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On an approximation property of Pisot numbers II

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Résumé. Soit q un nombre complexe, m un entier positif et $l_m(q) = \inf\{|P(q)|, P \in \mathbb{Z}_m[X], P(q) \neq 0\}$, où $\mathbb{Z}_m[X]$ désigne l'ensemble des polynômes à coefficients entiers de valeur absolue $\leqslant m$. Nous déterminons dans cette note le maximum des quantités $l_m(q)$ quand q décrit l'intervalle $]m, m+1[$. Nous montrons aussi que si q est un nombre non-réel de module > 1, alors q est un nombre de Pisot complexe si et seulement si $l_m(q) > 0$ pour tout m.

Abstract. Let q be a complex number, m be a positive rational integer and $l_m(q) = \inf\{|P(q)|, P \in \mathbb{Z}_m[X], P(q) \neq 0\}$, where $\mathbb{Z}_m[X]$ denotes the set of polynomials with rational integer coefficients of absolute value $\leqslant m$. We determine in this note the maximum of the quantities $l_m(q)$ when q runs through the interval $]m, m+1[$. We also show that if q is a non-real number of modulus > 1, then q is a complex Pisot number if and only if $l_m(q) > 0$ for all m.

1. Introduction

Let q be a complex number, m be a positive rational integer and $l_m(q) = \inf\{|P(q)|, P \in \mathbb{Z}_m[X], P(q) \neq 0\}$, where $\mathbb{Z}_m[X]$ denotes the set of polynomials with rational integer coefficients of absolute value $\leqslant m$ and not all 0. Initiated by P. Erdos et al. in [6], several authors studied the quantities $l_m(q)$, where q is a real number satisfying $1 < q < 2$. The aim of this note is to extend the study for a complex number q. Mainly we determine in the real case the maximum (resp. the infimum) of the quantities $l_m(q)$ when q runs through the interval $]m, m+1[$ (resp. the set of Pisot numbers in $]m, m+1[$). For the non-real case, we show that if q is of modulus > 1 then q is a complex Pisot number if and only if $l_m(q) > 0$ for all m. Recall that a Pisot number is a real algebraic integer > 1 whose conjugates are of modulus < 1. A complex Pisot number is a non-real algebraic integer of modulus > 1 whose conjugates except its complex conjugate are of

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modulus < 1. Note also that the conjugates, the minimal polynomial and the norm of algebraic numbers are considered here over the field of rationals. The set of Pisot numbers (resp. complex Pisot numbers) is usually noted S (resp. S_c). Let us now recall some known results for the real case.

THEOREM A. ([5], [7] and [9])
(i) If $q \in]1, \infty[$, *then* q *is a Pisot number if and only if* $l_m(q) > 0$ *for all* m;
(ii) if $q \in]1, 2[$, *then for any* $\varepsilon > 0$ *there exists* $P \in \mathbb{Z}_1[X]$ *such that* $|P(q)| < \varepsilon$.

THEOREM B. ([15])
(i) If q *runs through the set* $S \cap]1, 2[$, *then* $\inf l_1(q) = 0$;
(ii) if m *is fixed and* q *runs through the interval* $]1, 2[$, *then* $\max l_m(q) = l_m(A)$, *where* $A = \frac{1+\sqrt{5}}{2}$.

The values of $l_m(A)$ have been determined in [11].

In [3] P. Borwein and K. G. Hare gave an algorithm to calculate $l_m(q)$ for any Pisot number q (or any real number q satisfying $l_m(q) > 0$). The algorithm is based on the following points :

(i) From Theorem A (i), the set $\Omega(q, \varepsilon) = \cup_{d \geqslant 0} \Omega_d(q, \varepsilon)$, where ε is a fixed positive number and

$$\Omega_d(q, \varepsilon) = \{|P(q)|, P \in \mathbb{Z}_m[X], \partial P = d, 0 < |P(q)| < \varepsilon\},$$

is finite (∂P is the degree of P);

(ii) if $P \in \mathbb{Z}_m[X]$ and satisfies $|P(q)| < \frac{m}{q-1}$ and $\partial P \geqslant 1$, then P can be written $P(x) = xQ(x) + P(0)$ where $Q \in \mathbb{Z}_m[X]$ and $|Q(q)| < \frac{m}{q-1}$;

(iii) if $q \in]1, m+1[$, then $1 \in \Omega(q, \frac{m}{q-1})$ and $l_m(q)$ is the smallest element of the set $\Omega(q, \frac{m}{q-1})$ (if $q \in]m+1, \infty[$, then from Proposition 1 below we have $l_m(q) = 1$).

The algorithm consists in determining the sets $\Omega_d(q, \frac{m}{q-1})$ for $d \geqslant 0$ and the process terminates when $\cup_{k \leqslant d} \Omega_k(q, \frac{m}{q-1}) = \cup_{k \leqslant d+1} \Omega_k(q, \frac{m}{q-1})$ for some (the first) d. By (i) a such d exists. In this case, we have $\Omega(q, \frac{m}{q-1}) = \cup_{k \leqslant d} \Omega_k(q, \frac{m}{q-1})$ by (ii). For $d = 0$, we have $\Omega_d(q, \frac{m}{q-1}) = \{1, \dots, \min(m, E(\frac{m}{q-1}))\}$, where E is the integer part function. Suppose that the elements of $\Omega_d(q, \frac{m}{q-1})$ have been determined. Then, every polynomial P satisfying $|P(q)| \in \Omega_{d+1}(q, \frac{m}{q-1})$ is of the form $P(x) = xQ(x) + \eta$, where $|Q(q)| \in \Omega_d(q, \frac{m}{q-1})$ and $\eta \in \{-m, \dots 0, \dots, m\}$.

2. The real case

Let q be a real number. From the definition of the numbers $l_m(q)$, we have $l_m(q) = l_m(-q)$ and $0 \leqslant l_{m+1}(q) \leqslant l_m(q) \leqslant 1$, since the polynomial $1 \in \mathbb{Z}_m[X]$. Note also that if q is a rational integer (resp. if $|q| < 1$), then $l_m(q) = 1$ (resp. $l_m(q) \leqslant |q^n|$, where n is a rational integer, and $l_m(q) = 0$). It follows that without loss of generality, we can suppose $q > 1$. The next proposition is a generalization of Remark 2 of [5] and Lemma 8 of [7] :

Proposition 1.

(i) If $q \in [m+1, \infty[$, *then* $l_m(q) = 1$;

(ii) if $q \in]1, m+1[$, *then for any* $\varepsilon > 0$ *there exists* $P \in \mathbb{Z}_m[X]$ *such that* $|P(q)| < \varepsilon$.

Proof. (i) Let $q \in [m+1, \infty[$ and $P(x) = \varepsilon_0 x^d + \varepsilon_1 x^{d-1} + \ldots + \varepsilon_d \in \mathbb{Z}_m[X]$, where $d = \partial P \geqslant 1$ (if $d = 0$, then $|P(q)| \geqslant 1$). Then,

$$|P(q)| \geqslant \left|\varepsilon_0 q^d\right| - \left|\varepsilon_1 q^{d-1}\right| - \ldots - |\varepsilon_d| \geqslant f_{m,d}(q),$$

where the polynomial $f_{m,d}$ is defined by

$$f_{m,d}(x) = x^d - m(x^{d-1} + x^{d-2} + \ldots + x + 1).$$

It suffices now to show that $f_{m,d}(q) \geqslant 1$ and we use induction on d. For $d = 1$, we have $f_{m,d}(q) = q - m \geqslant m + 1 - m = 1$. Assume that $f_{m,d}(q) \geqslant 1$ for some $d \geqslant 1$. Then, from the recursive formula

$$f_{m,d+1}(x) = x f_{m,d}(x) - m$$

and the induction hypothesis we obtain

$$f_{m,d+1}(q) = q f_{m,d}(q) - m \geqslant q - m \geqslant 1.$$

(ii) Let $q \in]1, m+1[$. Then, the numbers $\xi_j = \varepsilon_0 + \varepsilon_1 q + \ldots + \varepsilon_n q^n$, where n is a non-negative rational integer and $\varepsilon_k \in \{0, 1, \ldots, m\}$, $0 \leqslant k \leqslant n$ satisfy $0 \leqslant \xi_j \leqslant m\frac{q^{n+1}-1}{q-1}$ for all $j \in \{1, 2, \ldots, (m+1)^{n+1}\}$ From the Pigeonhole principle, we obtain that there exist j and l such that $1 \leqslant j < l \leq (m+1)^{n+1}$ and

$$|\xi_j - \xi_l| \leqslant m\frac{q^{n+1} - 1}{((m+1)^{n+1} - 1)(q-1)}.$$

It follows that the polynomial $P \in \mathbb{Z}_m[X]$ defined by

$$P(q) = \xi_j - \xi_l$$

satisfies the relation $|P(q)| \leqslant m\frac{q^{n+1}-1}{((m+1)^{n+1}-1)(q-1)}$ and the result follows by choosing for any $\varepsilon > 0$, a rational integer n so that

$$\frac{m}{(q-1)}\frac{q^{n+1} - 1}{(m+1)^{n+1} - 1} < \varepsilon.$$

□

We cannot deduce from Proposition 1 (ii) that q is an algebraic integer when q satisfies $l_{E(q)}(q) > 0$ except for the case $E(q) = 1$. However, we have :

Proposition 2. *If* $l_{E(q)+1}(q) > 0$, *then* q *is a beta-number.*

Proof. Let $\sum_{n\geqslant 0} \frac{\varepsilon_n}{q^n}$ be the beta-expansion of q in basis q [13]. Then, q is said to be a beta-number if the subset $\{F_n(q), n \geqslant 1\}$ of the interval $[0,1[$, where

$$F_n(x) = x^n - \varepsilon_0 x^{n-1} - \varepsilon_1 x^{n-2} - \ldots - \varepsilon_{n-1},$$

is finite [12]. Here, the condition $l_{E(q)+1}(q) > 0$, implies trivially that q is a beta-number (as in the proof of Lemma 1.3 of [9]), since otherwise for any $\varepsilon > 0$ there exists n and m such that $n > m$, $0 < |F_n(q) - F_m(q)| < \varepsilon$ and $(F_n - F_m) \in \mathbb{Z}_{E(q)+1}[X]$. □

Remark 1. Recall that beta-numbers are algebraic integers, Pisot numbers are beta-numbers, beta-numbers are dense in the interval $]1, \infty[$ and the conjugates of a beta-number q are all of modulus $< \min(q, \frac{1+\sqrt{5}}{2})$ ([4], [12] and [14]). Note also that it has been proved in [8], that if $q \in]1, \frac{1+\sqrt{5}}{2}]$ and $l_{E(q)+1}(q) > 0$, then $q \in S$. The question whether Pisot numbers are the only numbers $q > 1$ satisfying $l_{E(q)}(q) > 0$, has been posed in [7] for the case $E(q) = 1$.

From Proposition 1 (resp. Theorem B) we deduce that $\inf l_m(q) = 0$ (resp. $\max l_1(q) = l_1(A)$) if q runs through the set $S \cap]1, m+1[$ (resp. the interval $]1, 2[$). Letting $A = A_1$, we have more generally :

Theorem 1.
(i) If q *runs through the set* $S \cap]m, m+1[$, *then* $\inf l_m(q) = 0$;
(ii) if q *runs through the interval* $]m, m+1[$, *then* $\max l_m(q) = l_m(A_m) = A_m - m$, *where* $A_m = \frac{m+\sqrt{m^2+4m}}{2}$.

Proof. (i) Let $q \in S \cap]m, m+1[$, such that its minimal polynomial $P \in \mathbb{Z}_m[X]$. Suppose moreover, that there exists a polynomial $Q \in \mathbb{Z}[X]$ satisfying $Q(q) > 0$ and $|Q(z)| < |P(z)|$ for $|z| = 1$ (choose for instance $q = A_m$ since $m < A_m < m+1$, $P(x) = x^2 - mx - m$ and $Q(x) = x^2 - 1$. In this case $|P(z)|^2 - |Q(z)|^2 = 2m^2 - 1 + m(m-1)(z + \frac{1}{z}) - (m-1)(z^2 + \frac{1}{z^2})$ and $|P(z)|^2 - |Q(z)|^2 \geqslant 2m^2 - 1 - 2m(m-1) - 2(m-1) = 1 > 0$).

From Rouché's theorem, we have that the roots of the polynomial

$$Q_n(x) = x^n P(x) - Q(x),$$

where n is a rational integer $\geqslant \partial P$, are all of modulus < 1 except only one root, say θ_n. Moreover, since $Q_n(q) < 0$, we deduce that $\theta_n > q$ and $\theta_n \in S$.

Now, from the equation

$$\theta_n^n P(\theta_n) - Q(\theta_n) = 0,$$

we obtain

$$|P(\theta_n)| = \frac{|Q(\theta_n)|}{\theta_n^n} \leqslant \frac{C_Q}{\theta_n^{n-\partial Q}} \leqslant \frac{C_Q}{q^{n-\partial Q}},$$

where C_Q is a constant depending only on the polynomial Q. As q is the only root > 1 of the polynomial P, from the last relation we obtain $\lim \theta_n = q$ and $\theta_n < m+1$ for n large. Moreover, since $l_m(\theta_n) \leqslant |P(\theta_n)|$, the last relation also yields

$$\lim l_m(\theta_n) = 0$$

and the result follows.

(ii) Note first that $m < A_m = \frac{m+\sqrt{m^2+4m}}{2} < m+1$ and $A_m^2 - mA_m - m = 0$. Let $q \in \,]m, m+1[$ and $q \neq A_m$. Then, $l_m(q) \leqslant q - m < A_m - m$ when $q < A_m$. Suppose now $q > A_m$ and $l_m(q) > 0$ (if $l_m(q) = 0$, then $l_m(q) < A_m - m$). Then, from Proposition 1 (ii), we know that for any $\varepsilon > 0$, there exists a polynomial $P \in \mathbb{Z}_m[X]$ such that $|P(q)| < \varepsilon$. Letting $\varepsilon = l_m(q)$, we deduce that there exist a positive rational integer d and $d+1$ elements, say η_i, of the set $\{-m, \dots, 0, \dots, m\}$ satisfying $\eta_0\eta_d \neq 0$ and

$$\eta_0 + \eta_1 q + \ldots + \eta_d q^d = 0.$$

Let t be the smallest positive rational integer such that $\eta_t \neq 0$. Then, from the last equation, we obtain

$$l_m(q) \leqslant \left|\eta_t + \eta_{t+1} q + \ldots + \eta_d q^{d-t}\right| = \left|\frac{\eta_0}{q^t}\right| \leqslant \frac{m}{q} < \frac{m}{A_m}$$

and

$$l_m(q) < \frac{m}{A_m} = A_m - m.$$

To prove the relation $l_m(A_m) = A_m - m$, we use the algorithm explained in the introduction. With the same notation, we have $\Omega_0(A_m, \frac{m}{A_m-1}) = \{1\}$, since $\frac{m}{A_m-1} = \frac{2m}{m-2+\sqrt{m^2+4m}} < \frac{5}{3}$. Let $P \in \mathbb{Z}_m[X]$. If $\partial P = 1$ and $|P(A_m)| \in \Omega_1(A_m, \frac{m}{A_m-1})$, then $P(x) = x - \varepsilon$, where $\varepsilon \in \{-m, \dots, 0, \dots, m\}$. A short computation shows that if $\varepsilon \neq m$, then $A_m - \varepsilon \geqslant A_m - (m-1) \geqslant \frac{m}{A_m-1}$. It follows that $\Omega_1(A_m, \frac{m}{A_m-1}) = \{A_m - m\}$ and if $\partial P = 2$ with $|P(A_m)| \in \Omega_2(A_m, \frac{m}{A_m-1})$, then $P(x) = x(x-m) - \varepsilon$. Since $A_m(A_m - m) = m$ and the inequality $|m - \varepsilon| < \frac{5}{3}$ holds only for $\varepsilon \in \{m-1, m\}$,we deduce that $P(A_m) = \pm 1$, $\Omega_2(A_m, \frac{m}{A_m-1}) = \{1\}$, $\Omega(A_m, \frac{m}{A_m-1}) = \Omega_0(A_m, \frac{m}{A_m-1}) \cup \Omega_1(A_m, \frac{m}{A_m-1}) = \{1, A_m - m\}$ and $l_m(A_m) = A_m - m$. $\square$

Corollary. *If q runs through the interval $]1, m+1[$ and is not a rational integer, then* $\max l_m(q) = l_m(A_m) = \frac{2}{1+\sqrt{1+\frac{4}{m}}}$.

Proof. From the relations $A_m = m\frac{1+\sqrt{1+\frac{4}{m}}}{2}$ and $l_m(A_m) = \frac{m}{A_m}$, we have

$$\frac{2}{l_m(A_m)} = 1 + \sqrt{1+\frac{4}{m}} > 1 + \sqrt{1+\frac{4}{m+1}} = \frac{2}{l_{m+1}(A_{m+1})}$$

and the sequence $l_m(A_m)$ is increasing with m (to $1 = \lim \frac{2}{1+\sqrt{1+\frac{4}{m}}}$). It follows that $l_{E(q)}(A_{E(q)}) \leq l_m(A_m)$ when $q \in]1, m+1[$. From Theorem 1 (ii), we have $l_{E(q)}(q) \leqslant l_{E(q)}(A_{E(q)})$ if q is not a rational integer. Furthermore, since $l_m(q) \leqslant l_{E(q)}(q)$ we deduce that $l_m(q) \leq l_m(A_m)$ and the result follows. □

Remark 2. From Theorem B (resp. Theorem 1) we have $\max l_{m+k}(q) = l_{m+k}(A_m)$ when q runs through the interval $]m, m+1[$, $m = 1$ and $k \geqslant 0$ (resp. $m \geqslant 1$ and $k = 0$). Recently [1], K. Alshalan and the author considered the case $m = 2$ and proved that if $k \in \{1, 3, 4, 5, 6\}$ (resp. if $k \in \{2, 7, 8, 9\}$), then $\max l_{2+k}(q) = l_{2+k}(1+\sqrt{2})$ (resp. $\max l_{2+k}(q) = l_{2+k}(\frac{3+\sqrt{5}}{2})$).

3. The non-real case

Let a be a complex number. As in the real case we have $l_m(a) = 0$ if $|a| < 1$. Since the complex conjugate of $P(a)$ is $P(\bar{a})$ for $P \in \mathbb{Z}_m[X]$, we have that $l_m(a) = l_m(\bar{a})$. Note also that if a is a non-real quadratic algebraic integer and if $P \in \mathbb{Z}_m[X]$ and satisfies $P(a) \neq 0$, then $|P(a)| \geqslant 1$, since $|P(a)|^2 = P(a)P(\bar{a})$ is the norm of the algebraic integer $P(a)$. It follows in this case that $l_m(a) = 1$.

Proposition 3.
(i) If $|a| \in [m+1, \infty[$, then $l_m(a) = 1$;
(ii) if $|a|^2 \in [1, m+1[$, then for any positive number ε, there exists $P \in \mathbb{Z}_m[X]$ such that $|P(a)| < \varepsilon$.

Proof. (i) The proof is identical to the proof of Proposition 1 (i).

(ii) Let $n \geqslant 0$ be a rational integer and $a^n = x_n + iy_n$, where x_n and y_n are real and $i^2 = -1$. Then, the pairs of real numbers

$$(X_j, Y_j) = (\varepsilon_0 x_0 + \varepsilon_1 x_1 + \ldots + \varepsilon_n x_n,\ \varepsilon_0 y_0 + \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n),$$

where $\varepsilon_k \in \{0, 1, \ldots, m\}$ for all $k \in \{0, 1, \ldots, n\}$, are contained in the rectangle $R = [m\sum_{x_k \leq 0} x_k, m\sum_{0 \leqslant x_k} x_k] \times [m\sum_{y_k \leq 0} y_k, m\sum_{0 \leqslant y_k} y_k]$. If we subdivide each one of two intervals $[m\sum_{x_k \leqslant 0} x_k, m\sum_{0 \leqslant x_k} x_k]$ and $[m\sum_{y_k \leqslant 0} y_k, m\sum_{0 \leqslant y_k} y_k]$ into N subintervals of equal length, then R will be divided into N^2 subrectangles.

Letting $N = (m+1)^{\frac{n+1}{2}} - 1$, where n is odd, then $N^2 < (m+1)^{n+1}$ and from the pigeonhole principle we obtain that there exist two points

(X_j, Y_j) and (X_k, Y_k) in the same subrectangle. It follows that there exist $\eta_0, \eta_1, \dots \eta_n \in \{-m, \dots, 0, \dots, m\}$ not all 0 such that

$$|X_j - X_k| = |\eta_0 x_0 + \eta_1 x_1 + \ldots + \eta_n x_n| \leqslant \frac{m \sum_{0 \leqslant k \leqslant n} |x_k|}{N},$$

$$|Y_j - Y_k| = |\eta_0 y_0 + \eta_1 y_1 + \ldots + \eta_n y_n| \leqslant \frac{m \sum_{0 \leqslant k \leqslant n} |y_k|}{N}$$

and the polynomial $P \in \mathbb{Z}_m[X]$ defined by

$$P(a) = (X_j - X_k) + i(Y_j - Y_k) = \eta_0 + \eta_1 a + \ldots + \eta_n a^n,$$

satisfies

$$|P(a)| \leqslant \frac{m}{N} \sqrt{(\sum_{0 \leqslant k \leqslant n} |x_k|)^2 + (\sum_{0 \leqslant k \leqslant n} |y_k|)^2}.$$

Since

$$\max(\sum_{0 \leqslant k \leqslant n} |x_k|\,, \sum_{0 \leqslant k \leqslant n} |y_k|) \leqslant \sum_{0 \leqslant k \leqslant n} \left| a^k \right| = n+1$$

(resp.

$$\max(\sum_{0 \leqslant k \leqslant n} |x_k|\,, \sum_{0 \leqslant k \leqslant n} |y_k|) \leqslant \sum_{0 \leqslant k \leqslant n} \left| a^k \right| = \frac{|a|^{n+1} - 1}{|a| - 1}),$$

when $|a| = 1$ (resp. when $|a| > 1$), from the last inequality we obtain

$$|P(a)| \leqslant \frac{m\sqrt{2}}{N}(n+1)$$

(resp.

$$|P(a)| \leqslant \frac{m\sqrt{2}}{N} \frac{|a|^{n+1} - 1}{|a| - 1})$$

and the result follows by choosing for any $\varepsilon > 0$ a rational integer n so that

$$(m\sqrt{2})(\frac{n+1}{\sqrt{(m+1)^{n+1}} - 1}) < \varepsilon$$

(resp.

$$(\frac{m\sqrt{2}}{|a| - 1})(\frac{|a|^{n+1} - 1}{\sqrt{(m+1)^{n+1}} - 1}) < \varepsilon).$$

□

Remark 3. The non-real quadratic algebraic integer $a = i\sqrt{m+1}$ satisfies $|a|^2 = m+1$, $l_m(a) = 1$ and is not a root of a polynomial $\in \mathbb{Z}_m[X]$, since its norm is $m+1$. Hence, Proposition 3 (ii) is not true for $|a|^2 = m+1$.

Now we obtain a characterization of the set S_c.

Theorem 2. *Let a be a non-real number of modulus > 1. Then, a is a complex Pisot number if and only if $l_m(a) > 0$ for all m.*

Proof. The scheme (resp. the tools) of the proof is (resp. are) the same as in [5] (resp. in [2] and [10]) with minor modifications. We prefer to give some details of the proof.

Let a be a complex Pisot number. If a is quadratic, then $l_m(a) = 1$ for all m. Otherwise, let $\theta_1, \theta_2, \ldots, \theta_s$ be the conjugates of modulus < 1 of a and let $P \in \mathbb{Z}_m[X]$ satisfying $P(a) \neq 0$. Then, for $k \in \{1, 2, \ldots, s\}$ we have

$$|P(\theta_k)| \leqslant m(|\theta_k|^{\partial P} + |\theta_k|^{\partial P - 1} + \ldots + |\theta_k| + 1) = m\frac{1 - |\theta_k|^{\partial P + 1}}{1 - |\theta_k|} \leqslant \frac{m}{1 - |\theta_k|}.$$

Furthermore, since the absolute value of the norm of the algebraic integer $P(a)$ is $\geqslant 1$, the last relation yields

$$|P(a)|^2 = |P(a)|\,|P(\bar{a})| \geqslant \frac{\prod_{1 \leqslant k \leqslant s}(1 - |\theta_k|)}{m^s}$$

and

$$l_m(a) \geqslant \sqrt{\frac{\prod_{1 \leqslant k \leqslant s}(1 - |\theta_k|)}{m^s}} > 0.$$

To prove the converse, note first that if a is a non-real number such that $l_m(a) > 0$ for all m, then a is an algebraic number by Proposition 3 (ii). In fact we have :

Lemma 1. *Let a be a non-real number of modulus > 1. If $l_m(a) > 0$ for all m, then a is an algebraic integer.*

Proof. As in the proof of Proposition 2, we look for a representation $a = \sum_{n \geqslant 0} \frac{\varepsilon_n}{a^n}$ of the number a in basis a where the absolute values of the rational integers ε_n are less than a constant c depending only on a. In fact from Lemma 1 of [2], such a representation exists with $c = E(\frac{1}{2} + \left|a^2\right| \frac{|a|+1}{|\sin t|})$, where $a = |a|\, e^{it}$. Then, the polynomials

$$F_n(x) = x^n - \varepsilon_0 x^{n-1} - \varepsilon_1 x^{n-2} - \ldots - \varepsilon_{n-1},$$

where $n \geqslant 1$, satisfy $F_n \in \mathbb{Z}_c[X]$ and

$$|F_n(a)| = \left|\sum_{k \geqslant 0} \frac{\varepsilon_{n+k}}{a^{k+1}}\right| \leqslant \frac{c}{|a| - 1}.$$

It follows that if $l_{2c}(a) > 0$, then the set $\{F_n(a), n \geqslant 1\}$ is finite. Consequently, there exists n and m such that $n > m$ and $F_n(a) = F_m(a)$, so that a is a root of the monic polynomial $(F_n - F_m) \in \mathbb{Z}_{2c}[X]$. □

To complete the proof of Theorem 2 it suffices to prove the next two results.

Lemma 2. *Let a be an algebraic integer of modulus > 1. If $l_m(a) > 0$ for all m, then a has no conjugate of modulus* 1.

Proof. Let $I_m = \{F \in \mathbb{Z}_m[X], F(x) = P(x)Q(x), Q \in \mathbb{Z}[X]\}$, where P is the minimal polynomial of a. Let $F \in I_m$ and define a sequence $F^{(k)}$ in $\mathbb{Z}_m[X]$ by the relations $F^{(0)} = F$ and $F^{(k+1)}(x) = \frac{F^{(k)}(x) - F^{(k)}(0)}{x}$,where k is a non-negative rational integer. Then, the polynomials $F^{(k)}$ satisfy $\left|F^{(k)}(a)\right| \leqslant \frac{m}{|a|-1}$. Indeed, we have $F^{(0)}(a) = 0$ and $\left|F^{(k+1)}(a)\right| \leqslant \frac{\left|F^{(k)}(a)\right| + \left|F^{(k)}(0)\right|}{|a|} \leqslant \frac{m}{|a|(|a|-1)} + \frac{m}{|a|} = \frac{m}{|a|-1}$, when $\left|F^{(k)}(a)\right| \leqslant \frac{m}{|a|-1}$. Let $R_F^{(k)} \in \mathbb{Z}[X]$ be the remainder of the euclidean division of the polynomial $F^{(k)}$ by P. Since P is irreducible and $\partial R_F^{(k)} < \partial P$, the set of polynomials $\{R_F^{(k)}, k \geqslant 0, F \in I_m\}$ is finite when the complex set $\{R_F^{(k)}(a), k \geqslant 0, F \in I_m\}$ is finite.

Suppose now that a has a conjugate of modulus 1. Then, from Proposition 2.5 of [10], there exists a positive rational integer c so that the set $\{ R_F^{(k)}, k \geqslant 0, F \in I_c\}$ is not finite. Hence, the bounded set $\{R_F^{(k)}(a) = F^{(k)}(a), k \geqslant 0, F \in I_c\}$ is not finite and for any $\varepsilon > 0$, there exist $F_1 \in I_c$ and $F_2 \in I_c$ such that $0 < \left|F_1^{(k)}(a) - F_2^{(j)}(a)\right| < \varepsilon$, where k and j are non-negative rational integers. Hence, $l_{2c}(a) = 0$, and this contradicts the assumption $l_m(a) > 0$ for all m. □

Lemma 3. *Let a be an algebraic integer of modulus > 1. If $l_m(a) > 0$ for all m, then a has no conjugate of modulus > 1 other than its complex conjugate.*

Proof. Let J_m be the set of polynomials $F \in \mathbb{Z}_m[X]$ satisfying $F(a) = \frac{S(\frac{1}{a})}{a}$, for some $S \in \mathbb{Z}_m[[X]]$ (the set of formal series with rational integers coefficients of absolute value $\leqslant m$). If the polynomials $F^{(k)}$ and $R_F^{(k)}$ are defined for $F \in J_m$ by the same way as in the precedent proof ($I_m \subset J_m$), we obtain immediately $F^{(k)} \in J_m$ and $|F(a)| = \left|\frac{S(\frac{1}{a})}{a}\right| \leqslant \frac{m}{|a|-1}$. Therefore, by the previous argument, the set $\{R_F^{(k)}, k \geqslant 0, F \in J_m\}$ is finite when $l_{2m}(a) > 0$.

Let α be a conjugate of modulus > 1 of a and let $S(x) = \sum_n s_n x^n \in \mathbb{Z}_m[[X]]$ satisfying $S(\frac{1}{a}) = 0$. Then, $S(\frac{1}{\alpha}) = 0$. Indeed, if $F(x) = s_0 x^n + s_1 x^{n-1} + \ldots + s_n$, then $F \in J_m$, $F(\alpha) = R_F^{(0)}(\alpha)$ and

$$S(\frac{1}{\alpha}) = \lim(s_0 + \frac{s_1}{\alpha} + \ldots + \frac{s_n}{\alpha^n}) = \lim \frac{F(\alpha)}{\alpha^n} = \lim \frac{R_F^{(0)}(\alpha)}{\alpha^n} = 0,$$

since the coefficients of the polynomial $R_F^{(0)}$ are bounded ($R_F^{(0)} \in \{ R_F^{(k)}, k \geqslant 0, F \in J_m\}$). It suffices now to find for $\alpha \notin \{a, \bar{a}\}$ a positive rational integer m and an element S of $\mathbb{Z}_m[[X]]$ satisfying $S(\frac{1}{a}) = 0$ and $S(\frac{1}{\alpha}) \neq 0$. In fact this follows from Proposition 7 of [2]. $\square$

Now from Theorem 1 we have the following analog :

Proposition 4.
(i) If a runs through the set $S_c \cap \{\ z\ ,\ \sqrt{m}\ <\ |z|\ <\ \sqrt{m+1}\ \}$, then $\inf l_m(a) = 0$;
(ii) if a runs through the annulus $\{\ z\ ,\ \sqrt{m}\ <\ |z|\ <\ \sqrt{m+1}\ \}$, then $\sup l_m(a) \geqslant l_m(i\sqrt{A_m}) = A_m - m$.

Proof. First we claim that if q is a real number > 1, then $l_m(q) = l_m(i\sqrt{q})$. Indeed, let $P \in \mathbb{Z}_m[X]$ such that

$$P(q) = \eta_0 + \eta_1 q + \ldots + \eta_{\partial P} q^{\partial P} \neq 0.$$

Then,

$$P(q) = \eta_0 - \eta_1 (i\sqrt{q})^2 + \ldots \pm \eta_{\partial P} (i\sqrt{q})^{2\partial P} = Q(i\sqrt{q}),$$

where $Q \in \mathbb{Z}_m[X]$ and $\partial Q = 2\partial P$. It follows that $|P(q)| \geqslant l_m(i\sqrt{q})$ and $l_m(q) \geqslant l_m(i\sqrt{q})$. Conversely, let $P \in \mathbb{Z}_m[X]$ such that

$$P(i\sqrt{q}) = \eta_0 + \eta_1(i\sqrt{q}) + \eta_2(i\sqrt{q})^2 + \ldots + \eta_{\partial P}(i\sqrt{q})^{\partial P} \neq 0.$$

Then, the polynomial R (resp. I) $\in \mathbb{Z}_m[X] \cup \{0\}$ defined by

$$R(q) = \frac{P(i\sqrt{q}) + P(-i\sqrt{q})}{2} = \eta_0 - \eta_2 q + \ldots \pm \eta_{2s} q^s,$$

where $0 \leqslant 2s \leqslant \partial P$, satisfies $|R(q)| \leqslant \left|P(i\sqrt{q})\right|$
(resp.

$$I(q) = \frac{P(i\sqrt{q}) - P(-i\sqrt{q})}{2i\sqrt{q}} = \eta_1 - \eta_3 q + \ldots \pm \eta_{2t+1} q^t$$

where $0 \leqslant 2t+1 \leqslant \partial P$, satisfies $|I(q)| \leqslant \left|\frac{P(i\sqrt{q})}{\sqrt{q}}\right| < \left|P(i\sqrt{q})\right|$).

Since $P(i\sqrt{q}) \neq 0$, at least one of the quantities $R(q)$ and $I(q)$ is $\neq 0$. It follows that $l_m(q) \leqslant \left|P(i\sqrt{q})\right|$ and $l_m(q) \leqslant l_m(i\sqrt{q})$.

Note also that if $q \in S$, then $i\sqrt{q} \in S_c$ and conversely if $i\sqrt{q} \in S_c$, where q is a real number, then $q \in S$. Hence, by Theorem 1 we have

$$0 \leqslant \inf l_m(a) \leqslant \inf l_m(i\sqrt{q}) = \inf l_m(q) = 0,$$

(resp.

$$l_m(i\sqrt{A_m}) = l_m(A_m) = \max l_m(q) = \max l_m(i\sqrt{q}) \leq \sup l_m(a),$$

when a runs through the set $S_c \cap \{z, \sqrt{m} < |z| < \sqrt{m+1}\}$ and q runs through the set $S \cap]m, m+1[$ (resp. when a runs through the annulus $\{z, \sqrt{m} < |z| < \sqrt{m+1}\}$ and q runs through the interval $]m, m+1[$). $\square$

Remark 4. The question of [7] cited in Remark 1, can also be extended to the non-real case : Are complex Pisot numbers the only non-real numbers a satisfying $l_{E(|a^2|)}(a) > 0$, $a^2 + 1 \neq 0$ and $a^2 - a + 1 \neq 0$?

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