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Young tableaux and other mutually describing sequences

Zoran Šunić

Department of Mathematics and Statistics
810 Oldfather Hall, University of Nebraska
Lincoln, NE 68588-0323, USA

zsunik@math.unl.edu

Abstract

We introduce a transformation on integer sequences for which the set of images is in bijective correspondence with the set of Young tableaux. We use this correspondence to show that the set of images, known as ballot sequences, is also the set of double points of our transformation.

In the second part, we introduce other transformations of integer sequences and show that, starting from any sequence, repeated applications of the transformations eventually produce a fixed point (a self-describing sequence) or a double point (a pair of mutually describing sequences).

Counting equal terms

Let $\mathcal{A}^+$ be the set of finite integer sequences $a = a_1a_2\dots$ with $1 \leq a_i \leq i$, for all indices. Define a transformation of sequences $\beta : \mathcal{A}^+ \to \mathcal{A}^+$ by

$$\beta(a)_i = \#\{j \mid j \leq i,\ a_j = a_i\}.$$

Thus $\beta(a)_i$ counts the number of terms in the sequence a that are equal to a_i and appear in the initial segment of a consisting of the first i positions. Therefore, in some sense, the sequence $\beta(a)$ describes the sequence a. It is easy to see that the only fixed point of β is the sequence 1. However, there are many double points, i.e., sequences a for which $\beta^2(a) = a$. If a is a double point so is $b = \beta(a)$, we have $a = \beta(b)$, and the sequences a and b mutually describe each other.

Theorem 1. *The set of double points of β of length n in $\mathcal{A}^+$*

- *corresponds bijectively to the set of Young tableaux of size n.*

- *is the set of images of the sequences of length n under β.*

- *is the set of ballot sequences of length n, i.e., sequences a of length n such that, for every initial segment a' of a and every positive integer x, the number of occurrences of the term x in a' is no smaller than the number of occurrences of $x+1$.*

By Young tableau of size n we mean a standard Young tableau, i.e., a left justified arrangement of the integers $1, 2, \ldots, n$ in several rows of non-increasing length such that all rows and columns have increasing terms (see [Sag90]). For example,

$$\begin{array}{cccccc} 1 & 2 & 5 & 8 & 10 & 12 \\ 3 & 6 & 7 \\ 4 & 9 & 11 \\ 13 \end{array} \tag{1}$$

is a Young tableau of size 13. Denote by $\mathcal{Y}_n$ the set of Young tableaux of size n and by $\mathcal{B}_n$ the set of ballot sequences of length n.

We first remind the reader of a known bijective correspondence between $\mathcal{Y}_n$ and $\mathcal{B}_n$ (see [Sag01, page 176]). For a tableau t in $\mathcal{Y}_n$ define a sequence $\sigma(t)$ of length n by

$$\sigma(t)_i = \text{the number of the row in which } i \text{ appears in } t.$$

In other words, the entries in the first row of t point to the positions in the sequence $\sigma(t)$ whose value is 1, the entries in the second row point to the positions in $\sigma(t)$ whose value is 2, etc. In the other direction, for a sequence $a = a_1 a_2 \ldots a_n$ in $\mathcal{B}_n$ define a tableau $\sigma'(a)$ by

$$\sigma'(a)_{i,j} = \text{the position number of the } j\text{-th occurrence of } i \text{ in } a.$$

Therefore, the first row of $\sigma'(a)$ consists of pointers, in increasing order, to the positions in the sequence a whose value is 1, the second row consists of pointers, in increasing order, to the positions whose value is 2, etc.

It is easy to see that, for every Young tableau t in $\mathcal{Y}_n$ and every ballot sequence a in $\mathcal{B}_n$, $\sigma(t)$ is a ballot sequence, $\sigma'(a)$ is a Young tableau, and $\sigma : \mathcal{Y}_n \to \mathcal{B}_n$ and $\sigma' : \mathcal{B}_n \to \mathcal{Y}_n$ are mutually inverse bijections.

For example, the table below gives the ballot sequence $a = \sigma(t)$ that corresponds to the Young tableau t in (1).

i	1	2	3	4	5	6	7	8	9	10	11	12	13
$\sigma(t)_i$	1	1	2	3	1	2	2	1	3	1	3	1	4

Lemma 1. *The restrictions of β and $\sigma\tau\sigma'$ to the set $\mathcal{B}_n$ of ballot sequences of length n are equal, where τ is transposition of tableaux.*

Proof. Let $a = a_1 a_2 \ldots a_m \ldots a_n$ be a sequence in $\mathcal{B}_n$ and let $a_m = i$ be the j-th occurrence of i in a. By definition,

$$\beta(a)_m = j.$$

On the other hand, $\sigma'(a)_{i,j} = m$, the transposition then gives $\tau\sigma'(a)_{j,i} = m$ and therefore

$$\sigma\tau\sigma'(a)_m = j.$$

□

Since τ acts as an involution on $\mathcal{Y}_n$, β acts as an involution on $\mathcal{B}_n$. Therefore, all ballot sequences are double points of β. This also shows that all ballot sequences are images under β. To prove Theorem 1 it remains to show that all images under β are ballot sequences. This is rather obvious. Indeed, the x-th occurrence of any term in a happens to the left of its $(x+1)$-th occurrence. Thus, in every prefix of b the number of occurrences of x is no smaller than the number of occurrences of $x+1$.

Other counts

For the other transformations we have in mind it is more pleasant to work with slightly different sequences than before. Let $\mathcal{A}$ be the set of all finite integer sequences $a = a_0a_1a_2\dots$ with $0 \le a_i \le i$, for all indices, and define $\mathcal{A}_n$ to be the set of sequences $a = a_0a_1a_2\dots a_n$ of length $n+1$ in $\mathcal{A}$.

Theorem 2. *Consider the following six transformations of sequences in $\mathcal{A}$, given by*

$$\begin{aligned}
\alpha_{eq}(a)_i &= \#\{j \mid j < i,\ a_j = a_i\},\\
\alpha_{neq}(a)_i &= \#\{j \mid j < i,\ a_j \neq a_i\},\\
\alpha_{geq}(a)_i &= \#\{j \mid j < i,\ a_j \geq a_i\},\\
\alpha_{l}(a)_i &= \#\{j \mid j < i,\ a_j < a_i\},\\
\alpha_{leq}(a)_i &= \#\{j \mid j < i,\ a_j \leq a_i\},\\
\alpha_{g}(a)_i &= \#\{j \mid j < i,\ a_j > a_i\}.
\end{aligned}$$

Starting from any sequence in $\mathcal{A}$, each of these transformations reaches a fixed or a double point after finitely many applications of the transformation. The following table gives the type of points that are reached, their number in $\mathcal{A}_n$ and a rough estimate of the number of steps needed to reach such a point.

	type of points	*number of points*	*steps needed*
α_{eq}	*double*	$(n+1)$*-th Young number*	*1*
α_{neq}	*fixed*	2^n	$O(n^2)$
α_{geq}	*double*	*unknown, at least* 2^n	$O(n^2)$
α_{l}	*fixed*	$(n+1)$*-th Catalan number*	$O(n^2)$
α_{leq}	*fixed*	*unique fixed point* $012\dots n$	$O(n^2)$
α_{g}	*fixed*	*unique fixed point* $000\dots 0$	$O(n)$

The two transformations that have double points have the sequence 0 *as their unique fixed point (which is counted as one double point).*

The proof is divided in six parts corresponding to the six transformations.

Properties of α_{eq}. The assertions about α_{eq} follow from Theorem 1. Indeed, the transformations α_{eq} and β are conjugated by the bijection from $\mathcal{A}$ to $\mathcal{A}^+$ that adds 1 to each term of the sequences in $\mathcal{A}$. □

Properties of α_{neq}. We claim that the set of fixed points of α_{neq} in $\mathcal{A}_n$ is the set $\mathcal{H}_n$ that consist of the 2^n sequences $a = a_0a_1 \dots a_n$ with $a_i = i$ or $a_i = a_{i-1}$, for $i = 1, \dots, n$.

The set of sequences in $\mathcal{A}_n$ can be ordered lexicographically. Namely, for $a = a_0a_1 \dots a_n$ and $b = b_0b_1 \dots b_n$, set $a < b$ if $a_i < b_i$ at the first index where a and b differ.

The statement of the theorem then follows from the fact that $\alpha_{neq}(a) = a$, for a in $\mathcal{H}_n$, and $\alpha_{neq}(a) > a$, for sequences a outside of $\mathcal{H}_n$. The proof is done by induction on n.

The claims are easily verified for $n = 0$ and $n = 1$. Assume that the claims are true for some $n \geq 1$.

Let

$$a = a_0a_1 \dots a_nx$$

be a sequence in $\mathcal{H}_{n+1}$. We consider two cases.

If $x = n + 1$ then

$$\#\{j \mid j < n+1,\ a_j \neq x\} = \#\{j \mid j < n+1,\ a_j \neq n+1\} = n+1 = x.$$

If $a_n = x$ then

$$\#\{j \mid j < n+1,\ a_j \neq x\} = \#\{j \mid j < n,\ a_j < a_n\} = a_n = x,$$

where the first equality comes from the fact that $a_n = x$ and the second from the inductive assumption.

Thus all sequences in $\mathcal{H}_n$ are fixed under α_{neq}, for all n.

Now, let

$$a = a_0a_1 \dots a_nx$$

be a sequence in $\mathcal{A}_{n+1}$ that is not in $\mathcal{H}_{n+1}$. If the proper initial segment

$$a' = a_0a_1 \dots a_n$$

is not fixed by α_{neq} we obtain the claim directly by the inductive assumption. So let us assume that a' is in $\mathcal{H}_n$ but a is not in $\mathcal{H}_{n+1}$.

We have

$$\#\{j \mid j < n+1,\ a_j \neq x\} = \#\{j \mid j < n,\ a_j \neq x\} + 1 \geq x + 1,$$

where the equality comes from the fact that $a_n \neq x$ and the inequality follows from the inductive assumption. Note that we could use the inductive assumption because $x \neq n + 1$ and therefore the sequence

$$a'' = a_0a_1 \dots a_{n-1}x$$

is in $\mathcal{A}_n$.

Thus, for all n and sequences a in $\mathcal{A}_n$ but not in $\mathcal{H}_n$, $\alpha_{neq}(a) > a$. □

Properties of α_{geq}. Define an *extreme sequence* in $\mathcal{A}_n$ to be a sequence $a = a_0a_1\dots a_n$ such that, for all indices, $a_i = 0$ or $a_i = i$. There are 2^n extreme sequences in $\mathcal{A}_n$ and they are all double points of α_{geq}.

We prove by induction on n that repeated applications of α_{geq} to the sequences in $\mathcal{A}_n$ eventually produce double points.

The statement is true for $n = 0$ and $n = 1$.

Assume that the statement is true for all sequences in $\mathcal{A}_n$.

Let

$$a = a_0a_1\dots a_nx$$

be a sequence in $\mathcal{A}_{n+1}$. By the inductive hypothesis, we may assume that the initial segment (prefix) $a_0a_1\dots a_n$ is already a double point of α_{geq}. Starting with the sequence a, we apply α_{geq}, α_{geq}^2 and α_{geq}^3, etc., and obtain consecutively the sequences

$$\begin{gathered} a_0a_1\dots a_nx \\ b_0b_1\dots b_ny \\ a_0a_1\dots a_nx' \\ b_0b_1\dots b_ny' \\ a_0a_1\dots a_nx'' \\ \dots \end{gathered}$$

If $x' \geq x$ then

$$\{j \mid j < n+1,\ a_j \geq x'\} \subseteq \{j < n+1 \mid a_j \geq x\}$$

and therefore

$$y' = \#\{j \mid j < n+1,\ a_j \geq x'\} \leq \#\{j \mid j < n+1,\ a_j \geq x\} = y.$$

By reversing the inequalities (including $\subseteq$) we also obtain that

$$x' \leq x \qquad \text{implies} \qquad y' \geq y.$$

Therefore, the infinite sequence $x, x', x'', \dots$ is either non-increasing or non-decreasing and since it takes values in the finite range between 0 and $n+1$ it must stabilize after no more than $n+1$ steps. □

Properties of α_l. This is proved in [Šun02]. All fixed points of α_l can be organized in a certain rooted labelled tree (called Catalan family tree) and the results follow from there. □

Properties of α_{leq}. If $012\dots nx$ is a sequence with $x \neq n+1$ then $\alpha_{leq}(012\dots nx) = 012\dots ny$, where $y = x+1$. □

Properties of α_g. If $0\dots0x$ is a sequence with $x \neq 0$ then $\alpha_g(0\dots0x) = 0\dots00$. □

As an example, we list the 10 double points of α_{geq} in $\mathcal{A}_3$, namely the four extreme pairs

$$0000 \leftrightarrow 0123, \quad 0003 \leftrightarrow 0120, \quad 0020 \leftrightarrow 0103, \quad 0023 \leftrightarrow 0100$$

and the only additional pair

$$0021 \leftrightarrow 0101.$$

One can verify directly that the sequence that counts the number of double points of α_{geq} in $\mathcal{A}_n$ starts as follows

$$1, 2, 4, 10, 26, 70, 216, \ldots$$

This sequence (actually only the first several terms) was submitted by the author to the On-Line Encyclopedia of Integer Sequences [Slo] on 06/24/2002 and it appears there as the sequence A071962. It is a new sequence that could not be found in the Encyclopedia before.

Concluding remarks

We note that the six transformations we defined are actually three pairs of transformations related by the *mirror involution* of $\mathcal{A}$, given by

$$\mu(a)_i = i - a_i.$$

Indeed, $\alpha_{eq} = \mu\alpha_{neq}$, $\alpha_{geq} = \mu\alpha_l$ and $\alpha_{leq} = \mu\alpha_g$. However, we did not use this fact in our considerations. In particular, we did not find a way to use it in order to count the double points of α_{geq} by relating them somehow to the fixed points of α_l counted by the Catalan numbers (see [Šun02]).

It was observed by Louis Shapiro that the Catalan numbers, the Young numbers and the powers of 2 count certain card shuffles in the paper by Robbins and Boelker [RB81], but the author could not find a connection to the periodic points of the sequence transformations introduced here.

One can easily define other sequence transformations that lead to periodic points (one rather general way of producing such is by using tree endomorphisms, as noted in [Šun02]). Apart from a precise description and enumeration of the periodic points, one may also try to describe and enumerate the sequences on the other side of the spectrum, namely the sequences that require maximal number of steps before a periodic point is reached.

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