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LOCALLY FINITE BOREL MEASURES IN RADOM SPACES

Abstract. For any locally finite Borel measure μ in a Radon space X we establish that the following properties are equivalent: (i) μ is semifinite; (ii) μ has a concassage; (iii) each atom of μ has finite μ-measure; (iv) μ is a Radon measure.

1. Introduction and preliminaries

Let X be a Hausdorff topological space. We shall denote by $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{B}$, respectively, the families of all open, compact and Borel subsets of X.

A *Borel measure* in X is a measure on $\mathcal{B}$. A Borel measure μ in X is called

(a) *locally finite* if each $x \in X$ has an open neighborhood V_x such that $\mu(V_x) < +\infty$;

(b) *semifinite* if $\mu(A) = \sup\{\mu(B) : A \supset B \in \mathcal{B}, \mu(B) < +\infty\}$ for each $A \in \mathcal{B}$;

(c) *Radon* if it is locally finite and $\mu(A) = \sup\{\mu(K) : A \supset K \in \mathcal{K}\}$ for each $A \in \mathcal{B}$.

The space X is said to be a *Radon* (resp. *strongly Radon*) *space* if each finite (resp. locally finite) Borel measure μ in X is a Radon measure. For a extensive treatment of Radon measures and Radon spaces, we refer to [3].

The *support* of a Borel measure μ in X is the set of all $x \in X$ such that $\mu(U) > 0$ for each open neighborhood U of x. It is clear that the support S of a Borel measure μ in X is a closed subset of X.

Let μ be a Borel measure in X. A *concassage* of μ is a disjoint family $\mathcal{D}$ of compact subsets of X such that

(a) $\mu(G \cap D) > 0$ for each $G \in \mathcal{G}$ and each $D \in \mathcal{D}$ with $G \cap D \neq \emptyset$;

(b) $\mu(A) = \sum_{D \in \mathcal{D}} \mu(A \cap D)$ for each $A \in \mathcal{B}$.

A set $A \in \mathcal{B}$ is called an *atom* of μ if $\mu(A) > 0$ and for each $B \in \mathcal{B}$ with $B \subset A$ either $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

Let μ be a Radon measure in X. We shall recall three know facts:

1.1. μ is semifinite;

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1.2. μ has a concassage (see e. g. ([2], Propositon 12.10));

1.3. each atom of μ has finite μ-measure (see ([1], Theorem 1)).

In this paper we establish the converse propositions for locally finite Borel measures in Radon spaces, and we deduce that a Radon space X is a strongly Radon space if and only if each locally finite Borel measure μ in X satisfies any of the conditions 1.1, 1.2 or 1.3.

2. The results

THEOREM 1. *Let X be a Radon space and let μ be a locally finite Borel measure in X. The following properties are equivalent:*

(i) μ is semifinite;

(ii) μ has a concassage;

(iii) each atom of μ has finite μ-measure;

(iv) μ is a Radon measure.

Proof. It is clear that *(ii)* $\Longrightarrow$ *(iii)* and that *(iv)* $\Longrightarrow$ *(i)*. We shall prove that *(i)* $\Longrightarrow$ *(ii)* and that *(iii)* $\Longrightarrow$ *(iv)*.

(i) $\Longrightarrow$ *(ii)*. By Zorn's lemma there is a maximal disjoint family $\mathcal{D}$ of compact subsets of X which satisfies the first condition of definition of concassage. In four etaps we shall prove that $\mathcal{D}$ also satisfies the second condition.

Let $B \in \mathcal{B}$ with $\mu(B) < +\infty$ and $B \cap \cup\mathcal{D} = \emptyset$, and suppose that $\mu(B) > 0$. The measure ν defined by $\nu(A) = \mu(A \cap B)$ for each $A \in \mathcal{B}$ is a finite Borel measure in the Radon space X, hence it is a Radon measure in X. Therefore

$$\mu(B) = \nu(B) = \sup\{\nu(K) : B \supset K \in \mathcal{K}\} = \sup\{\mu(K) : B \supset K \in \mathcal{K}\}$$

and there is $K \in \mathcal{K}$ such that $K \subset B$ and $\mu(K) > 0$. The restriction μ_K of ν to the family $\{A \in \mathcal{B} : A \subset K\}$ is a Radon measure in K. Let S be the support of μ_K. Then S is a compact subset of X and $S \cap \cup\, \mathcal{D} = \emptyset$, and adding S to $\mathcal{D}$ we obtain a contradiction to the maximality of $\mathcal{D}$. Thus $\mu(B) = 0$.

Let $K \in \mathcal{K}$. Since μ is locally finite, each $x \in X$ has a open neighborhood V_x with $\mu(V_x) < +\infty$ and a finite family $\{V_{x_1}, ..., V_{x_n}\}$ of these neighborhoods is a cover of K. Then $G = \cup_{i=1}^{n} V_{x_i}$ is an open set such that $K \subset G$ and $\mu(G) < +\infty$. Since $\mathcal{D}$ is a disjoint family, we have

$$\sum_{D \in \mathcal{D}} \mu(G \cap D) \leq \mu(G) < +\infty$$

hence the family

$$\mathcal{D}_0 = \{D \in \mathcal{D} : G \cap D \neq \emptyset\} = \{D \in \mathcal{D} : \mu(G \cap D) > 0\}$$

is countable. Since $(K \cap \cup\mathcal{D}_0) \cap (\cup\mathcal{D}) = \emptyset$, we have

$$\mu(K) = \mu(K \cap \cup\mathcal{D}_0) = \sum_{D \in \mathcal{D}_0} \mu(K \cap D) = \sum_{D \in \mathcal{D}} \mu(K \cap D).$$

Now, let $B \in \mathcal{B}$ wtih $\mu(B) < +\infty$. For each $K \in \mathcal{K}$ with $K \subset B$ we have

$$\mu(K) = \sum_{D \in \mathcal{D}} \mu(K \cap D) \leq \sum_{D \in \mathcal{D}} \mu(B \cap D)$$

and since the measure ν defined by $\nu(A) = \mu(A \cap B)$ for each $A \in \mathcal{B}$ is a Radon measure in X, as in the first etap we show that

$$\mu(B) = \sup\{\mu(K) : B \supset K \in \mathcal{K}\}$$

Consequently,

$$\mu(B) \leq \sum_{D \in \mathcal{D}} \mu(B \cap D)$$

Finally, let $A \in \mathcal{B}$. For each $B \in \mathcal{B}$ with $B \subset A$ and $\mu(B) < +\infty$ we have

$$\mu(B) \leq \sum_{D \in \mathcal{D}} \mu(B \cap D) \leq \sum_{D \in \mathcal{D}} \mu(A \cap D)$$

and since μ is semifinite,

$$\mu(A) = \sup\{\mu(B) : A \supset B \in \mathcal{B}, \mu(B) < +\infty\} \leq \sum_{D \in \mathcal{D}} \mu(A \cap D).$$

The reverse inequality is obvious because $\mathcal{D}$ is a disjoint family.

(iii)$\Longrightarrow$*(iv)*. As above we show that

$$\mu(A) = \sup\{\mu(K) : A \supset K \in \mathcal{K}\}$$

for each $A \in \mathcal{B}$ with $\mu(A) < +\infty$. Let $A \in \mathcal{B}$ with $\mu(A) = +\infty$. It suffices to prove that

$$\sup\{\mu(B) : A \supset B \in \mathcal{B},\ \mu(B) < +\infty\} = +\infty.$$

Proceeding towards a contradiction, let

$$\sup\{\mu(B) : A \supset B \in \mathcal{B},\ \mu(B) < +\infty\} = \alpha < +\infty$$

and for each $n \in \mathbb{N}$, let $A_n \in \mathcal{B}$ such that $A_n \subset A$ and

$$\alpha - \frac{1}{n} < \mu\left(A_n\right) \leq \alpha.$$

With no lost of generality, we can suppose that $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{B}$. Then we have

$$\mu\left(\cup_{n\in\mathbb{N}} A_n\right) = \alpha.$$

If $B \in \mathcal{B}$ is contained in $A \setminus \cup_{n\in\mathbb{N}} A_n$, we have either $\mu(B) = +\infty$ or $\mu(B) = 0$. Then $A \setminus \cup_{n\in\mathbb{N}} A_n$ is an atom of μ with infinite measure against *(iii)*.

□

COROLLARY 1. *A Radon space X is a strongly Radon space if and only if each locally finite Borel measure μ in X satisfies any of the following conditions:*

(i) μ is semifinite;

(ii) μ has a concassage;

(iii) each atom of μ has finite μ-measure.

References

[1] FERNÁNDEZ NOVOA J., *Atomos para las medidas de Radon*, Contribuciones Matemáticas, Libro homenaje al Prof. J. J. Etayo, 335–337, Complutense, Madrid, 1994.

[2] GARDNER R. J. AND PFEFFER W. F., *Borel measures*; in: "Handbook of Set Theoretic Topology", (eds. Kunen K. and Vaughan J. E.), 961–1043, North Holland, New York 1984.

[3] SCHWARTZ L., *Radon measures on Arbitrary Topological Spaces and Cylindrical measures*, Oxford Univ. Press, 1973.

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