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Bi-Strictly Cyclic Operators

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To the Memory of D.A. Herrero and His Pioneering Work on Algebras of Finite Strict Multiplicity.

Abstract. The genesis of this paper is the construction of a new operator that, when combined with a theorem of Herrero, settles a question of Herrero. Herrero proved that a strictly cyclic operator on an infinite dimensional Hilbert space is never triangular. He later asks whether the adjoint of a strictly cyclic operator is necessarily triangular. We settle the question by constructing an operator T for which both T and T^* are strictly cyclic. We make a detailed study of this *bi-strictly cyclic* operator which leads to theorems about general bi-strictly cyclic operators. We conclude the paper with a comparison of the operator space structures of the singly generated algebras $\mathcal{A}(S)$ and $\mathcal{A}(T)$, when S is strictly cyclic and T is bi-strictly cyclic.

Contents

1. Introduction

In this paper $\mathcal{H}$ will denote a complex infinite dimensional separable Hilbert space and $\mathcal{B}(\mathcal{H})$ will denote the algebra of operators on $\mathcal{H}$. With p denoting a complex polynomial, we say an operator $R \in \mathcal{B}(\mathcal{H})$ is *strictly cyclic* if there exists a vector $e \in \mathcal{H}$ for which the evaluation map

$$p(R) \overset{\varepsilon_e}{\mapsto} p(R)e$$

is bounded below and has dense range (as a mapping from the algebra generated by R into $\mathcal{H}$). In this situation we will say that e is a strict cyclic vector for R. This definition originated in A. Lambert's thesis [13] and was motivated by examples of

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weighted shift operators. A beautiful survey of weighted shift operators was written by A.L. Shields [20] and there you will find several of Lambert's results.

D.A. Herrero quickly became interested in the idea of strict cyclicity and was soon publishing myriad results on the subject, notable among which are the works [7], [8], and [9]. In a later paper titled *triangular strictly cyclic operators* (see [10]) Herrero proves that "the title refers to an empty class of operators". In remark (iii) of this paper one sees an example of a strictly cyclic operator R for which R^* fails to be triangular. In a private correspondence with the first author Herrero points out that this example is based on an error that appears in a preprint version of [11]. Thus the question of whether such an operator exists remained unresolved and Herrero asks "does T strictly cyclic imply that T^* is triangular?"

We will say that an operator T is *bi-strictly cyclic* if both T and T^* are strictly cyclic. After constructing such an operator we have that the answer to Herrero's question is "no", since there are no triangular strictly cyclic operators.

2. Two Operators

Let x be the column matrix with entries $(1, \frac{1}{2}, \frac{1}{3}, \dots)$ and let D be the diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The element of $\mathcal{B}(\mathbb{C} \oplus \mathcal{H})$ (relative to a chosen basis of $\mathcal{H}$) with the matrix

$$S = \begin{bmatrix} 0 & 0 \\ x & D \end{bmatrix}$$

was given in [12] as an example of a strictly cyclic operator that is *semitriangular;* it is an extension of a triangular operator by a finite rank operator. A bi-strictly cyclic operator is obtained by extending S one more dimension as follows. Let x^τ denote the transpose of x and let T be the element of $\mathcal{B}(\mathbb{C} \oplus \mathbb{C} \oplus \mathcal{H})$ given by the matrix

$$T = \begin{bmatrix} 0 & 0 & x^\tau \\ 0 & 0 & 0 \\ 0 & x & D \end{bmatrix}.$$

Theorem 2.1. *The operator T is bi-strictly cyclic.*

Proof. Our proof of strict cyclicity is a modification of the argument given in [12] that S is strictly cyclic. If p is a polynomial, and we write $p(t) = a + q(t)$ with $q(0) = 0$, then

$$p(T) = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b & w^\tau \\ 0 & 0 & 0 \\ 0 & w & q(D) \end{bmatrix},$$

where w is the column matrix with entries $(q(1), q(\frac{1}{2}), q(\frac{1}{3}), \dots)$. Let $|| \cdot ||_2$ denote the Hilbert-Schmidt norm of an operator, and let $e \in \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{H}$ be the vector

$e = (0, 1, 0)$. It follows that $p(T)e = (b, a, w)$ and using the Cauchy Schwartz inequality we have

$$\begin{aligned} ||p(T)|| &\leq |a| + ||q(T)||_2 \\ &= |a| + \sqrt{3\sum_{i=1}^{\infty} |q(\frac{1}{i})|^2 + |b|^2} \\ &\leq |a| + \sqrt{3}\sqrt{|w|^2 + |b|^2} \\ &\leq \sqrt{4}\sqrt{|a|^2 + |w|^2 + |b|^2} \\ &= 2||p(T)e||. \end{aligned}$$

Thus evaluation at e is bounded below.

We must now show that

$$\mathcal{M} \equiv \{\, p(T)e \mid p \text{ a polynomial } \,\}$$

is dense in $\mathbb{C} \oplus \mathbb{C} \oplus \mathcal{H}$. Assume that $(\alpha, \beta, v) \in \mathcal{M}^{\perp}$. Since $e \in \mathcal{M}$ we know that $\beta = 0$. Assume via contradiction that $\alpha \neq 0$. Then the vector $(-1, 0, \frac{-1}{\alpha}v)$ is orthogonal to $T^n e = (x^{\tau} D^{n-2}x, 0, D^{n-1}x)$ for all $n \geq 2$. It follows that

$$\langle D^{n-2}x, D(\frac{-1}{\alpha}v) - x\rangle = \langle D^{n-1}x, \frac{-1}{\alpha}v\rangle - \langle D^{n-2}x, x\rangle = 0$$

for all $n \geq 2$. Since x is a cyclic vector for D we have $D(\frac{-1}{\alpha}v) = x$ which contradicts the fact that x is not in the range of D. Thus $\alpha = 0$ and $v \perp D^n x$ for all $n \geq 0$, from which we see that $v = 0$.

To see that T^* is strictly cyclic, note that T is unitarily equivalent to T^* via the unitary

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I \end{bmatrix},$$

where I denotes the identity operator on $\mathcal{H}$. □

The hypothesis of bi-strict cyclicity forces a lot of structure on an operator, structure that can be seen by comparing S and T. As a first illustration we will prove that it is no accident T is equivalent to T^*. If $\mathcal{E}$ is a basis of $\mathcal{H}$ we will write $\tau_{\mathcal{E}}$ to indicate the transpose operator relative to $\mathcal{E}$.

Theorem 2.2. *Assume that R is a strictly cyclic operator. The following are equivalent:*

1. *R is bi-strictly cyclic*
2. *R is conjugate similar to R^**
3. *For any basis $\mathcal{E}$ of $\mathcal{H}$ we have R similar to $\tau_{\mathcal{E}}(R)$*

Proof. Assume that R is bi-strictly cyclic; we will construct a conjugate linear invertible operator $K : \mathcal{H} \to \mathcal{H}$ such that $R^* = KRK^{-1}$. Let e be a strict cyclic vector for R, let f be a strict cyclic vector for R^*, and let $\mathcal{A}(R)$ be the weakly closed algebra generated by R. It follows that the evaluation maps $\varepsilon_e : \mathcal{A}(R) \to \mathcal{H}$ and $\varepsilon_f : \mathcal{A}(R)^* \to \mathcal{H}$ are invertible operators (see [20]). Given $x \in \mathcal{H}$ define K by

$$Kx = \varepsilon_f(\varepsilon_e^{-1}x^*).$$

(The quantity in the parenthesis is intended to indicate the adjoint of the operator $\varepsilon_e^{-1}x$.) It follows that for every $B \in \mathcal{A}(R)$ one has $KBe = B^*f$. Since $\mathcal{A}(R)$ is commutative we have

$$KR(Ae) = K(RA)e = (RA)^*f = R^*A^*f = R^*K(Ae)$$

for all $A \in \mathcal{A}(R)$. Thus $KR = R^*K$ and R is conjugate similar to R^*.

Assume that R is conjugate similar to R^* and let K be a conjugate linear invertible operator with $R^* = KRK^{-1}$. Let $\mathcal{E}$ be any basis of $\mathcal{H}$ and let $\kappa_{\mathcal{E}} : \mathcal{H} \to \mathcal{H}$ be the map defined by

$$\langle \kappa_{\mathcal{E}} x, y \rangle = \langle y, x \rangle$$

for all $y \in \mathcal{E}$; thus $\kappa_{\mathcal{E}} x$ is the vector whose Fourier coefficients relative to $\mathcal{E}$ are the conjugates of the Fourier coefficients of x. Note that $\kappa_{\mathcal{E}}$ is a conjugate linear symmetry that behaves like the identity map on $\mathcal{E}$. These observations let us see that for any operator A

$$\tau_{\mathcal{E}}(A) = \kappa_{\mathcal{E}} A^* \kappa_{\mathcal{E}}.$$

If $G = \kappa_{\mathcal{E}} K$, then G is an invertible linear operator and

$$GRG^{-1} = \kappa_{\mathcal{E}} KRK^{-1} \kappa_{\mathcal{E}} = \tau_{\mathcal{E}}(R),$$

so R is similar to $\tau_{\mathcal{E}}(R)$.

Assume that R is strictly cyclic and R is similar to $\tau_{\mathcal{E}}(R)$ for some basis $\mathcal{E}$. Then $\tau_{\mathcal{E}}(R)$ must also be strictly cyclic, as must be $\kappa_{\mathcal{E}} \tau_{\mathcal{E}}(R) \kappa_{\mathcal{E}}$. It follows that R^* is strictly cyclic since $R^* = \kappa_{\mathcal{E}} \tau_{\mathcal{E}}(R) \kappa_{\mathcal{E}}$. □

3. Invariant Subspaces

In finite dimensions any cyclic operator is automatically bi-strictly cyclic, since an $n \times n$ complex matrix is similar to its transpose. Our previous result shows that a bi-strictly cyclic operator mimics the behavior of operators on finite dimensional spaces, which the next corollary further illustrates. The proof proceeds exactly like the case when R is an operator on a finite dimensional space (see [18] Theorem 4.6). Let Lat$R$ denote the lattice of invariant subspaces of R.

Corollary 3.1. *If R is bi-strictly cyclic, then* Lat$R$ *is self dual.*

We will call an operator R *hereditarily strictly cyclic* if the restriction of R to each invariant subspace is strictly cyclic (we take this definition from [19]). An example of a hereditarily strictly cyclic operator is the Donoghue operator (see [18] page 66 and [20]).

Theorem 3.2. *An operator on $\mathcal{H}$ cannot be both bi-strictly cyclic and hereditarily strictly cyclic.*

Proof. Assume by way of contradiction that R is bi-strictly cyclic and hereditarily strictly cyclic. It follows from Theorem 2.1 of [5] that Lat$R$ has the ascending chain condition. By our previous corollary Lat$R$ must also have the descending chain condition. It follows that a maximal chain of invariant subspaces has the form

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_n,$$

and since $\mathcal{H}$ is infinite dimensional at least one of the spaces $\mathcal{M}_i \ominus \mathcal{M}_{i-1}$ has dimension greater than one. But the compression of R to $\mathcal{M}_i \ominus \mathcal{M}_{i-1}$ is also strictly cyclic (since the restriction to $\mathcal{M}_i$ is strictly cyclic and by Lemma 1 of [19]) and hence has a non-trivial invariant subspace, which contradicts the maximality of the chain. $\square$

We will now make a detailed analysis of the invariant subspaces of S and of T, with the intention of revealing more secrets about bi-strictly cyclic operators. With this goal in sight we begin with a matricial characterization of $\mathcal{A}(S)$ and $\mathcal{A}(T)$, the weakly closed algebras generated by S and T (respectively). The following theorems will be stated for both S and T but only proved for T, since the proofs for S are the same.

We will think of elements of ℓ^2 as column vectors. Given $x \in \ell^2$ we will denote the transpose by x^τ and we let D_x denote the diagonal operator with x on its diagonal. We use I to denote an identity operator and rely on context to identify the space on which it acts.

Theorem 3.3. *We have*

$$\mathcal{A}(S) = \left\{ aI + \begin{bmatrix} 0 & 0 \\ x & D_x \end{bmatrix} \middle| a \in \mathbb{C}, x \in \ell^2 \right\}$$

and

$$\mathcal{A}(T) = \left\{ aI + \begin{bmatrix} 0 & b & x^\tau \\ 0 & 0 & 0 \\ 0 & x & D_x \end{bmatrix} \middle| a, b \in \mathbb{C}, x \in \ell^2 \right\}.$$

Proof. Let $\mathcal{A}$ denote the set above that we intend to prove equals $\mathcal{A}(T)$. It follows immediately that $\mathcal{A}$ is a commutative strictly cyclic algebra with strict cyclic vector $(0, 1, 0)$. Thus $\mathcal{A}$ is maximal abelian ([20] page 92) and weakly closed. Since $T \in \mathcal{A}$ it follows that $\mathcal{A}(T) \subset \mathcal{A}$. But $\mathcal{A}(T)$ is also a commutative strictly cyclic algebra, hence maximal abelian. Thus equality follows. $\square$

Many invariant subspaces are now visible; for example the range and kernel of any operator in $\mathcal{A}(T)$. This algebra has an abundance of rank one operators. Let E_0 be the nilpotent element with $b = 1$ and all other entries 0, and for $i \geq 1$ let

$$E_i = \begin{bmatrix} 0 & 1 & e_i^\tau \\ 0 & 0 & 0 \\ 0 & e_i & D_{e_i} \end{bmatrix},$$

where $e_1, e_2, \ldots$ is the standard basis of ℓ^2.

Theorem 3.4. *Both S and T are reflexive operators.*

Proof. Assume that $A \in$ algLat$T$. Since the range of E_i is in Lat$A$ for all $i \geq 0$ the matrix of A must look like

$$\begin{bmatrix} a & b & y_1 - a & y_2 - a & y_3 - a & \\ 0 & c & 0 & 0 & 0 & \\ 0 & x_1 & y_1 & 0 & 0 & \cdots \\ 0 & x_2 & 0 & y_2 & 0 & \\ 0 & x_3 & 0 & 0 & y_3 & \\ & & \vdots & & & \ddots \end{bmatrix}.$$

Since the kernel of E_i is also invariant, using this fact with $i \geq 1$ gives us $-c = x_i - y_i$. It follows that

$$A - cI = \begin{bmatrix} a-c & b & y_1-a & y_2-a & y_3-a & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & x_1 & x_1 & 0 & 0 & \cdots \\ 0 & x_2 & 0 & x_2 & 0 & \\ 0 & x_3 & 0 & 0 & x_3 & \\ & & \vdots & & & \ddots \end{bmatrix}.$$

Now $y_i - a = x_i + c - a$ is in the row of a bounded operator so it must be a null sequence, and similarly x_i is a null sequence, so it must be that $a = c$. It now follows from the previous theorem that $A \in \mathcal{A}(T)$. □

The reader has probably noticed that the nilpotent element appearing in $\mathcal{A}(T)$ is missing in $\mathcal{A}(S)$ (which is a clue foreshadowing a forthcoming theorem about general bi-strictly cyclic operators). It is possible to have ignored E_0 in the proof above since the range of E_0 may be obtained by intersecting the kernels of E_i and the kernel of E_0 is the span of the ranges of the E_i (for $i \geq 1$). The reader may verify that the operators E_i with $i \geq 1$ constitute all the rank one idempotents in $\mathcal{A}(T)$.

Theorem 3.5. *The lattice of invariant subspaces for T (resp. S) is the lattice generated by the ranges and kernels of rank one idempotents in $\mathcal{A}(T)$ (resp. $\mathcal{A}(S)$). Moreover, every invariant subspace of T may be obtained by spanning ranges of E_i or by intersecting kernels of E_i $(i \geq 0)$.*

Proof. By the remark preceding the theorem it suffices to prove the latter statement of the theorem. Assume that $\mathcal{M} \in \text{Lat}T$. Let $\mathcal{M}^+$ be the intersection of all kernels of E_i for which $\mathcal{M} \subset \text{Kernel}(E_i)$, and let $\mathcal{M}^-$ be the span of all ranges of E_i for which $\text{Range}(E_i) \subset \mathcal{M}$. We will prove that $\mathcal{M}^+ \ominus \mathcal{M}^-$ is at most one dimensional, in which case $\mathcal{M}$ must be either $\mathcal{M}^+$ or $\mathcal{M}^-$.

Since $\mathcal{M} \in \text{Lat}T$ and $E_i \in \mathcal{A}(T)$ we have $\mathcal{M} \in \text{Lat}E_i$ for all $i \geq 0$. Since E_i has rank one and $\mathcal{M}$ is invariant we must have either $\mathcal{M} \subset \text{Kernel}(E_i)$ or $\text{Range}(E_i) \subset \mathcal{M}$; thus the non-negative integers are partitioned into two subsets $\mathcal{I} = \{\, i \mid \mathcal{M} \subset \text{Kernel}(E_i)\,\}$ and $\mathcal{J} = \{\, i \mid \text{Range}(E_i) \subset \mathcal{M}\,\}$.

It will be convenient now to let $f_1, f_2, f_3, ...$ denote a basis relative to which the matrix of an element of $\mathcal{A}(T)$ has the form given in Theorem 3.3. With this notation and $i \geq 1$ we have $w \in \text{Kernel}(E_i)$ if and only if $\langle w, f_2\rangle = -\langle w, f_{i+2}\rangle$, and w is orthogonal to the range of E_i if and only if $\langle w, f_1\rangle = -\langle w, f_{i+2}\rangle$.

Assume that $w \in \mathcal{M}^+ \ominus \mathcal{M}^-$. If both $\mathcal{I}$ and $\mathcal{J}$ are infinite then $\langle w, f_2\rangle = -\langle w, f_{i+2}\rangle$ for all $i \in \mathcal{I}$ and $\langle w, f_1\rangle = -\langle w, f_{i+2}\rangle$ for all $i \in \mathcal{J}$. Since $\mathcal{I}$ and $\mathcal{J}$ partition the non-negative integers and the Fourier coefficients of w form a null sequence we conclude that the Fourier coefficients all vanish, thus $w = 0$ and $\mathcal{M}^+ = \mathcal{M}^-$. If $\mathcal{I}$ is infinite but $\mathcal{J}$ is finite and non-empty we conclude that $\langle w, f_2\rangle = -\langle w, f_{i+2}\rangle$ for all $i \in \mathcal{I}$, whence $\langle w, f_2\rangle = 0 = \langle w, f_{i+2}\rangle$ for all $i \in \mathcal{I}$, and $\langle w, f_1\rangle = -\langle w, f_{i+2}\rangle$ for all $i \in \mathcal{J}$, so $\mathcal{M}^+ \ominus \mathcal{M}^-$ is one dimensional. The remaining case is dealt with similarly. □

If R has a strict cyclic vector e then the evaluation map ε_e establishes a one to one correspondence between the closed ideals in $\mathcal{A}(R)$ and the elements of $\text{Lat}R$.

Thus knowing the maximal elements of Lat$R$ allows one to compute the radical of $\mathcal{A}(R)$. It is clear from the previous theorem that $\mathcal{A}(S)$ is semisimple and the radical of $\mathcal{A}(T)$ is the one dimensional span of E_0. (Note that the reflexivity of $\mathcal{A}(S)$ can now be deduced from the semisimplicity and from Theorem 5.2 of [14].) Thus the presence of the nilpotent in $\mathcal{A}(T)$ is a reflection of a non-empty radical, which is a property every bi-strictly cyclic operator shares.

Theorem 3.6. *Assume that R is a strictly cyclic operator and $\mathcal{A}(R)$ is semisimple. Then R^* is triangular.*

Proof. If $\mathcal{A}(R)$ is semisimple, then the set of eigenvectors for R^* spans $\mathcal{H}$ (see page 722 of [14]). It follows that the set of algebraic vectors of R^* is dense, hence R^* is triangular (see page 477 of [12]). □

Corollary 3.7. *If R is a bi-strictly cyclic operator on $\mathcal{H}$ then the radical of $\mathcal{A}(R)$ is non-zero. In particular, $\mathcal{A}(R)$ contains a non-zero quasinilpotent element.*

4. Singly Generated Strictly Cyclic Algebras

In this section we will investigate where strictly cyclic algebras reside in the category of operator spaces. We will assume the reader is familiar with the basic ideas of operator spaces and completely bounded maps. We refer the reader to the preliminary chapter of [2] for a recent exposition of the basics.

The notion of equivalence that is relevant to this discussion is that of complete isomorphism. We will say that the operator space $\mathcal{X}$ is completely isomorphic to the operator space $\mathcal{Y}$ if there exists a linear bijection $\varphi : \mathcal{X} \to \mathcal{Y}$ for which both φ and φ^{-1} are completely bounded.

Column Hilbert space is an object that plays a central role in the emerging theory of abstract operator spaces (see [1], [2], [3], and [17]). Its central role arises from the *ad hoc* manner that one assigns matrix norms on $\mathcal{B}(\mathcal{H})$; by identifying an $n \times n$ matrix of operators as an operator on the n-fold direct sum of $\mathcal{H}$ via matrix multiplication on the left. Given $e \in \mathcal{H}$ one obtains the evaluation map $\varepsilon_e : \mathcal{B}(\mathcal{H}) \to \mathcal{H}$ and is eventually confronted with the question of which operator norm assignments on $\mathcal{H}$ have the property that

$$||e|| = ||\varepsilon_e||_{cb}$$

for all $e \in \mathcal{H}$. The answer is that there are many; indeed, having found one matrix norm assignment that works any smaller system of norms will also work. However, there is a unique *largest* system of operator matrix norms one can endow on $\mathcal{H}$ so that $||e|| = ||\varepsilon_e||_{cb}$ for all $e \in \mathcal{H}$, and the resulting Hilbertian operator space is what we call column Hilbert space.

One obtains the column Hilbert space matrix norms by embedding $\mathcal{H}$ into $\mathcal{B}(\mathcal{H})$ in the following way; with $x, e \in \mathcal{H}$ let $x \otimes e$ denote the rank one operator defined by

$$x \otimes e\,(y) = \langle y, e \rangle x.$$

Now if $||e|| = 1$, then the map $x \mapsto x \otimes e$ is an isometry; the column Hilbert matrix norms are inherited from this embedding, i.e.

$$||(x_{ij})||_{col} = ||(x_{ij} \otimes e)||.$$

If one writes an orthonormal basis of $\mathcal{H}$ using e as the first basis vector and x_i are the corresponding Fourier coefficients of x, then the preceding embedding takes the matricial form

$$x \mapsto \begin{bmatrix} x_1 & 0 & 0 & \\ x_2 & 0 & 0 & \dots \\ x_3 & 0 & 0 & \\ & \vdots & & \ddots \end{bmatrix},$$

which is where the terminology arises from. The reader will verify that the matrix norms do not depend on the choice of the unit vector e.

If A_{ij} are operators on $\mathcal{H}$ and $e \in \mathcal{H}$ is a unit vector, then

$$||(\varepsilon_e A_{ij})||_{col} = ||(A_{ij} e \otimes e)|| = ||(A_{ij}) \begin{pmatrix} e \otimes e & 0 & \dots & 0 \\ 0 & e \otimes e & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & e \otimes e \end{pmatrix}|| \leq ||(A_{ij})||,$$

so ε_e is completely contractive. If one took another system of matrix norms on $\mathcal{H}$ that was larger than the column Hilbert space norms, so that

$$||(x_{ij})||_{col} < ||(x_{ij})||$$

for some choice of vectors $x_{ij} \in \mathcal{H}$, then one sees that

$$||(A_{ij})|| < ||(\varepsilon_e A_{ij})||$$

when $A_{ij} = x_{ij} \otimes e$, so the column matrix norm assignments are indeed the largest norms that ensure ε_e is completely contractive.

If R is a strictly cyclic operator with strict cyclic unit vector e, then we see that $\varepsilon_e : \mathcal{A}(R) \to \mathcal{H}_{col}$ is a complete contraction. We are interested in when ε_e^{-1} is completely bounded, i.e. when $\mathcal{A}(R)$ is completely isomorphic to column Hilbert space. By definition one obtains the cb norm of an operator $\varphi : \mathcal{H}_{col} \to \mathcal{B}(\mathcal{H})$ by

$$\sup ||(\varphi(x_{ij}))||$$

where one takes the supremum over matrices of all sizes and dimensions subject to $||(x_{ij})||_{col} \leq 1$. It is a noteworthy property of $\mathcal{H}_{col}$ that one obtains the cb norm of φ by just taking the supremum over row matrices, and better still, if the Hilbert space is finite dimensional with orthonormal basis $\{e_1, e_2, \dots, e_n\}$, then the cb norm is attained:

$$||\varphi||_{cb} = ||(\varphi(e_1)\ \varphi(e_2)\ \varphi(e_3) \cdots \varphi(e_n))||.$$

The proof of this fact is implicit in the proof of Theorem 3.11 in [1]. We feel it is an important and useful fact so we state the corresponding fact for an infinite dimensional Hilbert space and reproduce the proof below.

Theorem 4.1. *Let $\{e_1, e_2, \dots\}$ be any orthonormal basis of $\mathcal{H}$. Assume that $\varphi : \mathcal{H}_{col} \to \mathcal{B}(\mathcal{H})$ and let $A = (\varphi(e_1)\ \varphi(e_2)\ \varphi(e_3) \cdots)$. Then*

$$||\varphi||_{cb} = ||A||,$$

where the norm on the right is the operator norm on $\mathcal{B}(\mathcal{H}^\infty, \mathcal{H})$ if $A \in \mathcal{B}(\mathcal{H}^\infty, \mathcal{H})$, and it is ∞ if A is not the matrix of a bounded operator.

Proof. Since for every natural number n we have

$$||(e_1\ e_2\ e_3 \dots e_n)||_{col} = 1$$

it follows from the definition of the cb norm that

$$||A|| \leq ||\varphi||_{cb},$$

so if A is not the matrix of a bounded operator we are done. Assume then that $A \in \mathcal{B}(\mathcal{H}^\infty, \mathcal{H})$. Given $x \in \mathcal{H}_{col}$ with Fourier coefficients x_i, and letting I denote the identity operator on $\mathcal{H}$, note that the map

$$x \mapsto \begin{bmatrix} x_1 I \\ x_2 I \\ x_3 I \\ \vdots \end{bmatrix} \in \mathcal{B}(\mathcal{H}, \mathcal{H}^\infty)$$

is a complete isometry. Since φ can be viewed as multiplication on the left by A, i.e.

$$\varphi(x) = (\varphi(e_1)\ \varphi(e_2) \cdots) \begin{bmatrix} x_1 I \\ x_2 I \\ x_3 I \\ \vdots \end{bmatrix}.$$

It follows that $||\varphi||_{cb} \leq ||A||$. □

Corollary 4.2. *If R is strictly cyclic with strict cyclic vector e, and if $\{e_1, e_2, \dots\}$ is an orthonormal basis, then $\mathcal{A}(R)$ is completely isomorphic to column Hilbert space if and only if*

$$(\varepsilon_e^{-1}(e_1)\ \ \varepsilon_e^{-1}(e_2)\ \ \varepsilon_e^{-1}(e_3) \cdots)$$

is the matrix of a bounded operator.

Corollary 4.3. *We have that $\mathcal{A}(S)$ is completely isomorphic to $\mathcal{H}_{col}$.*

Proof. The matrix

$$\left(\begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \\ & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \\ 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & \\ & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \\ 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & \\ & \vdots & & \ddots \end{bmatrix} \cdots \right)$$

can be realized as a sum of three partial isometries, and thus it is the matrix of a bounded operator. □

Let R be an injective unilateral weighted shift with positive weight sequence $w_0, w_1, \dots$ relative to the basis $\{e_0, e_1, e_2, e_3, \dots\}$, and let

$$\beta(0) = 1, \quad \beta(1) = w_0, \quad \beta(2) = w_0 w_1, \quad \beta(3) = w_0 w_1 w_2, \quad \dots$$

be the associated β sequence (we follow the notation in [20]). Since $e = e_0$ is a cyclic vector for any injective unilateral weighted shift, the question of strict cyclicity is equivalent to the question of whether ε_e is bounded below on the algebra generated by R. This is equivalent to ε_e^{-1} being bounded on the linear span of the basis $\{e_i\}$, which is the point of view that contrasts well with our next theorem. Thus, by

Proposition 32 of [20], one has that ε_e^{-1} is bounded on the linear span of the basis $\{e_i\}$ if and only if

$$\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} a(k)b(n-k)\frac{\beta(n)}{\beta(k)\beta(n-k)}\right|^2 < \infty$$

for all $a, b \in \ell^2$.

A sufficient condition that R be strictly cyclic appears as equation (61) in Proposition 32 of [20], and that condition is

$$\sup_n \sum_{k=0}^{n}\left(\frac{\beta(n)}{\beta(k)\beta(n-k)}\right)^2 < \infty.$$

It was known at the time that (61) is necessary and sufficient for strict cyclicity if the weight sequence is monotone decreasing, but it was briefly an open question whether equation (61) was necessary and sufficient for general weight sequences. An example due to Fricke showed it not to be necessary [4].

Theorem 4.4. *With the notation of the previous paragraph we have*

$$||\varepsilon_e^{-1}||_{cb}^2 = \sup_n \sum_{k=0}^{n}\left(\frac{\beta(n)}{\beta(k)\beta(n-k)}\right)^2,$$

so $\mathcal{A}(R)$ is completely isomorphic to column Hilbert space if and only if equation (61) *of* [20] *is satisfied.*

Proof. For brevity let us write

$$\beta_{n,k} = \frac{\beta(n)}{\beta(k)\beta(n-k)}.$$

If W denotes the forward shift operator

$$We_i = e_{i+1}$$

and D_n is the diagonal operator with diagonal sequence $(\beta_{n,0}, \beta_{n+1,1}, \beta_{n+2,2}, \dots)$ for $n \geq 0$, then the reader will verify that

$$\varepsilon_e^{-1}(e_n) = W^n D_n$$

for all $n \geq 0$, with $e = e_0$. A moment's thought convinces us that

$$||(D_0 \;\; WD_1 \;\; W^2D_2 \;\; \cdots)|| = \sup_n \sqrt{\sum_{k=0}^{n} (\beta_{n,k})^2}.$$

□

Corollary 4.5. *If R is a strictly cyclic weighted shift with a monotonically decreasing weight sequence, then $\mathcal{A}(R)$ is completely isomorphic to column Hilbert space.*

Fricke's result may now be interpreted as the following statement.

Corollary 4.6. *There exists a unilateral strictly cyclic weighted shift such that $\mathcal{A}(R)$ is not completely isomorphic to column Hilbert space.*

The corollaries and the fact that equation (61) is equivalent to $\mathcal{A}(R)$ being completely isomorphic to column Hilbert space are not new results; both are implicit in [15]. However, the proof given above and the exact value for $||\varepsilon_e^{-1}||_{cb}^2$ are new.

In the highly referenced unpublished manuscript [6] Haagerup proves that contractive Shur product maps on $\mathcal{B}(\mathcal{H})$ are automatically completely contractive. Using the techniques in [16] one may construct examples of subspaces $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$ and Shur multipliers that are bounded when restricted to $\mathcal{L}$ but whose restrictions are not completely bounded. The preceding result provides us with another example.

Corollary 4.7. *There exist unilateral strictly cyclic weighted shifts R_1 and R_2 and a Shur multiplier*

$$\phi : \mathcal{A}(R_1) \to \mathcal{A}(R_2)$$

such that ϕ is a bounded bijection but ϕ is not completely bounded.

Proof. Let R_1 be the Donoghue operator, i.e. the shift with weights $(1/2)^i$, let R_2 be the shift in Fricke's example, and let $e = e_0$. It follows that $\mathcal{A}(R_1)$ is completely isomorphic to column Hilbert space via the evaluation map ε_e, while $\mathcal{A}(R_2)$ is not. Recall the matrix forms for the operators $\varepsilon_e^{-1}(e_i)$ and use that to convince yourself that both algebras can be realized as the set of all matrices of the form

$$\begin{bmatrix} x_0\beta_{0,0} & 0 & 0 & 0 & \\ x_1\beta_{1,0} & x_0\beta_{1,1} & 0 & 0 & \cdots \\ x_2\beta_{2,0} & x_1\beta_{2,1} & x_0\beta_{2,2} & 0 & \\ x_3\beta_{3,0} & x_2\beta_{3,1} & x_1\beta_{3,2} & x_0\beta_{3,3} & \\ & \vdots & & & \ddots \end{bmatrix}$$

as $\{x_i\}$ varies over all square summable sequences, while the $\beta_{i,j}$ for R_1 is different than that for R_2. Denote the doubly indexed β sequence for R_1 by $\beta^1_{i,j}$ and that for R_2 by $\beta^2_{i,j}$. Define a lower triangular matrix by setting $a_{ij} = 0$ if $j > i$ and

$$a_{ij} = \frac{\beta^2_{i,j}}{\beta^1_{i,j}}$$

when $i \geq j$. Then the Shur multiplier corresponding to the lower triangular matrix (a_{ij}) is a bounded bijection of $\mathcal{A}(R_1)$ onto $\mathcal{A}(R_2)$. Indeed, if we denote the restriction of ε_e to $\mathcal{A}(R_i)$ by ε_i then the indicated Shur multiplier is simply the composition $\varepsilon_2^{-1} \circ \varepsilon_1$. This map cannot be completely bounded; otherwise $\varepsilon_2^{-1} \circ \varepsilon_1$ would be a complete isomorphism since we know that $\varepsilon_1^{-1} \circ \varepsilon_2$ is completely bounded. □

The reader may wonder what any of this has to do with the title of the paper. While we know that there exists a strictly cyclic operator R such that $\mathcal{A}(R)$ is not column Hilbert space, we have no idea what space $\mathcal{A}(R)$ is. One glance at S is enough to guess that $\mathcal{A}(S)$ is completely isomorphic to column Hilbert space. One glance at T also suggests which Hilbertian operator space it is completely isomorphic to. It is the only example known of a strictly cyclic operator for which $\mathcal{A}(T)$ is identifiable and different from $\mathcal{H}_{col}$.

The dual of column Hilbert space is row Hilbert space $\mathcal{H}_{row}$, which acquires its matrix norms from the embedding

$$x \mapsto e \otimes x.$$

Note that it is impossible for a strictly cyclic operator to generate an algebra completely isomorphic to row Hilbert space. Indeed, if R is strictly cyclic and

$$\varphi : \mathcal{A}(R) \to \mathcal{H}_{col}$$

is any bounded map, then we may factor φ as $(\varphi \circ \varepsilon_e^{-1}) \circ \varepsilon_e$ viewing $\varphi \circ \varepsilon_e^{-1}$ as a map from column Hilbert space into itself. Now column Hilbert space is *homogeneous;* every bounded map from column Hilbert space into itself is completely bounded (the terminology comes from [17]). It follows that we have $\varphi \circ \varepsilon_e^{-1}$ completely bounded, and hence φ is completely bounded as a composite of completely bounded maps. Thus every bounded map from $\mathcal{A}(R)$ into $\mathcal{H}_{col}$ is completely bounded. The transpose map takes row Hilbert space isometrically onto column Hilbert space and it fails to be completely bounded, so $\mathcal{A}(R)$ cannot be row Hilbert space.

Looking at $\mathcal{A}(T)$ one sees a combination of row and column Hilbert space. Let $||\cdot||_{col \vee row}$ denote the smallest family of operator matrix norms that dominate both the column and the row matrix norms, and let $\mathcal{H}_{col} \vee \mathcal{H}_{row}$ denote the resulting Hilbertian operator space. That is, given $x_{ij} \in \mathcal{H}$ define

$$||(x_{ij})||_{col \vee row} = \max \{||(x_{ij})||_{col}, ||(x_{ij})||_{row}\}.$$

We like to think of this as the *join* of $\mathcal{H}_{col}$ and $\mathcal{H}_{row}$, which accounts for our notation. The reader will find other authors using different notation to describe the same space; for example, the same space is denoted by $R \cap C$ in [17].

We leave the proof of the following to the reader.

Theorem 4.8. *We have that $\mathcal{A}(T)$ is completely isomorphic to $\mathcal{H}_{col} \vee \mathcal{H}_{row}$.*

Once again we are led to a general theorem about bi-strictly cyclic operators. The discussion prior to the previous theorem shows that for any strictly cyclic operator R the operator space $\mathcal{A}(R)$ dominates column Hilbert space in the sense that every bounded map $\varphi : \mathcal{A}(R) \to \mathcal{H}_{col}$ is automatically completely bounded. Using the same reasoning and the homogeneity of $\mathcal{H}_{col} \vee \mathcal{H}_{row}$, one sees that every bounded map $\varphi : \mathcal{A}(T) \to \mathcal{H}_{col} \vee \mathcal{H}_{row}$ is completely bounded.

Theorem 4.9. *If R is a bi-strictly cyclic operator, then every bounded map from $\mathcal{A}(R)$ into either $\mathcal{H}_{col}$ or $\mathcal{H}_{row}$ is completely bounded. Equivalently, every bounded map into $\mathcal{H}_{col} \vee \mathcal{H}_{row}$ is completely bounded.*

Proof. Let e be a strict cyclic unit vector for R. In view of the discussion preceding this theorem we need only prove that bounded maps from $\mathcal{A}(R)$ into $\mathcal{H}_{row}$ are completely bounded, and since $\mathcal{H}_{row}$ is homogeneous it suffices to show that the evaluation map $\varepsilon_e : \mathcal{A}(R) \to \mathcal{H}_{row}$ is completely bounded. By Theorem 2.2 there exists a conjugate linear invertible map K such that $A^* = KAK^{-1}$ for all $A \in \mathcal{A}(R)$. It follows that given $A_{ij} \in \mathcal{A}(R)$ we have

$$||(A_{ij}^*)|| \leq ||K|| \, ||K^{-1}|| \, ||(A_{ij})||.$$

We have

$$\begin{aligned}
||(\varepsilon_e A_{ij})||_{row} &= ||(e \otimes A_{ij} e)|| \\
&= ||(e \otimes e\ A_{ij}^*)|| \\
&= || \begin{pmatrix} e\otimes e & 0 & \dots & 0 \\ 0 & e\otimes e & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & e\otimes e \end{pmatrix} (A_{ij}^*)|| \\
&\leq ||(A_{ij}^*)|| \\
&\leq ||K||\ ||K^{-1}||\ ||(A_{ij})||.
\end{aligned}$$

□

Corollary 4.10. *If R is bi-strictly cyclic, then $\mathcal{A}(R)$ cannot be completely isomorphic to $\mathcal{H}_{col}$.*

Proof. Bounded maps from column Hilbert space to row Hilbert space need not be completely bounded (e.g. the transpose map). □

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