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SOME RESULTS ON THE WELL-POSEDNESS OF AN INTEGRO-DIFFERENTIAL FRÉMOND MODEL FOR SHAPE MEMORY ALLOYS

Abstract. This note deals with the nonlinear three-dimensional Frémond model for shape memory alloys in the case when the heat flux law contains a thermal memory term. The abstract formulation of the initial and boundary value problem for the resulting system of PDE's is considered. Existence and uniqueness of the solutions can be proved by exploiting a time discretization semi-implicit scheme, combined with an *a priori estimate - passage to the limit procedure*, as well as by performing suitable contracting estimates on the solutions.

1. Introduction

This note is concerned with a mathematical model describing the thermo-mechanical evolution of a class of shape memory alloys (metallic alloys characterized by the possibility of recovering, after deformations, their original shape just by thermal means), in the case one takes into account some memory term in the heat flux law. We consider a three-dimensional initial-boundary value problem related to the thermo-mechanical model introduced by Frémond to describe the martensite-austenite phase transition in shape memory alloys (cf. [12, 13, 14, 15]). The difference between the problem we are investigating and the classical Frémond model is given by the fact that we do not refer to the standard Fourier law for the heat flux and, consequently, we deal with a different equation describing the energy balance.

The shape memory effect can be ascribed to a phase transition between two different configurations of the metallic lattice (martensite and austenite) and it results from the occurrence of an hysteretic behavior, shown as to a strong dependence of the load-deformation diagrams on temperature. The model proposed by Frémond describes the phenomenon from a macroscopic point of view and it can be applied to any dimension of space. Concerning the two phases, we recall that only two variants of martensite and one variant of austenite are considered and it is supposed they may coexist at each point. Hence, on account of the expression of the free-energy and by applying the conservation laws for energy and momentum (in the quasi-stationary case), one can deduce the constitutive equations of the model in accordance with the second principle of thermodynamics (cf. e.g. [4, 14]).

The unknowns of the resulting PDE's system are the absolute temperature θ, the vector of displacements $\mathbf{u}$, and two phase variables (χ_1, χ_2) that are linearly related to the volume fractions of martensite and austenite. Indeed, as it is assumed that no void nor overlapping can occur between the phases, denoting by (β_1, β_2) the volume fractions of martensitic variants and by β_3 the volumic fraction of austenite, it turns out physically consistent to require that

$$0 \le \beta_i \le 1,\ i = 1, 2, 3 \quad \text{and} \quad \sum_{i=1}^{3} \beta_i = 1. \tag{1}$$

Thus, we can fix as state variables (note that $\beta_3 = 1 - \beta_1 - \beta_2$)

$$\chi_1 := \beta_1 + \beta_2 \quad \text{and} \quad \chi_2 = \beta_2 - \beta_1. \tag{2}$$

We refer to [4] and references therein for a detailed argumentation on the mathematical derivation of the model as well as for a discussion on the mechanical aspects. Moreover, in [14, 15] it is shown that the model by Frémond predicts a behavior of the solutions which is in accordance with experimental results. Hence, we introduce a positive, bounded, and Lipschitz continuous function α, which vanishes over a critical temperature θ_c (the Curie temperature) with $\theta_c > \theta^*$, θ^* denoting the equilibrium temperature. Thus, by referring to a sample of shape memory isotropic material, located in a bounded smooth domain $\Omega \subset \mathbb{R}^3$, and after fixing a final time T, the system of PDE's describing the thermo-mechanical evolution, in $Q := \Omega \times (0, T)$, reads as follows

$$\begin{aligned} &(c_0 - \theta\alpha''(\theta)\chi_2 \operatorname{div}\mathbf{u})\partial_t\theta + \operatorname{div}\mathbf{q} = f + L\partial_t\chi_1 \\ &\quad +(\theta\alpha'(\theta) - \alpha(\theta)) \operatorname{div}\mathbf{u}\partial_t\chi_2 + \theta\alpha'(\theta)\chi_2\partial_t(\operatorname{div}\mathbf{u}), \end{aligned} \tag{3}$$

$$\operatorname{div}(-\nu\Delta(\operatorname{div}\mathbf{u})\mathbf{1} + \lambda \operatorname{div}\mathbf{u}\mathbf{1} + 2\mu\epsilon(\mathbf{u}) + \alpha(\theta)\chi_2\mathbf{1}) + \mathbf{s} = 0, \tag{4}$$

$$\zeta\partial_t \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \begin{pmatrix} l(\theta - \theta^*) \\ \alpha(\theta) \operatorname{div}\mathbf{u} \end{pmatrix} + \partial I_{\mathcal{K}}(\chi_1, \chi_2) \ni \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5}$$

where $\mathbf{1}$ denotes the identity matrix, $\epsilon(\mathbf{u})$ the linearized strain tensor, $\mathbf{q}$ the heat flux, f stands for an external heat source, $\mathbf{s}$ for the vector of the volume forces, and $\partial I_{\mathcal{K}}$ is the subdifferential of the indicator function $I_{\mathcal{K}}$ of a suitable convex subset $\mathcal{K}$ of $\mathbb{R}^2$ (a triangle with one of the vertices at the origin) and it accounts for the constraint on the phases (1) to attain only physically meaningful values. We just point out that c_0, μ, ν, λ, L, l, and ζ are strictly positive constants (see e.g. [14] for the mechanical meanings of the above constants). By virtue of (1) and (2), $\mathcal{K}$ can be taken i.e. as follows

$$\mathcal{K} := \{(\gamma_1, \gamma_2) \in \mathbb{R}^2 : 0 \le |\gamma_2| \le \gamma_1 \le 1\}. \tag{6}$$

For the reader's convenience, we also recall that $I_{\mathcal{K}}(\mathbf{y}) = 0$ if $\mathbf{y} \in \mathcal{K}$ and $I_{\mathcal{K}}(\mathbf{y}) = +\infty$ otherwise. Moreover, since $\mathcal{K}$ is a closed convex set, $\partial I_{\mathcal{K}}$ turns out a maximal monotone graph such that (cf. [5])

$$(y_1, y_2) \in \partial I_{\mathcal{K}}(\chi_1, \chi_2) \text{ if and only if } (\chi_1, \chi_2) \in \mathcal{K}$$

(7) $$\text{and } \sum_{i=1}^{2} y_i(x_i - \chi_i) \leq 0,\ \forall (x_1, x_2) \in \mathcal{K}.$$

In particular, we stress that if $(\chi_1, \chi_2) \notin \mathcal{K}$ then $\partial I_{\mathcal{K}}(\chi_1, \chi_2)$ is the empty set, if (χ_1, χ_2) belongs to the interior of $\mathcal{K}$ then $\partial I_{\mathcal{K}}(\chi_1, \chi_2) = (0, 0)$, and if (χ_1, χ_2) lies on the boundary of $\mathcal{K}$ then $\partial I_{\mathcal{K}}(\chi_1, \chi_2)$ coincides with the cone of normal vectors to $\mathcal{K}$ at point (χ_1, χ_2). We should also remark that, under the small perturbations assumption, equations (3) and (4) correspond to the balance laws for energy and momentum (in the quasi-stationary case), respectively, while the evolution of the phases (χ_1, χ_2) is governed by the inclusion (5) that could be rewritten as a pointwise variational inequality (cf. (7)). Finally, the system (3)-(5) has to be supplied by suitable initial and boundary conditions. In particular, we prescribe (natural) Cauchy conditions for θ and (χ_1, χ_2)

(8) $$\theta(0) = \theta^0, \quad \chi_1(0) = \chi_1^0, \quad \chi_2(0) = \chi_2^0,$$

and appropriate boundary conditions on θ and $\mathbf{u}$. We consider the boundary Γ of Ω be partied in Γ_0 and Γ_1 and we require that they are (measurable) sets with positive surface measures. Indicating by $\mathbf{n}$ the outer unit normal vector to the boundary Γ, we state

(9) $$\mathbf{q} \cdot \mathbf{n} = -h \quad \text{on } \Gamma \times (0, T),$$

(10) $$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_0 \times (0, T),$$

(11) $$((-\nu\Delta(\operatorname{div}\mathbf{u}) + \lambda \operatorname{div}\mathbf{u} + \alpha(\theta)\chi_2)\mathbf{1} + 2\mu\epsilon(\mathbf{u})) \cdot \mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_1 \times (0, T),$$

(12) $$\partial_{\mathbf{n}}(\operatorname{div}\mathbf{u}) = 0 \quad \text{on} \Gamma \times (0, T).$$

The above conditions mean that the heat flux h through the boundary (cf. (9)) and an external traction $\mathbf{g}$ (cf. (11)) are known, while neither displacements on Γ_0 (cf. (10)) nor double forces on Γ (cf. (12)) occur.

As we have already stressed, the standard Frémond model is given by equations (3)–(5) in which the heat flux $\mathbf{q}$ is assumed to fulfil the classical Fourier heat flux law, namely

(13) $$\mathbf{q}(x, t) = -k_0 \nabla\theta(x, t), \quad (x, t) \in Q,$$

with $k_0 > 0$. Here, we would like to discuss other possible different choices for the form of the heat flux $\mathbf{q}$ to combine with the energy balance (3). Indeed, our work is related to the problem of representing heat transported by conduction in which the heat pulses are transmitted by waves at a finite but possibly high speeds (cf. [18, 19] for a complete and detailed physical presentation of this subject). In the linearized theory, the heat flux is determined by an integral over the history of the temperature gradient weighted against a relaxation function $\widetilde{k}$ called *heat flux kernel*. More precisely, the heat flux $\mathbf{q}$ is assumed to be governed by the following relation

(14) $$\mathbf{q}(x, t) = -\int_{-\infty}^{t} \widetilde{k}(t - s)\nabla\theta(x, s)\, ds.$$

Let us point out that, in general, the thermal history of the material is assumed to be known up to the time $t = 0$, i.e. the term $\int_{-\infty}^{0} \widetilde{k}(t-s)\nabla\theta(x,s)\,ds$ is considered as a datum. Thus, in the sequel, we will denote by f (cf. (3)) a heat source term, accounting both for external thermal actions and for the past history of the temperature gradient. Many different constitutive models arise from different choices of the kernel $\widetilde{k}$ in (14). Note that if one considers $\widetilde{k}(s) = k_0\delta(s)$ (δ being the Dirac mass) one can recover the classical Fourier law (13). The Frémond model coupled with the Fourier law has been deeply investigated and existence as well as uniqueness of the solutions have been proved (cf., among the others, [6, 8, 9, 10, 11]). A different approach consists in taking into account a Jeffrey type kernel (formally derived from elasticity theory) that reads (cf. [18])

$$\widetilde{k}(s) = k_0\delta(s) + \frac{k_1}{\tau}\exp(-s/\tau), \tag{15}$$

in which an effective Fourier conductivity k_0 is explicitly acknowledged and k_1 is a positive constant. In particular, note that if $k_0 = 0$ then (15) reduces to the known Cattaneo-Maxwell heat flux law (cf. [7] and a mathematical discussion in [3]). Thus, in general, in (14) one could take $\widetilde{k}$ as follows

$$\widetilde{k}(s) = k_0\delta(s) + k(s), \tag{16}$$

where $k_0 \geq 0$ and k, in general, denotes a positive type (cf. [16]) and sufficiently smooth function. Observe that, in the case when k_0 is strictly positive and k is not identically zero, (16) is known as the Coleman-Gurtin heat flux law. In [2] we have investigated the thermo-mechanical Frémond model for shape memory alloys in the framework of Gurtin and Pipkin's theory (cf. [17]), which is characterized by the fact that no Dirac mass is considered in the kernel $\widetilde{k}$ ($k_0 = 0$ in (16)). In particular, by use of a fixed point argument and contracting estimates on the solutions, we have proved well-posedness of the initial and boundary value problem related to a slightly modified version of the PDE's system (3)-(5) , which is obtained by taking the equilibrium equation (4) , by linearizing the energy balance (3) (cf. [9] for a similar approximation)

$$c_0\partial_t\theta - k * \Delta\theta = f + L\partial_t\chi_1, \tag{17}$$

$*$ denoting the usual convolution product over $(0,t)$, and by adding a diffusive term in the variational inclusion describing the phases dynamics (cf. (5))

$$\zeta\,\partial_t \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} - \eta \begin{pmatrix} \Delta\chi_1 \\ \Delta\chi_2 \end{pmatrix} + \begin{pmatrix} l(\theta-\theta^*) \\ \alpha(\theta)\,\mathrm{div}\,\mathbf{u} \end{pmatrix} + \partial I_{\mathcal{K}}(\chi_1,\chi_2) \ni \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{18}$$

η being a strictly positive parameter. The reader can easily observe that (17) is obtained by neglecting the nonlinear terms in (3)and substituting $\mathbf{q}$ by (14) and (16) with $k_0 = 0$. In addition, we stress that, as it is assumed $k(0) > 0$, (17) turns out to be of hyperbolic type. On a second step, in [3] we have discussed the model in which the Cattaneo-Maxwell heat flux law is assumed, which corresponds to specify $k(s) = \frac{k_1}{\tau}\exp(-s/\tau)$ in (17). By letting diffusive dynamics for the phases (18), the resulting model turns out to be a singular perturbation of the standard Frémond's one, in which the heat flux is governed by the Fourier law and no diffusion for the phases is considered. Indeed, by performing a rigorous asymptotic analysis as the relaxation parameter τ and the diffusion parameter η tend to zero, we have proved that the resulting model converges to the classical Frémond model. This way of proceeding can be formally justified after observing that the Cattaneo-Maxwell relaxation kernel $k(s) = \frac{k_1}{\tau}\exp(-s/\tau)$ approximates, in some suitable sense, the measure $k_1\delta(s)$. Moreover, meaningful error estimates are established under some compatibility assumptions on the rates of convergence of the two parameters τ and η. Nonetheless, the presence of thermal memory forces us to deal with some mathematical difficulties strictly connected with this assumption and an existence result for the complete problem (cf. (3)-(5)) seems very hard to be proved. For this reason, in order to include thermal memory effects in the complete energy balance, we restrict ourselves to the case of the Coleman-Gurtin heat flux law, namely we consider the heat flux kernel k as in (16), but we assume that $k_0 > 0$. Thus, the resulting energy balance retains its parabolic behavior even if it accounts for thermal memory. As a consequence, we do not need to mollify the dynamics of the phases by introducing a diffusive term (cf. (5)and (18)). In this note an existence result is established for an abstract version of the resulting initial and boundary value problem by use of a semi-implicit time discretization scheme combined with an a priori estimate-passage to the limit procedure. In particular, let us stress the presence of a convolution product in the energy balance, following from (14) and (16), as one can easily check by specifying the term (div $\mathbf{q}$ in (3), as follows

$$\text{(19)} \qquad \operatorname{div}\mathbf{q}(x,t) = -k_0\Delta\theta(x,t) - k * \Delta\theta(x,t) - \int_{-\infty}^{0} k(t-s)\Delta\theta(x,s)\,ds,$$

for $(x,t) \in Q$; we recall that the term $-\int_{-\infty}^{0} k(t-s)\Delta\theta(x,s)\,ds$ has to be included in the energy balance as a datum in a given heat source f. Concerning the discretization procedure we have to point out that we treat convolution as an explicit term (cf. [1]). Finally, an uniqueness result is proved by use of suitable contracting estimates on the solutions of the problem, exploiting a similar argument as that introduced by Chemetov in [6].

2. Main results

We can now specify the abstract version of the problem we are dealing with and state the related main existence and uniqueness result. To this purpose, let $V \hookrightarrow H \hookrightarrow V'$ be an Hilbert triplet, with

$$H := L^2(\Omega), \qquad V := H^1(\Omega),$$

and identify, as usual, H with its dual space H'. Moreover, to write the variational formulation of (4), we introduce an appropriate Hilbert space $\mathbf{W}$ specified by

$$\mathbf{W} := \{\mathbf{v} \in (H^1(\Omega))^3 : \mathbf{v}_{|\Gamma_0} = \mathbf{0},\ \operatorname{div}\mathbf{v} \in H^1(\Omega)\},$$

endowed with the norm (cf. [8])

$$\|\mathbf{v}\|_{\mathbf{W}} := \left(\nu \int_\Omega |\nabla(\operatorname{div}\mathbf{v})|^2 + \sum_{i=1}^{3} \int_\Omega |\nabla v_i|^2 \right)^{1/2}, \quad \mathbf{v} = (v_1, v_2, v_3) \in \mathbf{W}.$$

Hence, we define a bilinear symmetric continuous form in $\mathbf{W} \times \mathbf{W}$ as follows: for $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{W}$, we set

$$a(\mathbf{v}, \mathbf{w}) := \int_\Omega \left(\nu \nabla(\operatorname{div}\mathbf{v}) \cdot \nabla(\operatorname{div}\mathbf{w}) + \lambda \operatorname{div}\mathbf{v} \operatorname{div}\mathbf{w} + 2\mu \sum_{i,j=1}^{3} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w}) \right)$$

with $\epsilon_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_{x_i} v_j + \partial_{x_j} v_i)$. Let us note that, thanks to the Korn's inequality, a turns out to be $\mathbf{W}$-coercive, namely there exists a positive constant C such that

$$a(\mathbf{v}, \mathbf{v}) \geq C \, \|\mathbf{v}\|^2_{\mathbf{W}}, \quad \forall\, \mathbf{v} \in \mathbf{W}.$$

We also outline that we could take $\mathcal{K}$ (cf. (6)) as any bounded, closed, and convex subset of $\mathbb{R}^2$ and then introduce the corresponding closed and convex subset of H^2

$$K := \{(\gamma_1, \gamma_2) \in H^2 : (\gamma_1, \gamma_2) \in \mathcal{K} \text{ a.e. in } \Omega\}.$$

Note that, by construction, there exists a positive constant c_K (depending on $\mathcal{K}$) such that if $(\gamma_1, \gamma_2) \in K$ there holds

$$(|\gamma_1(x)|^2 + |\gamma_2(x)|^2)^{1/2} \leq c_K, \quad \text{for a.e. } x \in \Omega. \tag{20}$$

In the following, we will denote by ∂I_K the subdifferential of the indicator function of the convex K, which turns out to be a maximal monotone operator in H, naturally induced by $\partial I_{\mathcal{K}}$ (cf. [5]). Hence, in order to write the abstract equivalent version of the problem given by (3)-(5) and (8)-(12), we introduce the following operators (cf. [2])

$$A : V \to V', \quad {}_{V'}\langle A v_1, v_2 \rangle_V = \int_\Omega \nabla v_1 \cdot \nabla v_2, \quad v_1, v_2 \in V,$$
$$\mathcal{H} : \mathbf{W} \to \mathbf{W}', \quad {}_{\mathbf{W}'}\langle \mathcal{H}\mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathbf{W}} = a(\mathbf{v}_1, \mathbf{v}_2), \quad \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{W},$$
$$\mathcal{B} : H \to \mathbf{W}', \quad {}_{\mathbf{W}'}\langle \mathcal{B} v, \mathbf{v} \rangle_{\mathbf{W}} = \int_\Omega v \operatorname{div}\mathbf{v}, \quad v \in H, \mathbf{v} \in \mathbf{W}.$$

Finally, concerning the data of the problem, we prescribe that

$$f \in L^2(0, T; L^2(\Omega)),$$
$$h \in H^1(0, T; L^2(\Gamma)),$$
$$\mathbf{g} \in H^1(0, T; L^2(\Gamma_1)^3),$$
$$\mathbf{s} \in H^1(0, T; L^2(\Omega)^3),$$
$$\Theta^0 \in H^1(\Omega), \tag{21}$$
$$(\chi_1^0, \chi_2^0) \in K, \tag{22}$$

so that, in the abstract formulation, we can introduce the corresponding functions

$$F \in L^2(0,T;H), \quad {}_{V'}\langle F,\, v\rangle_V = \int_\Omega f v,\ v \in V \tag{23}$$

$$H \in H^1(0,T;V'), \quad {}_{V'}\langle H,\, v\rangle_V = \int_\Gamma h v_{|\Gamma},\ v \in V, \tag{24}$$

$$\mathbf{G} \in H^1(0,T;\mathbf{W}'), \quad {}_{\mathbf{W}'}\langle \mathbf{G}, \mathbf{v}\rangle_{\mathbf{W}} = \int_{\Gamma_1} \mathbf{g}\cdot \mathbf{v}_{|\Gamma_1},\ \mathbf{v} \in \mathbf{W}, \tag{25}$$

$$\mathbf{S} \in H^1(0,T;H^3), \quad {}_{\mathbf{W}'}\langle \mathbf{S}, \mathbf{v}\rangle_{\mathbf{W}} = \int_\Omega \mathbf{s}\cdot \mathbf{v}, \mathbf{v} \in \mathbf{W}. \tag{26}$$

Moreover, we have to precise the assumptions on the kernel k in (19) and the function α. Precisely, we require that

$$k \in W^{1,1}(0,T), \tag{27}$$

and α fulfils

$$\alpha \in C^2(\mathbb{R}) \quad \text{and} \quad c_\alpha := \left\|\alpha''\right\|_{L^\infty(\mathbb{R})} \text{ is sufficiently small.} \tag{28}$$

Hence, as at high temperatures shape memory alloys present mostly an elastic behavior, $\alpha(\theta) = 0$ for $\theta \geq \theta_c$ and in addition we assume

$$\{\gamma \in \mathbb{R} : \alpha'(\gamma) \neq 0\} \subset [0, \theta_c]. \tag{29}$$

Note that, as a consequence, the functions of the variable θ in the nonlinear terms of (3) turn out continuous and uniformly bounded. Indeed, we observe that (28) and (29) imply

$$|\alpha'(\gamma)| \leq \theta_c c_\alpha, \quad |\gamma\alpha'(\gamma)| \leq \theta_c^2 c_\alpha, \quad \forall \gamma \in \mathbb{R}. \tag{30}$$

REMARK 1. As to concerns the constant c_α and the second of (28), it turns out necessary to assume some compatibility conditions (satisfied by physically realistic data) between the quantities involved in the model and the heat capacity of the system. Indeed, the coefficient of the temperature time derivative in the energy balance represents the specific heat of the solid-solid phase transition and it seems physically consistent to require it is positive everywhere. To this aim, later we will specify (28) by letting a suitable bound for c_α.

Now, we are in the position of stating the existence and uniqueness result referring to (3)-(5) and (8)-(12) combined with (19).

THEOREM 1. *Assume that (21)-(22), (23)-(26) and (27)-(29) hold. Then, there exists a unique quadruplet* $(\theta, \chi_1, \chi_2, \mathbf{u})$, *with*

$$\theta \in H^1(0,T;H) \cap L^\infty(0,T;V), \tag{31}$$

$$\chi_j \in W^{1,\infty}(0,T;H) \cap L^\infty(Q), \quad j = 1,2 \tag{32}$$

$$\mathbf{u} \in H^1(0,T;\mathbf{W}), \quad \operatorname{div}\mathbf{u} \in L^\infty(Q), \tag{33}$$

fulfilling

$$\theta(0) = \Theta^0, \tag{34}$$

$$(\chi_1(0), \chi_2(0)) = (\chi_1^0, \chi_2^0), \tag{35}$$

and satisfying, almost everywhere in $(0, T)$*,*

$$\begin{aligned}&(c_0 - \theta\alpha''(\theta)\chi_2 \operatorname{div}\mathbf{u})\partial_t\theta + k_0 A\theta + k * A\theta = F + H + L\partial_t\chi_1\\ &\quad+(\theta\alpha'(\theta) - \alpha(\theta)) \operatorname{div}\mathbf{u}\partial_t\chi_2 + \theta\alpha'(\theta)\chi_2\partial_t(\operatorname{div}\mathbf{u}) \quad \textit{in} \quad V'\end{aligned} \tag{36}$$

$$\zeta\partial_t \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \partial I_K(\chi_1, \chi_2) \ni \begin{pmatrix} -l(\theta - \theta^*) \\ -\alpha(\theta) \operatorname{div}\mathbf{u} \end{pmatrix} \quad \textit{in} \quad H^2 \tag{37}$$

$$\mathcal{H}\mathbf{u} + \mathcal{B}(\alpha(\theta)\chi_2) = \mathbf{S} + \mathbf{G} \quad \textit{in } \mathbf{W}'. \tag{38}$$

In particular, the boundedness result in (33) for div $\mathbf{u}$ follows from the next lemma, which can be proved by use of the Lax-Milgram theorem and exploiting standard estimates and regularity results on elliptic equations (cf. [9]).

LEMMA 1. *Let* θ*,* χ_2 *belong to* $L^2(Q)$ *such that* $|\chi_2| \le c_K$ *a.e. in* Q*. Then, under assumptions (25), (26), (28), and (29), there exists a unique solution* $\mathbf{u} \in L^\infty(0, T; \mathbf{W})$ *solving the resulting equation (38). Moreover, the following bound holds*

$$\|\operatorname{div}\mathbf{u}\|_{L^\infty(Q)} \le c_1, \tag{39}$$

for a constant c_1 *depending only on* Ω*,* C*,* $\|\alpha\|_{L^\infty(\mathbb{R})}$ *and the convex* $\mathcal{K}$*.*

In particular, the previous lemma allows us to specify hypothesis (28) (cf. Remark 1). Indeed, in order to get positivity of the coefficient of the temperature time derivative in the energy balance (36), by virtue of (20), (28), (29), and (39), it is now clear that it is sufficient to ask for a constant c_α sufficiently small in the sense that there holds (cf. [14, 15])

$$(c_0 - \theta\alpha''(\theta)\chi_2 \operatorname{div}\mathbf{u}) \ge c_2 := c_0 - \theta_c c_\alpha c_K c_1 > 0. \tag{40}$$

Let us in addition note that the specific heat turns out bounded

$$|c_0 - \theta\alpha''(\theta)\chi_2 \operatorname{div}\mathbf{u}| \le c_0 + \theta_c c_\alpha c_K c_1.$$

Finally, we have also to introduce a technical assumption, we need to exploit basic a priori estimates on the solutions of the problem (see, i.e., [8] for similar proceeding). Thus, we require that there holds

$$(\theta_c(\theta_c + 1)c_\alpha c_K)^2 \le c_2(\lambda + 2\mu/3). \tag{41}$$

Let us note that both (40) and (41) are in accordance with experiments (see [20]).

3. Proof of Theorem 1

Existence result stated by the Theorem 1 can be proved by applying a semi-implicit time discretization scheme combined with an a priori estimate - passage to the limit procedure. For the sake of synthesis we only outline the proof but omit the details for which we refer to [4]. We first introduce the time step of our backward finite differences scheme $\tau := T/N$, N being a fixed positive integer. Hence, the time discrete scheme for the problem (31)-(38) relies on the approximation of (36)-(38) by

$$\begin{aligned}&(c_0 - \Theta^{i-1}\alpha''(\Theta^{i-1})X_2^{i-1}\operatorname{div}\mathbf{U}^{i-1})\frac{\Theta^i - \Theta^{i-1}}{\tau} + k_0 A\Theta^i + (k * \mathcal{I}_\tau A\theta_\tau)^i \\ &= L\frac{X_1^i - X_1^{i-1}}{\tau} + (\Theta^{i-1}\alpha'(\Theta^{i-1}) - \alpha(\Theta^{i-1}))\operatorname{div}\mathbf{U}^{i-1}\frac{X_2^i - X_2^{i-1}}{\tau} \\ &+\Theta^{i-1}\alpha'(\Theta^{i-1})X_2^{i-1}\frac{\operatorname{div}\mathbf{U}^i - \operatorname{div}\mathbf{U}^{i-1}}{\tau} + F^i + H^i \quad \text{in } V' \end{aligned} \tag{42}$$

$$\zeta\begin{pmatrix}\frac{X_1^i - X_1^{i-1}}{\tau} \\ \frac{X_2^i - X_2^{i-1}}{\tau}\end{pmatrix} + \partial I_K(X_1^i, X_2^i) \ni \begin{pmatrix} -l(\Theta^i - \theta^*) \\ -\alpha(\Theta^i)\operatorname{div}\mathbf{U}^{i-1}\end{pmatrix} \quad \text{in } H^2 \tag{43}$$

$$\mathcal{H}U^i + \mathcal{B}(\alpha(\Theta^i)X_2^i) = \mathbf{G}^i + \mathbf{S}^i \quad \text{in } \mathbf{W}', \tag{44}$$

where $\mathcal{I}_\tau$ in (42) denotes the one step backward translation operator (i.e. $\mathcal{I}_\tau a(t) = a(t-\tau)$) and θ_τ the piecewise constant function related to the vector of solutions Θ^i by

$$\theta_\tau(t) = \Theta^i, \quad \text{if } t \in ((i-1)\tau, i\tau], \tag{45}$$

for $i = 1, ..., N$. Note that the term $(k * \mathcal{I}_\tau A\theta_\tau)^i$ turns out to be explicit in the scheme (see [1]). Finally, F^i, H^i, $\mathbf{G}^i$, and $\mathbf{S}^i$ stand for suitable time independent functions discretizing the data F, H, $\mathbf{G}$, and $\mathbf{S}$ (i.e. $F^i = \tau^{-1}\int_{(i-1)\tau}^{i\tau} F(s)\,ds$). Thus, if we let $\Theta^0 = \theta^0$, $X_{i0} = \chi_i^0$ for $i = 1, 2$, and $\mathbf{U}^0$ the corresponding unique solution of (44) written for $i = 0$ (cf. Lemma 1), by use of a fixed point theorem we are able to prove existence of a discrete solution for (42)-(44) for any $i \geq 1$, at least for τ sufficiently small. Henceforth, we perform suitable estimates on the discrete solutions independent of the parameter τ, in order to pass to the limit as $\tau \searrow 0$ by use of weak and weak star compactness arguments or by direct Cauchy proof. To this aim, let us introduce the following notation: given a $N+1$-vector of time independent functions $(a^0, ..., a^N)$ we term by a_τ the related piecewise constant function a_τ (cf. (45)) and by $\widetilde{a}_\tau$ the piecewise linear in time interpolation function, namely

$$\widetilde{a}_\tau(t) = a^i + \frac{a^i - a^{i-1}}{\tau}(t - i\tau), \quad t \in [(i-1)\tau, \tau]. \tag{46}$$

Thus, if we use the above notation, it is straightforward to rewrite the discrete system (42)-(44), as follows

$$(c_0 - \mathcal{I}_\tau(\theta_\tau\alpha''(\theta_\tau)\chi_{2\tau}\mathrm{div}\mathbf{u}_\tau))\partial_t\widetilde{\theta}_\tau + k_0 A\theta_\tau + (k * \mathcal{I}_\tau(A\theta_\tau))_\tau =$$
$$H_\tau + F_\tau + L\partial_t\widetilde{\chi}_{1\tau} + \mathcal{I}_\tau((\theta_\tau\alpha'(\theta_\tau) - \alpha(\theta_\tau))\mathrm{div}\mathbf{u}_\tau)\partial_t\widetilde{\chi}_{2\tau}$$
$$+\mathcal{I}_\tau(\theta_\tau\alpha'(\theta_\tau)\chi_{2\tau})\partial_t\mathrm{div}\widetilde{\mathbf{u}}_\tau \tag{47}$$

$$\zeta\begin{pmatrix}\partial_t\widetilde{\chi}_{1\tau}\\ \partial_t\widetilde{\chi}_{2\tau}\end{pmatrix} + \partial I_K(\chi_{1\tau},\chi_{2\tau}) \ni \begin{pmatrix}-l(\theta_\tau - \theta^*)\\ -\alpha(\theta_\tau)\mathcal{I}_\tau\mathrm{div}\mathbf{u}_\tau\end{pmatrix} \tag{48}$$

$$\mathcal{H}\mathbf{u}_\tau + \mathcal{B}(\alpha(\theta_\tau)\chi_{2\tau}) = \mathbf{G}\tau + \mathbf{S}\tau \tag{49}$$

with

$$\widetilde{\theta}_\tau(0) = \theta^0, \quad \widetilde{\chi}_{i\tau}(0) = \chi_i^0 \quad i = 1, 2. \tag{50}$$

Hence, by exploiting suitable a priori estimates on the system (42)-(44), we can prove that there exists $\sigma > 0$ such that for $\tau \le \sigma$ the following bounds hold independently of τ

$$\|\widetilde{\theta}_\tau\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\theta_\tau\|_{L^\infty(0,T;V)} \le c \tag{51}$$

$$\sum_{i=1}^{2}\|\widetilde{\chi}_{i\tau}\|_{H^1(0,T;H)\cap L^\infty(Q)} + \|\widetilde{\chi}_{i\tau}\|_{L^\infty(Q)} \le c \tag{52}$$

$$\|\widetilde{\mathbf{u}}_\tau\|_{H^1(0,T;\mathbf{W})} + \|\mathbf{u}_\tau\|_{L^\infty(0,T;\mathbf{W})} + \|\mathrm{div}\,\mathbf{u}_\tau\|_{L^\infty(Q)} \le c. \tag{53}$$

The reader can refer to [8] and [10] for a detailed presentation of an estimating procedure as that we have used to prove (51)-(53) and to [1] for a possible argument to handle the convolution product $(k * \mathcal{I}_\tau(A\theta_\tau))_\tau$. Thus, by use of compactness arguments from (51)-(53), and (45), (46), we can deduce up to subsequences the following convergence results, as $\tau \searrow 0$

$$\widetilde{\theta}_\tau \overset{*}{\rightharpoonup} \theta \text{ in } H^1(0,T;H)\cap L^\infty(0,T;V),$$
$$\widetilde{\theta}_\tau \to \theta \text{ in } C^0([0,T];H) \tag{54}$$

$$\theta_\tau \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(0,T;V), \quad \theta_\tau \to \theta \text{ in } L^\infty(0,T;H) \tag{55}$$

$$\widetilde{\chi}_{j\tau} \overset{*}{\rightharpoonup} \chi_j \text{ in } H^1(0,T;H)\cap L^\infty(Q), \quad \chi_{j\tau} \overset{*}{\rightharpoonup} \chi_j \text{ in } L^\infty(Q), \quad j = 1, 2 \tag{56}$$

$$\widetilde{\mathbf{u}}_\tau \rightharpoonup \mathbf{u} \text{ in } H^1(0,T;\mathbf{W}), \quad \mathbf{u}_\tau \overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0,T;\mathbf{W}),$$
$$\mathrm{div}\,\mathbf{u}_\tau \overset{*}{\rightharpoonup} \mathrm{div}\,\mathbf{u} \text{ in } L^\infty(Q). \tag{57}$$

In addition, by direct Cauchy arguments, we are able to infer that

$$\widetilde{\chi}_{j\tau} \to \chi_j \text{ in } C^0([0,T];H), \quad \chi_{j\tau} \to \chi_j \text{ in } L^\infty(0,T;H). \tag{58}$$

Thus, by (54)-(58), and thanks to the Lebesgue dominated convergence theorem, we are allowed to pass to the limit in (47)-(49) and get existence of a solution for the limit system of PDE's (36)-(38). In particular, let us stress that by the above argumentation (cf. (54), (58)) the Cauchy conditions (34) and (35) are eventually satisfied by virtue of (50). Henceforth, the regularity result (32) (cf. (56)) can be proved thanks to the monotonicity of ∂I_K and the regularity results (31) and (33). Finally, concerning uniqueness, we base our proof on a contradiction argument which is similar as that introduced in [6]. Now, we outline the proof and stress some mathematical devices we have used to exploit the contracting estimates and get uniqueness. We first consider two solutions $\mathcal{S}_1 = (\theta_1, \chi_{11}, \chi_{21}, \mathbf{u}_1)$ and $\mathcal{S}_2 = (\theta_2, \chi_{12}, \chi_{22}, \mathbf{u}_2)$, write the corresponding equations (37) and (38), take the difference and test by $(\chi_{11} - \chi_{11}, \chi_{21} - \chi_{22})$ and $\mathbf{u}_1 - \mathbf{u}_2$, respectively. After integrating in time, we perform standard estimates as that detailed i.e. in [6, 9]. Hence, to deal with the energy balance, we have to rewrite (36) in a more convenient form

$$\begin{aligned}&\partial_t(c_0\theta - L\chi_1 + (\alpha(\theta) - \theta\alpha'(\theta))\chi_2 \operatorname{div}\mathbf{u}) + k_0 A\theta + k * A\theta\\ &\quad = F + H + \alpha(\theta)\chi_2\partial_t(\operatorname{div}\mathbf{u}).\end{aligned} \tag{59}$$

Thus, we write (59) for $\mathcal{S}_1$ and $\mathcal{S}_2$, integrate in time, take the difference, and test it by $\theta_1 - \theta_2$. After integrating once more over $(0, t)$, and by use of some integration by parts, due to (40) we get

$$\begin{aligned}&c_2\|\theta_1 - \theta_2\|^2_{L^2(0,t;H)} + \frac{k_0}{2}\|1 * \nabla(\theta_1 - \theta_2)(t)\|^2_H\\ &\leq \int_0^t \int_\Omega \big(L(\chi_{11} - \chi_{12}) - (\alpha(\theta_2) - \theta_2\alpha'(\theta_2)) \operatorname{div}\mathbf{u}_1(\chi_{21} - \chi_{22})\\ &\quad -(\alpha(\theta_2) - \theta_2\alpha'(\theta_2))\chi_{22}(\operatorname{div}\mathbf{u}_1 - \operatorname{div}\mathbf{u}_2)\big)(\theta_1 - \theta_2)\\ &\quad + \int_0^t \int_\Omega (1 * (\alpha(\theta_1)\chi_{21} - \alpha(\theta_2)\chi_{22})\partial_t \operatorname{div}\mathbf{u}_1\\ &\quad +1 * \alpha(\theta_2)\chi_{22}\partial_t(\operatorname{div}\mathbf{u}_1 - \operatorname{div}\mathbf{u}_2))(\theta_1 - \theta_2)\\ &\quad - \int_0^t \int_\Omega (1 * (k * \nabla(\theta_1 - \theta_2))) \cdot \nabla(\theta_1 - \theta_2).\end{aligned} \tag{60}$$

We note that

$$\begin{aligned}&-(\alpha(\theta_2) - \theta_2\alpha'(\theta_2))\chi_{22}(\operatorname{div}\mathbf{u}_1 - \operatorname{div}\mathbf{u}_2) + 1 * \alpha(\theta_2)\chi_{22}\partial_t(\operatorname{div}\mathbf{u}_1 - \operatorname{div}\mathbf{u}_2)\\ &= \theta_2\alpha'(\theta_2)\chi_{22}(\operatorname{div}\mathbf{u}_1 - \operatorname{div}\mathbf{u}_2) - 1 * \partial_t(\alpha(\theta_2)\chi_{22})(\operatorname{div}\mathbf{u}_1 - \operatorname{div}\mathbf{u}_2),\end{aligned} \tag{61}$$

and

$$\begin{aligned}&\int_0^t \int_\Omega 1 * (k * \nabla(\theta_1 - \theta_2)) \cdot \nabla(\theta_1 - \theta_2)\\ &= \int_\Omega (1 * (k * \nabla(\theta_1 - \theta_2)))(t) \cdot (1 * \nabla(\theta_1 - \theta_2))(t)\\ &\quad - \int_0^t \int_\Omega (k * \nabla(\theta_1 - \theta_2)) \cdot (1 * \nabla(\theta_1 - \theta_2)).\end{aligned} \tag{62}$$

Thus, thanks to the regularity of the solutions (cf. (31)-(33)), and (61)-(62), by use of the Hölder's inequality, well-known properties on convolution product, and exploiting in particular (30) and (41), we finally can prove that there exists $\hat{t} \in [0, T]$ such that the following inequality holds at least for $t \in (0, \hat{t})$

$$\begin{aligned}
&\|\theta_1 - \theta_2\|^2_{L^2(0,t;H)} + \|1 * \nabla(\theta_1 - \theta_2)(t)\|^2_H + \sum_{i=1}^{2} \|(\chi_{i1} - \chi_{i2})(t)\|^2_{L^2(0,t;H)} \\
&+ \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L^2(0,t;\mathbf{W})} + \|\operatorname{div}\mathbf{u}_1 - \operatorname{div}\mathbf{u}_2\|^2_{L^2(0,t;V)} \\
&\leq c\left(1 + \|1 * \nabla(\theta_1 - \theta_2)\|^2_{L^2(0,t;H)} + \sum_{j=1}^{2} \left\|\chi_{j1} - \chi_{j2}\right\|^2_{L^2(0,t;H)}\right).
\end{aligned}$$

Hence, it is straightforward to apply the Gronwall lemma to deduce

$$\theta_1 = \theta_2, \quad \chi_{11} = \chi_{12}, \quad \chi_{21} = \chi_{22}, \quad \mathbf{u}_1 = \mathbf{u}_2,$$

a.e. in $\Omega \times (0, \hat{t})$. Hence, since we can iterate our argument on the interval $(\hat{t}, 2\hat{t})$ and so on, we get uniqueness on the whole interval $(0, T)$, which concludes the proof of the Theorem 1.

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