Symmetry, Integrability and Geometry: Methods and Applications SIGMA 7 (2011), 112, 16 pages

Resolutions of Identity for Some Non-Hermitian Hamiltonians. II. Proofs*

Andrey V. SOKOLOV

V.A. Fock Department of Theoretical Physics, Saint-Petersburg State University, 198504 Saint-Petersburg, Russia

E-mail: *avs_avs@rambler.ru*

Received August 06, 2011, in final form November 25, 2011; Published online December 05, 2011
http://dx.doi.org/10.3842/SIGMA.2011.112

Abstract. This part is a continuation of the Part I where we built resolutions of identity for certain non-Hermitian Hamiltonians constructed of biorthogonal sets of their eigen- and associated functions for the spectral problem defined on entire axis. Non-Hermitian Hamiltonians under consideration are taken with continuous spectrum and the following cases are examined: an exceptional point of arbitrary multiplicity situated on a boundary of continuous spectrum and an exceptional point situated inside of continuous spectrum. In the present work the rigorous proofs are given for the resolutions of identity in both cases.

Key words: non-Hermitian quantum mechanics; supersymmetry; exceptional points; resolution of identity

2010 Mathematics Subject Classification: 81Q60; 81R15; 47B15

1 Introduction

This part is a continuation of the Part I [1] where resolutions of identity for certain non-Hermitian Hamiltonians were constructed of biorthogonal sets of their eigen- and associated functions. The spectral problem was defined on entire axis. Non-Hermitian Hamiltonians were taken with continuous spectrum and they were endowed with an exceptional point of arbitrary multiplicity situated on a boundary of continuous spectrum or an exceptional point situated inside of continuous spectrum. In the present work (Part II) the detailed rigorous proofs are given for the resolutions of identity in both cases. Moreover the reductions of the derived resolutions of identity under narrowing of the classes of employed test functions in the Gel'fand triple [2] are built. In Section 2 the definitions of the employed spaces of test functions and distributions are given. In Section 3 the proofs of the initial resolution of identity and of its reduced forms for restricted spaces of test functions are elaborated for an exceptional point of arbitrary multiplicity situated on a boundary of continuous spectrum. In Section 4 the analogous proofs of resolutions of identity are presented for an exceptional point situated inside of continuous spectrum.

2 Definition of spaces of test functions and distributions

In this paper we shall use the following spaces of test functions and distributions.

Let $CL_\gamma = C^\infty_{\mathbb{R}} \cap L^2(\mathbb{R}; (1+|x|)^\gamma)$, $\gamma \in \mathbb{R}$, be the space of test functions. The sequence $\varphi_n(x) \in CL_\gamma$, $n = 1, 2, 3, \ldots$ is called convergent in CL_γ to $\varphi(x) \in CL_\gamma$,

$$\lim_{n\to+\infty}{}_\gamma\ \varphi_n(x) = \varphi(x)$$

*This paper is a contribution to the Proceedings of the Workshop "Supersymmetric Quantum Mechanics and Spectral Design" (July 18–30, 2010, Benasque, Spain). The full collection is available at http://www.emis.de/journals/SIGMA/SUSYQM2010.html if

$$\lim_{n\to+\infty}\int_{-\infty}^{+\infty}|\varphi_n(x)-\varphi(x)|^2(1+|x|)^\gamma dx=0,$$

and for any $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$ and any $l = 0, 1, 2, \dots$,

$$\lim_{n\to+\infty}\max_{[x_1,x_2]}\left|\varphi_n^{(l)}(x)-\varphi^{(l)}(x)\right|=0.$$

We shall denote the value of a functional f on $\varphi \in CL_\gamma$ conventionally as (f, φ). A linear functional f is called continuous if for any sequence $\varphi_n \in CL_\gamma$, $n = 1, 2, 3, \dots$ convergent in CL_γ to zero the equality

$$\lim_{n\to+\infty}(f,\varphi_n)=0$$

is valid. The space of distributions over CL_γ, i.e. of linear continuous functionals over CL_γ, is denoted CL'_γ. The sequence $f_n \in CL'_\gamma$, $n = 1, 2, 3, \dots$ is called convergent in CL'_γ to $f \in CL'_\gamma$,

$$\operatorname*{lim'_\gamma}_{n\to+\infty} f_n = f,$$

if for any $\varphi \in CL_\gamma$ the relation takes place,

$$\lim_{n\to+\infty}(f_n,\varphi)=(f,\varphi).$$

A functional $f \in CL'_\gamma$ is called regular if there is $f(x) \in L^2(\mathbb{R}; (1 + |x|)^{-\gamma})$ such that for any $\varphi \in CL_\gamma$ the equality

$$(f,\varphi)=\int_{-\infty}^{+\infty}f(x)\varphi(x)\,dx$$

holds. In this case we shall identify the distribution $f \in CL'_\gamma$ with the function $f(x) \in L^2(\mathbb{R}; (1+|x|)^{-\gamma})$. In virtue of the Bunyakovskii inequality,

$$\left|\int_{-\infty}^{+\infty}f(x)\varphi(x)\,dx\right|^2\leqslant\int_{-\infty}^{+\infty}\frac{|f^2(x)|\,dx}{(1+|x|)^\gamma}\int_{-\infty}^{+\infty}\left|\varphi^2(x)\right|(1+|x|)^\gamma\,dx,$$

it is evident that $L_2(\mathbb{R}; (1 + |x|)^{-\gamma}) \subset CL'_\gamma$ and this inclusion is continuous.

For any $\gamma_1 < \gamma_2$ there is a continuous inclusion $CL_{\gamma_2} \subset CL_{\gamma_1}$. Let us also notice that the Dirac delta function $\delta(x - x')$ belongs to CL'_γ for any $\gamma \in \mathbb{R}$.

3 Proofs of resolutions of identity for the model Hamiltonians with exceptional point of arbitrary multiplicity at the bottom of continuous spectrum

3.1 Proof of the biorthogonality relations between eigenfunctions for continuous spectrum

Let us start proofs by proving of the biorthogonality relation

$$\int_{-\infty}^{+\infty}\left[k^n\psi_n(x;k)\right]\left[(k')^n\psi_n(x;-k')\right]dx=(k')^{2n}\delta(k-k') \tag{3.1}$$

between eigenfunctions $\psi_n(x; k)$ for the continuous spectrum of the Hamiltonian h_n, $n = 0, 1, 2, \dots$ (see (2.17) of Part I). Proof of this biorthogonality relation (3.1) is based on the following Lemmas 3.1–3.3.

Lemma 3.1. *Suppose that the functions $\psi_n(x;k)$, $n=0,1,2,\dots$ are defined by the formula* (2.6) *of Part* I *for any $x\in\mathbb{R}$, $k\in\mathbb{C}$, $k\neq 0$ and fixed $z\in\mathbb{C}$, $\operatorname{Im} z\neq 0$. Then for any $n=1,2,3,\dots$, $R>0$, $k\in\mathbb{C}$ and $k'\in\mathbb{C}$ the following relation holds,*

$$\int_{-R}^{R}\big[k^n\psi_n(x;k)\big]\big[(k')^n\psi_n(x;-k')\big]\,dx=(k')^{2n}\frac{\sin R(k-k')}{\pi(k-k')}$$
$$-i\sum_{l=0}^{n-1}(k')^{2l}\big[k^{n-1-l}\psi_{n-1-l}(x;k)\big]\big[(k')^{n-l}\psi_{n-l}(x;-k')\big]\Big|_{-R}^{R}. \tag{3.2}$$

Proof. Let us check first that

$$\int_{-R}^{R}\big[k^n\psi_n(x;k)\big]\big[(k')^n\psi_n(x;-k')\big]\,dx=-i\big[k^{n-1}\psi_{n-1}(x;k)\big]\big[(k')^n\psi_n(x;-k')\big]\Big|_{-R}^{R}$$
$$+(k')^2\int_{-R}^{R}\big[k^{n-1}\psi_{n-1}(x;k)\big]\big[(k')^{n-1}\psi_{n-1}(x;-k')\big]\,dx. \tag{3.3}$$

This equality can be derived with the help of (2.3), (2.6) and (2.9) of Part I and integration by parts,

$$\begin{aligned}
&\int_{-R}^{R}\big[k^n\psi_n(x;k)\big]\big[(k')^n\psi_n(x;-k')\big]\,dx\\
&\quad=i\int_{-R}^{R}\left\{\left(-\partial+\frac{n}{x-z}\right)\big[k^{n-1}\psi_{n-1}(x;k)\big]\right\}\big[(k')^n\psi_n(x;-k')\big]\,dx\\
&\quad=-i\big[k^{n-1}\psi_{n-1}(x;k)\big]\big[(k')^n\psi_n(x;-k')\big]\Big|_{-R}^{R}\\
&\qquad+i\int_{-R}^{R}\big[k^{n-1}\psi_{n-1}(x;k)\big]\left\{\left(\partial+\frac{n}{x-z}\right)\big[(k')^n\psi_n(x;-k')\big]\right\}dx\\
&\quad=-i\big[k^{n-1}\psi_{n-1}(x;k)\big]\big[(k')^n\psi_n(x;-k')\big]\Big|_{-R}^{R}\\
&\qquad+\int_{-R}^{R}\big[k^{n-1}\psi_{n-1}(x;k)\big]\big\{q_n^-q_n^+\big[(k')^{n-1}\psi_{n-1}(x;-k')\big]\big\}\,dx\\
&\quad=-i\big[k^{n-1}\psi_{n-1}(x;k)\big]\big[(k')^n\psi_n(x;-k')\big]\Big|_{-R}^{R}\\
&\qquad+\int_{-R}^{R}\big[k^{n-1}\psi_{n-1}(x;k)\big]\big\{h_{n-1}\big[(k')^{n-1}\psi_{n-1}(x;-k')\big]\big\}\,dx\\
&\quad=-i\big[k^{n-1}\psi_{n-1}(x;k)\big]\big[(k')^n\psi_n(x;-k')\big]\Big|_{-R}^{R}\\
&\qquad+(k')^2\int_{-R}^{R}\big[k^{n-1}\psi_{n-1}(x;k)\big]\big[(k')^{n-1}\psi_{n-1}(x;-k')\big]\,dx.
\end{aligned}$$

The equality (3.2) follows from (3.3) by induction, in view of the relation

$$\int_{-R}^{R}\psi_0(x;k)\psi_0(x;-k')\,dx=\frac{1}{2\pi}\int_{-R}^{R}e^{ix(k-k')}\,dx=\frac{\sin R(k-k')}{\pi(k-k')}.$$

Lemma 3.1 is proved. ■

Lemma 3.2. *For any $k'\in\mathbb{R}$, $z\in\mathbb{C}$, $\operatorname{Im} z\neq 0$, $j=0,1,2,\dots$, $l=0,1,2,\dots$, $m=0,1,2,\dots$ and $\gamma>1+2l-2m$ the following relation holds,*

$$\operatorname*{lim'_\gamma}_{x\to\pm\infty}\frac{k^le^{ix(k-k')}}{(1+k^2)^{m/2}(x-z)^j}=0.$$

Proof. It is true that

$$\frac{k^l e^{ix(k-k')}}{(1+k^2)^{m/2}(x-z)^j} \in L^2(\mathbb{R}; (1+|k|)^{-\gamma}) \subset CL'_\gamma, \qquad \gamma > 1+2l-2m.$$

Thus, to prove Lemma 3.2, it is sufficient to prove that for any $\varphi(k) \in CL_\gamma$ the function $k^l\varphi(k)/(1+k^2)^{m/2}$ belongs to $L^1_{\mathbb{R}}$ in view of the Riemann theorem. The latter is valid by virtue of the Bunyakovskii inequality:

$$\left(\int_{-\infty}^{+\infty} \frac{|k^l\varphi(k)|}{(1+k^2)^{m/2}}\,dk\right)^2 \leqslant \int_{-\infty}^{+\infty} |\varphi^2(k)|(1+|k|)^\gamma\,dk \int_{-\infty}^{+\infty} \frac{k^{2l}\,dk}{(1+k^2)^m(1+|k|)^\gamma} < +\infty,$$

where the condition $\gamma > 1+2l-2m$ is taken into account. Lemma 3.2 is proved. ■

Corollary 3.1. *In the conditions of Lemma 3.1, in view of (2.6) of Part I by virtue of Lemma 3.2 for any $m = 0,1,2,\ldots$, $n = 1,2,3,\ldots$, $l = 0,\ldots,n-1$ and $\gamma > -2l-2m+2n-1$, the following relation holds,*

$$\lim_{x\to\pm\infty}{}'_\gamma\, (k')^{2l}\left[\frac{k^{n-1-l}\psi_{n-1-l}(x;k)}{(1+k^2)^{m/2}}\right]\left[\frac{(k')^{n-l}\psi_{n-l}(x;-k')}{(1+(k')^2)^{m/2}}\right] = 0.$$

Lemma 3.3. *For any $k' \in \mathbb{R}$, $m = 0,1,2,\ldots$, $n = 0,1,2,\ldots$ and $\gamma > -2m-1$ the following relation is valid,*

$$\lim_{R\to+\infty}{}'_\gamma\, \frac{(k')^{2n}}{(1+k^2)^{m/2}(1+(k')^2)^{m/2}}\,\frac{\sin R(k-k')}{\pi(k-k')} = \frac{(k')^{2n}}{(1+(k')^2)^m}\delta(k-k').$$

The proof of Lemma 3.3 is quite analogous to the one for Lemma 3.6 from Section 3.2.
Validity of the biorthogonality relation (3.1) is a corollary of the following theorem.

Theorem 3.1. *Suppose that the functions $\psi_n(x;k)$, $n = 0,1,2,\ldots$ are defined by the formula (2.6) of Part I for any $x \in \mathbb{R}$, $k \in \mathbb{C}$, $k \neq 0$ and fixed $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$. Then for any $k' \in \mathbb{R}$, $m = 0,1,2,\ldots$, $n = 0,1,2,\ldots$ and $\gamma > -2m+2n-1$ the following relation takes place,*

$$\lim_{R\to+\infty}{}'_\gamma \int_{-R}^{R} \left[\frac{k^n\psi_n(x;k)}{(1+k^2)^{m/2}}\right]\left[\frac{(k')^n\psi_n(x;-k')}{(1+(k')^2)^{m/2}}\right] dx = \frac{(k')^{2n}}{(1+(k')^2)^m}\,\delta(k-k'). \tag{3.4}$$

The statement of Theorem 3.1 follows from Lemmas 3.1 and 3.3 and from Corollary 3.1.

Remark 3.1. The parameter m in Theorem 3.1 regulates the class of test functions for which the biorthogonality relation (3.4) takes place. One can prove as well this relation for any fixed m for test functions from a wider class than in Theorem 3.1 with the help of the technique of Theorem 3.3 and Remark 3.2 from Section 3.2.

3.2 Proofs of the resolutions of identity

Proof of the initial resolution of identity (2.18) of Part I is based on the following Lemmas 3.4–3.6.

Lemma 3.4. *Suppose that*

(1) *the functions $\psi_n(x;k)$, $n = 0,1,2,\ldots$ are defined by the formula (2.6) of Part I for any $x \in \mathbb{R}$, $k \in \mathbb{C}$, $k \neq 0$ and fixed $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$;*

(2) *$\mathcal{L}(A)$ is a path in complex k plane, made of the segment $[-A, A]$ by its deformation near the point $k = 0$ upwards or downwards and the direction of $\mathcal{L}(A)$ is specified from $-A$ to A.*

Then for any $n = 1, 2, 3, \ldots$, $x \in \mathbb{R}$ and $x' \in \mathbb{R}$ the following relation holds,

$$\int_{\mathcal{L}(A)} \psi_n(x;k)\psi_n(x';-k)\,dk$$
$$= \sum_{l=0}^{n-1}\left(\frac{x'-z}{x-z}\right)^l \frac{\psi_{n-1-l}(x;k)\psi_{n-l}(x';-k)}{i(x-z)}\Big|_{-A}^{A} + \left(\frac{x'-z}{x-z}\right)^n \frac{\sin A(x-x')}{\pi(x-x')}. \tag{3.5}$$

Proof. Let us check first that

$$\int_{\mathcal{L}(A)} \psi_n(x;k)\psi_n(x';-k)\,dk$$
$$= \frac{\psi_{n-1}(x;k)\psi_n(x';-k)}{i(x-z)}\Big|_{-A}^{A} + \frac{x'-z}{x-z}\int_{\mathcal{L}(A)} \psi_{n-1}(x;k)\psi_{n-1}(x';-k)\,dk. \tag{3.6}$$

This equality can be derived with the help of (2.3), (2.6) and (2.9) of Part I and of integration by parts:

$$\int_{\mathcal{L}(A)} \psi_n(x;k)\psi_n(x';-k)\,dk = \int_{\mathcal{L}(A)} \frac{e^{ikz}}{i(x-z)}\left[\left(\frac{\partial}{\partial k}-\frac{n}{k}\right)\left(e^{-ikz}\psi_{n-1}(x;k)\right)\right]\psi_n(x';-k)\,dk$$
$$= \frac{\psi_{n-1}(x;k)\psi_n(x';-k)}{i(x-z)}\Big|_{-A}^{A} - \int_{\mathcal{L}(A)} \frac{e^{-ikz}\psi_{n-1}(x;k)}{i(x-z)}\left[\left(\frac{\partial}{\partial k}+\frac{n}{k}\right)\left(e^{ikz}\psi_n(x';-k)\right)\right]dk$$
$$= \frac{\psi_{n-1}(x;k)\psi_n(x';-k)}{i(x-z)}\Big|_{-A}^{A}$$
$$+ i\frac{x'-z}{x-z}\int_{\mathcal{L}(A)} \frac{e^{-ikz}\psi_{n-1}(x;k)}{x'-z}\left[\left(\frac{\partial}{\partial k}+\frac{n}{k}\right)\left(e^{ikz}\psi_n(x';-k)\right)\right]dk$$
$$= \frac{\psi_{n-1}(x;k)\psi_n(x';-k)}{i(x-z)}\Big|_{-A}^{A}$$
$$+ i\frac{x'-z}{x-z}\int_{\mathcal{L}(A)} e^{-ikz}\psi_{n-1}(x;k)\left[\left(\frac{1}{k}\frac{\partial}{\partial x'}+\frac{n}{k(x'-z)}\right)\left(e^{ikz}\psi_n(x';-k)\right)\right]dk$$
$$= \frac{\psi_{n-1}(x;k)\psi_n(x';-k)}{i(x-z)}\Big|_{-A}^{A}$$
$$+ \frac{x'-z}{x-z}\int_{\mathcal{L}(A)} \frac{1}{k^2}\psi_{n-1}(x;k)\left[\left(\frac{\partial}{\partial x'}+\frac{n}{x'-z}\right)\left(-\frac{\partial}{\partial x'}+\frac{n}{x'-z}\right)\psi_{n-1}(x';-k)\right]dk$$
$$= \frac{\psi_{n-1}(x;k)\psi_n(x';-k)}{i(x-z)}\Big|_{-A}^{A} + \frac{x'-z}{x-z}\int_{\mathcal{L}(A)} \frac{1}{k^2}\psi_{n-1}(x;k)\left[h_{n-1}\psi_{n-1}(x';-k)\right]dk$$
$$= \frac{\psi_{n-1}(x;k)\psi_n(x';-k)}{i(x-z)}\Big|_{-A}^{A} + \frac{x'-z}{x-z}\int_{\mathcal{L}(A)} \psi_{n-1}(x;k)\psi_{n-1}(x';-k)\,dk.$$

The equality (3.5) follows from (3.6) by induction in view of the relation

$$\int_{\mathcal{L}(A)} \psi_0(x;k)\psi_0(x';-k)\,dk = \frac{1}{2\pi}\int_{-A}^{A} e^{ik(x-x')}\,dk = \frac{\sin A(x-x')}{\pi(x-x')}. \tag{3.7}$$

Lemma 3.4 is proved. ■

Lemma 3.5. *For any $x' \in \mathbb{R}$, $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$, $l = 0, 1, 2, \ldots$, $m = 1, 2, 3, \ldots$ and $\gamma > 1 - 2m$ the following relation takes place,*

$$\lim_{k\to\pm\infty}{}'_{\gamma} \frac{e^{ik(x-x')}}{k^l(x-z)^m} = 0.$$

Proof. It is true that

$$\frac{e^{ik(x-x')}}{k^l(x-z)^m} \in L^2(\mathbb{R};(1+|x|)^{-\gamma}) \subset CL'_\gamma, \qquad \gamma > 1-2m.$$

Thus, in view of the Riemann theorem, in order to prove the lemma, it is sufficient to prove that for any $\varphi(x) \in CL_\gamma$ the fraction $\varphi(x)/(x-z)^m$ belongs to $L^1_{\mathbb{R}}$. The latter is valid by virtue of the Bunyakovskii inequality:

$$\left(\int_{-\infty}^{+\infty} \frac{|\varphi(x)|}{|x-z|^m}\,dx\right)^2 \leqslant \int_{-\infty}^{+\infty} |\varphi^2(x)|(1+|x|)^\gamma\,dx \int_{-\infty}^{+\infty} \frac{dx}{|x-z|^{2m}(1+|x|)^\gamma} < +\infty,$$

where the condition $\gamma > 1-2m$ is taken into account. Lemma 3.5 is proved. ■

Corollary 3.2. *In the conditions of Lemma 3.4, in view of* (2.6) *from Part* I *by virtue of Lemma 3.5 for any $n = 1, 2, 3, \ldots$, $l = 0, \ldots, n-1$ and $\gamma > -2l-1$, the following relation holds,*

$$\lim_{k\to\pm\infty}{}'_\gamma \left(\frac{x'-z}{x-z}\right)^l \frac{\psi_{n-1-l}(x;k)\psi_{n-l}(x';-k)}{i(x-z)} = 0.$$

Lemma 3.6. *For any $x' \in \mathbb{R}$, $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$, $n = 0, 1, 2, \ldots$ and $\gamma > -2n-1$ the following relation is valid,*

$$\lim_{A\to+\infty}{}'_\gamma \left(\frac{x'-z}{x-z}\right)^n \frac{\sin A(x-x')}{\pi(x-x')} = \delta(x-x').$$

Proof. It is true that

$$\left(\frac{x'-z}{x-z}\right)^n \frac{\sin A(x-x')}{\pi(x-x')} \in L^2(\mathbb{R};(1+|x|)^{-\gamma}) \subset CL'_\gamma, \qquad \gamma > -2n-1.$$

Thus, to prove the lemma, it is sufficient to prove that for any $\varphi(x) \in CL_\gamma$, $\gamma > -2n-1$ the equality

$$\lim_{A\to+\infty} \int_{-\infty}^{+\infty} \frac{\sin A(x-x')}{\pi(x-x')} \left(\frac{x'-z}{x-z}\right)^n \varphi(x)\,dx = \varphi(x')$$

takes place. For this purpose let us consider the function

$$\psi(x) = \left(\frac{x'-z}{x-z}\right)^n \varphi(x).$$

By virtue of the Bunyakovskii inequality for arbitrary $\delta > 0$,

$$\left[\left(\int_{-\infty}^{x'-\delta} + \int_{x'+\delta}^{+\infty}\right) \frac{|\psi(x)|}{|x-x'|}\,dx\right]^2 \leqslant \left(\int_{-\infty}^{x'-\delta} + \int_{x'+\delta}^{+\infty}\right) |\varphi^2(x)|(1+|x|)^\gamma\,dx$$
$$\times \left(\int_{-\infty}^{x'-\delta} + \int_{x'+\delta}^{+\infty}\right) \frac{|x'-z|^{2n}\,dx}{|x-z|^{2n}|x-x'|^2(1+|x|)^\gamma} < +\infty$$

the following inclusion is valid,

$$\frac{\psi(x)}{x-x'} \in L^1(\mathbb{R} \setminus (x'-\delta, x'+\delta)), \qquad \delta > 0,$$

and, moreover, it is evident that

$$\frac{\psi(x)-\psi(x')}{x-x'} \in L^1([x'-\delta, x'+\delta]), \qquad \delta > 0.$$

Hence, by virtue of the Riemann theorem,

$$\lim_{A\to+\infty}\int_{-\infty}^{+\infty}\frac{\sin A(x-x')}{\pi(x-x')}\left(\frac{x'-z}{x-z}\right)^n\varphi(x)\,dx$$
$$= \lim_{A\to+\infty}\left[\psi(x')\int_{x'-\delta}^{x'+\delta}\frac{\sin A(x-x')}{\pi(x-x')}\,dx\right] = \frac{2}{\pi}\varphi(x')\int_0^{+\infty}\frac{\sin t}{t}\,dt = \varphi(x').$$

Thus, Lemma 3.6 is proved. ■

Validity of the resolution of identity (2.18) of Part I in CL'_γ for any $\gamma > -1$ is a corollary of the following theorem.

Theorem 3.2. *Suppose that*

(1) *the functions* $\psi_n(x;k)$, $n = 0,1,2,\dots$ *are defined by the formula* (2.6) *of Part* I *for any* $x \in \mathbb{R}$, $k \in \mathbb{C}$, $k \neq 0$ *and fixed* $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$*;*

(2) $\mathcal{L}(A)$ *is a path in complex* k *plane, made of the segment* $[-A, A]$, $A > 0$ *by its deformation near the point* $k = 0$ *upwards or downwards and the direction of* $\mathcal{L}(A)$ *is specified from* $-A$ *to* A.

Then for any $\gamma > -1$, $x' \in \mathbb{R}$ *and* $n = 0,1,2,\dots$ *the following relation holds,*

$$\operatorname*{lim'_\gamma}_{A\to+\infty}\int_{\mathcal{L}(A)}\psi_n(x;k)\psi_n(x';-k)\,dk = \delta(x-x').$$

The statement of Theorem 3.2 follows from Lemmas 3.4 and 3.6 and from Corollary 3.2.

The applicability of the resolution of identity (2.18) of Part I for some bounded and slowly increasing test functions is based on the next theorem.

Theorem 3.3. *Suppose that*

(1) *the functions* $\psi_n(x;k)$, $n = 0,1,2,\dots$ *are defined by the formula* (2.6) *of Part* I *for any* $x \in \mathbb{R}$, $k \in \mathbb{C}$, $k \neq 0$ *and fixed* $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$*;*

(2) $\mathcal{L}(A)$ *is a path in complex* k *plane, made of the segment* $[-A, A]$, $A > 0$ *by its deformation near the point* $k = 0$ *upwards or downwards and the direction of* $\mathcal{L}(A)$ *is specified from* $-A$ *to* A*;*

(3) *the function* $\eta(x) \in C^\infty_{\mathbb{R}}$, $\eta(x) \equiv 0$ *for any* $x \leqslant 1$, $\eta(x) \in [0,1]$ *for any* $x \in [1,2]$ *and* $\eta(x) \equiv 1$ *for any* $x \geqslant 2$.

Then for any $\varkappa \in [0,1)$, $k_0 \in \mathbb{R}$, $x' \in \mathbb{R}$ *and* $n = 0,1,2,\dots$ *the following relation is valid,*

$$\lim_{A\to+\infty}\int_{-\infty}^{+\infty}\left[\int_{\mathcal{L}(A)}\psi_n(x;k)\psi_n(x';-k)\,dk\right]\left[\eta(\pm x)e^{ik_0x}|x|^\varkappa\right]dx = \eta(\pm x')e^{ik_0x'}|x'|^\varkappa. \tag{3.8}$$

Proof. In the case $n = 0$ in view of (3.7) the proof can be easily realized in the same way as for Theorem 2 from Appendix B of [4]. Thus, we present the proof for the case $n = 1,2,3,\dots$ with upper signs in (3.8) only, taking into account that the proof for the case with lower signs is quite similar. In order to prove Theorem 3.3 in this case we employ Lemmas 3.4 and 3.6, Corollary 3.2 and the fact that

$$\eta(x)e^{ik_0x}|x|^{\varkappa} \in CL_\gamma, \qquad -3 < \gamma < -1 - 2\varkappa. \tag{3.9}$$

Then it is sufficient to prove that

$$\lim_{A\to+\infty}\int_{-\infty}^{+\infty}\left[\frac{\psi_{n-1}(x;k)\psi_n(x';-k)}{i(x-z)}\Big|_{-A}^{A}\right]\left[\eta(x)e^{ik_0x}|x|^{\varkappa}\right]dx = 0.$$

In turn, to prove the latter, in view of (2.6) from Part I, (3.9) and Lemma 3.5, it is sufficient to prove that

$$\lim_{A\to+\infty}\int_{-\infty}^{+\infty}\left[\frac{e^{\pm iA(x-x')}}{x-z}\right]\left[\eta(x)e^{ik_0x}|x|^{\varkappa}\right]dx = 0. \tag{3.10}$$

The equality (3.10) follows from the Riemann theorem and the chain of transformations,

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\left[\frac{e^{\pm iA(x-x')}}{x-z}\right]\left[\eta(x)e^{ik_0x}|x|^{\varkappa}\right]dx = e^{ik_0x'}\int_{-\infty}^{+\infty}\frac{|t+x'|^{\varkappa}\eta(t+x')}{t+x'-z}e^{i(k_0\pm A)t}\,dt\\
&\quad = e^{ik_0x'}\int_{-\infty}^{+\infty}\frac{|t+x'|^{\varkappa}\eta(t+x')}{t+x'-z}d\frac{e^{i(k_0\pm A)t}}{i(k_0\pm A)}\\
&\quad = -\frac{e^{ik_0x'}}{i(k_0\pm A)}\int_{-\infty}^{+\infty}e^{i(k_0\pm A)t}d\frac{|t+x'|^{\varkappa}\eta(t+x')}{t+x'-z}, \qquad A > |k_0|,
\end{aligned}$$

derived with help of integration by parts. Thus, Theorem 3.3 is proved. ■

Remark 3.2. Theorems 3.2 and 3.3 provide the validity of the resolution of identity (2.18) from Part I for test functions which are linear combinations of functions $\eta(\pm x)e^{ik_0x}|x|^{\varkappa}$, in general, with different $\varkappa \in [0,1)$ and $k_0 \in \mathbb{R}$ and functions from CL_γ, in general, with different $\gamma > -1$. In particular, these theorems guarantee applicability of (2.18) from Part I for the eigenfunctions (2.6) from Part I and for the associated function (2.2) from Part I (in the case of even n) of the Hamiltonian h_n, $n = 0, 1, 2, \ldots$.

Remark 3.3. The first of resolutions of identity (2.19) of Part I follows from (2.18) of Part I and Lemma 3.4. The second of resolutions of identity (2.19) of Part I and (2.35) of Part I can be derived from the first of resolutions of identity (2.19) of Part I with the help of calculation of the substitution $|_{-\varepsilon}^{\varepsilon}$ and of identical transformations. The resolution of identity (2.20) of Part I follows trivially from (2.19) of Part I.

The resolutions of identity (2.21) and (2.36) of Part I are corollaries of the resolutions of identity (2.19) and (2.35) of Part I respectively and of the following Lemma 3.7.

Lemma 3.7. *For any $\gamma > -1$ and $x' \in \mathbb{R}$ the relation holds,*

$$\mathop{\lim\nolimits'_\gamma}_{\varepsilon\downarrow 0}\frac{\sin\varepsilon(x-x')}{x-x'} = 0.$$

Proof of Lemma 3.7 is analogous to the proof of Lemma 2 from Appendix B of [4].

The resolution of identity (2.37) of Part I is a corollary of the resolution of identity (2.36) of Part I and of the following Lemma 3.8.

Lemma 3.8. *For any $\gamma > 1$, $x' \in \mathbb{R}$ and $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$ the relation takes place,*

$$\lim_{\varepsilon\downarrow 0}{}'_{\gamma} \frac{\sin^2 \frac{\varepsilon}{2}(x-x')}{\varepsilon(x-z)(x'-z)} = 0.$$

Proof of Lemma 3.8 is analogous to the proof of Lemma 3 from Appendix of [3].

The resolution of identity (2.38) of Part I is a corollary of the resolution of identity (2.37) of Part I and of the following Lemmas 3.9 and 3.10.

Lemma 3.9. *For any $\gamma > 3$, $x' \in \mathbb{R}$ and $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$ the relation is valid,*

$$\lim_{\varepsilon\downarrow 0}{}'_{\gamma} \frac{(x-x')\sin^2 \frac{\varepsilon}{4}(x-x')\sin\frac{\varepsilon}{2}(x-x')}{\varepsilon^2(x-z)^2(x'-z)^2} = 0.$$

Proof. It is true that

$$\frac{(x-x')\sin^2 \frac{\varepsilon}{4}(x-x')\sin\frac{\varepsilon}{2}(x-x')}{\varepsilon^2(x-z)^2(x'-z)^2} \in L^2(\mathbb{R};(1+|x|)^{-\gamma}) \subset CL_{\gamma}, \qquad \gamma > 3.$$

Thus, to prove the lemma it is sufficient to establish that for any $\varphi(x) \in CL_{\gamma}$, $\gamma > 3$, the relation

$$\lim_{\varepsilon\downarrow 0} \int_{-\infty}^{+\infty} \frac{(x-x')\sin^2 \frac{\varepsilon}{4}(x-x')\sin\frac{\varepsilon}{2}(x-x')}{\varepsilon^2(x-z)^2(x'-z)^2}\varphi(x)\,dx = 0$$

is valid. But its validity follows from the chain of inequalities,

$$\begin{aligned}
&\left|\int_{-\infty}^{+\infty} \frac{(x-x')\sin^2 \frac{\varepsilon}{4}(x-x')\sin\frac{\varepsilon}{2}(x-x')}{\varepsilon^2(x-z)^2(x'-z)^2}\,\varphi(x)\,dx\right|^2 \\
&\leqslant \int_{-\infty}^{+\infty} \frac{\sin^4 \frac{\varepsilon}{4}(x-x')\sin^2\frac{\varepsilon}{2}(x-x')}{\varepsilon^4|x-x'|^{\alpha}}\,dx \int_{-\infty}^{+\infty} \frac{|x-x'|^{\alpha+2}|\varphi^2(x)|}{|x-z|^4|x'-z|^4}\,dx \\
&= \varepsilon^{\alpha-5} \int_{-\infty}^{+\infty} \frac{\sin^4 \frac{t}{4}\,\sin^2\frac{t}{2}}{|t|^{\alpha}}\,dt \int_{-\infty}^{+\infty} \frac{|x-x'|^{\alpha+2}|\varphi^2(x)|}{|x-z|^4|x'-z|^4}\,dx \\
&\leqslant \varepsilon^{\alpha-5} \sup_{x\in\mathbb{R}}\left[\frac{|x-x'|^{\alpha+2}}{|x-z|^4|x'-z|^4(1+|x|)^{\gamma}}\right] \int_{-\infty}^{+\infty} \frac{\sin^4 \frac{t}{4}\,\sin^2\frac{t}{2}}{|t|^{\alpha}}\,dt \int_{-\infty}^{+\infty} |\varphi^2(x)|(1+|x|)^{\gamma}\,dx, \\
&5 < \alpha < \min\{7, \gamma+2\},
\end{aligned}$$

derived with the help of the Bunyakovskii inequality. Lemma 3.9 is proved. ∎

Lemma 3.10. *For any $\gamma > 3$, $x' \in \mathbb{R}$ and $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$ the relation takes place,*

$$\lim_{\varepsilon\downarrow 0}{}'_{\gamma} \frac{[\varepsilon(x-x') - 2\sin\frac{\varepsilon}{2}(x-x')]^2}{\varepsilon^3(x-z)^2(x'-z)^2} = 0.$$

Proof. It is true that

$$\frac{[\varepsilon(x-x') - 2\sin\frac{\varepsilon}{2}(x-x')]^2}{\varepsilon^3(x-z)^2(x'-z)^2} \in L^2(\mathbb{R};(1+|x|)^{-\gamma}) \subset CL_{\gamma}, \qquad \gamma > 3.$$

Thus, to prove the lemma it is sufficient to establish that for any $\varphi(x) \in CL_{\gamma}$, $\gamma > 3$, the relation

$$\lim_{\varepsilon\downarrow 0} \int_{-\infty}^{+\infty} \frac{[\varepsilon(x-x') - 2\sin\frac{\varepsilon}{2}(x-x')]^2}{\varepsilon^3(x-z)^2(x'-z)^2}\,\varphi(x)\,dx = 0$$

is valid. But its validity follows from the chain of inequalities,

$$\begin{aligned}
&\left|\int_{-\infty}^{+\infty} \frac{[\varepsilon(x-x') - 2\sin\frac{\varepsilon}{2}(x-x')]^2}{\varepsilon^3(x-z)^2(x'-z)^2}\varphi(x)\,dx\right|^2 \\
&\leqslant \int_{-\infty}^{+\infty} \frac{[\varepsilon(x-x') - 2\sin\frac{\varepsilon}{2}(x-x')]^4}{\varepsilon^6|x-x'|^\alpha}\,dx \int_{-\infty}^{+\infty} \frac{|x-x'|^\alpha|\varphi^2(x)|}{|x-z|^4|x'-z|^4}\,dx \\
&= \varepsilon^{\alpha-7}\int_{-\infty}^{+\infty} \frac{[t-2\sin\frac{t}{2}]^4}{|t|^\alpha}\,dt \int_{-\infty}^{+\infty} \frac{|x-x'|^\alpha|\varphi^2(x)|}{|x-z|^4|x'-z|^4}\,dx \\
&\leqslant \varepsilon^{\alpha-7}\sup_{x\in\mathbb{R}}\left[\frac{|x-x'|^\alpha}{|x-z|^4|x'-z|^4(1+|x|)^\gamma}\right]\int_{-\infty}^{+\infty} \frac{[t-2\sin\frac{t}{2}]^4}{|t|^\alpha}\,dt \int_{-\infty}^{+\infty} |\varphi^2(x)|(1+|x|)^\gamma\,dx, \\
&7 < \alpha < \min\{13, \gamma+4\},
\end{aligned}$$

derived with the help of the Bunyakovskii inequality. Lemma 3.10 is proved. ■

Remark 3.4. Let us consider the functionals

$$\operatorname*{lim''_\gamma}_{\varepsilon\downarrow 0} \frac{(x-x')\sin^2\frac{\varepsilon}{4}(x-x')\sin\frac{\varepsilon}{2}(x-x')}{\varepsilon^2(x-z)^2(x'-z)^2}, \qquad \operatorname*{lim''_\gamma}_{\varepsilon\downarrow 0} \frac{[\varepsilon(x-x') - 2\sin\frac{\varepsilon}{2}(x-x')]^2}{\varepsilon^3(x-z)^2(x'-z)^2}, \tag{3.11}$$

each of which is defined by a related expression in the set

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{+\infty} \frac{(x-x')\sin^2\frac{\varepsilon}{4}(x-x')\sin\frac{\varepsilon}{2}(x-x')}{\varepsilon^2(x-z)^2(x'-z)^2}\varphi(x)\,dx, \\
&\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{+\infty} \frac{[\varepsilon(x-x') - 2\sin\frac{\varepsilon}{2}(x-x')]^2}{\varepsilon^3(x-z)^2(x'-z)^2}\varphi(x)\,dx,
\end{aligned} \tag{3.12}$$

for all test functions $\varphi(x) \in CL_\gamma$, $\gamma \in \mathbb{R}$, for which the limit from (3.12) corresponding to (3.11) exists. It follows from Lemmas 3.9 and 3.10 that these functionals are trivial (equal to zero) for any $\gamma > 3$, but at the same time in view of the formulae (2.39) and (2.40) from [1] these functionals are nontrivial (different from zero) for any $\gamma < 3$. By virtue of Lemmas 3.9 and 3.10 the restrictions of the functionals (3.11) on the standard space $\mathcal{D}(\mathbb{R}) \subset CL_\gamma$, $\gamma \in \mathbb{R}$ are equal to zero. Hence, the supports of these functionals for any $\gamma \in \mathbb{R}$ do not contain any finite real number. On the other hand, one can represent a test function $\varphi(x) \in CL_\gamma$, $\gamma \in \mathbb{R}$ for any $R > 0$ as a sum of two functions from CL_γ in the form

$$\varphi(x) = \eta(|x|-R)\varphi(x) + [1-\eta(|x|-R)]\varphi(x), \qquad R > 0, \tag{3.13}$$

where $\eta(x) \in C^\infty_{\mathbb{R}}$, $\eta(x) \equiv 1$ for any $x < 0$, $\eta(x) \in [0,1]$ for any $x \in [0,1]$ and $\eta(x) \equiv 0$ for any $x > 1$. In view of Lemmas 3.9 and 3.10 the values of the functionals (3.11) for $\varphi(x)$ are equal to their values for the second term of (3.13) for any arbitrarily large $R > 0$. Hence, the values of the functionals (3.11) for a test function depend only on the behavior of this function in any arbitrarily close (in the conformal sense) vicinity of the infinity and are independent of values of the function in any finite interval of real axis. In this sense the supports of the functionals (3.11) for any $\gamma < 3$ consist of the unique element which is the infinity. At last, since (i) for any $\varphi(x) \in CL_\gamma$ and $\gamma \in \mathbb{R}$ the relation

$$\operatorname*{lim_\gamma}_{R\to+\infty} \eta(|x|-R)\varphi(x) = \varphi(x)$$

holds; (ii) the restrictions of the functionals (3.11) on $\mathcal{D}(\mathbb{R})$ are zero for any $\gamma \in \mathbb{R}$ and (iii) the functionals (3.11) are nontrivial for any $\gamma < 3$, so the latter functionals are discontinuous for any $\gamma < 3$.

4 Proofs of resolutions of identity for the model Hamiltonian with exceptional point inside of continuous spectrum

Proof of the initial resolution of identity (3.7) of Part I is based on the following Lemmas 4.1–4.3.

Lemma 4.1. *Suppose that*

(1) *the functions* $\psi(x;k)$, $\psi_0(x)$ *and* $\psi_1(x)$ *are defined by the formulas* (3.1) *and* (3.2) *of Part* I *for fixed* $\alpha>0$, $z\in\mathbb{C}$, $\operatorname{Im} z\neq 0$ *and any* $x\in\mathbb{R}$, $k\in\mathbb{C}$, $k\neq\pm\alpha$*;*

(2) $\mathcal{L}(A)$ *is an integration path in complex* k *plane, obtained from the segment* $[-A,A]$, $A>\alpha$ *by its simultaneous deformation near the points* $k=-\alpha$ *and* $k=\alpha$ *upwards or downwards and the direction of* $\mathcal{L}(A)$ *is specified from* $-A$ *to* A.

Then for any $x,x'\in\mathbb{R}$ *and* $A>\alpha$ *the following relation is valid,*

$$\begin{aligned}\int_{\mathcal{L}(A)}&\psi(x;k)\psi(x';-k)\,dk\\&=\frac{\sin A(x-x')}{\pi(x-x')}-\frac{1}{2\pi\alpha}\left\{\frac{\cos[(A+\alpha)(x-x')]}{A+\alpha}+\frac{\cos[(A-\alpha)(x-x')]}{A-\alpha}\right\}\psi_0(x)\psi_0(x')\\&-\frac{1}{\pi}\left[\int_{A-\alpha}^{A+\alpha}\cos t(x-x')\frac{dt}{t}\right][\psi_0(x)\psi_1(x')+\psi_1(x)\psi_0(x')].\end{aligned}\tag{4.1}$$

Proof. With the help of (3.1) and (3.2) of Part I and certain identical transformations one can rearrange the left-hand part of (4.1) to the form,

$$\begin{aligned}\int_{\mathcal{L}(A)}&\psi(x;k)\psi(x';-k)\,dk=\frac{1}{2\pi}\int_{\mathcal{L}(A)}e^{ik(x-x')}\,dk\\&-\frac{1}{4\pi\alpha}\psi_0(x)\psi_0(x')\left[\int_{\mathcal{L}(A)}\frac{\partial}{\partial k}\left(\frac{e^{i(k-\alpha)(x-x')}}{k-\alpha}\right)dk+\int_{\mathcal{L}(A)}\frac{\partial}{\partial k}\left(\frac{e^{i(k+\alpha)(x-x')}}{k+\alpha}\right)dk\right]\\&+\frac{1}{2\pi}[\psi_0(x)\psi_1(x')+\psi_1(x)\psi_0(x')]\left[\int_{\mathcal{L}(A)}\frac{e^{i(k-\alpha)(x-x')}}{k-\alpha}dk-\int_{\mathcal{L}(A)}\frac{e^{i(k+\alpha)(x-x')}}{k+\alpha}dk\right],\end{aligned}$$

where from the equality (4.1) follows trivially. Lemma 4.1 is proved. ■

Lemma 4.2. *In the conditions of Lemma* 4.1 *for any* $x'\in\mathbb{R}$ *and* $\gamma>-1$ *the following relation holds,*

$$\operatorname*{lim'_{\gamma}}_{k\to\pm\infty}\left[\frac{e^{ik(x-x')}}{k}\psi_0(x)\right]=0.$$

Proof of Lemma 4.2 in view of (3.2) of Part I is quite similar to the proof of a more complicated Lemma 3.2 from Section 3.1.

Lemma 4.3. *In the conditions of Lemma* 4.1 *for any* $x'\in\mathbb{R}$ *and* $\gamma>-1$ *the following relation takes place,*

$$\operatorname*{lim'_{\gamma}}_{A\to+\infty}\left\{\left[\int_{A-\alpha}^{A+\alpha}\cos t(x-x')\,\frac{dt}{t}\right][\psi_0(x)\psi_1(x')+\psi_1(x)\psi_0(x')]\right\}=0.$$

Proof. Let us use the estimate (B12) from [4],

$$\left|\int_{A-\alpha}^{A+\alpha}\cos t(x-x')\,\frac{dt}{t}\right|\leqslant\frac{AC}{[1+(A-\alpha)|x-x'|](A-\alpha)},$$
$$x,x'\in\mathbb{R},\qquad A>\alpha,\qquad C=2\sup_{\xi>0}\left|(1+\xi)\frac{\sin\xi}{\xi}\right|.$$

Therefrom it follows that

$$\left|\left[\int_{A-\alpha}^{A+\alpha}\cos t(x-x')\,\frac{dt}{t}\right][\psi_0(x)\psi_1(x')+\psi_1(x)\psi_0(x')]\right|\leqslant\frac{AD}{[1+(A-\alpha)|x-x'|](A-\alpha)},$$
$$x,x'\in\mathbb{R},\qquad A>\alpha,\qquad D=2C\sup_{x,x'\in\mathbb{R}}|\psi_0(x)\psi_1(x')|,\tag{4.2}$$

where D is a finite constant by virtue of (3.2) of Part I. The statement of Lemma 4.3 is valid in view of the following chain of inequalities obtained with the help of (4.2) and the Bunyakovskii inequality,

$$\begin{aligned}
&\left|\int_{-\infty}^{+\infty}\left\{\left[\int_{A-\alpha}^{A+\alpha}\cos t(x-x')\,\frac{dt}{t}\right][\psi_0(x)\psi_1(x')+\psi_1(x)\psi_0(x')]\right\}\varphi(x)\,dx\right|^2\\
&\quad\leqslant\frac{A^2D^2}{(A-\alpha)^2}\int_{-\infty}^{+\infty}|\varphi^2(x)|(1+|x-x'|)^{\gamma}\,dx\int_{-\infty}^{+\infty}\frac{dx}{(1+|x-x'|)^{\gamma}[1+(A-\alpha)|x-x'|]^2}\\
&\quad=\frac{2A^2D^2}{(A-\alpha)^3}\int_{-\infty}^{+\infty}|\varphi^2(x)|(1+|x-x'|)^{\gamma}\,dx\int_0^{+\infty}\frac{dt}{[1+t/(A-\alpha)]^{\gamma}(1+t)^2}\\
&\quad\leqslant\frac{2A^2D^2}{(A-\alpha)^3}\int_{-\infty}^{+\infty}|\varphi^2(x)|(1+|x-x'|)^{\gamma}\,dx\\
&\qquad\times\begin{cases}\displaystyle\int_0^{+\infty}\frac{dt}{(1+t)^{2+\gamma}}, & -1<\gamma<0,\ A\geqslant\alpha+1,\\ \displaystyle\int_0^{+\infty}\frac{dt}{(1+t)^2}, & \gamma\geqslant 0\end{cases}\to 0,\qquad A\to+\infty,
\end{aligned}$$

where $\varphi(x)$ is any function from CL_γ, $\gamma>-1$. Lemma 4.3 is proved. ■

Validity of the resolution of identity (3.7) of Part I in CL'_γ for any $\gamma>-1$ is a corollary of the following theorem.

Theorem 4.1. *Suppose that*

1. *the function $\psi(x;k)$ is defined by the formula* (3.1) *of Part* I *for fixed $\alpha>0$, $z\in\mathbb{C}$, $\operatorname{Im}z\neq 0$ and any $x\in\mathbb{R}$, $k\in\mathbb{C}$, $k\neq\pm\alpha$;*
2. *$\mathcal{L}(A)$ is an integration path in complex k plane, obtained from the segment $[-A,A]$, $A>\alpha$ by its simultaneous deformation near the points $k=-\alpha$ and $k=\alpha$ upwards or downwards and the direction of $\mathcal{L}(A)$ is specified from $-A$ to A.*

Then for any $\gamma>-1$ and $x'\in\mathbb{R}$ the following relation holds,

$$\operatorname*{lim'_\gamma}_{A\to+\infty}\int_{\mathcal{L}(A)}\psi(x;k)\psi(x';-k)\,dk=\delta(x-x').$$

Theorem 4.1 follows from Lemmas 4.1–4.3 and from the case $n=0$ of Lemma 3.6.

Proof of the resolution of identity (3.7) of Part I for some bounded and slowly increasing test functions is based on the following lemma.

Lemma 4.4. *In the conditions of Lemma 4.1 for any $x \in \mathbb{R}$, $x' \in \mathbb{R}$ and $A > \alpha$ the inequalities take place,*

$$\left|\int_{A-\alpha}^{A+\alpha} \cos t(x-x')\frac{dt}{t} - \left\{\frac{\sin[(A+\alpha)(x-x')]}{(A+\alpha)(x-x')} - \frac{\sin[(A-\alpha)(x-x')]}{(A-\alpha)(x-x')}\right\}\right| \leqslant \frac{6}{(A-\alpha)^2(x-x')^2}, \tag{4.3}$$

$$|\psi_0(x)| \leqslant \frac{(2\alpha)^{3/2}}{|\sin 2\alpha x + 2\alpha(x-z)|}, \tag{4.4}$$

$$\left|\psi_0(x) - \sqrt{2\alpha}\frac{\cos\alpha x}{x-z}\right| \leqslant \frac{\sqrt{2\alpha}}{|x-z||\sin 2\alpha x + 2\alpha(x-z)|} \tag{4.5}$$

and

$$\left|\psi_1(x) - \frac{1}{\sqrt{2\alpha}}\sin\alpha x\right| \leqslant \frac{1}{\sqrt{2\alpha}|\sin 2\alpha x + 2\alpha(x-z)|}. \tag{4.6}$$

Proof. The inequality (4.3) can be derived with the help of integration by parts,

$$\begin{aligned}
&\left|\int_{A-\alpha}^{A+\alpha} \cos t(x-x')\frac{dt}{t} - \left\{\frac{\sin[(A+\alpha)(x-x')]}{(A+\alpha)(x-x')} - \frac{\sin[(A-\alpha)(x-x')]}{(A-\alpha)(x-x')}\right\}\right| \\
&= \left|\int_{A-\alpha}^{A+\alpha} \frac{\sin t(x-x')}{x-x'}\frac{dt}{t^2}\right| = \left|2\int_{A-\alpha}^{A+\alpha}\frac{1}{t^2}d\frac{\sin^2[t(x-x')/2]}{(x-x')^2}\right| \\
&= \left|2\left\{\frac{\sin^2[(A+\alpha)(x-x')/2]}{(A+\alpha)^2(x-x')^2} - \frac{\sin^2[(A-\alpha)(x-x')/2]}{(A-\alpha)^2(x-x')^2}\right\} + 4\int_{A-\alpha}^{A+\alpha}\frac{\sin^2[t(x-x')/2]}{(x-x')^2}\frac{dt}{t^3}\right| \\
&\leqslant \frac{4}{(A-\alpha)^2(x-x')^2} + \frac{4}{(x-x')^2}\int_{A-\alpha}^{A+\alpha}\frac{dt}{t^3} \leqslant \frac{6}{(A-\alpha)^2(x-x')^2}.
\end{aligned}$$

The inequality (4.4) follows trivially from (3.2) of Part I. The inequalities (4.5) and (4.6) can be obtained with the help of (3.2) of Part I,

$$\begin{aligned}
&\left|\psi_0(x) - \sqrt{2\alpha}\frac{\cos\alpha x}{x-z}\right| = \frac{\sqrt{2\alpha}|\sin 2\alpha x\cos\alpha x|}{|(x-z)[\sin 2\alpha x + 2\alpha(x-z)]|} \leqslant \frac{\sqrt{2\alpha}}{|x-z|\,|\sin 2\alpha x + 2\alpha(x-z)|}, \\
&\left|\psi_1(x) - \frac{1}{\sqrt{2\alpha}}\sin\alpha x\right| = \frac{|\cos 2\alpha x\cos\alpha x|}{\sqrt{2\alpha}|\sin 2\alpha x + 2\alpha(x-z)|} \leqslant \frac{1}{\sqrt{2\alpha}|\sin 2\alpha x + 2\alpha(x-z)|}.
\end{aligned}$$

Lemma 4.4 is proved. ■

The applicability of the resolution of identity (3.7) of Part I for some bounded and slowly increasing test functions is based on the next theorem.

Theorem 4.2. *Suppose that*

(1) *the function $\psi(x;k)$ is defined by the formula (3.1) of Part I for fixed $\alpha > 0$, $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$ and any $x \in \mathbb{R}$, $k \in \mathbb{C}$, $k \neq \pm\alpha$;*

(2) *$\mathcal{L}(A)$ is an integration path in complex k plane, obtained from the segment $[-A, A]$, $A > \alpha$ by its simultaneous deformation near the points $k = -\alpha$ and $k = \alpha$ upwards or downwards and the direction of $\mathcal{L}(A)$ is specified from $-A$ to A;*

(3) *the function $\eta(x) \in C_{\mathbb{R}}^{\infty}$, $\eta(x) \equiv 0$ for any $x \leqslant 1$, $\eta(x) \in [0, 1]$ for any $x \in [1, 2]$ and $\eta(x) \equiv 1$ for any $x \geqslant 2$.*

Then for any $\varkappa \in [0,1)$, $k_0 \in \mathbb{R}$ *and* $x' \in \mathbb{R}$ *the following relation holds,*

$$\lim_{A\to+\infty}\int_{-\infty}^{+\infty}\left[\int_{\mathcal{L}(A)}\psi(x;k)\psi(x';-k)\,dk\right]\left[\eta(\pm x)e^{ik_0x}|x|^{\varkappa}\right]dx=\eta(\pm x')e^{ik_0x'}|x'|^{\varkappa}.$$

Proof of Theorem 4.2 is quite analogous to the proof of Theorem 2 from Appendix B of [4] and it is based on the inequalities from Lemma 4.4.

Remark 4.1. Theorems 4.1 and 4.2 provide the validity of the resolution of identity (3.7) of Part I for test functions which are linear combinations of functions $\eta(\pm x)e^{ik_0x}|x|^{\varkappa}$, in general, with different $\varkappa \in [0,1)$ and $k_0 \in \mathbb{R}$ and functions from CL_γ, in general, with different $\gamma > -1$. In particular, these theorems guarantee applicability of (3.7) of Part I to the eigenfunctions $\psi(x;k)$ and to the associated function $\psi_1(x)$ of the Hamiltonian h (see Part I).

The resolutions of identity (3.8) and (3.9) of Part I are corollaries of the resolution of identity (3.7) of Part I and of the following Lemma 4.5.

Lemma 4.5. *Suppose that*

(1) *the functions* $\psi(x;k)$, $\psi_0(x)$ *and* $\psi_1(x)$ *are defined by the formulas* (3.1) *and* (3.2) *of Part* I *for fixed* $\alpha > 0$, $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$ *and any* $x \in \mathbb{R}$, $k \in \mathbb{C}$, $k \neq \pm\alpha$*;*

(2) $\mathcal{L}_\pm(k_0;\varepsilon)$ *with fixed* $k_0 \in \mathbb{R}$ *and* $\varepsilon > 0$ *is an integration path in complex* k *plane defined by*

$$k = k_0 + \varepsilon[\cos(\pi-\vartheta) \pm i\sin(\pi-\vartheta)], \qquad 0 \leqslant \vartheta \leqslant \pi,$$

where the upper (*lower*) *sign corresponds to the upper* (*lower*) *index of* $\mathcal{L}_\pm(k_0;\varepsilon)$, *and the direction of* $\mathcal{L}_\pm(k_0;\varepsilon)$ *is specified from* $\vartheta = 0$ *to* $\vartheta = \pi$.

Then for any $x, x' \in \mathbb{R}$ *and* $\varepsilon \in (0,\alpha)$ *the following relation is valid,*

$$\begin{aligned}
&\left(\int_{\mathcal{L}_\pm(-\alpha;\varepsilon)}+\int_{\mathcal{L}_\pm(\alpha;\varepsilon)}\right)\psi(x;k)\psi(x';-k)\,dk\\
&\quad=\frac{2}{\pi}\cos\alpha(x-x')\frac{\sin\varepsilon(x-x')}{x-x'}-\frac{1}{\pi\alpha}\psi_0(x)\psi_0(x')\left\{\frac{1}{\varepsilon}\left[1-2\sin^2\frac{\varepsilon}{2}(x-x')\right]\right.\\
&\quad\quad\left.-\frac{\varepsilon}{4\alpha^2-\varepsilon^2}\cos 2\alpha(x-x')\cos\varepsilon(x-x')-\frac{2\alpha}{4\alpha^2-\varepsilon^2}\sin 2\alpha(x-x')\sin\varepsilon(x-x')\right\}\\
&\quad\quad-\frac{1}{\pi}[\psi_0(x)\psi_1(x')+\psi_1(x)\psi_0(x')]\int_{2\alpha-\varepsilon}^{2\alpha+\varepsilon}\cos t(x-x')\frac{dt}{t}.
\end{aligned}\tag{4.7}$$

(4.7) follows trivially from the same representation of the integrand $\psi(x;k)\psi(x';-k)$ as in the proof of Lemma 4.1.

Proof of the resolution of identity (3.10) of Part I is based on the following Lemmas 4.6 and 4.7.

Lemma 4.6. *In the conditions of Lemma* 4.5 *for any* $x' \in \mathbb{R}$ *and* $\gamma > -1$ *the following relation takes place,*

$$\mathop{\lim\nolimits'_\gamma}_{\varepsilon\downarrow 0}\left\{\psi_0(x)\left[\varepsilon\cos 2\alpha(x-x')\cos\varepsilon(x-x')+2\alpha\sin 2\alpha(x-x')\sin\varepsilon(x-x')\right]\right\}=0.$$

Proof. The fact that for any $\gamma > -1$ the relation

$$\lim_{\varepsilon\downarrow 0}{}'_{\gamma} \left\{\psi_0(x)\big[2\alpha\sin 2\alpha(x-x')\sin\varepsilon(x-x')\big]\right\} = 0$$

holds follows from Lemma 3.7 in view of (3.2) of Part I. Hence, to prove Lemma 4.6, it is sufficient to show that for any $\gamma > -1$ the relation

$$\lim_{\varepsilon\downarrow 0}{}'_{\gamma} \left\{\psi_0(x)\big[\varepsilon\cos 2\alpha(x-x')\cos\varepsilon(x-x')\big]\right\} = 0 \tag{4.8}$$

is valid. It is true that

$$\psi_0(x)\big[\varepsilon\cos 2\alpha(x-x')\cos\varepsilon(x-x')\big] \in L^2(\mathbb{R};(1+|x|)^{-\gamma}) \subset CL'_{\gamma}, \qquad \gamma > -1.$$

Thus, to prove (4.8) it is sufficient to establish that for any $\varphi(x) \in CL_{\gamma}$, $\gamma > -1$, the relation

$$\lim_{\varepsilon\downarrow 0} \int_{-\infty}^{+\infty} \psi_0(x)\big[\varepsilon\cos 2\alpha(x-x')\cos\varepsilon(x-x')\big]\varphi(x)\,dx = 0$$

holds. But in view of (3.2) of Part I its validity follows from the chain of inequalities,

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} \psi_0(x)\,\big[\varepsilon\cos 2\alpha(x-x')\cos\varepsilon(x-x')\big]\,\varphi(x)\,dx\right|^2 &\leqslant \varepsilon^2\left(\int_{-\infty}^{+\infty}|\psi_0(x)\varphi(x)|\,dx\right)^2\\ \leqslant \varepsilon^2\int_{-\infty}^{+\infty}\frac{|\psi_0^2(x)|}{(1+|x|)^{\gamma}}\,dx\int_{-\infty}^{+\infty}|\varphi^2(x)|(1+|x|)^{\gamma}\,dx &\to 0, \qquad \varepsilon\downarrow 0,\end{aligned}$$

derived with the help of the Bunyakovskii inequality. Thus, Lemma 4.6 is proven. ∎

Lemma 4.7. *In the conditions of Lemma 4.5 for any $x' \in \mathbb{R}$ and $\gamma > -1$ the following relation holds,*

$$\lim_{\varepsilon\downarrow 0}{}'_{\gamma} \left\{[\psi_0(x)\psi_1(x') + \psi_1(x)\psi_0(x')]\int_{2\alpha-\varepsilon}^{2\alpha+\varepsilon}\cos t(x-x')\frac{dt}{t}\right\} = 0.$$

Proof of Lemma 4.7 with the help of the estimate from Lemma 3 from Appendix B of [4] and the Bunyakovskii inequality is quite analogous to the proof of Lemma 4 from Appendix B of [4].

Corollary 4.1. *The resolution of identity* (3.10) *of Part* I *follows from the resolution of identity* (3.8) *of Part* I *and from Lemmas 4.6, 4.7 and 3.7.*

The resolution of identity (3.11) of Part I is a corollary of the resolution of identity (3.10) of Part I and of the following Lemma 4.8.

Lemma 4.8. *In the conditions of Lemma 4.5 for any $x' \in \mathbb{R}$ and $\gamma > 1$ the following relation takes place,*

$$\lim_{\varepsilon\downarrow 0}{}'_{\gamma} \left\{\psi_0(x)\left[\frac{1}{\varepsilon}\sin^2\frac{\varepsilon}{2}(x-x')\right]\right\} = 0.$$

Proof of Lemma 4.8 is analogous to the proof of Lemma 3 from Appendix of [3].

Remark 4.2. Let us consider the functional

$$\lim_{\varepsilon\downarrow 0}{}^{\prime\prime}_{\gamma} \left[\frac{2}{\pi\varepsilon\alpha}\sin^2\frac{\varepsilon}{2}(x-x')\psi_0(x)\psi_0(x')\right], \tag{4.9}$$

where $\psi_0(x)$ is the eigenfunction (3.2) of Part I, which is defined by the expression

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{+\infty}\left[\frac{2}{\pi\varepsilon\alpha}\sin^2\frac{\varepsilon}{2}(x-x')\,\psi_0(x)\psi_0(x')\right]\varphi(x)\,dx \tag{4.10}$$

for all test functions $\varphi(x)\in CL_\gamma$, $\gamma\in\mathbb{R}$, for which the limit (4.10) exists. It follows from Lemma 4.8 that the functional (4.9) is trivial (equal to zero) for any $\gamma>1$, but at the same time, in view of the formula (3.12) from [1], this functional is nontrivial (different from zero) for any $\gamma<1$. By virtue of Lemma 4.8 the restriction of the functional (4.9) on the standard space $\mathcal{D}(\mathbb{R})\subset CL_\gamma$, $\gamma\in\mathbb{R}$ is equal to zero. Hence, the support of this functional for any $\gamma\in\mathbb{R}$ does not contain any finite real number. On the other hand, one can represent any test function $\varphi(x)\in CL_\gamma$, $\gamma\in\mathbb{R}$ for any $R>0$ as a sum of two functions from CL_γ in the form

$$\varphi(x)=\eta(|x|-R)\varphi(x)+[1-\eta(|x|-R)]\varphi(x),\qquad R>0, \tag{4.11}$$

where $\eta(x)\in C^\infty_{\mathbb{R}}$, $\eta(x)\equiv 1$ for any $x<0$, $\eta(x)\in[0,1]$ for any $x\in[0,1]$ and $\eta(x)\equiv 0$ for any $x>1$. In view of Lemma 4.8 the value of the functional (4.9) for $\varphi(x)$ is equal to its value for the second term of (4.11) for any arbitrarily large $R>0$. Hence, the value of the functional (4.9) for a test function depends only on the behavior of this function in any arbitrarily close (in the conformal sense) vicinity of the infinity and is independent of values of the function in any finite interval of real axis. In this sense the support of the functional (4.9) for any $\gamma<1$ consists of the unique element which is the infinity. At last, since (i) for any $\varphi(x)\in CL_\gamma$ and $\gamma\in\mathbb{R}$ the relation

$$\lim_{R\to+\infty}{}_{\gamma}\ \eta(|x|-R)\varphi(x)=\varphi(x)$$

holds; (ii) the restriction of the functional (4.9) on $\mathcal{D}(\mathbb{R})$ is zero for any $\gamma\in\mathbb{R}$ and (iii) the functional (4.9) is nontrivial for any $\gamma<1$, so the functional (4.9) for any $\gamma<1$ is discontinuous.

Acknowledgments

This work was supported by Grant RFBR 09-01-00145-a and by the SPbSU project 11.0.64.2010.

References

[1] Andrianov A.A., Sokolov A.V., Resolutions of identity for some non-Hermitian Hamiltonians. I. Exceptional point in continuous spectrum, *SIGMA* **7** (2011), 111, 19 pages, arXiv:1107.5911.

[2] Gel'fand I.M., Vilenkin N.J., Generalized functions, Vol. 4, Some applications of harmonic analysis, Academic Press, New York, 1964.

[3] Sokolov A.V., Andrianov A.A., Cannata F., Non-Hermitian quantum mechanics of non-diagonalizable Hamiltonians: puzzles with self-orthogonal states, *J. Phys. A: Math. Gen.* **39** (2006), 10207–10227, quant-ph/0602207.

[4] Andrianov A.A., Cannata F., Sokolov A.V., Spectral singularities for non-Hermitian one-dimensional Hamiltonians: puzzles with resolution of identity, *J. Math. Phys.* **51** (2010), 052104, 22 pages, arXiv:1002.0742.