonhomogeneous one on $J^1\pi$.

The paper is organized as follows. In Section 2 we introduce the principal geometric tools induced by a time-dependent semispray on $J^1\pi$ and characterize Lagrangian vector fields with respect to semi-basic 1-forms. Section 3 is dedicated to the application of the Spencer theory to the study of formal integrability of the partial differential operator (PDO) $P = (d_J, d_h)$. The most important results are Theorems 3 and 4. Section 3.3 presents classes of semisprays for which the obstruction in Theorem 4 is automatically satisfied. For these classes, the PDO P is formally integrable, and hence these semisprays will be Lagrangian SODE.

2 Preliminaries

2.1 The first-order jet bundle $J^1\pi$

The appropriate geometric setting for the study of time-dependent SODE is the affine jet bundle $(J^1\pi, \pi_{10}, M)$ [39]. We consider an $(n+1)$ -dimensional, real, smooth manifold M , which is fibred over \mathbb{R} , $\pi : M \rightarrow \mathbb{R}$, and represents the space-time. The first jet bundle of π is denoted by $\pi_{10} : J^1\pi \rightarrow M$, $\pi_{10}(j_t^1\phi) = \phi(t)$, for ϕ a local section of π and $j_t^1\phi$ the first-order jet of ϕ at t . A local coordinate system $(t, x^i)_{i \in \{1, \dots, n\}}$ on M induces a local coordinate system on $J^1\pi$, denoted by (t, x^i, y^i) . Submersion π_{10} induces a *natural foliation* on $J^1\pi$ such that (t, x^i) are transverse coordinates for this foliation, while (y^i) are coordinates for the leaves of the foliation. Throughout the paper we consider Latin indices $i \in \{1, \dots, n\}$ and Greek indices $\alpha \in \{0, \dots, n\}$, using the notation $(x^\alpha) = (t = x^0, x^i)$.

In this article we use the Frölicher–Nijenhuis theory [21, 23, 25] of derivations of vector-valued differential forms on the first jet bundle $J^1\pi$. We adopt the following notations: $C^\infty(J^1\pi)$ for the ring of smooth functions on $J^1\pi$, $\mathfrak{X}(J^1\pi)$ for the C^∞ module of vector fields on $J^1\pi$ and $\Lambda^k(J^1\pi)$ for the C^∞ module of k -forms on $J^1\pi$. The C^∞ module of (r, s) -type vector fields on $J^1\pi$ is denoted by $\mathcal{T}_s^r(J^1\pi)$ and the tensor algebra on $J^1\pi$ is denoted by $\mathcal{T}(J^1\pi)$. The graded algebra of differential forms on $J^1\pi$ is written as $\Lambda(J^1\pi) = \bigoplus_{k \in \{1, \dots, 2n+1\}} \Lambda^k(J^1\pi)$. We denote by $S^k(J^1\pi)$ the space of symmetric $(0, k)$ tensors on $J^1\pi$ and by $\Psi(J^1\pi) = \bigoplus_{k \in \{1, \dots, 2n+1\}} \Psi^k(J^1\pi)$ the graded algebra of vector-valued differential forms on $J^1\pi$. Throughout the paper we assume that all objects are C^∞ -smooth where defined.

A parametrized curve on M is a section of π : $\gamma : \mathbb{R} \rightarrow M$, $\gamma(t) = (t, x^i(t))$. Its *first-order jet prolongation* $J^1\gamma : t \in \mathbb{R} \rightarrow J^1\gamma(t) = (t, x^i(t), dx^i/dt) \in J^1\pi$ is a section of the fibration $\pi_1 := \pi \circ \pi_{10} : J^1\pi \rightarrow \mathbb{R}$.

Let $VJ^1\pi$ be the *vertical subbundle* of $TJ^1\pi$, $VJ^1\pi = \{\xi \in TJ^1\pi, D\pi_{10}(\xi) = 0\} \subset TJ^1\pi$. The fibers $V_u J^1\pi = \text{Ker } D_u \pi_{10}$, $u \in J^1\pi$ determine a regular, n -dimensional, integrable vertical distribution. Remark that $V_u J^1\pi = \text{spann}\{\partial/\partial y^i\}$ and its annihilators are the *contact 1-forms* $\delta x^i = dx^i - y^i dt$, $i \in \{1, \dots, n\}$ and *basic 1-forms* λdt , $\lambda \in C^\infty(J^1\pi)$. The *vertical endomorphism* $J = \frac{\partial}{\partial y^i} \otimes \delta x^i$ is a vector-valued 1-form on $J^1\pi$, with $\text{Im } J = V(J^1\pi)$, $V(J^1\pi) \subset \text{Ker } J$ and $J^2 = 0$.

Its Frölicher–Nijenhuis tensor is given by

$$N_J = \frac{1}{2}[J, J] = -\frac{\partial}{\partial y^i} \otimes \delta x^i \wedge dt = -J \wedge dt.$$

Consequently, $d_J^2 = d_{N_J} = -d_{J \wedge dt} \neq 0$ and therefore d_J -exact forms on $J^1\pi$ may not be d_J -closed. Here d_J is the *exterior derivative* with respect to the vertical endomorphisms.

Remark 1. For $A \in \Psi^1(J^1\pi)$ a vector-valued 1-form, the exterior derivative with respect to A is a derivation of degree 1 given by $d_A = i_A \circ d - d \circ i_A$.

A k -form ω on $J^1\pi$, $k \geq 1$, is called *semi-basic* if it vanishes whenever one of the arguments is vertical.

A vector-valued k -form A on $J^1\pi$ is called semi-basic if it takes values in the vertical bundle and it vanishes whenever one of the arguments is vertical.

A semi-basic k -form satisfies the relation $i_J \theta = 0$ and locally can be expressed as $\theta = \theta_0 dt + \theta_i \delta x^i$. For example, contact 1-forms δx^i are semi-basic 1-forms.

If a vector-valued k -form A is semi-basic, then $J \circ A = 0$ and $i_J A = 0$. The vertical endomorphism J is a vector-valued, semi-basic 1-form.

Locally, a semi-basic k -form θ has the next form

$$\theta = \frac{1}{k!} \theta_{i_1 \dots i_k}(x^\alpha, y^j) \delta x^{i_1} \wedge \dots \wedge \delta x^{i_k} + \frac{1}{(k-1)!} \tilde{\theta}_{i_1 \dots i_{k-1}}(x^\alpha, y^j) \delta x^{i_1} \wedge \dots \wedge \delta x^{i_{k-1}} \wedge dt.$$

For simplicity, we denote by T^* the vector bundle of 1-forms on $J^1\pi$, by T_v^* the vector bundle of semi-basic 1-forms on $J^1\pi$ and by $\Lambda^k T_v^*$ the vector bundle of semi-basic k -forms on $J^1\pi$. We also denote by $\Lambda_v^k = \text{Sec}(\Lambda^k T_v^*)$ the $C^\infty(J^1\pi)$ -module of sections of $\Lambda^k T_v^*$ and by $S^k T^*$ the vector bundle of symmetric tensors of $(0, k)$ -type on $J^1\pi$. $S^1 T^*$ will be identified with T^* .

A *semispray* is a globally defined vector field S on $J^1\pi$ such that

$$J(S) = 0 \quad \text{and} \quad dt(S) = 1.$$

The integral curves of a semispray are first-order jet prolongations of sections of $\pi \circ \pi_{10} : J^1\pi \rightarrow \mathbb{R}$. Locally, a semispray has the form

$$S = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} - 2G^i(x^\alpha, y^j) \frac{\partial}{\partial y^i}, \tag{1}$$

where functions G^i , called the semispray coefficients, are locally defined on $J^1\pi$.

A parametrized curve $\gamma : I \rightarrow M$ is a *geodesic* of S if $S \circ J^1\gamma = \frac{d}{dt}(J^1\gamma)$.

In local coordinates, $\gamma(t) = (t, x^i(t))$ is a geodesic of the semispray S given by (1) if and only if it satisfies the system of SODE

$$\frac{d^2x^i}{dt^2} + 2G^i \left(t, x, \frac{dx}{dt} \right) = 0. \quad (2)$$

Therefore such a system of time-dependent SODE can be identified with a semispray on $J^1\pi$.

Canonical nonlinear connection. A *nonlinear connection* on $J^1\pi$ is an $(n+1)$ -dimensional distribution $H : u \in J^1\pi \mapsto H_u \subset T_u J^1\pi$, supplementary to $VJ^1\pi$: $\forall u \in J^1\pi, T_u J^1\pi = H_u \oplus V_u$.

A semispray S induces a nonlinear connection on $J^1\pi$, given by the *almost product structure* $\Gamma = -\mathcal{L}_S J + S \otimes dt$, $\Gamma^2 = \text{Id}$. The *horizontal projector* that corresponds to this almost product structure is $h = \frac{1}{2}(\text{Id} - \mathcal{L}_S J + S \otimes dt)$ and the *vertical projector* is $v = \text{Id} - h$.

The horizontal subspace is spanned by S and by $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_j^i \frac{\partial}{\partial y^j}$, where $N_j^i = \frac{\partial G^i}{\partial y^j}$. In this paper we prefer to work with the following adapted basis and cobasis:

$$\left\{ S, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\}, \quad \{dt, \delta x^i, \delta y^i\}, \quad (3)$$

with δx^i the *contact 1-forms* and $\delta y^i = dy^i + N_\alpha^i dx^\alpha$, $N_0^i = 2G^i - N_j^i y^j$. Functions N_j^i and N_0^i are the coefficients of the nonlinear connection induced by the semispray S .

With respect to basis and cobasis (3), the horizontal and vertical projectors are locally expressed as $h = S \otimes dt + \frac{\delta}{\delta x^i} \otimes \delta x^i$, $v = \frac{\partial}{\partial y^i} \otimes \delta y^i$. We consider the $(1,1)$ -type tensor field $\mathbb{F} = h \circ \mathcal{L}_S h - J$, which corresponds to the almost complex structure in the autonomous case. It satisfies $\mathbb{F}^3 + \mathbb{F} = 0$, which means that it is an $f(3,1)$ structure. It can be expressed locally as $\mathbb{F} = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^j} \otimes \delta x^i$.

Curvature. The following properties for the torsion and curvature of the nonlinear connection induced by the semispray are proved in [5].

The weak torsion tensor field of the nonlinear connection Γ vanishes: $[J, h] = 0$, which is equivalent also with $[J, \Gamma] = 0$.

The curvature tensor $R = N_h$ of the nonlinear connection Γ is a vector-valued semi-basic 2-form, locally given by

$$R = \frac{1}{2}[h, h] = \frac{1}{2}R_{ij}^k \frac{\partial}{\partial y^k} \otimes \delta x^i \wedge \delta x^j + R_i^j \frac{\partial}{\partial y^j} \otimes dt \wedge \delta x^i, \quad (4)$$

where

$$R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}$$

and

$$R_j^i = 2 \frac{\partial G^i}{\partial x^j} - \frac{\partial G^i}{\partial y^k} \frac{\partial G^k}{\partial y^j} - S \left(\frac{\partial G^i}{\partial y^j} \right). \quad (5)$$

The *Jacobi endomorphism* is defined as

$$\Phi = v \circ \mathcal{L}_S h = \mathcal{L}_S h - \mathbb{F} - J. \quad (6)$$

Jacobi endomorphism Φ is a semi-basic, vector-valued 1-form and satisfies $\Phi^2 = 0$. Locally, can be expressed as $\Phi = R_j^i \frac{\partial}{\partial y^j} \otimes \delta x^i$, where R_j^i are given by (5).

The Jacobi endomorphism and the curvature of the nonlinear connection are related by the following formulae:

$$\Phi = i_S R, \quad (7)$$

$$[J, \Phi] = 3R + \Phi \wedge dt. \quad (8)$$

Remark that $R = 0$ if and only if $\Phi = 0$.

Definition 1. A semispray S is called *isotropic* if its Jacobi endomorphism has the form

$$\Phi = \lambda J, \quad (9)$$

where $\lambda \in C^\infty(J^1\pi)$.

Next we express the isotropy condition (9) for a semispray in terms of the curvature tensor R .

Proposition 1. *A semispray S is isotropic if and only if its curvature tensor R has the form*

$$R = \alpha \wedge J,$$

where α is a semi-basic 1-form on $J^1\pi$.

Proof. Suppose that S is an isotropic SODE. Then there exists $\lambda \in C^\infty(J^1\pi)$ such that $\Phi = \lambda J$. From (8) it results

$$3R = [J, \lambda J] - \Phi \wedge dt,$$

$$[J, \lambda J] = (d_J \lambda) \wedge J - d\lambda \wedge J^2 + \lambda[J, J] \Rightarrow R = \frac{1}{3} (d_J \lambda) \wedge J - \lambda J \wedge dt = \alpha \wedge J,$$

with $\alpha = \frac{1}{3} d_J \lambda + \lambda dt \in T_v^*$.

In the above calculus we used the formula [23]

$$[K, gL] = (d_K g) \wedge L - dg \wedge KL + g[K, L],$$

for K, L vector-valued one-forms on $J^1\pi$ and $g \in C^\infty(J^1\pi)$.

For the converse, suppose that $R = \alpha \wedge J$, with $\alpha \in T_v^*$. Formula (7) implies $\Phi = i_S(\alpha \wedge J) = (i_S \alpha)J - \alpha \wedge i_S J = (i_S \alpha)J$. ■

2.2 Lagrangian semisprays

In this subsection we recall some basic notions about Lagrangian semisprays.

Definition 2.

- 1) A smooth function $L \in C^\infty(J^1\pi)$ is called a *Lagrangian function*.
- 2) The Lagrangian L is regular if the (0, 2) type tensor with local components

$$g_{ij}(x^\alpha, y^k) = \frac{\partial^2 L}{\partial y^i \partial y^j}$$

has rank n on $J^1\pi$. The tensor $g = g_{ij} \delta x^i \otimes \delta x^j$ is called the *metric tensor* of the Lagrangian L .

Remark 2. More exactly, [39], a function $L \in C^\infty(J^1\pi)$ is called a Lagrangian density on π . If Ω is a volume form on \mathbb{R} , the corresponding Lagrangian is the semi-basic 1-form $L\pi_1^* \Omega$ on $J^1\pi$. Using a fixed volume form on \mathbb{R} , for example dt , it is natural to consider the function L as a (first-order) Lagrangian.

For the particular choice of dt as volume form on \mathbb{R} , the *Poincaré–Cartan 1-form* of the Lagrangian L is $\theta_L := Ldt + d_J L$. The Lagrangian L is regular if and only if the *Poincaré–Cartan 2-form* $d\theta_L$ has maximal rank $2n$ on $J^1\pi$.

For a detailed exposition on the regularity conditions for Lagrangians see [27].

The *geodesics* of a semispray S , given by the system of SODE (2), coincide with the solutions of the *Euler–Lagrange equations*

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0$$

if and only if

$$g_{ij} \left(t, x, \frac{dx}{dt} \right) \left(\frac{d^2 x^i}{dt^2} + 2G^i \left(t, x, \frac{dx}{dt} \right) \right) = \frac{d}{dt} \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i}. \quad (10)$$

Therefore, for a semispray S , there exists a Lagrangian function L such that (10) holds true if and only if $S \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0$, which can be further expressed as

$$\mathcal{L}_S \theta_L = dL \Leftrightarrow i_S d\theta_L = 0. \quad (11)$$

Definition 3. A semispray S is called a *Lagrangian semispray* (or a Lagrangian vector field) if there exists a Lagrangian function L , locally defined on $J^1\pi$, that satisfies (11).

In [5] it has been shown that a semispray S is a Lagrangian semispray if and only if there exists a semi-basic 1-form $\theta \in \Lambda_v^1$ with $\text{rank}(d\theta) = 2n$ on $J^1\pi$, such that $\mathcal{L}_S \theta$ is closed. This represents a reformulation, in terms of semi basic 1-forms, of the result in terms of 2-forms obtained by Crampin et al. in [15]. The characterization of Lagrangian higher order semisprays in terms of a closed 2-form appears also in [2].

Based on this result we can obtain the following reformulation in terms of semi-basic 1-forms of the known Helmholtz conditions [5, Lemma 4.2, Lemma 4.3, Theorem 4.5, Theorem 5.1].

Theorem 1. *A semispray S is a Lagrangian vector field if and only if there exists a semi-basic 1-form $\theta \in \Lambda_v^1$, with $\text{rank}(d\theta) = 2n$ on $J^1\pi$, such that*

$$d_J \theta = 0, \quad d_h \theta = 0. \quad (12)$$

Proof. In order to make this paper self contained, we give a direct proof of this theorem.

Suppose that S is a Lagrangian semispray. It results that there exists a regular Lagrangian L on $J^1\pi$ with $\mathcal{L}_S \theta_L = dL$, or equivalently $i_S d\theta_L = 0$, where $\theta_L = Ldt + d_J L$ is its Poincaré 1-form. Evidently θ_L is a semi-basic 1-form with $\text{rank}(d\theta_L) = 2n$ on $J^1\pi$. We will prove that $d_J \theta_L = d_h \theta_L = 0$.

Indeed, $d_J \theta_L = d_J L \wedge dt + L d_J dt + d_J^2 L = i_J dL \wedge dt - d_{J \wedge dt} L = i_J dL \wedge dt - i_{J \wedge dt} dL = 0$.

From the formula $i_J \mathcal{L}_S - \mathcal{L}_S i_J = i_{\Gamma - S \otimes dt}$ and $i_J \mathcal{L}_S d\theta_L = 0$ we obtain $\mathcal{L}_S d_J \theta_L + i_{\Gamma} d\theta_L - i_{S \otimes dt} d\theta_L = 0$. We also compute $i_{\Gamma} d\theta_L = i_{2h - \text{Id}} d\theta_L = 2i_h d\theta_L - 2d\theta_L = 2d_h \theta_L$. Therefore $2d_h \theta_L = i_{S \otimes dt} d\theta_L = -i_S d\theta_L \wedge dt = 0$.

Conversely, suppose that there exists a semi-basic 1-form $\theta \in \Lambda_v^1$, with $\text{rank}(d\theta) = 2n$ on $J^1\pi$, such that $d_J \theta = 0$, $d_h \theta = 0$. In order to prove that S is a Lagrangian vector field, we will first show that $\mathcal{L}_S \theta = d(i_S \theta)$.

The hypothesis $d_J \theta = 0$ implies $\theta = (i_S \theta)dt + d_J(i_S \theta)$. Indeed, $d_J i_S + i_S d_J = \mathcal{L}_{JS} - i_{[S, J]} = i_{h - S \otimes dt - v} \Rightarrow d_J(i_S \theta) = i_h \theta - (i_S \theta)dt - i_v \theta = \theta - (i_S \theta)dt$.

Next, from $d_h i_S + i_S d_h = \mathcal{L}_{hS} - i_{[S, h]}$ and $d_h \theta = 0$ it results that $d_h i_S \theta = \mathcal{L}_S \theta - i_{\mathbb{F} + J + \Phi} \theta = \mathcal{L}_S \theta - i_{\mathbb{F}} \theta$.

From $i_{\mathbb{F}} \theta = i_{\mathbb{F}}((i_S \theta)dt + d_J(i_S \theta)) = i_{\mathbb{F}}(d_J(i_S \theta))$ and $i_{\mathbb{F}} d_J - d_J i_{\mathbb{F}} = d_{J \circ \mathbb{F}} - i_{[\mathbb{F}, J]} \Rightarrow i_{\mathbb{F}}(d_J(i_S \theta)) = d_v(i_S \theta)$. It results that $\mathcal{L}_S \theta = d_h(i_S \theta) + d_v(i_S \theta) = d(i_S \theta)$.

Consider $L = i_S \theta$. Then θ is the Poincaré–Cartan 1-form of L and $\mathcal{L}_S \theta_L = dL$. From $\text{rank}(d\theta) = 2n$ on $J^1\pi$ it results that L is a regular Lagrangian and S is a Lagrangian vector field. ■

In the next section we discuss the formal integrability of these Helmholtz conditions using two sufficient conditions provided by Cartan–Kähler theorem.

3 Formal integrability for the nonautonomus inverse problem of the calculus of variations

In order to study the integrability conditions of the set of differential equations (12), we associate to it a linear partial differential operator and study its formal integrability, using Spencer's technique. The approach in this work follows the one developed in [7] for studying the projective metrizable problem for autonomous sprays. For the basic notions of formal integrability theory of linear partial differential operators see [7, 23].

Consider T_v^* the vector bundle of semi-basic 1-forms on $J^1\pi$ and Λ_v^1 the module of sections of T_v^* . For $\theta \in \Lambda_v^1$ and $k \geq 1$ we denote by $j_u^k\theta$ the k th order jet of θ at the base point u in $J^1\pi$. The bundle of k th order jets of sections of T_v^* is denoted by $J^kT_v^*$. The projection $\pi_0 : J^kT_v^* \rightarrow J^1\pi$ is defined by $\pi_0(j_u^k\theta) = u$. If $l > k$, one defines the projections π_k as follows: $\pi_k(j_u^l\theta) = j_u^k\theta$ and $J^lT_v^*$ is also a fibred manifold over $J^kT_v^*$.

If $f_1, \dots, f_k \in C^\infty(J^1\pi)$ are functions vanishing at $u \in J^1\pi$ and $\theta \in \Lambda_v^1$, we define $\epsilon : \mathcal{S}^kT^* \otimes T_v^* \rightarrow J^kT_v^*$ by $\epsilon(df_1 \odot \dots \odot df_k \otimes \theta)_u = j_u^k(f_1 \cdots f_k \theta)$, where \odot is the symmetric product. Then the sequence

$$0 \rightarrow \mathcal{S}^kT^* \otimes T_v^* \xrightarrow{\epsilon} J^kT_v^* \xrightarrow{\pi_{k-1}} J^{k-1}T_v^* \rightarrow 0$$

is exact.

Consider the *linear partial differential operator* of order one

$$P : \Lambda_v^1 \rightarrow \Lambda_v^2 \oplus \Lambda_v^2, \quad P = (d_J, d_h). \quad (13)$$

Remark that $P(\theta)$ can be expressed in terms of first-order jets of θ , for any $\theta \in \Lambda_v^1$, and therefore it induces a morphism between vector bundles:

$$p^0(P) : J^1T_v^* \rightarrow \Lambda^2T_v^* \oplus \Lambda^2T_v^*, \quad p^0(P)(j_u^1\theta) = P(\theta)_u, \quad \forall \theta \in \Lambda_v^1.$$

We also consider the *lth order jet prolongations* of the differential operator P , $l \geq 1$, which will be identified with the morphisms of vector bundles over M ,

$$p^l(P) : J^{l+1}T_v^* \rightarrow J^l(\Lambda^2T_v^* \oplus \Lambda^2T_v^*), \quad p^l(P)(j_u^{l+1}\theta) = j_u^l(P(\theta)), \quad \forall \theta \in \Lambda_v^1.$$

Remark that for a semi-basic 1-form $\theta = \theta_\alpha \delta x^\alpha$, its first-order jet $j^1\theta = \frac{\delta\theta_\alpha}{\delta x^\beta} \delta x^\beta \otimes \delta x^\alpha + \frac{\partial\theta_\alpha}{\partial y^i} \delta y^i \otimes \delta x^\alpha$ determines the local coordinates $(x^\alpha, y^i, \theta_\alpha, \theta_{\alpha\beta}, \theta_{\alpha i})$ on $J^1T_v^*$. In this work all contravariant or covariant indices, related to vertical components of tensor fields will be underlined>.

Consider $\theta = \theta_\alpha \delta x^\alpha$, a semi-basic 1-form on $J^1\pi$. Then

$$\begin{aligned} d\theta &= \left(\frac{\partial\theta_i}{\partial t} - \theta_j N_i^j - \frac{\delta\theta_0}{\delta x^i} \right) \delta x^0 \wedge \delta x^i + \left(\theta_i - \frac{\partial\theta_0}{\partial y^i} \right) \delta x^0 \wedge \delta y^i \\ &\quad + \frac{1}{2} \left(\frac{\delta\theta_j}{\delta x^i} - \frac{\delta\theta_i}{\delta x^j} \right) \delta x^i \wedge \delta x^j + \left(\frac{\partial\theta_j}{\partial y^i} \right) \delta y^i \wedge \delta x^j, \\ d_J\theta &= \left(\theta_i - \frac{\partial\theta_0}{\partial y^i} \right) \delta x^0 \wedge \delta y^i + \frac{1}{2} \left(\frac{\partial\theta_i}{\partial y^j} - \frac{\partial\theta_j}{\partial y^i} \right) \delta x^j \wedge \delta x^i, \\ d_h\theta &= \left(\frac{\partial\theta_i}{\partial t} - \theta_j N_i^j - \frac{\delta\theta_0}{\delta x^i} \right) \delta x^0 \wedge \delta x^i + \frac{1}{2} \left(\frac{\delta\theta_i}{\delta x^j} - \frac{\delta\theta_j}{\delta x^i} \right) \delta x^j \wedge \delta x^i. \end{aligned}$$

Using these formulae we obtain

$$\begin{aligned} p^0(P)(j^1\theta) &= \left(\left(\theta_i - \frac{\partial\theta_0}{\partial y^i} \right) \delta x^0 \wedge \delta y^i + \frac{1}{2} \left(\frac{\partial\theta_i}{\partial y^j} - \frac{\partial\theta_j}{\partial y^i} \right) \delta x^j \wedge \delta x^i, \right. \\ &\quad \left. \left(\frac{\partial\theta_i}{\partial t} - \theta_j N_i^j - \frac{\delta\theta_0}{\delta x^i} \right) \delta x^0 \wedge \delta x^i + \frac{1}{2} \left(\frac{\delta\theta_i}{\delta x^j} - \frac{\delta\theta_j}{\delta x^i} \right) \delta x^j \wedge \delta x^i \right). \end{aligned}$$

The *symbol* of P is the vector bundle morphism $\sigma^1(P) : T^* \otimes T_v^* \rightarrow \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*$ defined by the first-order terms of $p^0(P)$. More exactly, $\sigma^1(P) = p^0(P) \circ \epsilon$.

For $A \in T^* \otimes T_v^*$, $A = A_{\alpha\beta} \delta x^\alpha \otimes \delta x^\beta + A_{i\beta} \delta y^i \otimes \delta x^\beta$, we compute

$$\begin{aligned}\sigma^1(d_J)A &= -A_{i0} \delta x^0 \wedge \delta x^i + \frac{1}{2} (A_{ji} - A_{ij}) \delta x^j \wedge \delta x^i, \\ \sigma^1(d_h)A &= (A_{0i} - A_{i0}) \delta x^0 \wedge \delta x^i + \frac{1}{2} (A_{ji} - A_{ij}) \delta x^j \wedge \delta x^i\end{aligned}$$

and hence

$$\begin{aligned}\sigma^1(P)A &= (\tau_J A, \tau_h A), \quad (\tau_J A)(X, Y) = A(JX, Y) - A(JY, X), \\ (\tau_h A)(X, Y) &= A(hX, Y) - A(hY, X),\end{aligned}$$

for $X, Y \in \mathfrak{X}(J^1\pi)$. In the above formulae τ_J, τ_h are *alternating operators* [7].

Remark 3. The alternating operators are defined in general as follows. For $K \in \Psi^k(J^1\pi)$, a vector-valued k -form, we consider $\tau_K : \Psi^1(J^1\pi) \otimes \Psi^l(J^1\pi) \rightarrow \Psi^{l+k}(J^1\pi)$,

$$(\tau_K B)(X_1, \dots, X_{l+k}) = \frac{1}{l!k!} \sum_{\sigma \in S_{l+k}} \varepsilon(\sigma) B(K(X_{\sigma(1)}, \dots, X_{\sigma(k)}), X_{\sigma(k+1)}, \dots, X_{\sigma(l+k)}), \quad (14)$$

where $X_1, \dots, X_{l+k} \in \mathfrak{X}(J^1\pi)$ and S_{l+k} is the permutation group of $\{1, \dots, l+k\}$. The restriction of τ_K to $\Psi^{l+1}(J^1\pi)$, is a derivation of degree $(k-1)$ and it coincides with the *inner product* i_K .

The *first-order prolongation of the symbol* of P is the vector bundle morphism $\sigma^2(P) : S^2 T^* \otimes T_v^* \rightarrow T^* \otimes (\Lambda^2 T_v^* \oplus \Lambda^2 T_v^*)$ that verifies

$$i_X(\sigma^2(P)B) = \sigma^1(P)(i_X B), \quad \forall B \in S^2 T^* \otimes T_v^*, \quad \forall X \in \mathfrak{X}(J^1\pi).$$

Therefore

$$\begin{aligned}\sigma^2(P)B &= (\sigma^2(d_J)B, \sigma^2(d_h)B), \\ (\sigma^2(d_J)B)(X, Y, Z) &= B(X, JY, Z) - B(X, JZ, Y), \\ (\sigma^2(d_h)B)(X, Y, Z) &= B(X, hY, Z) - B(X, hZ, Y).\end{aligned}$$

In local coordinates we obtain the following formulae.

If $B \in S^2 T^* \otimes T_v^*$, then it has the local decomposition

$$\begin{aligned}B &= B_{\alpha\beta\gamma} \delta x^\alpha \otimes \delta x^\beta \otimes \delta x^\gamma + B_{i\alpha\beta} \delta y^i \otimes \delta x^\alpha \otimes \delta x^\beta \\ &\quad + B_{\alpha i\beta} \delta x^\alpha \otimes \delta y^i \otimes \delta x^\beta + B_{ij\alpha} \delta y^i \otimes \delta y^j \otimes \delta x^\alpha,\end{aligned} \quad (15)$$

with

$$B_{\alpha\beta\gamma} = B_{\beta\alpha\gamma}, \quad B_{ij\alpha} = B_{ji\alpha}, \quad B_{i0\alpha} = B_{0i\alpha}, \quad B_{ij\alpha} = B_{ji\alpha}. \quad (16)$$

The first-order prolongation of the symbol of P is given by

$$\begin{aligned}\sigma^2(d_J)B &= B_{\alpha i 0} \delta x^\alpha \otimes \delta x^i \wedge \delta x^0 + B_{ij 0} \delta y^i \otimes \delta x^j \wedge \delta x^0 \\ &\quad + \frac{1}{2} (B_{\alpha ij} - B_{\alpha ji}) \delta x^\alpha \otimes \delta x^i \wedge \delta x^j + \frac{1}{2} (B_{ij k} - B_{ik j}) \delta y^i \otimes \delta x^j \wedge \delta x^k, \\ \sigma^2(d_h)B &= \frac{1}{2} (B_{\alpha\beta\gamma} - B_{\alpha\gamma\beta}) \delta x^\alpha \otimes \delta x^\beta \wedge \delta x^\gamma + \frac{1}{2} (B_{i\alpha\beta} - B_{i\beta\alpha}) \delta y^i \otimes \delta x^\alpha \wedge \delta x^\beta.\end{aligned}$$

For each $u \in J^1\pi$, we consider

$$\begin{aligned} g_u^k(P) &= \text{Ker } \sigma_u^k(P), \quad k \in \{1, 2\}, \\ g_u^1(P)_{e_1 \dots e_j} &= \{A \in g_u^1(P) \mid i_{e_1} A = \dots = i_{e_j} A = 0\}, \quad j \in \{1, \dots, n\}, \end{aligned}$$

where $\{e_1, \dots, e_n\}$ is a basis of $T_u(J^1\pi)$. Such a basis is called *quasi-regular* if it satisfies

$$\dim g_u^2(P) = \dim g_u^1(P) + \sum_{j=1}^n \dim g_u^1(P)_{e_1 \dots e_j}.$$

Definition 4. The symbol $\sigma^1(P)$ is called *involutive* at u in $J^1\pi$ if there exists a quasi-regular basis of $T_u J^1\pi$.

A first-order jet $j_u^1\theta \in J^1T_v^*$ is a *first-order formal solution* of P at u in $J^1\pi$ if $p^0(P)(\theta)_u = 0$.

For $l \geq 1$, a $(1+l)$ th order jet $j_u^{1+l}\theta \in J_u^{1+l}T_v^*$ is a $(1+l)$ th order formal solution of P at u in $J^1\pi$ if $p^l(P)(\theta)_u = 0$.

For any $l \geq 0$, consider $R_u^{1+l}(P) = \ker p_u^l(P)$ the space of $(1+l)$ th order formal solutions of P at u . We denote also $\bar{\pi}_{l,u} : R_u^{1+l}(P) \rightarrow R_u^l(P)$ the restriction of $\pi_{l,u} : J_u^{1+l}(T_v^*) \rightarrow J_u^l(T_v^*)$ to $R_u^{1+l}(P)$.

Definition 5. The partial differential operator P is called *formally integrable* at u in $J^1\pi$ if $R^{1+l}(P) = \bigcup_{u \in J^1\pi} R_u^{1+l}(P)$ is a vector bundle over $J^1\pi$, for all $l \geq 0$, and the map $\bar{\pi}_{l,u} : R_u^{1+l}(P) \rightarrow R_u^l(P)$ is onto for all $l \geq 1$.

The fibred submanifold $R^1(P)$ of $\pi_0 : J_u^1(T_v^*) \rightarrow J^1\pi$ is called *the partial differential equation corresponding to the first-order PDO P* . A solution of the operator P on an open set $U \subset J^1\pi$ is a section $\theta \in \Lambda_v^1$ defined on U such that $P\theta = 0 \Leftrightarrow p^0(P)(j_u^1\theta) = 0, \forall u \in U$.

The Cartan–Kähler theorem [23] takes the following form for the particular case of first-order PDO.

Theorem 2. *Let P be a first-order linear partial differential operator with $g^2(P)$ a vector bundle over $R^1(P)$. If $\bar{\pi}_1 : R^2(P) \rightarrow R^1(P)$ is onto and the symbol $\sigma^1(P)$ is involutive, then P is formally integrable.*

3.1 The involutivity of the symbol of P

In this subsection we prove that the operator P satisfies one of the two sufficient conditions for formal integrability, provided by Cartan–Kähler theorem: the involutivity of the symbol $\sigma^1(P)$.

Theorem 3. *The symbol $\sigma^1(P)$ of the PDO $P = (d_J, d_h)$ is involutive.*

Proof. First we determine $g^1(P) = \{A \in T^* \otimes T_v^* \mid \sigma^1(P)A = 0\}$, and compute the dimension of its fibers. We obtain

$$g_u^1(P) = \{A = A_{\alpha\beta}\delta x^\alpha \otimes \delta x^\beta + A_{i\beta}\delta y^i \otimes \delta x^\beta \mid A_{i0} = 0, A_{ji} = A_{ij}, A_{\alpha\beta} = A_{\beta\alpha}\}.$$

From $A_{i0} = 0$ and $A_{ji} = A_{ij}$ it results that A_{ij} contribute with $n(n+1)/2$ components to the dimension of $g_u^1(P)$, and from $A_{\alpha\beta} = A_{\beta\alpha}$ it follows that $A_{\alpha\beta}$ contribute with $(n+1)(n+2)/2$ components to the dimension of $g_u^1(P)$. So

$$\dim g_u^1(P) = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} = (n+1)^2.$$

Next we determine $g^2(P) = \{B \in S^2T^* \otimes T_v^* \mid \sigma^2(P)B = 0\}$.

If $B \in S^2T^* \otimes T_v^*$ has the local components (15), then $B \in g^2(P)$ if and only if the following relations are satisfied:

$$\begin{aligned} B_{\alpha i 0} &= 0, & B_{\underline{i}j0} &= 0, & B_{\alpha ij} &= B_{\alpha ji}, \\ B_{\underline{i}jk} &= B_{\underline{i}kj}, & B_{\alpha\beta\gamma} &= B_{\alpha\gamma\beta}, & B_{i\alpha\beta} &= B_{i\beta\alpha}. \end{aligned} \quad (17)$$

From the relations (16) and (17) it results that $B \in g^2(P)$ if and only if its local components $B_{\alpha\beta\gamma}$, $B_{\underline{i}jk}$, $B_{\underline{i}kj}$, $B_{\underline{i}jk}$ are totally symmetric and the rest are vanishing. Therefore $B_{\alpha\beta\gamma}$ contribute with $(n+1)(n+2)(n+3)/6$ components to the dimension of $g_u^2(P)$, and $B_{\underline{i}jk}$, $B_{\underline{i}kj}$ with $n(n+1)(n+2)/6$ components each of them. It results

$$\dim g_u^2(P) = \frac{(n+1)(n+2)(n+3)}{6} + 2 \frac{n(n+1)(n+2)}{6} = \frac{(n+1)^2(n+2)}{2}.$$

Consider

$$\mathcal{B} = \{h_0 = S, h_1, \dots, h_n, v_1 = Jh_1, \dots, v_n = Jh_n\}$$

a basis in $T_u J^1\pi$ with $h_0 = S, h_1, \dots, h_n$ horizontal vector fields. For any $A \in g^1(P)$, we denote

$$A(h_\alpha, h_\beta) = a_{\alpha\beta}, \quad A(v_i, h_\alpha) = b_{\underline{i}\alpha}.$$

Because A is semi-basic in the second argument it follows that these are the only components of A .

Since $A \in \text{Ker } \sigma^1(d_J)$ it follows that $A_{i0} = 0$, $A_{ij} = A_{ji} \Rightarrow b_{i0} = 0$, $b_{ij} = b_{\underline{j}i}$. Because $A \in \text{Ker } \sigma^1(d_h)$ it results that $A_{0i} = A_{i0}$, $A_{ij} = A_{ji}$ and hence $a_{0i} = a_{i0}$, $a_{ij} = a_{ji}$.

Consider $j \in \{1, \dots, n\}$ arbitrarily fixed and

$$\begin{aligned} \tilde{\mathcal{B}} &= \{e_0 = S + h_j + v_n, e_1 = h_1, e_2 = h_2 + v_1, \dots, \\ &e_i = h_i + v_{i-1}, \dots, e_n = h_n + v_{n-1}, v_1, \dots, v_n\} \end{aligned}$$

a new basis in $T_u J^1\pi$. If we denote

$$A(e_\alpha, e_\beta) = \tilde{a}_{\alpha\beta}, \quad A(v_i, e_\alpha) = \tilde{b}_{i\alpha},$$

a simple computation and the fact that A is semi-basic in the second argument determine

$$\begin{aligned} \tilde{a}_{00} &= a_{00} + 2a_{0j} + a_{jj} + b_{\underline{n}j}, \\ \tilde{a}_{ik} &= a_{ik} + b_{\underline{i-1},k} \neq \tilde{a}_{ki} = a_{ki} + b_{\underline{k-1},i}, \\ \tilde{a}_{i0} &= a_{i0} + a_{ij} + b_{\underline{i-1},j} \neq \tilde{a}_{0i} = a_{0i} + a_{ji} + b_{\underline{n}i}, \\ \tilde{b}_{ik} &= b_{ik} = \tilde{b}_{ki}, \quad \tilde{b}_{i0} = b_{ij}. \end{aligned}$$

It can be seen that all the independent components of A in the basis \mathcal{B} can be obtained from the components of A in the basis $\tilde{\mathcal{B}}$, and hence we can use the later for determining the dimensions of $(g_u^1)_{e_0 e_1 \dots e_k}$.

If $A \in (g_u^1)_{e_0}$ it results $\tilde{a}_{0\alpha} = 0$, so using this new basis we impose $n+1$ supplementary independent restrictions. It follows that $\dim(g_u^1)_{e_0} = (n+1)^2 - (n+1) = (n+1)n$.

If $A \in (g_u^1)_{e_0 e_1}$ it results that together with the previous restrictions we impose also $\tilde{a}_{1\alpha} = 0$, so another independent $n+1$ restrictions. Hence $\dim(g_u^1)_{e_0 e_1} = (n+1)n - (n+1) = (n+1)(n-1)$.

In general $\dim(g_u^1)_{e_0 e_1 \dots e_k} = (n+1)(n-k)$, $\forall k \in \{1, \dots, n\}$. Hence $\dim(g_u^1)_{e_0 e_1 \dots e_n} = 0 \Rightarrow \dim(g_u^1)_{e_0 \dots e_n v_1 \dots v_k} = 0$, $\forall k \in \{1, \dots, n\}$,

$$\begin{aligned} \dim(g_u^1) + \sum_{k=0}^n \dim(g_u^1)_{e_0 e_1 \dots e_k} &= (n+1)^2 + (n+1)[n + n-1 + \dots + 1] \\ &= \frac{(n+1)^2(n+2)}{2} = \dim g_u^2(P). \end{aligned}$$

We proved that $\tilde{\mathcal{B}}$ is a quasi-regular basis, hence the symbol $\sigma^1(P)$ is involutive. ■

3.2 First obstruction to the inverse problem

In this subsection we determine necessary and sufficient conditions for $\bar{\pi}_1$ to be onto. We will obtain only one obstruction for the integrability of the operator P . The obstruction is due to the curvature tensor of the nonlinear connection induced by the semispray.

Theorem 4. *A first-order formal solution $\theta \in \Lambda_v^1$ of the system $d_J\theta = 0$, $d_h\theta = 0$ can be lifted into a second-order solution, which means that $\bar{\pi}_1 : R^2(P) \rightarrow R^1(P)$ is onto, if and only if*

$$d_R\theta = 0,$$

where R is the curvature tensor (4).

Proof. We use a known result from [23, Proposition 1.1].

If \mathcal{K} is the cokernel of $\sigma^2(P)$,

$$\mathcal{K} = \frac{T^* \otimes (\Lambda^2 T_v^* \oplus \Lambda^2 T_v^*)}{\text{Im} \sigma^2(P)},$$

there exists a morphism $\varphi : R^1(P) \rightarrow \mathcal{K}$ such that the sequence

$$R^2(P) \xrightarrow{\bar{\pi}_1} R^1(P) \xrightarrow{\varphi} \mathcal{K}$$

is exact. In particular $\bar{\pi}_1$ is onto if and only if $\varphi = 0$.

After defining φ , we will prove that for $\theta \in \Lambda_v^1$, with $j_u^1\theta \in R_u^1(P)$, a first-order formal solution of P at $u \in J^1\pi$, we have that $\varphi_u\theta = 0$ if and only if $(d_R\theta)_u = 0$.

The construction of the morphism φ is represented in the next diagram by dashed arrows. We denote $F = \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*$.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & g^2(P) & \longrightarrow & S^2 T^* \otimes T_v^* \xrightarrow{\sigma^2(P)} & T^* \otimes F & \xrightarrow{\tau} K \longrightarrow 0 \\
 & & \downarrow & & \downarrow \varepsilon & \uparrow \varepsilon & \downarrow \varepsilon \\
 0 & \longrightarrow & R^2(P) & \xrightarrow{i} & J^2 T_v^* & \xrightarrow{p^0(\nabla)} J^1 F & \\
 & & \downarrow \bar{\pi}_1 & & \uparrow \pi_1 & \downarrow \pi_0 & \\
 0 & \longrightarrow & R^1(P) & \xrightarrow{i} J^1 T_v^* & \xrightarrow{p^0(P)} & F & \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
 \end{array}$$

Remark that $\dim T^* = 2n + 1$, $\dim T_v^* = n + 1$, $\dim \Lambda^2 T_v^* = \frac{(n+1)n}{2}$, $\dim S^2 T^* = \frac{(2n+1)(2n+2)}{2}$, $\dim T^* \otimes (\Lambda^2 T_v^* \oplus \Lambda^2 T_v^*) = (2n + 1)n(n + 1)$. Therefore

$$\begin{aligned}
 \dim \mathcal{K} &= \dim [T^* \otimes (\Lambda^2 T_v^* \oplus \Lambda^2 T_v^*)] - \dim (\text{Im} \sigma^2(P)) \\
 &= \dim [T^* \otimes (\Lambda^2 T_v^* \oplus \Lambda^2 T_v^*)] - [\dim (S^2 T^* \otimes T_v^*) - \dim (\ker \sigma^2(P))] \\
 &= \frac{(n-1)n(n+1)}{2} = 3 \binom{n+1}{3}.
 \end{aligned}$$

It results from this that

$$\mathcal{K} \simeq \oplus^{(3)} \Lambda^3 T_v^*.$$

Next we define $\tau : T^* \otimes (\Lambda^2 T_v^* \oplus \Lambda^2 T_v^*) \rightarrow \oplus^{(3)} \Lambda^3 T_v^*$ such as the next sequence is exact:

$$0 \rightarrow g^2(P) \xrightarrow{i} \mathcal{S}^2 T^* \otimes T_v^* \xrightarrow{\sigma^2(P)} T^* \otimes (\Lambda^2 T_v^* \oplus \Lambda^2 T_v^*) \xrightarrow{\tau} \oplus^{(3)} \Lambda^3 T_v^* \rightarrow 0. \quad (18)$$

For $B_1, B_2 \in T^* \otimes \Lambda^2 T_v^*$, we define $\tau(B_1, B_2) = (\tau_1(B_1, B_2), \tau_2(B_1, B_2), \tau_3(B_1, B_2))$, where $\tau_i : T^* \otimes (\Lambda^2 T_v^* \oplus \Lambda^2 T_v^*) \rightarrow \Lambda^3 T_v^*$, $i \in \{1, 2, 3\}$, are given by

$$\tau_1(B_1, B_2) = \tau_J B_1, \quad \tau_2(B_1, B_2) = \tau_h B_2, \quad \tau_3(B_1, B_2) = \tau_h B_1 + \tau_J B_2.$$

Using the definition (14) of the alternating operators τ_J, τ_h , we prove that $\tau \circ \sigma^2(P) = 0$.

Indeed, using that any $B \in \mathcal{S}^2 T^* \otimes T_v^*$ is symmetric in the first two arguments, it follows that $(\tau \circ \sigma^2(P))(B) = \tau(\sigma^2(d_J)B, \sigma^2(d_h)B) = (\tau_J \sigma^2(d_J)B, \tau_h \sigma^2(d_h)B, \tau_h \sigma^2(d_J)B + \tau_J \sigma^2(d_h)B) = 0, \forall B \in \mathcal{S}^2 T^* \otimes T_v^*$. For example,

$$\begin{aligned} \tau_J(\sigma^2(d_J)B)(X, Y, Z) &= [\sigma^2(d_J)B(JX, Y, Z) - \sigma^2(d_J)B(JY, X, Z) + \sigma^2(d_J)B(JZ, X, Y)] \\ &= [B(JX, JY, Z) - B(JX, JZ, Y) - B(JY, JX, Z) + B(JY, JZ, X) \\ &\quad + B(JZ, JX, Y) - B(JZ, JY, X)] = 0, \quad \forall X, Y, Z \in \mathfrak{X}(J^1\pi). \end{aligned}$$

The relation $\tau \circ \sigma^2(P) = 0$ implies that $\text{Im}(\sigma^2(P)) \subseteq \text{Ker } \tau$. Using that τ is onto (τ_J, τ_h are both onto) it results that $\dim[\text{Im}(\sigma^2(P))] = \dim(\text{Ker } \tau)$ and hence $\text{Im}(\sigma^2(P)) = \text{Ker } \tau$ and the sequence (18) is exact.

The last step before defining $\varphi : R^1(P) \rightarrow \mathcal{K}$ is to consider a linear connection ∇ on $J^1\pi$ such that $\nabla J = 0$. It means that ∇ preserve semi-basic forms and ∇ can be considered as a connection in the fiber bundle $\Lambda^2 T_v^* \oplus \Lambda^2 T_v^* \rightarrow J^1\pi$. As a first-order PDO we can identify ∇ with the bundle morphism $p^0(\nabla) : J^1(\Lambda^2 T_v^* \oplus \Lambda^2 T_v^*) \rightarrow T^* \otimes (\Lambda^2 T_v^* \oplus \Lambda^2 T_v^*)$.

We will also use two derivations of degree 1 introduced in [7], defined by $\mathcal{D}_J = \tau_J \nabla, \mathcal{D}_h = \tau_h \nabla$. Both derivations $\mathcal{D}_J, \mathcal{D}_h$ preserve semi-basic forms and $d_J - \mathcal{D}_J, d_h - \mathcal{D}_h$ are algebraic derivations. It means that if $\omega \in \Lambda^k(J^1\pi)$ vanishes at some point $u \in J^1\pi$, then $(\mathcal{D}_J \omega)_u = (d_J \omega)_u$ and $(\mathcal{D}_h \omega)_u = (d_h \omega)_u$ [7, Lemma 2.1].

Now we are able to define $\varphi : R^1(P) \rightarrow \mathcal{K}$ such that the sequence

$$R^2(P) \xrightarrow{\bar{\pi}_1} R^1(P) \xrightarrow{\varphi} \mathcal{K}$$

is exact.

Let $\theta \in \Lambda_v^1$ such that $j_u^1 \theta \in R_u^1(P) \subset J_u^1 T_v^*$ is a first-order formal solution of P at $u \in J^1\pi$, which means that $(d_J \theta)_u = (d_h \theta)_u = 0$.

Consider

$$\varphi_u \theta = \tau_u \nabla P \theta = \tau_u (\nabla d_J \theta, \nabla d_h \theta).$$

Using the fact that $d_J - \tau_J \nabla$ and $d_h - \tau_h \nabla$ are algebraic derivations and $(d_J \theta)_u = (d_h \theta)_u = 0$ it results $d_J(d_J \theta)_u = \tau_J \nabla(d_J \theta)_u$ and $d_h(d_h \theta)_u = \tau_h \nabla(d_h \theta)_u$.

We will compute the three components of the map φ . It follows that

$$\begin{aligned} \tau_1(\nabla P \theta)_u &= \tau_1(\nabla d_J \theta, \nabla d_h \theta)_u = \tau_J(\nabla d_J \theta)_u \\ &= (d_J^2 \theta)_u = \frac{1}{2} (d_{[J, J]} \theta)_u = - (d_{J \wedge dt} \theta)_u = - (d_J \theta)_u \wedge dt = 0, \\ \tau_2(\nabla P \theta)_u &= \tau_2(\nabla d_J \theta, \nabla d_h \theta)_u = \tau_h(\nabla d_h \theta)_u = (d_h^2 \theta)_u = \frac{1}{2} (d_{[h, h]} \theta)_u = (d_R \theta)_u, \end{aligned}$$

where R is given by (4),

$$\begin{aligned}\tau_3(\nabla P\theta)_u &= \tau_3(\nabla d_J\theta, \nabla d_h\theta)_u = \tau_h(\nabla d_J\theta)_u + \tau_J(\nabla d_h\theta)_u \\ &= (d_h d_J\theta)_u + (d_J d_h\theta)_u = (d_{[h,J]}\theta)_u = 0.\end{aligned}$$

Hence $\varphi = 0$ if and only if $d_R\theta = 0$. ■

Remark 4. Locally, $d_R\theta$ has the following form:

$$\begin{aligned}R &= \underbrace{\frac{1}{2}R_{ij}^k \frac{\partial}{\partial y^k} \otimes \delta x^i \wedge \delta x^j - \Phi \wedge dt}_{\tilde{R}=\frac{1}{3}[J,\Phi]} \Rightarrow d_R\theta = d_{\tilde{R}}\theta - d_\Phi\theta \wedge dt, \\ d_\Phi\theta &= R_i^j \left(\theta_j - \frac{\partial\theta_0}{\partial y^j} \right) dt \wedge \delta x^i + \frac{1}{2!} \left(\frac{\partial\theta_j}{\partial y^k} R_i^k - \frac{\partial\theta_i}{\partial y^k} R_j^k \right) \delta x^j \wedge \delta x^i \Rightarrow \\ d_R\theta &= \frac{1}{3!} \left(a_{il} R_{jk}^l + a_{jl} R_{ki}^l + a_{kl} R_{ij}^l \right) \delta x^i \wedge \delta x^j \wedge \delta x^k + \frac{1}{2!} \left(a_{jk} R_i^k - a_{ik} R_j^k \right) dt \wedge \delta x^i \wedge \delta x^j,\end{aligned}$$

where we denoted $a_{ij} = \frac{\partial\theta_i}{\partial y^j}$.

Hence $d_R\theta = 0$ if and only if $a_{il} R_{jk}^l + a_{jl} R_{ki}^l + a_{kl} R_{ij}^l = 0$ and $a_{jk} R_i^k - a_{ik} R_j^k = 0$. The first identity represents the algebraic Bianchi identity for the curvatures of the nonlinear connection. The second identity is one of the classical Helmholtz condition for the multiplier matrix a_{ij} . These obstructions appear also in [2].

It can be seen that for $n = 2$ the formula of $d_R\theta$ becomes

$$d_R\theta = \frac{1}{2!} (a_{jk} R_i^k - a_{ik} R_j^k) dt \wedge \delta x^i \wedge \delta x^j = -d_\Phi\theta \wedge dt.$$

Therefore, for $n = 2$, the obstruction is equivalent with $d_\Phi\theta \wedge dt = 0$.

3.3 Classes of Lagrangian time-dependent SODE

We present now some classes of semisprays for which the obstruction in Theorem 4 is automatically satisfied. Therefore the PDO P is formally integrable, and hence these semisprays will be Lagrangians SODEs. These classes of semisprays are:

- flat semisprays, $R = 0 \Leftrightarrow \Phi = 0$;
- arbitrary semisprays on 2-dimensional manifolds;
- isotropic semisprays, $\Phi = \lambda J$, for λ a smooth function on $J^1\pi$.

All these classes of semisprays were already studied in the articles cited in the introduction.

In the flat case, the obstruction is automatically satisfied.

If $\dim M = 1$ then for a semi-basic 1-form θ on $J^1\pi$, $d_R\theta$ is a semi-basic 3-form on $J^1\pi$. Because $\dim \Lambda^3(T_v^*) = (n+1)n(n-1)/6$ and it is zero if $n = 1$, $d_R\theta$ will necessarily vanish.

We consider now the last case, of isotropic semisprays.

Proposition 2. *Any isotropic semispray is a Lagrangian second-order vector field.*

Proof. Assume now that S is an isotropic SODE and θ a semi-basic 1-form on $J^1\pi$ such that $(d_J\theta)_u = (d_h\theta)_u = 0$, for some $u \in J^1\pi$,

$$(d_R\theta)_u = d_{\alpha \wedge J}\theta = \alpha \wedge d_J\theta + (-1)^2 d\alpha \wedge i_J\theta = 0.$$

We used that $i_J\theta = 0$, $(d_J\theta)_u = 0$ and the formula [23]

$$d_{\omega \wedge K}\pi = \omega \wedge d_K\pi + (-1)^{q+k} d\omega \wedge i_K\pi,$$

for ω a q -form on $J^1\pi$ and K a vector-valued k -form on $J^1\pi$.

Since $(d_R\theta)_u$ vanishes, S is a Lagrangian semispray. ■

Next we give some simple examples of Lagrangian semisprays, corresponding to the above general classes.

We start with the semispray expressed by the SODE

$$\frac{d^2x^1}{dt^2} + f\left(t, \frac{dx^2}{dt}\right) = 0, \quad \frac{d^2x^2}{dt^2} + g(t) = 0,$$

with f an arbitrary smooth function depending only on t and $y^2 = \frac{dx^2}{dt}$, and g an arbitrary smooth function depending only on t . The only possible nonvanishing local component of the Jacobi endomorphism is

$$R_2^1 = -S(N_2^1) = -\frac{1}{2} \frac{\partial^2 f}{\partial t \partial y^2} + g(t) \frac{\partial^2 f}{\partial (y^2)^2}.$$

Hence, if $\frac{\partial^2 f}{\partial t \partial y^2} = 2g(t) \frac{\partial^2 f}{\partial (y^2)^2}$ the semispray is flat ($\Phi = 0$). This example is a generalization of the one given by Douglas [20, (8.14)].

If $\frac{\partial^2 f}{\partial t \partial y^2} \neq 2g(t) \frac{\partial^2 f}{\partial (y^2)^2}$, then the semispray is isotropic

$$\Phi = \begin{pmatrix} 0 & -\frac{1}{2} \frac{\partial^2 f}{\partial t \partial y^2} + g(t) \frac{\partial^2 f}{\partial (y^2)^2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \frac{\partial^2 f}{\partial t \partial y^2} + g(t) \frac{\partial^2 f}{\partial (y^2)^2} \\ 0 \end{pmatrix} J.$$

Another example of isotropic (or flat) semispray is the one given by the SODE

$$\frac{d^2x^1}{dt^2} + f(t, x^2) = 0, \quad \frac{d^2x^2}{dt^2} + g(t) = 0,$$

with f an arbitrary smooth function depending only on t and x^2 , and g an arbitrary smooth function depending only on t . All the local coefficients of the associated nonlinear connection are vanishing. Evidently

$$\Phi = \begin{pmatrix} 0 & \frac{\partial f}{\partial x^2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x^2} \\ 0 \end{pmatrix} J.$$

This example was treated in [2, (6.1)] and [20, (15.4)] for $f(x^2) = -x^2$ and $g = 0$. The first paper also presents all the Lagrangians corresponding to the given SODE.

Consider also the semispray given by the SODE

$$\frac{d^2x^1}{dt^2} + 2\frac{dx^2}{dt} = 0, \quad \frac{d^2x^2}{dt^2} - \left(\frac{dx^2}{dt}\right)^2 = 0.$$

Evidently

$$\Phi = \begin{pmatrix} 0 & y^2 \\ 0 & 0 \end{pmatrix},$$

hence the semispray is isotropic.

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