

On temporally completely monotone functions for Markov processes

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Abstract: Any negative moment of an increasing Lamperti process $(Y_t, t \geq 0)$ is a completely monotone function of t . This property enticed us to study systematically, for a given Markov process $(Y_t, t \geq 0)$, the functions f such that the expectation of $f(Y_t)$ is a completely monotone function of t . We call these functions temporally completely monotone (for Y). Our description of these functions is deduced from the analysis made by Ben Saad and Janßen, in a general framework, of a dual notion, that of completely excessive measures. Finally, we illustrate our general description in the cases when Y is a Lévy process, a Bessel process, or an increasing Lamperti process.

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Contents

1	Introduction, motivations	254
1.1	Lamperti's correspondence	254
1.2	Perpetuity and remainder variables	254
1.3	Temporally completely monotone functions	255
1.4	Representation of TCM functions	256
1.5	Organization	256
2	A general framework	257
3	A representation theorem	259

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4	The Markovian set-up	265
5	Lévy processes	267
6	Bessel processes	271
7	On increasing Lamperti processes	273
7.1	General facts	273
7.2	Pseudo-stable (increasing) processes	275
7.3	Lamperti processes associated to the exponential compound Poisson processes	280
	References	285

1. Introduction, motivations

1.1. Lamperti's correspondence

In 1972, J. Lamperti [16] established an extremely interesting correspondence between:

- on one hand, real-valued Lévy processes $(\xi_t, t \geq 0)$, and
- on the other hand, Feller processes $(X_u, u \geq 0)$ taking values in $(0, \infty)$, and, furthermore, satisfying the scaling property:

$$\forall c > 0, \quad (X_{x,cu}, u \geq 0) \stackrel{(\text{law})}{=} (cX_{\frac{x}{c},u}, u \geq 0) \quad (1.1)$$

where $(X_{y,u}, u \geq 0)$ denotes the Markov process X starting at y .

Likewise, $(\xi_t, t \geq 0)$ denotes the Lévy process starting at 0, and for $a \in \mathbb{R}$,

$$(\xi_{a,t}, t \geq 0) \stackrel{(\text{law})}{=} (a + \xi_t, t \geq 0).$$

In the particular case where: $\int_0^\infty \exp(\xi_s) ds = \infty$ a.s., Lamperti's correspondence may be presented very simply in the form of either of the following identities:

$$\exp(\xi_{a,t}) = X_{\exp a, A_t}, \quad \text{or:} \quad \log(X_{x,u}) = \xi_{\log x, H_u}, \quad (1.2)$$

$$\text{where:} \quad A_t = \int_0^t \exp(\xi_{a,s}) ds, \quad \text{and:} \quad H_u = \int_0^u \frac{1}{X_{x,v}} dv.$$

Put simply, Lamperti's correspondence expresses the fact that the independence and homogeneity properties of the increments of the Lévy process $(\xi_t, t \geq 0)$ translate, via (1.2), into the scaling property (1.1) of the process X .

1.2. Perpetuity and remainder variables

In the particular case where $(\xi_t, t \geq 0)$ is a subordinator, the law of the perpetuity:

$$I \equiv I(\xi) = \int_0^\infty \exp(-\xi_t) dt$$

has been of great interest for a number of “real-world” problems, and many properties of this law have been obtained; see, e.g., Bertoin-Yor [2, 3], Salminen-Yor [22]. In fact, the studies on this topic are now exploding; see e.g. a number of recent papers by P. Patie ([19]), A. Kuznetsov et al ([15]), J.C. Pardo et al ([20]). Among other results, the law of I is characterized by its integral moments:

$$\mathbb{E}[I^n] = \frac{n!}{\Phi(1) \cdots \Phi(n)}, \quad n \geq 1, \quad (1.3)$$

where $(\Phi(s), s \geq 0)$ is the Laplace-Bernstein exponent of (ξ_t) :

$$\mathbb{E}[\exp(-s \xi_t)] = \exp(-t \Phi(s)).$$

In fact, continuing to quote Bertoin-Yor [2], the relation (1.3) may be complemented as follows: the standard exponential variable \mathbf{e} may be factorized as:

$$\mathbf{e} \stackrel{(\text{law})}{=} I \cdot R, \quad (1.4)$$

with I and R independent, and R (or rather its law) is characterized by:

$$\mathbb{E}_1 \left(\frac{1}{X_t} \right) = \mathbb{E}[\exp(-t R)], \quad t \geq 0. \quad (1.5)$$

($\forall x > 0$, \mathbb{P}_x indicates the law of the process $(X_{x,u})_{u \geq 0}$, and \mathbb{P} a generic probability.)

Combining (1.3) and (1.4), the integral moments of R are seen to be given by:

$$\mathbb{E}[R^n] = \Phi(1) \cdots \Phi(n). \quad (1.6)$$

More generally, for every $p > 0$, Bertoin-Yor [2] show the existence of a variable R_p , taking values in \mathbb{R}_+ , such that:

$$\mathbb{E}_1 \left(\frac{1}{X_t^p} \right) = \mathbb{E}[\exp(-t R_p)], \quad t \geq 0. \quad (1.7)$$

The motivation of the present paper stems essentially from (1.7).

1.3. Temporally completely monotone functions

Since the variables $(R_p, p > 0)$ play such an important rôle in the computations of the laws of exponential functionals associated with the Lévy process ξ , it seemed of some interest, a generic E -valued Markov process $(Y_t, t \geq 0)$ being given, to study systematically the functions f on E such that, for every $y \in E$, the function:

$$t \longrightarrow \mathbb{E}_y[f(Y_t)]$$

is a completely monotone function on $(0, \infty)$. We say that such a function f is *temporally completely monotone* for Y , and we write for short: f is TCM (for Y). For example, from (1.7) and the scaling property of X , the function: $\phi_p(x) = x^{-p}$ is TCM for X .

In the present paper, our goal is, as much as possible, to determine the TCM functions of a general Markov process, or, at least, to find some remarkable properties of these functions, and to treat some significant examples.

1.4. Representation of TCM functions

In a general context, a natural question is to determine whether the TCM functions may be represented as integrals of extremal TCM functions, in analogy with the classical Bernstein representation theorem for completely monotone functions on $(0, \infty)$.

Actually, this kind of question has already been investigated in the analytic-probabilistic literature, under various formulations.

In particular, Itô-Suzuki [13] and Ben Saad-Janßen [1] studied a *dual* problem, namely the integral representation of completely superharmonic ([13]) or completely excessive ([1]) *measures*. In [13], the method consists, modulo some additional hypotheses, in applying the classical Choquet representation theorem, whereas in [1], the authors use a result of Gettoor [10] on the representation of pseudo-kernels as kernels (and the classical Bernstein theorem) to give a simple proof of a general representation theorem of completely excessive measures.

Completely superharmonic *functions* (or ultrapotentials) were studied by Bezná [4] in the set-up of resolvents in duality, fulfilling an absolute continuity hypothesis (Meyer's celebrated Hypothesis (L)). Here again, the method consists, modulo additional hypotheses, in applying Choquet's theorem.

Our "strategy" to obtain an integral representation of TCM (or completely excessive) functions is: given a completely excessive function, to associate with it a completely excessive measure, and to use the theorem in [1]. For this, we shall need an absolute continuity hypothesis (similarly as in [4]), and, following [24] closely, to prove under this hypothesis the existence of a dual semi-group.

1.5. Organization

The above mentioned relation with the paper [1] led us to the following organization of our paper:

- Section 2 consists in the study of general properties of completely excessive functions, in a very general framework. In particular, we determine the extremal rays of the cone of completely excessive functions.
- In Section 3, under an absolute continuity hypothesis, we state a representation theorem of completely excessive functions, which is deduced from the corresponding theorem in [1] for the completely excessive measures.
- In Section 4, we express, in a Markovian set-up, the TCM property in terms of complete superharmonicity with respect to the extended generator \mathcal{L} of Y . Said a little roughly, f is TCM iff for any $n \in \mathbb{N}$,

$$(-1)^n \mathcal{L}^n f \geq 0.$$

- Sections 5, 6 and 7 are devoted to the general description of the TCM functions when Y is respectively a Lévy process, a Bessel process, and finally an increasing Lamperti process X , as featured in (1.2) when ξ is a subordinator.

2. A general framework

The following very general framework shall be made more and more particular, as we progress throughout our study.

Let (E, \mathcal{E}) denote a measurable space. We also use the notation \mathcal{E} for the set of measurable functions from E into \mathbb{R} ; we note \mathcal{E}^+ , resp. $\overline{\mathcal{E}}^+$, for the set of measurable functions from E into $\mathbb{R}_+ = [0, \infty)$, resp. $\overline{\mathbb{R}}_+ = [0, \infty]$.

In the sequel, $(P_t, t > 0)$ denotes a *measurable semi-group*, that is a family of kernels on (E, \mathcal{E}) such that:

- (SG) $\forall s > 0, \forall t > 0, \quad P_{t+s} = P_t P_s,$
- (M) $\forall f \in \mathcal{E}^+, \quad (t, x) \longrightarrow P_t f(x)$ is measurable on $(0, \infty) \times E.$

Definition 2.1. A function $f \in \mathcal{E}^+$ is said to be *completely excessive* relatively to (P_t) if, for every $x \in E$, the function: $t \longrightarrow P_t f(x)$ is completely monotone and $f(x) = \lim_{t \downarrow 0} \uparrow P_t f(x).$

We denote by \mathcal{T} the set of completely excessive functions.

For $\lambda \geq 0$, we denote:

$$\mathcal{T}_\lambda = \{f \in \mathcal{E}^+; \forall t > 0, \forall x \in E, \quad P_t f(x) = e^{-\lambda t} f(x)\}$$

Obviously, $\mathcal{T}_\lambda \subset \mathcal{T}.$

Finally, if \mathcal{C} is a convex cone contained in $\mathcal{E}^+.$ we denote as $\text{ex}(\mathcal{C})$ the set of extremal rays of $\mathcal{C}.$ If $f \in \mathcal{E}^+$ and $f \not\equiv 0,$ we denote by $\rho(f)$ the ray generated by $f:$ $\rho(f) = \{r f; r \geq 0\}.$

Proposition 2.1. *There is the equality between sets:*

$$\text{ex}(\mathcal{T}) = \bigcup_{\lambda \geq 0} \text{ex}(\mathcal{T}_\lambda).$$

Proof. 1) Let $f \in \mathcal{T}.$ Then, for every $t > 0$ and every $x \in E,$

$$\lim_{s \rightarrow 0} P_{s+t} f(x) = \lim_{s \rightarrow 0} P_t (P_s f)(x) = P_t f(x), \text{ by monotone convergence.}$$

Hence, if $f \in \mathcal{T}$ and $t > 0,$ then both $P_t f$ and $(f - P_t f)$ belong to $\mathcal{T}.$ Consequently, if $f \not\equiv 0$ and $\rho(f) \in \text{ex}(\mathcal{T}),$ then, for every $t > 0,$ $P_t f = \lambda(t) f,$ with λ a completely monotone function such that $\lim_{t \rightarrow 0} \lambda(t) = 1.$ Moreover, λ satisfies:

$$\forall s, t > 0, \quad \lambda(s+t) = \lambda(s) \lambda(t).$$

Hence, there exists $\mu \geq 0$ such that: $\lambda(t) = \exp(-\mu t),$ and $f \in \mathcal{T}_\mu.$ Since $\rho(f) \in \text{ex}(\mathcal{T}),$ a fortiori, $\rho(f) \in \text{ex}(\mathcal{T}_\mu).$ Consequently:

$$\text{ex}(\mathcal{T}) \subset \bigcup_{\lambda \geq 0} \text{ex}(\mathcal{T}_\lambda).$$

- 2) Conversely, assume that $f \neq 0$ and $\rho(f) \in \text{ex}(\mathcal{T}_\mu)$ for some $\mu \geq 0$. Assume moreover that: $f = g + h$, with g and h belonging to \mathcal{T} . The extremality of exponential functions in the set of completely monotone functions entails that, for every $x \in E$, there exists $c(x) \geq 0$, such that:

$$\forall t > 0, \quad P_t g(x) = c(x) e^{-\mu t} f(x).$$

Letting t go to 0, we obtain: $c f = g$, hence $g \in \mathcal{T}_\mu$. Likewise, $h \in \mathcal{T}_\mu$, and the extremality of $\rho(f)$ in \mathcal{T}_μ implies that:

$$\exists \alpha_1, \alpha_2 \geq 0, \quad g = \alpha_1 f \quad \text{and} \quad h = \alpha_2 f,$$

which proves that $\rho(f) \in \text{ex}(\mathcal{T})$. □

We now define the *potential kernel* V by:

$$\forall f \in \mathcal{E}^+, \quad Vf = \int_0^\infty (P_t f) dt.$$

Proposition 2.2. For $\lambda > 0$,

$$\mathcal{T}_\lambda = \{f \in \mathcal{E}^+; f = \lambda Vf\}.$$

Proof. Clearly, if $f \in \mathcal{T}_\lambda$, then $Vf = \frac{1}{\lambda} f$.

Suppose $f \in \mathcal{E}^+$ and $f = \lambda Vf$. Then, for $t > 0$,

$$P_t f = \lambda P_t Vf = \lambda \int_t^\infty (P_s f) ds.$$

Hence, for every $x \in E$, there exists $c(x)$ such that:

$$\int_t^\infty P_s f(x) ds = c(x) e^{-\lambda t},$$

and therefore:

$$c(x) = Vf(x) \quad \text{and} \quad P_t f = \lambda \int_t^\infty (P_s f) ds = \lambda e^{-\lambda t} Vf = e^{-\lambda t} f.$$

□

At this stage, a natural question arises: is it possible to represent the elements of \mathcal{T} as integrals of extremal elements, which, by Proposition 2.1, belong to $\bigcup_{\lambda \geq 0} (\mathcal{T}_\lambda)$? In other words: is the subset of \mathcal{T} presented in the following (easy) proposition, equal to \mathcal{T} ? In the next section, we shall show that this actually holds under additional hypotheses.

Proposition 2.3. We suppose that, for every $t > 0$ and $x \in E$, the measure $P_t(x, dy)$ is σ -finite. Assume that there exist a σ -finite Borel measure σ on $[0, \infty)$ and a family $(f_s, s \geq 0)$ such that:

- i) $\forall s \geq 0, f_s \in \mathcal{T}_s,$
- ii) $(s, x) \in [0, \infty) \times E \longrightarrow f_s(x)$ is measurable,
- iii) $\forall x \in E, f(x) := \int f_s(x) \sigma(ds) < \infty.$

Then $f \in \mathcal{T}.$

Proof. Obviously, $f \in \mathcal{E}^+$ and, by Fubini's theorem, we obtain directly:

$$\forall t > 0, \forall x \in E, P_t f(x) = \int e^{-st} f_s(x) \sigma(ds),$$

which shows that $f \in \mathcal{T}.$ □

3. A representation theorem

We now seek a representation theorem, of the kind of Bernstein's theorem, for the elements of \mathcal{T} , i.e. a converse of Proposition 2.3. We shall use a theorem due to Ben Saad and Janßen [1] which bears on measures. For this purpose, we need to introduce a reference measure, and a dual semi-group, which will allow to use the functions - measures duality. Thus, in this section, we are compelled to make further hypotheses on the semi-group (P_t) , namely the following absolute continuity hypotheses:

($\sigma\mathbf{F}$) There exists a sequence $(A_n, n \geq 0)$ of measurable sets of E such that:

$$E = \bigcup_{n \geq 0} A_n \quad \text{and} \quad \forall t > 0, \forall n \geq 0, P_t 1_{A_n} \in \mathcal{E}^+.$$

(\mathbf{AC}) There exists a σ -finite measure m on (E, \mathcal{E}) such that:

$$\forall t > 0, \forall x \in E, P_t(x, dy) \ll m(dy).$$

The following theorem is stated in [9, Lemma 2.1] without proof, and is proven in [24, Note added in proof, Theorem 4] in the case of a non-homogeneous semi-group. However, in these two papers, no reference is made as to the measurability with respect to t . This motivated us to give a complete proof of this theorem.

Theorem 3.1. *There exists a nonnegative measurable function*

$$p : (t, x, y) \in (0, \infty) \times E \times E \longrightarrow p(t, x, y)$$

such that:

$$\forall t > 0, \forall x \in E, P_t(x, dy) = p(t, x, y) m(dy)$$

and

$$\forall x, y \in E, \forall t, s > 0, p(t + s, x, y) = \int p(t, x, z) p(s, z, y) m(dz).$$

Proof. 1) According to $(\sigma\mathbf{F})$, the measure $P_t(x, dy) m(dx)$ is σ -finite on $E \times E$ and, from (\mathbf{AC}) , it is absolutely continuous with respect to $m \otimes m$. In particular, for every $n \geq 1$, there exists a measurable function $a_n(x, y)$ such that:

$$P_{1/n}(x, dy) m(dx) = a_n(x, y) m(dx)m(dy).$$

We set, for $n \geq 1$ and $t \in (\frac{1}{n}, \frac{1}{n-1}]$,

$$q(t, x, y) = \int P_{t-\frac{1}{n}}(x, dz) a_n(z, y).$$

Using again $(\sigma\mathbf{F})$, one sees that q is measurable on $(0, \infty) \times E \times E$ and, for $f \in \mathcal{E}^+$,

$$\int q(t, x, y) f(y) m(dy) = \int P_{t-\frac{1}{n}}(x, dz) \int a_n(z, y) f(y) m(dy).$$

Now,

$$\int a_n(z, y) f(y) m(dy) = P_{\frac{1}{n}} f(z) \quad m(dz)\text{-a.e.}$$

Therefore, by (\mathbf{AC}) ,

$$\int q(t, x, y) f(y) m(dy) = \int P_{t-\frac{1}{n}}(x, dz) P_{\frac{1}{n}} f(z) = P_t f(x).$$

Thus,

$$\forall t > 0, \forall x \in E, \quad q(t, x, y) m(dy) = P_t(x, dy).$$

2) Let $p \in \mathbb{N}$. We set, for $t > \frac{1}{p}$,

$$\pi_p(t, x, y) = \int q\left(t - \frac{1}{p}, x, z\right) q\left(\frac{1}{p}, z, y\right) m(dz).$$

As previously,

$$\pi_p(t, x, y) m(dy) = P_t(x, dy)$$

and therefore, for every $t > \frac{1}{p}$ and $x \in E$,

$$\pi_p(t, x, y) = q(t, x, y) \quad m(dy)\text{-a.e.}$$

By Fubini's theorem, for m -a.e. y ,

$$\pi_p(t, x, y) = q(t, x, y) \quad m(dx)\text{-a.e.}$$

Set, for $0 < n < p$,

$$\pi_{p,n}(x, y) = \pi_p\left(\frac{1}{n}, x, y\right)$$

and $\tilde{E} = \{y \in E; \forall 0 < n < p, \pi_{p,n}(x, y) = q\left(\frac{1}{n}, x, y\right) \quad m(dx)\text{-a.e.}\}$.

Then, $m(\tilde{E}^c) = 0$.

3) Let $\frac{1}{t} < n < p$, $x \in E$ and $y \in \tilde{E}$. Then,

$$\begin{aligned}
\pi_p(t, x, y) &= P_{t-\frac{1}{p}} \left(q \left(\frac{1}{p}, \bullet, y \right) \right) (x) \\
&= P_{t-\frac{1}{n}} P_{\frac{1}{n}-\frac{1}{p}} \left(q \left(\frac{1}{p}, \bullet, y \right) \right) (x) \\
&= P_{t-\frac{1}{n}} (\pi_{p,n}(\bullet, y)) (x) \\
&= P_{t-\frac{1}{n}} \left(q \left(\frac{1}{n}, \bullet, y \right) \right) (x) \quad (\text{since } y \in \tilde{E}) \\
&= \pi_n(t, x, y).
\end{aligned}$$

We set, if $y \in \tilde{E}$, $p(t, x, y) = \pi_n(t, x, y)$ for any $n > \frac{1}{t}$, and, if $y \notin \tilde{E}$, $p(t, x, y) = 0$. Then, the function p is measurable on $(0, \infty) \times E \times E$, and

$$\forall t > 0, \forall x \in E, \quad P_t(x, dy) = p(t, x, y) m(dy).$$

4) Let $0 < s$, $0 < t$, $x \in E$, and $y \in \tilde{E}$. Then

$$\begin{aligned}
&\int p(t, x, z) p(s, z, y) m(dz) \\
&= \int p(t, x, z) \pi_n(s, z, y) m(dz) \quad \text{for } n > \frac{1}{s}, \text{ since } y \in \tilde{E} \\
&= \int \int p(t, x, z) q \left(s - \frac{1}{n}, z, v \right) q \left(\frac{1}{n}, v, y \right) m(dz) m(dv) \\
&= \int q \left(t + s - \frac{1}{n}, x, v \right) q \left(\frac{1}{n}, v, y \right) m(dv) \\
&= \pi_n(t + s, x, y) = p(t + s, x, y) \quad \text{since } y \in \tilde{E}.
\end{aligned}$$

Moreover, the desired equality is obvious if $y \notin \tilde{E}$ since then, both sides are equal to 0. □

Corollary 3.1. *We set:*

$$\hat{P}_t(x, dy) = p(t, y, x) m(dy).$$

Then, $(\hat{P}_t, t > 0)$ is a measurable semi-group and

$$\forall f, g \in \bar{\mathcal{E}}^+, \quad \int (\hat{P}_t f) g dm = \int f (P_t g) dm.$$

Remark 3.1. Recall that, if (X_t) is a linear diffusion taking values in an interval $I \subset \mathbb{R}$ and m is a speed measure for X , then there exists a continuous density function: $(t, x, y) \rightarrow p(t, x, y)$, which is, moreover, symmetric in x and y ; see Itô-Mc Kean ([12, p. 149]) for a proof. Hence, in this case, $P_t = \hat{P}_t$.

In order to apply the results in [1], we need a further topological hypothesis, e.g.:

(T) (E, \mathcal{E}) is a Polish space, endowed with its Borel σ -field.

Definition 3.1. 1. A function $f \in \overline{\mathcal{E}}^+$ is said to be *excessive* if

$$\forall t > 0, P_t f \leq f \quad \text{and} \quad \forall x \in E, \lim_{t \downarrow 0} \uparrow P_t f(x) = f(x).$$

2. A σ -finite measure μ is said to be *excessive* if

$$\forall t > 0, \mu \widehat{P}_t \leq \mu \quad \text{and} \quad \lim_{t \downarrow 0} \uparrow \mu \widehat{P}_t = \mu.$$

(We recall that if μ is a measure and P a kernel, then μP denotes the measure: $\mu P(dy) = \int \mu(dx) P(x, dy)$.)

These two notions are compared in the following lemma.

Lemma 3.1. *A measure μ is excessive if and only if there exists an excessive function f such that:*

$$f < \infty \text{ } m\text{-a.e.} \quad \text{and} \quad \mu = f \, dm.$$

Proof. 1) Assume μ is excessive. As the semi-group (\widehat{P}_t) admits densities, one has: $\mu \ll m$. Let g denote a density of μ with respect to m . Then, for every $h \in \mathcal{E}^+$,

$$\int (P_t g) h \, dm = \int (\widehat{P}_t h) g \, dm = \int h \, d(\mu \widehat{P}_t) \leq \int h \, d\mu = \int h g \, dm.$$

Hence, $P_t g \leq g$ m -a.e. and consequently:

$$\forall s > 0, P_{t+s} g \leq P_s g.$$

We set:

$$f = \lim_{n \uparrow \infty} \uparrow P_{1/n} g.$$

Then, for every $h \in \mathcal{E}^+$,

$$\int h \, d\mu = \lim_{n \uparrow \infty} \uparrow \int h \, d(\mu \widehat{P}_{1/n}) = \lim_{n \uparrow \infty} \uparrow \int (P_{1/n} g) h \, dm = \int f h \, dm.$$

Thus, f is also a density of μ and, as μ is σ -finite, then $f < \infty$ m -a.e. Moreover, f is clearly an excessive function.

2) Conversely, suppose that f is an excessive function and $f < \infty$ m -a.e. Then the measure $\mu = f \, dm$ is σ -finite and, for $t > 0$, $\mu \widehat{P}_t = (P_t f) \, dm$, which entails that μ is an excessive measure. □

Lemma 3.2. *If f and g are two excessive functions, then*

$$f = g \quad m\text{-a.e.} \quad \implies \quad \hat{f} = \hat{g}.$$

In particular, if μ is an excessive measure, there exists a unique excessive function f which is a density of μ . This function f will be called the excessive representant of μ .

Proof. If $f = g$ m -a.e., then, for every $t > 0$, $P_t f = P_t g$, and the desired result follows directly. \square

We borrow the following definition from [1].

Definition 3.2. A family of measures $(\mu_t, t > 0)$ is called a *completely monotone family of measures* if

- i) $\mu_t(f_0) < \infty$ for all $t > 0$, for some strictly positive function $f_0 \in \mathcal{E}^+$,
- ii) for every $f \in \mathcal{E}^+$ such that $f \leq f_0$, $t \rightarrow \mu_t(f)$ is completely monotone.

We denote by \mathcal{M} the set of excessive measures μ such that $(\mu\hat{P}_t, t > 0)$ is a completely monotone family of measures. (In [1], the elements of \mathcal{M} are called *completely excessive measures* (with respect to (\hat{P}_t)).

Lemma 3.3. *If $f \in \mathcal{T}$, then $\mu = f \, dm$ belongs to \mathcal{M} .*

Conversely, if $\mu \in \mathcal{M}$, and if its excessive representant $f \in \mathcal{E}^+$, then $f \in \mathcal{T}$.

Proof. 1) Assume $f \in \mathcal{T}$. In particular, f is excessive and finite, hence $\mu = f \, dm$ is excessive. There exists $f_0 > 0$ such that: $\int f_0 f \, dm < \infty$. If $h \in \mathcal{E}^+$ and $h \leq f_0$, then:

$$\mu\hat{P}_t(h) = \int (P_t f) h \, dm \leq \int f f_0 \, dm < \infty$$

and $t \rightarrow \int (P_t f) h \, dm$ is completely monotone. Thus, $\mu \in \mathcal{M}$.

2) Conversely, let $\mu \in \mathcal{M}$, and denote by f its excessive representant. Assuming f is in \mathcal{E}^+ , we obtain:

$$\forall s > 0, \forall x \in E, \quad P_{t+s} f(x) = \int P_t f(y) p(s, x, y) \, m(dy) = \mu\hat{P}_t(p(s, x, \bullet))$$

and therefore,

$$P_{t+s} f(x) = \lim_{n \uparrow \infty} \uparrow \mu\hat{P}_t[p(s, x, \bullet) \wedge n f_0] \leq f(x) < \infty.$$

Hence, the function: $t \rightarrow P_{t+s} f(x)$ is completely monotone, and it remains to let s decrease to 0 to conclude. \square

For $\lambda \geq 0$, we denote by $\overline{\mathcal{T}}_\lambda$ the set of functions $f \in \overline{\mathcal{E}}^+$ such that:

$$\forall t > 0, \forall x \in E, \quad P_t f(x) = e^{-\lambda t} f(x) \quad \text{and} \quad m(f = \infty) = 0.$$

We also denote by \mathcal{M}_λ the set of σ -finite measures μ such that:

$$\forall t > 0, \quad \mu\hat{P}_t = e^{-\lambda t} \mu.$$

Lemma 3.4. *The two following properties are equivalent:*

- i) $\mu \in \mathcal{M}_\lambda$.
- ii) $\mu = f \, dm$, with $f \in \overline{\mathcal{T}}_\lambda$.

Proof. 1) If $\mu \in \mathcal{M}_\lambda$, then μ is an excessive measure. Let f denote its excessive representant. The equality:

$$\forall t > 0, \quad P_t f = e^{-\lambda t} f \quad m\text{-a.e.}$$

holds, and since f and $P_t f$ are excessive, by Lemma 3.2, the equality holds everywhere.

2) Conversely, if $\mu = f \, dm$ with $f \in \overline{\mathcal{T}}_\lambda$, then μ is σ -finite and

$$\forall h \in \mathcal{E}^+, \quad \mu \widehat{P}_t(h) = \int (P_t f) h \, dm = e^{-\lambda t} \int f h \, dm = e^{-\lambda t} \int h \, d\mu$$

and hence $\mu \in \mathcal{M}_\lambda$. □

Theorem 2.2 in [1] yields a representation of elements of \mathcal{M} as integrals of elements of $(\mathcal{M}_\lambda, \lambda \geq 0)$. We deduce from that Theorem the following representation result for elements of \mathcal{T} .

Theorem 3.2. *Let $f \in \mathcal{E}^+$. Then, $f \in \mathcal{T}$ if and only if there exist a σ -finite Borel measure σ on $[0, \infty)$ and a family $(f_s, s \geq 0)$ such that:*

- i) $\forall s \geq 0, \quad f_s \in \overline{\mathcal{T}}_s$,
- ii) $(s, x) \in [0, \infty) \times E \longrightarrow f_s(x)$ is measurable,
- iii) $\forall x \in E, \quad f(x) = \int f_s(x) \sigma(ds)$.

Proof. 1) Let $f \in \mathcal{T}$. Then, by Lemma 3.3, $\mu = f \, dm \in \mathcal{M}$. According to [1, Theorem 2.2], there exist a σ -finite Borel measure σ on $[0, \infty)$ and a family $(\mu_s, s \geq 0)$ such that:

- a) $\forall s \geq 0, \quad \mu_s \in \mathcal{M}_s$,
- b) $\forall h \in \mathcal{E}^+, \quad s \longrightarrow \mu_s(h)$ is Borel,
- c) $\mu = \int \mu_s \sigma(ds)$.

By a) and Lemma 3.4, one has:

$$a') \quad \mu_s = f_s \, dm \text{ with } f_s \in \overline{\mathcal{T}}_s,$$

and by b) and c), one obtains:

$$b') \quad \forall h \in \mathcal{E}^+, \quad s \longrightarrow \int f_s h \, dm \text{ is Borel,}$$

$$c') \quad \forall h \in \mathcal{E}^+, \quad \int h(x) f(x) \, m(dx) = \int \left[\int f_s h \, dm \right] \sigma(ds).$$

On the other hand, since $f \in \mathcal{E}^+$, there exists a sequence $(A_n, n \geq 0)$ in \mathcal{E} such that: $\cup_n A_n = E$, and

$$\forall n, \quad \int_{A_n} f \, dm < \infty.$$

Therefore, by c'), there exists a Borel set $N \subset [0, \infty)$ such that:

$$\sigma(N) = 0 \quad \text{and} \quad \forall n, \forall s \notin N, \quad \int_{A_n} f_s \, dm < \infty.$$

By monotone class, we deduce therefrom that, for every $\mathcal{E} \otimes \mathcal{E}$ -measurable function φ ,

$$(s, x) \in N^c \times E \longrightarrow \int f_s(y) \varphi(x, y) \, m(dy)$$

is measurable, and we set, for $s \in N$, $f_s = 0$. Then, for $t > 0$,

$$(s, x) \in [0, \infty) \times E \longrightarrow \int f_s(y) p(t, x, y) \, m(dy) = e^{-st} f_s(x)$$

is measurable, which shows that property ii) is satisfied.

Now, by Fubini's theorem,

$$\forall h \in \mathcal{E}^+, \quad \int h(x) f(x) \, m(dx) = \int h(x) \left[\int f_s(x) \, \sigma(ds) \right] m(dx),$$

which entails:

$$f(x) = \int f_s(x) \, \sigma(ds) \quad m\text{-a.e.},$$

and therefore,

$$\forall t > 0, \forall x \in E, \quad P_t f(x) = \int e^{-st} f_s(x) \, \sigma(ds).$$

Property iii) follows, letting t go to 0.

- 2) Conversely, if $f \in \mathcal{E}^+$ satisfies properties i), ii) and iii), then, for every $t > 0$ and $x \in E$,

$$P_t f(x) = \int e^{-st} f_s(x) \, \sigma(ds),$$

which entails that $f \in \mathcal{T}$.

□

4. The Markovian set-up

We still assume here:

(T) (E, \mathcal{E}) is a Polish space, endowed with its Borel σ -field.

We consider an homogeneous Markov process $X = ((X_t)_{t \geq 0}; (\mathcal{F}_t)_{t \geq 0}; (\mathbb{P}_x)_{x \in E})$ taking values in (E, \mathcal{E}) , with its transition semi-group $(P_t)_{t \geq 0}$, which satisfies **(SG)**, **(M)** and $P_t 1 = 1$ for every $t \geq 0$.

In such a Markovian set-up, the completely excessive functions (i.e. the elements of \mathcal{T}) will also be called *temporally completely monotone* functions or, in short, TCM functions (see Section 1).

We now define the extended generator \mathcal{L} of this process. This notion was introduced by Kunita [14], and then used by many authors, with definitions slightly changing from one author to another (see for example [18, 21], ...). The following definition seems to be well adapted to our situation.

Definition 4.1. The *extended generator* \mathcal{L} which is associated to X , is defined through its graph in the following way: let $f, g \in \mathcal{E}$. Then f belongs to the domain $D(\mathcal{L})$ of \mathcal{L} and $g = \mathcal{L}f$ if:

- (a) $\forall t \geq 0, \forall x \in E, \int_0^t P_s(|g|)(x) ds < \infty,$
- (b) $\left(f(X_t) - f(X_0) - \int_0^t g(X_s) ds, t \geq 0 \right)$ is a $((\mathcal{F}_t)_{t \geq 0}; (\mathbb{P}_x)_{x \in E})$ martingale.

We easily obtain the following equivalent definition.

Proposition 4.1. Let $f, g \in \mathcal{E}$. Then $f \in D(\mathcal{L})$ and $g = \mathcal{L}f$ if and only if:

- (a') $\forall t \geq 0, \forall x \in E, P_t(|f|)(x) < \infty$ and $\int_0^t P_s(|g|)(x) ds < \infty,$
- (b') $\forall t \geq 0, \forall x \in E, P_t f(x) = f(x) + \int_0^t P_s g(x) ds.$

Proof. This follows from the definitions and the Markov property. \square

Corollary 4.1. Let $f \in \mathcal{E}^+$ and $\lambda \geq 0$. Then, f belongs to \mathcal{T}_λ if and only if: f belongs to $D(\mathcal{L})$ and $\mathcal{L}f = -\lambda f$.

Proof. Let $f \in \mathcal{E}^+$. Then, by Proposition 4.1,

$$\begin{aligned} & f \in D(\mathcal{L}) \quad \text{and} \quad \mathcal{L}f = -\lambda f \\ \iff & \forall t \geq 0, \forall x \in E, P_t f(x) < \infty \quad \text{and} \\ & P_t f(x) = f(x) - \lambda \int_0^t P_s f(x) ds \\ \iff & \forall t \geq 0, \forall x \in E, P_t f(x) = e^{-\lambda t} f(x). \end{aligned}$$

\square

By iteration, we may define \mathcal{L}^n , for every integer $n \geq 1$, and we write \mathcal{L}^0 for the identity in \mathcal{E} .

Proposition 4.2. The following properties are equivalent:

- i) $f \in \bigcap_{n \geq 0} D(\mathcal{L}^n)$ and $\forall n \geq 0, (-1)^n \mathcal{L}^n f \geq 0$.
- ii) $f \in \mathcal{T}$ and $\forall n \geq 1, \forall x \in E, \lim_{t \downarrow 0} \uparrow (-1)^n \frac{d^n [P_t f(x)]}{dt^n} < \infty$.

Moreover, if these properties are satisfied,

$$\forall x \in E, \lim_{t \downarrow 0} \uparrow (-1)^n \frac{d^n [P_t f(x)]}{dt^n} = (-1)^n \mathcal{L}^n f(x).$$

Proof. 1) Suppose first that f satisfies property i). We see, by induction, that, for every $n \geq 0$,

$$\forall t > 0, \forall x \in E, \frac{d^n [P_t f(x)]}{dt^n} = P_t [\mathcal{L}^n f](x)$$

$$\text{and } \forall x \in E, \lim_{t \downarrow 0} \uparrow (-1)^n \frac{d^n [P_t f(x)]}{dt^n} = (-1)^n \mathcal{L}^n f(x).$$

Hence, f satisfies property ii).

2) Conversely, suppose that f satisfies property ii). We set, for $n \geq 0$,

$$g_n(x) = \lim_{t \rightarrow 0} \frac{d^n [P_t f(x)]}{dt^n}.$$

We see, by induction, that, for every $n \geq 0$,

$$f \in D(\mathcal{L}^n), \mathcal{L}^n f = g_n \text{ and } \forall t > 0, \forall x \in E, \frac{d^n [P_t f(x)]}{dt^n} = P_t g_n(x).$$

Hence, f satisfies property i). □

Definition 4.2. A function f which satisfies property i) in Proposition 4.2 is called a *completely superharmonic* function.

We denote by \mathcal{S} the set of completely superharmonic functions. Then, there are the inclusions:

$$\bigcup_{\lambda \geq 0} \mathcal{T}_\lambda \subset \mathcal{S} \subset \mathcal{T}.$$

The goal of the following sections is to find as many temporally completely monotone functions as possible, which are associated to some particular Markov processes.

5. Lévy processes

In this section, we assume that (E, \mathcal{E}) is the space \mathbb{R}^n endowed with its Borel σ -field. We consider an \mathbb{R}^n -valued Lévy process: $(X_t, t \geq 0)$, whose semi-group

of measures is $(\alpha_t, t \geq 0)$, that is: $\alpha_t(dx) = \mathbb{P}_0(X_t \in dx)$. The associated Markov semi-group is:

$$P_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}_0[f(x + X_t)] = \int f(x + y) \alpha_t(dy) = (f * \check{\alpha}_t)(x),$$

where $*$ denotes convolution, and $\check{\mu}$ is the image of μ by: $x \rightarrow -x$. The dual semi-group of (P_t) , with respect to the Lebesgue measure, is defined by:

$$\widehat{P}_t f(x) = (f * \alpha_t)(x) = \int f(x - y) \alpha_t(dy).$$

Note that, if μ is a ≥ 0 Radon measure, then: $\mu \widehat{P}_t = \mu * \check{\alpha}_t$.

Throughout the sequel of this section, we work under the following hypothesis:

(S) For every $t > 0$, the closed group generated by the support of α_t is \mathbb{R}^n .

(We note that a theorem of Tortrat [23] asserts the existence of a closed semi-group $S \subset \mathbb{R}^n$, and of a point $a \in \mathbb{R}^n$ such that, for every $t > 0$, $\text{supp}(\alpha_t) = a t + S$.)

Theorem 5.1. *Let f be a nonnegative continuous function. Then, $f \in \mathcal{T}$ if and only if there exist a σ -finite Borel measure σ on $[0, \infty)$ and a family $(f_s, s \geq 0)$ of continuous functions such that:*

i) $\forall s \geq 0$, there exists a measure $\theta_s(du)$ on \mathbb{R}^n , which is carried by:

$$\left\{ u \in \mathbb{R}^n; \forall t > 0, \int \exp(u \cdot x) \alpha_t(dx) = e^{-st} \right\}$$

and such that:

$$\forall x \in \mathbb{R}^n, \quad f_s(x) = \int \exp(u \cdot x) \theta_s(du),$$

ii) $(s, x) \in [0, \infty) \times \mathbb{R}^n \rightarrow f_s(x)$ is measurable,

iii) $\forall x \in \mathbb{R}^n, \quad f(x) = \int f_s(x) \sigma(ds)$.

Proof. 1) Let $f \in \mathcal{T}$. Moreover we assume that f is a continuous function.

Then, the family of measures: $((f * \check{\alpha}_t)(x) dx, t > 0)$ is completely monotone in the sense of Definition 3.2. The representation theorem [1, Theorem 2.2] applies, and there exist a σ -finite Borel measure σ on $[0, \infty)$ and a family $(\mu_s, s \geq 0)$ of ≥ 0 Radon measures on \mathbb{R}^n , such that:

a) $\forall s \geq 0, \forall t > 0, \quad \mu_s * \check{\alpha}_t = e^{-st} \mu_s,$

b) $\forall h \in \mathcal{E}^+, \quad s \rightarrow \mu_s(h)$ is Borel,

c) $\forall h \in \mathcal{E}^+, \int f(x) h(x) dx = \int \left[\int h(x) \mu_s(dx) \right] \sigma(ds)$.

Taking into account hypothesis **(S)**, we deduce from the above property a) and from the classical Choquet-Deny theorem ([7],[8]) that, for every $s \geq 0$, μ_s has a continuous density f_s which is defined from a measure θ_s as described in i).

Let φ be a nonnegative continuous function with compact support such that $\int \varphi(x) dx = 1$. By property b), we see that, for every $p \geq 1$,

$$(s, x) \longrightarrow p^n \int \varphi(p(x - y)) f_s(y) dy \quad \text{is measurable.}$$

Then, letting p go to ∞ , we obtain property ii).

Likewise, property c) entails that, for every $p \geq 1$ and $x \in \mathbb{R}^n$:

$$p^n \int f(y) \varphi(p(x - y)) dy = p^n \iint \varphi(p(x - y)) f_s(y) dy \sigma(ds). \quad (5.1)$$

By Fatou's lemma, we have:

$$\forall x \in \mathbb{R}^n, \quad \int f_s(x) \sigma(ds) \leq f(x) < \infty.$$

It is then easy to see, from the expression of f_s as a Laplace transform, that the function:

$$x \in \mathbb{R}^n \longrightarrow \int f_s(x) \sigma(ds)$$

is continuous. Hence, letting p go to ∞ in (5.1), we obtain property iii).

2) Conversely, if a nonnegative continuous function f satisfies properties i), ii) and iii), then, for every $s \geq 0$, $f_s \in \mathcal{T}_s$ and Proposition 2.3 applies. □

Corollary 5.1. *Suppose that the Lévy process X is symmetric, which means:*

$$\forall t > 0, \quad \alpha_t = \check{\alpha}_t.$$

Then, the only continuous TCM functions are the nonnegative constants.

Proof. Suppose that, for some $t > 0$,

$$\int \exp(u \cdot x) \alpha_t(dx) = e^{-st}.$$

Then, by symmetry,

$$\int \cosh(u \cdot x) \alpha_t(dx) = e^{-st}.$$

Since α_t is a probability, we get that, necessarily: $s = 0$, and

$$\text{supp}(\alpha_t) \subset \{x \in \mathbb{R}^n; u \cdot x = 0\}.$$

By property **(S)**, this entails: $u = 0$. Thus, if f is a continuous TCM function, then Theorem 5.1 applies with $\theta_s = 0$ for $s > 0$, and θ_0 is carried by $\{0\}$. This yields the desired result. □

Corollary 5.2. *Suppose that X is a subordinator (considered here as taking values in \mathbb{R}). In other words, we assume:*

$$n = 1 \quad \text{and} \quad \forall t \geq 0, \quad \text{supp}(\alpha_t) \subset \mathbb{R}_+.$$

Then, a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}_+$ is TCM if and only if there exists a measure κ on \mathbb{R}_+ such that:

$$\forall x \in \mathbb{R}, \quad f(x) = \int e^{-sx} \kappa(ds)$$

(in particular, $\kappa(ds)$ admits exponential moments of any order).

Moreover, any continuous TCM function is completely superharmonic, i.e. belongs to \mathcal{S} .

Proof. 1) Let f be a continuous TCM function. We apply Theorem 5.1, and we keep its notation. We denote by Φ the Bernstein function which is the Laplace exponent of X . We have, for $t > 0$,

$$\begin{aligned} \int \exp(ux) \alpha_t(dx) &= e^{-st} \\ \iff u \leq 0 \quad \text{and} \quad \Phi(-u) &= s. \end{aligned}$$

Therefore, for $0 \leq s < \Phi(\infty)$,

$$\forall x \in \mathbb{R}, \quad f_s(x) = c(s) e^{-\Phi^{-1}(s)x},$$

where c denotes a nonnegative Borel function and Φ^{-1} denotes the inverse function, for the composition operation, of the strictly increasing function Φ . Therefore,

$$\forall x \in \mathbb{R}, \quad f(x) = \int c(\Phi(s)) e^{-sx} \tilde{\sigma}(ds),$$

where $\tilde{\sigma}$ denotes the image of the measure σ , appearing in Theorem 5.1, by Φ^{-1} . We may then set:

$$\kappa = c(\Phi(s)) \tilde{\sigma}(ds).$$

2) Conversely, suppose:

$$\forall x \in \mathbb{R}, \quad f(x) = \int e^{-sx} \kappa(ds).$$

Then, for $t > 0$ and $n \geq 0$,

$$\frac{d^n P_t f(x)}{dt^n} = (-1)^n \int e^{-sx} e^{-t\Phi(s)} (\Phi(s))^n \kappa(ds).$$

Since $\Phi(s) = O(s)$ when s tends to ∞ , we deduce from Proposition 4.2:

$$f \in \bigcap_{n \geq 0} D(\mathcal{L}^n) \quad \text{and}$$

$$\forall n \geq 0, \forall x \in \mathbb{R}, \quad \mathcal{L}^n f(x) = (-1)^n \int e^{-sx} (\Phi(s))^n \kappa(ds).$$

Therefore, $f \in \mathcal{S}$.

□

Remark 5.1. Usually, a subordinator is considered as defining a Markov process on $(0, \infty)$ (instead of \mathbb{R} as here). In Section 7 (see Theorem 7.3), we state a representation theorem for the corresponding TCM functions (on $(0, \infty)$ this time).

6. Bessel processes

Here, we consider the Bessel process with dimension $\delta \geq 2$ as a Markov process taking values in $(0, \infty)$. It is well-known that its semi-group admits densities with respect to the Lebesgue measure; hence, the representation theorem of Section 3 applies. We shall obtain therefrom:

Theorem 6.1. *The only TCM functions are the nonnegative constants.*

Proof. 1) We first assume $\delta > 2$, and we denote as usual: $\nu = \frac{\delta}{2} - 1 > 0$. A simple computation shows that the potential kernel is given by:

$$Vf(x) = \frac{1}{\nu} \left[x^{-2\nu} \int_0^x f(y) y^{2\nu+1} dy + \int_x^\infty f(y) y dy \right]. \quad (6.1)$$

If $\lambda > 0$ and $f \in \overline{\mathcal{T}}_\lambda$, then f is a.e. finite and

$$\forall x > 0, \quad Vf(x) = \frac{1}{\lambda} f(x).$$

By (6.1), f is a C^∞ -function and

$$f'' + \frac{2\nu + 1}{x} f' = -2\lambda f. \quad (6.2)$$

Now, (6.2) admits the following solution:

$$\varphi_\lambda(x) = \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{\lambda x^2}{2}\right)^n}{n! \Gamma\left(\frac{\delta}{2} + n\right)} = \left(\frac{\lambda x^2}{2}\right)^{\frac{2-\delta}{4}} J_\nu\left(\sqrt{2\lambda} x\right)$$

where J_ν is the Bessel function of index ν . It follows that φ_λ admits a sequence of zeros (with sign changes):

$$0 < a_1 < a_2 < \dots < a_n < \dots$$

In fact, with the usual notation $j_{\nu,n}$, $a_n = \frac{1}{\sqrt{2\lambda}} j_{\nu,n}$ (see Lebedev [17]).

Let φ be a solution of (6.2). We set, on (a_n, a_{n+1}) , $\varphi(x) = C(x) \varphi_\lambda(x)$. One sees that, if the function C is not constant, then C has a sign change in (a_n, a_{n+1}) . Finally, we obtain: $\overline{\mathcal{T}}_\lambda = \{0\}$, for every $\lambda > 0$.

2) Suppose now that $f \in \mathcal{T}_0$. Then, by Corollary 4.1, for every $x > 0$, $(f(X_t), t \geq 0)$ is a \mathbb{P}_x -martingale. For the sake of completeness, we shall now give a full proof of the fact that f is constant. We denote, for $r > 0$, by T_r the hitting

time of $\{r\}$. We know that, if $0 < a < x < b < +\infty$, then $T_a \wedge T_b$ is finite \mathbb{P}_x -a.s. Actually, since we have assumed $\delta > 2$, if $x < b$, then T_b is finite \mathbb{P}_x -a.s. One has, for $x < b$,

$$\mathbb{E}_x[f(X_{T_b \wedge n})] = f(x),$$

hence, $f(x) \geq f(b) \mathbb{P}_x(T_b \leq n)$ and, letting n go to ∞ , $f(x) \geq f(b)$. Thus, f is a decreasing function. Consequently, if $0 < a < x < b < +\infty$, then, for every $t \geq 0$, $f(X_{T_a \wedge T_b \wedge t}) \leq f(a)$. We then obtain by dominated convergence, and using classical results (see, for example, [21]):

$$f(x) = f(b) \frac{a^{-2\nu} - x^{-2\nu}}{a^{-2\nu} - b^{-2\nu}} + f(a) \frac{x^{-2\nu} - b^{-2\nu}}{a^{-2\nu} - b^{-2\nu}}.$$

We deduce therefrom that there exist constants $C \geq 0$ and $D \geq 0$ such that:

$$\forall x > 0, \quad f(x) = C x^{-2\nu} + D.$$

Now, $\mathbb{E}_x[(X_t)^{-2\nu}]$ is not constant with respect to t , as it converges to 0 as $t \rightarrow \infty$. Therefore, $C = 0$ and $f = D$.

3) We now assume: $\delta = 2$. The densities of the semi-group are given by:

$$p(t, x, y) = \frac{1}{t} \exp\left(-\frac{x^2}{2t}\right) y \exp\left(-\frac{y^2}{2t}\right) I_0\left(\frac{xy}{t}\right), \tag{6.3}$$

where I_0 denotes the modified Bessel function of index 0. In particular, I_0 is an increasing function on \mathbb{R}_+ and $I_0(0) = 1$.

Suppose $\lambda > 0$ and $f \in \overline{\mathcal{T}}_\lambda$. We deduce from (6.3) that, for $t > 0$ and $x > 0$,

$$f(x) \geq \frac{e^{\lambda t}}{t} \exp\left(-\frac{x^2}{2t}\right) \int_0^\infty y \exp\left(-\frac{y^2}{2t}\right) f(y) dy.$$

Letting t go to ∞ , one sees that $f = 0$ a.e. and therefore, $f = 0$. Finally, we obtain: $\overline{\mathcal{T}}_\lambda = \{0\}$, for every $\lambda > 0$.

4) Suppose now that $f \in \mathcal{T}_0$. Then, by Corollary 4.1, for every $x > 0$, $(f(X_t), t \geq 0)$ is a \mathbb{P}_x -martingale. By (6.3), we see that $f(x) = P_t f(x)$ is the product of the function $\frac{1}{t} \exp(-\frac{x^2}{2t})$ by an increasing function of x . In particular, f is a locally bounded function. Consequently, we obtain in the same way as before that, if $0 < a < x < b < +\infty$,

$$f(x) = f(b) \frac{\log x - \log a}{\log b - \log a} + f(a) \frac{\log b - \log x}{\log b - \log a}.$$

We deduce therefrom that there exist constants C and D such that:

$$\forall x > 0, \quad f(x) = C \log x + D.$$

Since $f \geq 0$, necessarily $C = 0$ and $D \geq 0$.

5) To conclude, it then suffices to apply Theorem 3.2.

□

To complement our understanding of Theorem 6.1, let us look closer at the function of t which plays an essential role in the above proof, at the end of point 2), namely:

$$\rho_x^{(\nu)}(t) := \mathbb{E}_x^{(\nu)} \left[\left(\frac{x}{X_t} \right)^{2\nu} \right],$$

where $\mathbb{P}_x^{(\nu)}$ denotes the law of the Bessel process with index ν issued from x . The following identities are part of the folklore on Bessel processes (see, e.g., Revuz-Yor [21, Chapter VIII]): for $\nu > 0$ and $x > 0$,

$$\mathbb{E}_x^{(\nu)} \left[\left(\frac{x}{X_t} \right)^{2\nu} \right] = \mathbb{P}_x^{(-\nu)}(T_0 > t) = \mathbb{P}_0^{(\nu)}(\Lambda_x > t),$$

where T_0 , resp. Λ_x , denotes the first hitting time of 0, resp. the last passage time at x , for X .

Also well-known is the fact that, under $\mathbb{P}_0^{(\nu)}$: $\Lambda_x \stackrel{(\text{law})}{=} x^2/2\gamma_\nu$, where γ_ν denotes a gamma variable with parameter ν . It is now easily seen that $\mathbb{P}(\frac{1}{\gamma_\nu} > t)$ cannot be written as

$$\mathbb{E}[e^{-tR}] \equiv \mathbb{P}(\mathbf{e} > tR)$$

for some r.v. R , and \mathbf{e} a standard exponential. Indeed, this would imply the equality in law between γ_ν and R/\mathbf{e} , which is impossible since γ_ν admits positive moments of all orders, and R/\mathbf{e} moments of, at most, positive order $p < 1$. Of course, we might also see directly that:

$$\mathbb{P}\left(\frac{1}{\gamma_\nu} > t\right) \equiv \frac{1}{\Gamma(\nu)} \int_0^{1/t} e^{-u} u^{\nu-1} du$$

is not completely monotone, but the preceding argument is elementary and quite convincing.

7. On increasing Lamperti processes

In this section, in order to illustrate our preceding general discussion, we study two particular families of increasing Lamperti processes.

First of all, we fix the notation and recall some general facts.

7.1. General facts

Let us consider a subordinator starting from 0: $(\xi_t, t \geq 0)$, whose Laplace exponent is denoted by Φ and whose infinitesimal generator is denoted by L . Following Lamperti [16], we associate with ξ an increasing Feller process $(X_u, u \geq 0)$ which takes values in $(0, \infty)$, and, furthermore, satisfies the scaling property:

$$\forall c > 0, \quad (X_{x,cu}, u \geq 0) \stackrel{(\text{law})}{=} (cX_{\frac{x}{c},u}, u \geq 0)$$

where $(X_{y,u}, u \geq 0)$ denotes the Markov process X starting at y . Such correspondence is one to one. In the sequel, we adopt the notation and definitions of Section 4, the Markov process under consideration being here the Lamperti process X , and $E = (0, \infty)$. In particular, we denote by \mathcal{L} the extended generator of X . The following intertwining relation holds (see, for example, [5]):

$$\mathcal{L}f(x) = \frac{1}{x} L(f \circ \exp)(\log x), \quad Lg(x) = e^x \mathcal{L}(g \circ \log)(e^x). \tag{7.1}$$

The following result comes from Bertoin-Yor [2].

Proposition 7.1. *Set, for $p > 0$, $\phi_p(x) = x^{-p}$. Then,*

$$\forall p > 0, \quad \mathcal{L}\phi_p = -\Phi(p) \phi_{p+1}.$$

Consequently, for every $p > 0$, ϕ_p is a completely superharmonic function ($\phi_p \in \mathcal{S}$) and, a fortiori, ϕ_p is a TCM function.

Proof. According to [2], the process

$$\left(X_t^{-p} + \Phi(p) \int_0^t X_s^{-p-1} ds, \quad t \geq 0 \right)$$

is a uniformly integrable martingale. It then suffices to use Definition 4.1 to identify $\mathcal{L}\phi_p$.

We also may use relation (7.1), which yields:

$$\mathcal{L}\phi_p(x) = \frac{1}{x} L(e^{-p \bullet})(\log x) = \frac{1}{x} (-\Phi(p) x^{-p}) = -\Phi(p) \phi_{p+1}(x).$$

□

The following proposition improves upon the above result.

Proposition 7.2. *Completely monotone functions on $(0, \infty)$ are completely superharmonic for any increasing Lamperti process.*

Proof. 1) One has:

$$\Phi(\lambda) = a \lambda + \int (1 - e^{-\lambda t}) \nu(dt)$$

for some $a \geq 0$ and some measure ν on $(0, \infty)$ such that: $\int (t \wedge 1) \nu(dt) < \infty$.

We set:

$$\forall y \geq 0, \quad \bar{\nu}(y) = \int_{(y, \infty)} \nu(dt).$$

2) Let f be a completely monotone function on $(0, \infty)$. Then, for every $n \geq 1$ and $x > 0$,

$$\begin{aligned} (-1)^n \int_0^\infty \dots \int_0^\infty f^{(n)}(x \exp(y_1 + \dots + y_n)) \exp(y_1 + 2y_2 + \dots + ny_n) dy_1 \dots dy_n \\ = x^{-n} (f(x) - f(\infty)) < \infty. \end{aligned}$$

3) By (7.1),

$$\mathcal{L}f(x) = a f'(x) + \int \frac{f(x e^y) - f(x)}{x} \nu(dy).$$

Hence, by integration by parts, if $f \in C^1((0, \infty))$ and if:

$$\int_0^\infty |f'(x e^y)| e^y dy < \infty,$$

then, $f \in D(\mathcal{L})$ and:

$$\forall x > 0, \quad \mathcal{L}f(x) = \int_{\mathbb{R}_+} f'(x e^y) e^y \hat{\nu}(dy)$$

with $\hat{\nu}(dy) = \bar{\nu}(y) dy + a \delta(dy)$, $\delta(dy)$ denoting the Dirac measure at 0.

4) Now, let f be a completely monotone function. By what precedes, we obtain by induction that, for every $n \geq 1$, $f \in D(\mathcal{L}^n)$ and for every $x > 0$, $\mathcal{L}^n f(x)$ is equal to:

$$\int \dots \int f^{(n)}(x \exp(y_1 + \dots + y_n)) \exp(y_1 + 2y_2 + \dots + ny_n) \hat{\nu}(dy_1) \dots \hat{\nu}(dy_n).$$

Therefore, f is completely superharmonic for X . □

Remark 7.1. Suppose that $\xi_t = t$, i.e. the subordinator ξ is deterministic. Then, the associated Lamperti process is merely given by:

$$\forall x > 0, \quad \forall u \geq 0, \quad X_{x,u} = x + u.$$

Therefore, the TCM functions for X are exactly the completely monotone functions on $(0, \infty)$. Thus, according to the above Proposition 7.2, completely monotone functions are the only functions which are TCM for any increasing Lamperti process (see also property 2 in Proposition 7.5).

7.2. Pseudo-stable (increasing) processes

Definition 7.1. For $0 < \alpha < 1$, we call *pseudo-stable increasing process of index α* the $(0, \infty)$ -valued process:

$$(X_{x,u}^{(\alpha)}; x > 0, u \geq 0) := ((x^{1/\alpha} + \tau_u^{(\alpha)})^\alpha; x > 0, u \geq 0)$$

where $(\tau_t^{(\alpha)}, t \geq 0)$ denotes the α -stable subordinator started at 0, defined from the Bernstein function $F_\alpha(s) = s^\alpha$:

$$\mathbb{E}[\exp(-s \tau_t^{(\alpha)})] = e^{-t s^\alpha}, \quad s > 0, t \geq 0.$$

Clearly, $(X_{x,u}^{(\alpha)}; x > 0, u \geq 0)$ is an increasing Lamperti process, whose associated subordinator is denoted $(\xi_t^{(\alpha)}, t \geq 0)$. The next theorem describes the Lévy measure and the Laplace exponent of $\xi^{(\alpha)}$.

Theorem 7.1. 1. Let $\nu^{(\alpha)}$ be the Lévy measure of $\xi^{(\alpha)}$. Then:

$$\nu^{(\alpha)}(dv) = \frac{1}{\Gamma(1-\alpha)} e^{v/\alpha} (e^{v/\alpha} - 1)^{-\alpha-1} dv.$$

2. Let $\Phi^{(\alpha)}$ be the Bernstein function which is the Laplace exponent of $\xi^{(\alpha)}$. Then:

$$\Phi^{(\alpha)}(s) = \frac{\Gamma(\alpha(s+1))}{\Gamma(\alpha s)}, \quad s > 0. \quad (7.2)$$

Proof. 1) As is well known, the Lévy measure of $(\tau_t^{(\alpha)}, t \geq 0)$ is:

$$\sigma^{(\alpha)}(dy) = \frac{\alpha}{\Gamma(1-\alpha)} y^{-1-\alpha} dy. \quad (7.3)$$

We denote by L (resp. \mathcal{L}) the generator of $\xi^{(\alpha)}$ (resp. $X^{(\alpha)}$). By Definition 7.1 and (7.3), we have clearly:

$$\mathcal{L}f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty [f((x^{1/\alpha} + y)^\alpha) - f(x)] y^{-1-\alpha} dy.$$

Therefore, by (7.1),

$$Lg(x) = \frac{\alpha}{\Gamma(1-\alpha)} e^x \int_0^\infty [g(\alpha \log(e^{x/\alpha} + y)) - g(x)] y^{-1-\alpha} dy.$$

By the change of variable: $y = e^{x/\alpha} (e^{v/\alpha} - 1)$, one obtains:

$$Lg(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty [g(x+v) - g(x)] e^{v/\alpha} (e^{v/\alpha} - 1)^{-\alpha-1} dv,$$

which yields property 1.

2) By property 1,

$$\Phi^{(\alpha)}(s) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-sv}) e^{v/\alpha} (e^{v/\alpha} - 1)^{-\alpha-1} dv.$$

By integration by parts, one gets:

$$\Phi^{(\alpha)}(s) = \frac{s}{\Gamma(1-\alpha)} \int_0^\infty e^{-sv} (e^{v/\alpha} - 1)^{-\alpha} dv.$$

By the change of variable: $y = e^{-v/\alpha}$, we obtain:

$$\begin{aligned} \Phi^{(\alpha)}(s) &= \frac{\alpha s}{\Gamma(1-\alpha)} \int_0^1 y^{\alpha(s+1)-1} (1-y)^{-\alpha} dy \\ &= \frac{\alpha s}{\Gamma(1-\alpha)} B(1-\alpha, \alpha(s+1)) = \frac{\alpha s}{\Gamma(1-\alpha)} \frac{\Gamma(1-\alpha)\Gamma(\alpha(s+1))}{\Gamma(\alpha s + 1)} \\ &= \frac{\Gamma(\alpha(s+1))}{\Gamma(\alpha s)}. \end{aligned}$$

□

As a corollary to formula (7.2), we may identify the law of $R^{(\alpha)}$, as introduced in formula (1.5), with the help of the moment formula (1.6).

Corollary 7.1. For $0 < \alpha < 1$, one has:

$$R^{(\alpha)} \stackrel{(\text{law})}{=} (\gamma_\alpha)^\alpha \quad (7.4)$$

where γ_α denotes a gamma variable with index α , i.e.:

$$\mathbb{P}(\gamma_\alpha \in dt) = \frac{1}{\Gamma(\alpha)} e^{-t} t^{\alpha-1} dt.$$

Proof. From (7.2), and the moments formula (1.6), we get:

$$\mathbb{E}[(R^{(\alpha)})^n] = \frac{\Gamma(\alpha(n+1))}{\Gamma(\alpha)}$$

which is precisely the moments formula for $(\gamma_\alpha)^\alpha$. Hence, there is the identity in law (7.4). \square

In a companion paper [11], we shall develop further relationship between the above Theorem 7.1 and its Corollary on one hand, and on the other hand the paper by Bertoin-Yor [2], where formulae (1.3), (1.4), and (1.6) were first established.

Remark 7.2. Below we give an alternative proof of Theorem 7.1, showing directly property 2. This proof is based on Bertoin-Yor [2] (see Proposition 7.1). We have:

$$\mathbb{E}_1[(X_u^{(\alpha)})^{-s}] = \mathbb{E}[(1 + \tau_u^{(\alpha)})^{\alpha s}].$$

Now, for $r > 0$ and $x > 0$,

$$x^{-r} = \frac{1}{\Gamma(r)} \int_0^\infty e^{-xv} v^{r-1} dv.$$

Therefore:

$$\begin{aligned} \mathbb{E}_1[(X_u^{(\alpha)})^{-s}] &= \frac{1}{\Gamma(\alpha s)} \int_0^\infty \mathbb{E}[e^{-v \tau_u^{(\alpha)}}] e^{-v} v^{\alpha s-1} dv \\ &= \frac{1}{\Gamma(\alpha s)} \int_0^\infty e^{-uv^\alpha} e^{-v} v^{\alpha s-1} dv. \end{aligned}$$

Hence:

$$\begin{aligned} \frac{d}{du} \mathbb{E}_1[(X_u^{(\alpha)})^{-s}] &= \frac{-1}{\Gamma(\alpha s)} \int_0^\infty e^{-uv^\alpha} e^{-v} v^{\alpha(s+1)-1} dv \\ &= -\frac{\Gamma(\alpha(s+1))}{\Gamma(\alpha s)} \mathbb{E}_1[(X_u^{(\alpha)})^{-s-1}]. \end{aligned}$$

By Proposition 7.1, we obtain, by identification,

$$\Phi^{(\alpha)}(s) = \frac{\Gamma(\alpha(s+1))}{\Gamma(\alpha s)}, \quad s > 0.$$

In the next theorem, we characterize the TCM functions for $X^{(\alpha)}$.

Theorem 7.2. *The TCM functions for the pseudo-stable increasing process of index α are the functions: $f(x) = h(x^{1/\alpha})$, with h a completely monotone function. Moreover, for the pseudo-stable increasing process of index α , any TCM function is completely superharmonic.*

Proof. 1) By definition, we need to prove that the TCM functions for the process $(x + \tau_t^{(\alpha)}; x > 0, t \geq 0)$ considered as a Markov process on $(0, \infty)$ are the completely monotone functions. Note that, unfortunately, Corollary 5.2 cannot be used, since in this corollary, the subordinator is considered as a Markov process on \mathbb{R} . Let us fix $\alpha \in (0, 1)$, and consequently, we shall delete α from our notation. The semi-group which is associated to the process $(x + \tau_t; x > 0, t \geq 0)$ is given, for $x > 0$ and $t > 0$, by:

$$P_t f(x) = \mathbb{E}[f(x + \tau_t)] = \int_0^\infty f(x + y) p_t(y) dy$$

where p_t denotes the continuous density of τ_t (which exists, see e.g. Zolotarev [25]). Thus, for the sequel of this proof, we are in the situation of Section 3, the notation of which we preserve; in particular, $E = (0, \infty)$, m is the Lebesgue measure dx on $(0, \infty)$, and the TCM functions are characterized by Theorem 3.2.

2) Consequently, we need to identify, for $s \geq 0$, the Borel functions f , which are ≥ 0 , finite a.e., and which satisfy:

$$\forall t > 0, \forall x > 0, \quad P_t f(x) = e^{-st} f(x),$$

i.e: the functions $f \in \overline{\mathcal{T}}_s$. We introduce:

$$\begin{aligned} \varepsilon(x) &= \int_0^\infty e^{-t} p_t(x) dt \text{ if } x > 0, \\ &= 0 \text{ if } x \leq 0. \end{aligned}$$

Now, ε is a probability density on \mathbb{R} , carried by \mathbb{R}_+ , and it is well-known (see e.g. Chaumont-Yor [6, Exercise 4.21]) that:

$$\forall x > 0, \quad \varepsilon(x) = \frac{\sin(\pi \alpha)}{\pi} \int_0^\infty e^{-xy} \frac{y^\alpha}{1 + 2y^\alpha \cos(\alpha\pi) + y^{2\alpha}} dy.$$

Thus, ε is a completely monotone function on $(0, \infty)$; in particular, it is continuous, and > 0 for every $x > 0$.

Assume that $f \in \overline{\mathcal{T}}_s$. Then:

$$\forall x > 0, \quad \int_0^\infty f(x + y) \varepsilon(y) dy = \frac{f(x)}{1 + s}.$$

It easily follows that $f \in \mathcal{L}_{loc}^1(m)$. We now consider $\mu = f dm$, which is a Radon measure on $(0, \infty)$. We then have:

$$\mu(dx) = \left[(s + 1) \int_{(x, \infty)} \varepsilon(y - x) \mu(dy) \right] m(dx). \tag{7.5}$$

3) In order to find all ≥ 0 Radon measures μ on $(0, \infty)$ which solve equation (7.5), it is possible to apply the same method as that presented by Deny [8] in order to prove the Choquet-Deny theorem on a group. Here, we just show how we may determine the extremal solutions of (7.5), which is the essential point of the proof.

Let μ be a solution of (7.5), and set, for $0 < r < a$,

$$\mu_{a,r}(dx) = \left[(s + 1) \int_{(x+a-r, x+a+r)} \varepsilon(y - x) \mu(dy) \right] m(dx). \tag{7.6}$$

By a simple computation, we obtain that $\mu_{a,r}$ solves (7.5). Since $\mu_{a,r} \leq \mu$, then $\mu - \mu_{a,r}$ also solves (7.5). Therefore, if $\mu \neq 0$ is an extremal solution of (7.5), there exists $c(a, r) \geq 0$ such that:

$$\mu_{a,r} = c(a, r) \mu. \tag{7.7}$$

We deduce from (7.6) that $[(s + 1) \int_{a-r}^{a+r} \varepsilon(y) dy]^{-1} \mu_{a,r}$ vaguely converges, when r tends to 0, to the measure μ_a defined by:

$$\forall \varphi \in C_c^+((0, \infty)), \quad \int \varphi(x) \mu_a(dx) = \int \varphi(x - a) \mu(dx),$$

and by (7.7), there exists $c(a) \geq 0$ such that: $\mu_a = c(a) \mu$. By definition, c is obviously continuous on $(0, \infty)$ and satisfies:

$$\forall a, b > 0, \quad c(a + b) = c(a) c(b).$$

Therefore, there exists $u \in \mathbb{R}$ such that:

$$\forall a > 0, \quad c(a) = e^{u a}.$$

On the other hand, since μ satisfies (7.5), then μ admits a density h and:

$$\forall a > 0, \quad h(a + x) = e^{u a} h(x) \quad dx\text{-a.e.}$$

Then, by (7.5), we may take, as density of μ ,

$$\tilde{h}(x) := (s + 1) \int_x^\infty \varepsilon(y - x) h(y) dy = e^{u x} (s + 1) \int_0^\infty h(z) \varepsilon(z) dz.$$

Finally, there exists $c > 0$ such that $\mu(dx) = c e^{u x} dx$ and, using again (7.5), we obtain:

$$(s + 1) \int_0^\infty e^{u y} \varepsilon(y) dy = 1,$$

which is equivalent to $u = -s^{1/\alpha}$. Hence, the convex cone of solutions of (7.5) admits only one extremal ray, which is:

$$\{c \exp(-s^{1/\alpha} x) dx; c \geq 0\}.$$

Taking back again the Choquet-Deny method, we find that these are the only solutions of (7.5). Finally, $\overline{\mathcal{T}}_s = \{c \exp(-s^{1/\alpha} x) dx; c \geq 0\}$.

- 4) Now, Theorem 3.2 allows to conclude that the TCM functions for the process $(x + \tau_t^{(\alpha)}; x > 0, t \geq 0)$ are exactly the completely monotone functions.
- 5) Finally, suppose that f is a TCM function for the pseudo-stable increasing process of order α . Then, there exists a measure κ on \mathbb{R}_+ such that:

$$\forall x > 0, \quad f(x) = \int e^{-s x^{1/\alpha}} \kappa(ds).$$

Then, for $t > 0$ and $n \geq 0$,

$$\frac{d^n P_t f(x)}{dt^n} = (-1)^n \int e^{-s x^{1/\alpha}} e^{-t s^\alpha} s^{n\alpha} \kappa(ds).$$

Hence, by Proposition 4.2, $f \in \bigcap_{n \geq 0} D(\mathcal{L}^n)$ and

$$\forall n \geq 0, \forall x \in \mathbb{R} \quad \mathcal{L}^n f(x) = (-1)^n \int e^{-s x^{1/\alpha}} s^{n\alpha} \kappa(ds).$$

Therefore, $f \in \mathcal{S}$.

□

Remark 7.3. In the previous proof, we showed that the TCM functions for the process $(x + \tau_t^{(\alpha)}; x > 0, t \geq 0)$ considered as a Markov process on $(0, \infty)$ are the completely monotone functions. This result extends, with the same proof, replacing $(\tau_t^{(\alpha)}, t \geq 0)$ by a large class of subordinators. For example, we may state the following theorem.

Theorem 7.3. Consider $(\xi_t, t \geq 0)$, a subordinator started at 0. We assume that:

- i) for every $t > 0$, ξ_t admits a density p_t with respect to the Lebesgue measure on $(0, \infty)$,
- ii) for every compact K of $(0, \infty)$, $\inf_{x \in K} \varepsilon(x) > 0$, with

$$\varepsilon(x) = \int_0^\infty e^{-t} p_t(x) dt.$$

Then the TCM functions for the process $(x + \xi_t; x > 0, t \geq 0)$ considered as a Markov process on $(0, \infty)$, are the completely monotone functions. Moreover, every TCM function is completely superharmonic.

Note that for ii) to be satisfied, it suffices that, for every $t > 0$, p_t is a lower semi-continuous, strictly positive function.

7.3. Lamperti processes associated to the exponential compound Poisson processes

In this subsection, we consider the Lamperti increasing processes X which are associated to the compound Poisson processes ξ with exponential Lévy measure,

i.e. whose Bernstein functions are:

$$\Phi(x) = C \int_0^\infty (1 - e^{-xt}) \exp(-\theta x) dx = \frac{C}{\theta} \frac{x}{x + \theta}$$

(with $C > 0$, and $\theta > 0$). In the sequel, we fix, to simplify: $C = \theta$.

These processes are studied in detail in [5], and lead themselves to many explicit computations, so that they are of much use in Applied Probability.

We denote by $\mathcal{L}^{(\theta)}$, resp. $\mathcal{T}^{(\theta)}$, the extended generator, resp. the set of TCM functions, for the Lamperti process X . We also set, in view of Corollary 4.1,

$$\forall \lambda \geq 0, \quad \mathcal{T}_\lambda^{(\theta)} = \{f \in \mathcal{E}^+; f \in D(\mathcal{L}^{(\theta)}) \text{ and } \mathcal{L}^{(\theta)} f = -\lambda f\}.$$

Proposition 7.3. *Let $\lambda > 0$. Then, $\mathcal{T}_\lambda^{(\theta)}$ is generated by*

$$1_{\{1/\lambda\}}(x) \quad \text{and} \quad 1_{(0,1/\lambda)}(x) (1 - \lambda x)^{\theta-1}.$$

$\mathcal{T}_0^{(\theta)}$ consists of the nonnegative constants.

Proof. By (7.1), one has:

$$\begin{aligned} \forall x > 0, \quad \mathcal{L}^{(\theta)} f(x) &= \frac{\theta}{x} \int_0^\infty [f(x e^y) - f(x)] e^{-\theta y} dy \\ &= \frac{\theta}{x} \int_1^\infty [f(x v) - f(x)] v^{-\theta-1} dv. \end{aligned}$$

Hence, if $\lambda \geq 0$ and $f \in \mathcal{E}^+$, then $f \in \mathcal{T}_\lambda^{(\theta)}$ if and only if:

$$\forall x > 0, \quad \theta x^{\theta-1} \int_x^\infty f(y) y^{-\theta-1} dy = \left(\frac{1}{x} - \lambda\right) f(x). \quad (7.8)$$

Setting:

$$h(x) = \int_x^\infty f(y) y^{\theta-1} dy,$$

the above equation (7.8) is equivalent to:

$$\forall x > 0, \quad (\lambda x - 1) x^{\theta+1} h'(x) - \theta x^\theta h(x) = 0.$$

Solving this differential equation, we obtain:

$$\begin{aligned} h(x) &= 0 \quad \text{for } x \geq 1/\lambda, \\ &= c x^{-\theta} (1 - \lambda x)^\theta \quad \text{for } 0 < x < 1/\lambda \end{aligned}$$

with $c \geq 0$, and hence,

$$\begin{aligned} f(x) &= 0 \quad \text{for } x > 1/\lambda, \\ &= a \quad \text{for } x = 1/\lambda, \\ &= b(1 - \lambda x)^{\theta-1} \quad \text{for } 0 < x < 1/\lambda \end{aligned}$$

with $a, b \geq 0$, which is the announced result. \square

Note that Theorem 3.2 does not apply here, as the semi-group does not satisfy hypothesis (AC) of Section 3. However, the following proposition is a straightforward consequence of Proposition 2.3 and of the above Proposition 7.3.

Proposition 7.4. *Let \mathcal{K}_θ be the set of functions f from $(0, \infty)$ into \mathbb{R}_+ for which there exists a measure $\sigma(d\lambda)$ on \mathbb{R}_+ such that:*

$$\forall x > 0, \quad f(x) = \int_{[0,1/x)} (1 - \lambda x)^{\theta-1} \sigma(d\lambda).$$

Then: $\mathcal{K}_\theta \subset \mathcal{T}^{(\theta)}$.

Remark 7.4. In fact, the elements of \mathcal{K}_θ are completely superharmonic. Indeed, we may use the characterization given in Proposition 4.2. If

$$f(x) = \int_{[0,1/x)} (1 - \lambda x)^{\theta-1} \sigma(d\lambda),$$

then

$$P_t f(x) = \int_{[0,1/x)} e^{-\lambda t} (1 - \lambda x)^{\theta-1} \sigma(d\lambda)$$

and hence:

$$(-1)^n (\mathcal{L}^{(\theta)})^n f(x) = \int_{[0,1/x)} \lambda^n (1 - \lambda x)^{\theta-1} \sigma(d\lambda) \leq x^{-n} f(x) < \infty.$$

We now introduce some further notation:

For $\lambda \geq 0$ and $\gamma > 0$, we set:

$$h_\lambda^{(\gamma)}(x) = 1_{[0,1/\lambda)}(x) (1 - \lambda x)^{\gamma-1}.$$

We denote, for $\theta < \eta$, by $\Lambda_{\theta,\eta}$ the kernel:

$$\Lambda_{\theta,\eta} f(x) = \frac{1}{B(\theta, \eta - \theta)} \int_0^1 f\left(\frac{x}{u}\right) u^{\theta-1} (1 - u)^{\eta-\theta-1} du.$$

In the two following propositions, we establish some relations between the spaces \mathcal{K}_θ .

Proposition 7.5. 1. *The map: $\theta \rightarrow \mathcal{K}_\theta$ is decreasing.*

2. *The set: $\bigcap_{\theta>0} \mathcal{K}_\theta$ is the set of completely monotone functions on $(0, \infty)$.*

Proof. 1) An easy computation yields, for $0 < \theta < \eta$,

$$h_\lambda^{(\eta)}(x) = \frac{1}{B(\theta, \eta - \theta)} \int_0^1 h_{\lambda/u}^{(\theta)}(x) u^{\theta-1} (1 - u)^{\eta-\theta-1} du. \tag{7.9}$$

This entails property 1.

2) One has, for $a > 0$,

$$\forall x > 0, \quad e^{-ax} = \frac{a^\theta}{\Gamma(\theta)} \int_0^{1/x} (1 - \lambda x)^{\theta-1} \lambda^{-\theta-1} e^{-a/\lambda} d\lambda.$$

This entails that any completely monotone function belongs to \mathcal{K}_θ .

3) Now let $f \in \bigcap_{\theta > 0} \mathcal{K}_\theta$. Then, for every pair of integers $1 \leq p < n$, there exists a measure σ_n on \mathbb{R}_+ such that:

$$\forall x > 0, \quad f(x) = \int [(1 - \lambda x)^+]^n \sigma_n(d\lambda).$$

Therefore, f is of C^p class, and:

$$\forall x > 0, \quad f^{(p)}(x) = (-1)^p n(n-1) \cdots (n-p+1) \int [(1 - \lambda x)^+]^{n-p} \lambda^p \sigma_n(d\lambda).$$

Consequently;

$$\forall x > 0, \quad (-1)^p f^{(p)}(x) \geq 0,$$

and f is completely monotone. □

Proposition 7.6. 1. For $0 < \theta < \eta$ and $\lambda \geq 0$,

$$h_\lambda^{(\eta)} = \Lambda_{\theta, \eta} h_\lambda^{(\theta)}.$$

2. For $0 < \theta < \eta$,

$$\Lambda_{\theta, \eta} \mathcal{K}_\theta \subset \mathcal{K}_\eta.$$

Proof. 1) One has:

$$h_{\lambda/u}^{(\theta)}(x) = h_\lambda^{(\theta)}(x/u).$$

Property 1 then follows from (7.9).

2) Suppose $f \in \mathcal{K}_\theta$. Then, there exists a measure $\sigma(d\lambda)$ on \mathbb{R}_+ such that:

$$\forall x > 0, \quad f(x) = \int h_\lambda^{(\theta)}(x) \sigma(d\lambda).$$

By property 1,

$$\Lambda_{\theta, \eta} f(x) = \int h_\lambda^{(\eta)}(x) \sigma(d\lambda) \leq \int h_\lambda^{(\theta)}(x) \sigma(d\lambda) < \infty$$

and property 2 follows. □

Remark 7.5. Suppose $0 < \theta < \eta$. If $f \in \mathcal{K}_\eta$, then there exists a measure $\sigma(d\lambda)$ on \mathbb{R}_+ such that:

$$\forall x > 0, \quad f(x) = \int h_\lambda^{(\eta)}(x) \sigma(d\lambda).$$

By property 1 in Proposition 7.6, $f = \Lambda_{\theta,\eta}g$ with

$$g(x) = \int h_\lambda^{(\theta)}(x) \sigma(d\lambda).$$

However, if $\theta < 1$, it does not follow that: $g(x) < \infty$ for every $x > 0$, and therefore, g does not necessarily belong to \mathcal{K}_θ . Nevertheless, this shows that, if $1 \leq \theta < \eta$, then:

$$\Lambda_{\theta,\eta} \mathcal{K}_\theta = \mathcal{K}_\eta.$$

Concerning the spaces $\mathcal{T}^{(\theta)}$, we have:

Proposition 7.7. *Suppose $0 < \theta < \eta$ and $f \in \mathcal{T}^{(\theta)}$. If $\Lambda_{\theta,\eta}f \in \mathcal{E}^+$, then $\Lambda_{\theta,\eta}f \in \mathcal{T}^{(\eta)}$.*

Proof. We denote by $(P_t^{(\theta)})$ and $(P_t^{(\eta)})$ the semi-groups of the Lamperti processes. From [5], for $0 < \theta < \eta$ the following intertwining relation holds:

$$P_t^{(\eta)} \Lambda_{\theta,\eta} = \Lambda_{\theta,\eta} P_t^{(\theta)}. \tag{7.10}$$

Let $f \in \mathcal{T}^{(\theta)}$ and suppose $\Lambda_{\theta,\eta}f \in \mathcal{E}^+$. Then:

$$\Lambda_{\theta,\eta} P_t^{(\theta)} f(x) = \int P_t^{(\theta)} f(y) \Lambda_{\theta,\eta}(x, dy) \leq \Lambda_{\theta,\eta}f(x) < \infty.$$

Therefore, $t \rightarrow \Lambda_{\theta,\eta} P_t^{(\theta)} f(x)$ is completely monotone and

$$\lim_{t \downarrow 0} \uparrow \Lambda_{\theta,\eta} P_t^{(\theta)} f(x) = \Lambda_{\theta,\eta}f(x).$$

Finally, we deduce from (7.10) that $\Lambda_{\theta,\eta}f$ belongs to $\mathcal{T}^{(\eta)}$. □

Proposition 7.8. *Every ≥ 0 decreasing function belongs to $\mathcal{T}^{(\theta)}$ for every $0 < \theta \leq 1$. Conversely, every function f in $\mathcal{T}^{(1)}$, which is right-continuous, locally with bounded variation, and such that $\lim_{x \rightarrow \infty} f(x)$ exists, is a decreasing function.*

Proof. 1) By definition, \mathcal{K}_1 consists of the ≥ 0 right-continuous functions, which are decreasing, and, by property 1 in Proposition 7.5, $\mathcal{K}_1 \subset \mathcal{K}_\theta$ for $0 < \theta \leq 1$. Moreover, by Proposition 7.3, any ≥ 0 function with countable support, belongs to $\mathcal{T}^{(\theta)}$ for every $0 < \theta$. The first part of the proposition follows therefrom.

2) We know, from [5], that:

$$P_t^{(1)} f(x) = f(x) \exp(-t/x) + t \int_x^\infty f(y) \exp(-t/y) \frac{1}{y^2} dy.$$

Under the indicated conditions, we get by integration by parts:

$$P_t^{(1)} f(x) = f(\infty) - \int_{(x,\infty)} \exp(-t/y) df(y).$$

Hence, if $f \in \mathcal{T}^{(1)}$, then: $-df(y) \geq 0$. □

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