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On the Truncated Kernel Function

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Abstract

We study properties of the *truncated kernel function* γ_2 defined on integers $n \geq 2$ by $\gamma_2(n) = \gamma(n)/P(n)$, where $\gamma(n) = \prod_{p|n} p$ is the well-known *kernel function* and $P(n)$ is the largest prime factor of n. In particular, we show that the maximal order of $\gamma_2(n)$ for $n \leq x$ is $(1+o(1))x/\log x$ as $x \to \infty$ and that $\sum_{n\leq x} 1/\gamma_2(n) = (1+o(1))\eta x/\log x$, where $\eta = \zeta(2)\zeta(3)/\zeta(6)$. We further show that, given any positive real number $u < 1$, $\lim_{x\to\infty} \frac{1}{x}\#\{n \leq x : \gamma_2(n) < x^u\} = \lim_{x\to\infty} \frac{1}{x}\#\{n \leq x : n/P(n) < x^u\} = 1 - \rho(1/(1-u))$, where ρ is the Dickman function. We also show that $n/P(n)$ can very often be

much larger than $\gamma_2(n)$, namely by proving that, given any $c \in [1, \xi)$, where ξ is the unique solution to $\xi \log 2 = \log(1+\xi) + \xi \log(1+1/\xi)$, then

$$\#\{n \le x : \gamma_2(n) \ge n/(c \log n)\} = o\left(\#\{n \le x : n/P(n) \ge n/(c \log n)\}\right) \qquad (x \to \infty).$$

1 Introduction

Let $\gamma(n) = \prod_{p|n} p$ for $n \ge 2$ and $\gamma(1) = 1$ be the traditional *kernel function*, also at times called the *largest squarefree divisor function*. This arithmetic function has been extensively studied. For instance, it was shown by Cohen [4] that

$$\sum_{n \le x} \gamma(n) = c_0\, x^2 + O(x^{3/2} \log x), \tag{1}$$

where

$$c_0 = \frac{1}{2} \prod_p \left(1 - \frac{1}{p(p+1)}\right) \approx 0.352. \tag{2}$$

It was later shown by De Koninck and Sitaramachandrarao [5] that, given any positive integer k,

$$\sum_{n \le x} \frac{1}{n \log \gamma(n)} = \log\log x + \sum_{m=0}^{k-1} \frac{b_m}{\log^m x} + O\left(\frac{1}{\log^k x}\right),$$

where the b_m's are computable constants.

The kernel function is at the heart of several problems in number theory, one of which is the famous $abc-$conjecture (stated below). Although simple in appearance, the kernel function remains intriguing and in some aspects very mysterious.

As an example of its intriguing character, let us mention the search for an asymptotic formula for the sum of its reciprocal values up to a given number x, namely the sum

$$I(x) := \sum_{n \leqslant x} \frac{1}{\gamma(n)}.$$

At first, since $\gamma(n) = n$ when n is squarefree and since this occurs for approximately $6x/\pi^2$ of all integers up to x, one might think that $I(x)$ would be near $\sum_{n \leqslant x} 1/n \sim \log x$. But, in 1962, N. G. de Bruijn [1] proved that, as $x \to \infty$,

$$\log I(x) = (1 + o(1)) \sqrt{\frac{8 \log x}{\mathrm{loglog}\, x}},$$

the asymptotic behavior of $I(x)$ itself remaining at that time a total mystery. Three years later, W. Schwarz [11] came up with the unexpected formula

$$I(x) = (1 + o(1)) \frac{1}{2^{1/4}\sqrt{4\pi}} \left(\frac{\mathrm{loglog}\, x}{\log x}\right)^{1/4} Q(x) \qquad (x \to \infty),$$

where

$$Q(x) = \min_{0<\sigma<\infty} x^\sigma \kappa(\sigma) \qquad \text{with} \quad \kappa(\sigma) := \sum_{n=1}^{\infty} \frac{1}{n^\sigma \gamma(n)} = \prod_p \left(1 + \frac{1}{p(p^\sigma - 1)}\right),$$

and was thus able to prove the asymptotic formula

$$I(x) = (1 + o(1)) \frac{1}{2^{1/4}\sqrt{4\pi}} \left(\frac{\log\log x}{\log x}\right)^{1/4} \exp\{-R(\log x)\} \qquad (x \to \infty),$$

where, by setting $\varphi(\sigma) := \log \kappa(\sigma)$ and $\Phi(\sigma) := \varphi'(\sigma)$, the function $R(u)$ is defined by the relation

$$-R(u) = \varphi(\Phi^{-1}(-u)) + u\Phi^{-1}(-u).$$

The study of the kernel function can be seen as a way to investigate the multiplicative structure of integers. Now, the quantity $\gamma(n)$ is easy to grasp when it is fairly large compared with n, say in the neighborhood of n/k for a fixed $k \geq 1$. However, when $\gamma(n)$ is very small compared with n, say when $\gamma(n) \leq \sqrt{n}$ or even worst when $\gamma(n) \leq \log n$, then its behavior depends on the small prime factors of n (since $P(n)$ is small) and is therefore very hard to grasp, thus for instance explaining the difficulty in obtaining the above asymptotic value of $I(x)$. This motivates us to study the *truncated kernel function* γ_2 defined by $\gamma_2(1) = 1$ and for each integer $n \geq 2$ by $\gamma_2(n) = \gamma(n)/P(n)$, where $P(n)$ stands for the largest prime factor of n (with $P(1) = 1$). This new function is very similar to $\gamma(n)$ in many respects but it also has the advantage of being simpler to investigate, essentially because the distribution of its small values no longer depends on $P(n)$ and is therefore much easier to understand.

We will study the global and local behavior of $\gamma_2(n)$ and also compare it with that of $\gamma(n)$ and of $n/P(n)$. Although $\gamma_2(n)$ is not a multiplicative function, in general its behavior is more easily understood than that of $\gamma(n)$, as is the case when comparing the corresponding sums of their reciprocal values. However, as we will see, it turns out to be quite the opposite in many aspects.

2 Main results

Our first two theorems provide information about the global behavior of $\gamma_2(n)$.

Theorem 1. *With c_0 defined as in (2), we have*

$$\sum_{n \leq x} \gamma_2(n) = (1 + o(1)) 2c_0 x^2 \delta(x) \qquad (x \to \infty), \tag{3}$$

where

$$\delta(x) = \int_2^x \frac{1}{t^2} \rho\left(\frac{\log x}{\log t}\right) \mathrm{d}t \tag{4}$$

and ρ is the well-known Dickman function defined as the (unique) continuous solution to the differential-difference equation

$$u\rho'(u) = -\rho(u - 1) \qquad (u > 1)$$

satisfying the initial condition $\rho(u) = 1$ for $0 \leq u \leq 1$.

Theorem 2. *For all $x \geq 2$,*

$$\sum_{n \leq x} \frac{1}{\gamma_2(n)} = c_2 \frac{x}{\log x} + O\left(\frac{x \log\log x}{\log^2 x}\right), \tag{5}$$

where $c_2 = \dfrac{\zeta(2)\zeta(3)}{\zeta(6)} \approx 1.9436$ (here ζ stands for the Riemann Zeta Function).

The next theorem allows us to study the distribution of $\gamma_2(n)$, in particular by showing that it behaves like the function $n/P(n)$ almost everywhere.

Theorem 3. *Given any positive real number $u < 1$,*

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : \gamma_2(n) < x^u\} = \lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : n/P(n) < x^u\} = 1 - \rho\left(\frac{1}{1-u}\right).$$

Given a real number $h \geq 1$, let us set

$$\Psi_h(x, y) = \#\{n \leq x : P(n) \leq y,\ \gamma(n) < n/h\}.$$

Observe that $\gamma_2(n)$ can be written as the product of two functions, namely $\gamma_2(n) = \dfrac{\gamma(n)}{n} \cdot \dfrac{n}{P(n)}$. The next result shows that $\gamma_2(n)$ is the product of two basic functions which are statistically independent almost everywhere.

Theorem 4. *Given real numbers $u > 0$ and $h \geq 1$, then*

$$\Psi_h(x, x^{1/u}) = (1 + o(1)) x\, D(h)\, \rho(u) \qquad (x \to \infty), \tag{6}$$

where

$$D(h) := \lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : \gamma(n) < n/h\},$$

a number whose existence is established in Lemma 13.

While $\gamma_2(n)$ and $n/P(n)$ are independent almost everywhere, this is not true for extremal values. Indeed, our next theorem shows that the maximal value of $\gamma_2(n)$ is smaller than the maximal value of $n/P(n)$, while Theorem 6 shows that $n/P(n)$ takes on large values more often than $\gamma_2(n)$.

Theorem 5. *As $x \to \infty$,*

$$\max_{n \leq x} \gamma_2(n) = (1 + o(1)) \frac{x}{\log x}. \tag{7}$$

Theorem 6. *Let ξ be the unique solution of the equation*

$$\xi \log 2 = \log(1 + \xi) + \xi \log(1 + 1/\xi),$$

so that $\xi \approx 3.403$. Fix any number $c \in [1, \xi)$ and set

$$\begin{aligned} A(x) &:= \# \left\{ n \leq x : \gamma_2(n) \geq \frac{n}{c \log n} \right\}, \\ B(x) &:= \# \left\{ n \leq x : \frac{n}{P(n)} \geq \frac{n}{c \log n} \right\}. \end{aligned}$$

Then,

$$A(x) = o(B(x)) \qquad (x \to \infty).$$

The following theorem is essentially the counterpart to the last two theorems, that is that there are many more integers n with a small $\gamma_2(n)$ than there are integers n with a small $P(n)$.

Theorem 7. *Given a fixed positive integer* k*,*

$$\#\{n \le x : n/P(n) \le k\} = C(k)\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

where $C(k) := \sum_{j=1}^{k} \frac{1}{j}$*, and*

$$\#\{n \le x : \gamma_2(n) \le k\} = E(k)\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

where $E(k) := \sum_{\substack{m\ge 1\\ \gamma(m)\le k}} \frac{1}{m} = \sum_{j=1}^{k} \frac{\mu^2(j)}{j} \prod_{p|j}\left(1-\frac{1}{p}\right)^{-1} = \sum_{j=1}^{k}\frac{\mu^2(j)}{\phi(j)}.$

3 Preliminary results and notations

We say that a function $\lambda : [1,+\infty[\to [0,+\infty[$ is *slowly increasing* if, for each constant $c > 0$,

$$\lim_{x\to\infty} \lambda(cx)/\lambda(x) = 1.$$

Proposition 8. *Let* f *be a non negative multiplicative function for which there exist a positive real number* k *and a slowly increasing function* λ *such that, as* $x \to \infty$*,*

$$\sum_{n\le x} f(n) = (1+o(1))\, x\, (\log^{k-1} x)\, \lambda(\log x),$$

and such that for all real numbers $u > 1$*, as* $y \to \infty$*,*

$$\sum_{y<p<y^u} \frac{f(p)}{p} = (1+o(1))\, k\, \log u.$$

Then, as $x = y^u \to \infty$*,*

$$\sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} f(n) = (1+o(1))\,\Gamma(k)\, u^{1-k}\rho_k(u) \sum_{n\leqslant x} f(n),$$

uniformly for u *bounded, where* $\rho_k(u)$ *is the continuous solution to the differential difference equation with delayed argument, defined by*

$$\begin{aligned} \rho_k(u) &= 0 && (u \leqslant 0),\\ \rho_k(u) &= \frac{u^{k-1}}{\Gamma(k)} && (0 < u \leqslant 1),\\ u\rho_k'(u) &= (k-1)\rho_k(u) - k\rho_k(u-1) && (u > 1), \end{aligned}$$

where Γ *stands for the Gamma function.*

Proof. This result is due to de Bruijn and van Lint [3]. □

Proposition 9. *As $x \to \infty$,*

$$\sum_{n \leqslant x} \frac{1}{P(n)} = \left(1 + O\left(\sqrt{\frac{\operatorname{loglog} x}{\log x}}\right)\right) x\delta(x), \tag{8}$$

$$\sum_{n \leqslant x} \frac{\mu^2(n)}{P(n)} = \left(\frac{6}{\pi^2} + O\left(\sqrt{\frac{\operatorname{loglog} x}{\log x}}\right)\right) \sum_{n \leqslant x} \frac{1}{P(n)}, \tag{9}$$

$$\sum_{\substack{n \leqslant x \\ P(n)^2 \mid n}} \frac{1}{P(n)} = x \exp\left\{-\sqrt{4 \log x \operatorname{loglog} x}\left(1 + O\left(\frac{\operatorname{logloglog} x}{\operatorname{loglog} x}\right)\right)\right\} = x\delta(x)^{\sqrt{2}+o(1)}, \tag{10}$$

where $\delta(x)$ is the function defined in (4).

Proof. A proof of (8) was established by Erdős, Ivić, and Pomerance [6], while (9) can be found in Ivić [9], and (10) in Ivić and Pomerance [10]. □

Remark 10. Using the estimate

$$\rho(u) = \exp\{-u(\log u + \operatorname{loglog} u - 1 + o(1))\} \qquad (u \to \infty) \tag{11}$$

(see the book of Tenenbaum [12]), one can show that the function $\delta(x)$ defined in (4) is slowly increasing and satisfies

$$\delta(x) = \exp\left\{-(1 + o(1))\sqrt{2 \log x \operatorname{loglog} x}\right\} \quad (x \to \infty), \tag{12}$$

so that

$$\delta(x) = L_0(x)^{-1+o(1)} \quad (x \to \infty),$$

where

$$L_0(x) := \exp\{\sqrt{2 \log x \log \log x}\}, \tag{13}$$

Lemma 11. *As $x \to \infty$,*

$$\int_1^x t\, \rho\left(\frac{\log t}{\log y}\right) \mathrm{d}t = \int_y^x t\, \rho\left(\frac{\log t}{\log y}\right) \mathrm{d}t + O(y) = O(x),$$

uniformly for $2 \leqslant y \leqslant x$.

Proof. We first estimate the maximum value of

$$g(t) := t\, \rho\left(\frac{\log t}{\log y}\right) \tag{14}$$

for $y \leqslant t \leqslant x$ and fixed $y \in [2, x]$. For this, consider $h(t) := \log g(t)$ and solve $h'(t) = 0$ for t.

In view of (11), we have

$$h'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\log t}{\log y}(\log\log t - \log\log y + \log\log\log t - \log\log\log y - 1) + \log t\right)$$

so that

$$\log\log t - \log\log y = \log y,$$

in which case,

$$t = y^y.$$

Substituting this value of t in (14), we get in view of (11),

$$g(t) = \rho(y)y^y \ll e^{-y\log y}y^y = 1,$$

which completes the proof of the lemma. □

Lemma 12. *Uniformly for $x \geq y \geq 2$,*

$$\log\Psi(x,y) = Z\left\{1 + O\left(\frac{1}{\log y} + \frac{1}{\log\log x}\right)\right\},$$

where

$$Z = \frac{\log x}{\log y}\log\left(1 + \frac{y}{\log x}\right) + \frac{y}{\log y}\log\left(1 + \frac{\log x}{y}\right).$$

Proof. This result is due to de Bruijn [2]. □

Lemma 13. *Given any real number $h \geq 1$, the limit*

$$D(h) := \lim_{x\to\infty}\frac{1}{x}\#\{n \leq x : \gamma(n) < n/h\} \tag{15}$$

exists and moreover, as h becomes large,

$$D(h) \ll \frac{1}{\sqrt{h}}. \tag{16}$$

Proof. Writing each positive integer n as $n = st$, where s is powerful and t squarefree with $(s,t) = 1$, we have that the condition $\frac{\gamma(n)}{n} < \frac{1}{h}$ is equivalent to $\frac{\gamma(s)}{s} < \frac{1}{h}$. In light of these facts, we have that

$$\sum_{\substack{n\leq x\\ \frac{\gamma(n)}{n}<\frac{1}{h}}} 1 = \sum_{\substack{s\leq x\\ \frac{\gamma(s)}{s}<\frac{1}{h}\\ s\ \text{powerful}}}\ \sum_{\substack{t\leq x/s\\ (t,s)=1}}\mu^2(t) = (1+o(1))\frac{6}{\pi^2}x\sum_{\substack{s\leq x\\ \frac{\gamma(s)}{s}<\frac{1}{h}\\ s\ \text{powerful}}}\frac{1}{s\prod_{p|s}\left(1+\frac{1}{p}\right)}. \tag{17}$$

Since $\displaystyle\sum_{\substack{s\ \text{powerful}\\ \frac{\gamma(s)}{s}<\frac{1}{h}}}\frac{1}{s\prod_{p|s}\left(1+\frac{1}{p}\right)}$ is clearly a convergent sequence, we can define $D(h)$ as

$$D(h) = \sum_{\substack{s\ \text{powerful}\\ \frac{\gamma(s)}{s}<\frac{1}{h}}}\frac{1}{s\prod_{p|s}\left(1+\frac{1}{p}\right)}.$$

Now, recall the following result proved by S. Golomb [7] in 1970:

$$S(y) := \#\{n \le y : n \text{ powerful}\} = (c_3 + o(1))\sqrt{y} \qquad (y \to \infty),$$

where $c_3 = \zeta(3/2)/\zeta(2) \approx 2.173$. Finally, in order to prove (16), we only need to observe that, since the condition $\frac{\gamma(s)}{s} < \frac{1}{h}$ certainly implies that $s > \frac{s}{\gamma(s)} > h$, we have

$$\sum_{\substack{s>x \\ s \text{ powerful} \\ \frac{\gamma(s)}{s}<\frac{1}{h}}} \frac{1}{s} < \sum_{\substack{s>h \\ s \text{ powerful}}} \frac{1}{s} \ll \int_h^\infty \frac{1}{t} \mathrm{d}\, S(t) = \int_h^\infty \frac{S(t)}{t}\,\mathrm{d}t + \int_h^\infty \frac{S(t)}{t^2}\,\mathrm{d}t \ll \frac{1}{\sqrt{h}}.$$

□

4 The proof of the theorems

4.1 The proof of Theorem 1

We first write

$$S_2(x) := \sum_{n \le x} \gamma_2(n) = \sum_{\substack{n \le x \\ P(n) \| n}} \frac{\gamma(n)}{P(n)} + \sum_{\substack{n \le x \\ P(n)^2 | n}} \frac{\gamma(n)}{P(n)} = \Sigma_1 + \Sigma_2,$$

say. Since it follows from (10) of Proposition 9 that

$$\Sigma_2 \leqslant \sum_{\substack{n \leqslant x \\ P(n)^2 | n}} \frac{n}{P(n)} \leqslant x \sum_{\substack{n \leqslant x \\ P(n)^2 | n}} \frac{1}{P(n)} \ll x^2\,\delta(x)^{\sqrt{2}+o(1)} \qquad (x \to \infty), \tag{18}$$

we only need to estimate Σ_1.

We first observe that the true order of Σ_1 is $x^2\,\delta(x)$ and in fact that

$$\frac{\Sigma_1}{(x^2 \delta(x))/2} \in \left[\frac{6}{\pi^2}, 1\right]$$

since it is easily shown that, as $x \to \infty$,

$$(1 + o(1))\frac{6}{\pi^2}\frac{x^2}{2}\,\delta(x) \leqslant \sum_{\substack{n \leqslant x \\ P(n) \| n}} \frac{\gamma(n)}{P(n)} \leqslant (1 + o(1))\frac{x^2}{2}\,\delta(x). \tag{19}$$

Indeed, on the one hand,

$$\sum_{\substack{n \le x \\ P(n) \| n}} \frac{\gamma(n)}{P(n)} \leqslant \sum_{\substack{n \leqslant x \\ P(n) \| n}} \frac{n}{P(n)} = (1 + o(1))\frac{x^2}{2}\,\delta(x) \qquad (x \to \infty),$$

by way of (8) and partial summation. On the other hand, using the trivial observation $\gamma(n) \geqslant \mu^2(n)\, n$ valid for all $n \geqslant 1$, we have

$$\sum_{\substack{n\le x\\ P(n)\| n}} \frac{\gamma(n)}{P(n)} \geqslant \sum_{\substack{n\leqslant x\\ P(n)\| n}} \frac{\mu^2(n)\, n}{P(n)} = \sum_{n\leqslant x} \frac{\mu^2(n) n}{P(n)} = (1+o(1)) \frac{6}{\pi^2}\frac{x^2}{2}\,\delta(x) \qquad (x\to\infty),$$

where first we used (9) and partial summation and thereafter estimate (8) of Proposition 9.

In order to estimate Σ_1, we shall first prove that

$$G(x,y) := \sum_{\substack{n\le x\\ P(n)\leqslant y}} \gamma(n) = (1+o(1))\, c_1 x^2\ \rho(u) \qquad (x\to\infty), \tag{20}$$

where $u = \dfrac{\log x}{\log y}$ and ρ is the Dickman function.

Let $f(n) := \gamma(n)/n$. First, it is an easy matter to derive from (1) that

$$\sum_{n\le x} \frac{\gamma(n)}{n} = c_1 x + O(x^{1/2}\log x), \tag{21}$$

where $c_1 = 2c_0$.

On the other hand, using Mertens' formula, we have that

$$\sum_{y<p<y^u} \frac{f(p)}{p} = \sum_{y<p<y^u} \frac{1}{p} = \log u + O(1) \qquad (y\to\infty). \tag{22}$$

Hence, it follows from (22) and (21) that f satisfies the conditions of Proposition 8 with $k=1$, yielding

$$\sum_{\substack{n\le x\\ P(n)\le y}} \frac{\gamma(n)}{n} = (1+o(1))\, c_1\ \rho(u)\ x \qquad (x\to\infty). \tag{23}$$

Using partial summation and Lemma 11, we get, as $x\to\infty$,

$$\begin{aligned}
\sum_{\substack{n\le x\\ P(n)\le y}} \gamma(n) &= \sum_{\substack{n\leqslant x\\ P(n)\le y}} \frac{\gamma(n)}{n} n \\
&= (1+o(1))\, c_1\rho(u)x^2 - c_1(1+o(1)) \int_1^x \rho\left(\frac{\log t}{\log y}\right) t\, \mathrm{d}t \\
&= (1+o(1))\, c_1\rho(u)x^2 + O(x),
\end{aligned}$$

which proves (20).

Getting back to the definition of Σ_1, we obtain

$$\begin{aligned}
\Sigma_1 &= \sum_{p\leqslant x} \sum_{\substack{mp\leqslant x\\ P(m)<p}} \frac{\gamma(m)\, p}{p} = \sum_{p\leqslant x} \sum_{\substack{m\leqslant x/p\\ P(m)<p}} \gamma(m) = \sum_{p\leqslant x} \sum_{\substack{m\leqslant x/p\\ P(m)\leqslant p}} \gamma(m) - \sum_{p\leqslant x} \sum_{\substack{m\leqslant x/p\\ P(m)=p}} \gamma(m) \\
&= \sum_{p\leqslant x} G\left(\frac{x}{p}, p\right) - \sum_{\substack{mp\leqslant x\\ P(mp)^2|mp}} \frac{\gamma(mp)}{P(mp)} \\
&= T_1 - T_2,
\end{aligned} \tag{24}$$

where, by (20),

$$T_1 = (1+o(1))c_1x^2 \sum_{p\leqslant x} \frac{1}{p^2}\rho\left(\frac{\log x}{\log p}-1\right) \qquad (x\to\infty)$$

while, by (18),

$$T_2 \ll x^2\delta(x)^{\sqrt{2}+o(1)} \qquad (x\to\infty).$$

Following an argument used by Ivić and Pomerance [10], we obtain that

$$T_1 = c_1(1+o(1))\,x^2\int_2^x \frac{1}{t^2\log t}\rho\left(\frac{\log x}{\log t}-1\right)\mathrm{d}t = c_1(1+o(1))\,x^2\delta(x) \qquad (x\to\infty), \quad (25)$$

while clearly

$$T_2 = o\left(x^2\delta(x)\right) \qquad (x\to\infty). \quad (26)$$

Substituting (25) and (26) in (24), (3) follows.

4.2 The proof of Theorem 2

Let K be a fixed large integer. Then,

$$\sum_{n\le x}\frac{1}{\gamma_2(n)} = \sum_{k\le K}\frac{\mu^2(k)}{k}\sum_{\substack{n\le x\\ \gamma_2(n)=k}}1 + \sum_{K<k\le x}\frac{\mu^2(k)}{k}\sum_{\substack{n\le x\\ \gamma_2(n)=k}}1 = S_1(x;K)+S_2(x;K), \quad (27)$$

say.

Using the fact that, for any squarefree integer k,

$$\sum_{\substack{n\ge 1\\ \gamma_2(n)=k}}\frac{1}{n} = \frac{1}{k}\prod_{p|k}\left(1+\frac{1}{p}+\frac{1}{p^2}+\ldots\right) = \frac{1}{\phi(k)},$$

we easily obtain that

$$S_2(x;K) \le \sum_{K<k\le x}\frac{\mu^2(k)}{k}\sum_{\substack{n\le x\\ \gamma_2(n)=k}}\frac{x}{n} \le x\sum_{k>K}\frac{\mu^2(k)}{k\phi(k)} < x\sum_{k>K}\frac{1}{k^{3/2}} < \frac{x}{K^{1/2}}, \quad (28)$$

where we used the trivial inequality $\phi(k) > k^{1/2}$ valid for all $k\ge 7$.

On the other hand, in light of estimate (47), which is proved in section 4.7, we get

$$\begin{aligned} S_1(x;K) &= \sum_{k\le K}\frac{\mu^2(k)}{k}\left(\sum_{\substack{n\le x\\ \gamma_2(n)\le k}}1 - \sum_{\substack{n\le x\\ \gamma_2(n)\le k-1}}1\right)\\ &= \sum_{k\le K}\frac{\mu^2(k)}{k}\left((E(k)-E(k-1))\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right)+O(\pi(k))\right)\\ &= \left(\sum_{k\le K}\frac{\mu^2(k)}{k\phi(k)}\right)\frac{x}{\log x}+O\left(\frac{x\log K}{\log^2 x}\right)+O\left(\frac{K}{\log K}\right). \quad (29)\end{aligned}$$

Again using the fact that $\phi(k) < k^{1/2}$ for all $k \geq 7$, we have

$$\sum_{k \leq K} \frac{\mu^2(k)}{k\phi(k)} = \sum_{k=1}^{\infty} \frac{\mu^2(k)}{k\phi(k)} - \sum_{k > K} \frac{\mu^2(k)}{k\phi(k)} = c_2 + O\left(\frac{1}{\sqrt{K}}\right). \tag{30}$$

Choosing $K = \log^4 x$ and using (28), (29) and (30) in (27), estimate (5) follows, thereby completing the proof of Theorem 2.

4.3 The proof of Theorem 3

Since, for each integer $n \geq 2$, we have $\gamma_2(n) = \dfrac{\gamma(n)}{P(n)} \leq \dfrac{n}{P(n)}$, it follows that

$$\begin{aligned}
\sum_{\substack{n \leq x \\ \gamma_2(n) < x^u}} 1 &\geq \sum_{\substack{n \leq x \\ \frac{n}{P(n)} < x^u}} 1 = \sum_{\substack{n \leq x \\ \frac{P(n)}{n} > \frac{1}{x^u}}} 1 \\
&= [x] - \sum_{\substack{n \leq x \\ P(n) \leq n/x^u}} 1 \geq [x] - \sum_{\substack{n \leq x \\ P(n) \leq x^{1-u}}} 1 \\
&= x\left(1 - \rho\left(\frac{1}{1-u}\right) + o(1)\right) \qquad (x \to \infty).
\end{aligned} \tag{31}$$

On the other hand, let $\varepsilon > 0$ be an arbitrarily small number and choose k large enough so that, using estimate (16) of Lemma 13, we can claim that

$$\lim_{x \to \infty} \frac{1}{x} \#\{n \leq x : \gamma(n)/n < 1/k\} < \varepsilon. \tag{32}$$

Then, using (32), provided x is large enough, we have

$$\sum_{\substack{n \leq x \\ \gamma_2(n) < x^u}} 1 = \sum_{\substack{n \leq x \\ \gamma_2(n) < x^u \\ \frac{\gamma(n)}{n} < \frac{1}{k}}} 1 + \sum_{\substack{n \leq x \\ \gamma_2(n) < x^u \\ \frac{\gamma(n)}{n} \geq \frac{1}{k}}} 1 < \varepsilon x + \sum_{\substack{n \leq x \\ \frac{n}{P(n)} < kx^u}} 1, \tag{33}$$

where the last sum was obtained using the fact that

$$\frac{n}{kP(n)} \leq \frac{\gamma(n)}{P(n)} = \gamma_2(n) < x^u.$$

Hence, from (33), we conclude that

$$\sum_{\substack{n \leq x \\ \gamma_2(n) < x^u}} 1 < \varepsilon x + [x] - \sum_{\substack{n \leq x \\ P(n) \leq x^{1-u}}} 1 = x\left(1 - \rho\left(\frac{1}{1-u}\right) + o(1)\right) \qquad (x \to \infty). \tag{34}$$

Combining estimates(31) and (34) completes the proof of Theorem 3.

4.4 The proof of Theorem 4

First, we can assume that $P(n)\|n$ since we know from Proposition 9 that the number of positive integers $n \le x$ such that $P^2(n)|n$ is no larger than $x\delta(x)^{\sqrt{2}+1} = o(x)$.

We will use the same pattern of proof as the one that Granville [8] used to prove that $\psi(x, x^{1/u}) = (1 + o(1))x\rho(u)$ as $x \to \infty$.

First, in light of the definition of $D(h)$ given in (15), it is clear that

$$\Psi_h(x, x^{1/u}) = (1 + o(1))xD(h)\rho(u) \quad \text{for all } 0 < u \le 1 \qquad (x \to \infty). \tag{35}$$

Let us now consider the case where $u \in [1, 2]$. Then, with the assumption $P(n)\|n$, we have

$$\sum_{\substack{n\le x \\ P(n)\le x^{1/u} \\ \frac{\gamma(n)}{n}<\frac{1}{h}}} 1 = \sum_{\substack{n\le x \\ \frac{\gamma(n)}{n}<\frac{1}{h}}} 1 - \sum_{\substack{n\le x \\ x^{1/u}<P(n)\le x \\ \frac{\gamma(n)}{n}<\frac{1}{h}}} 1 = \Sigma_1 - \Sigma_2, \tag{36}$$

say.

Again, it follows from the definition of $D(h)$ that

$$\Sigma_1 = (1 + o(1))D(h)x \qquad (x \to \infty). \tag{37}$$

On the other hand, writing $n = mp$ with $P(m) < p$, then, because $u \in [1, 2]$, the condition $P(m) < p$ is automatically satisfied, while the assumption $P(n)\|n$ guarantees that the condition $\frac{\gamma(n)}{n} < \frac{1}{h}$ is equivalent to the condition $\frac{\gamma(m)}{m} < \frac{1}{h}$, implying that

$$\begin{aligned}
\Sigma_2 &= \sum_{x^{1/u}<p\le x} \sum_{\substack{m\le \frac{x}{p} \\ P(m)\le p \\ \frac{\gamma(m)}{m}<\frac{1}{h}}} 1 = \sum_{x^{1/u}<p\le x} \sum_{\substack{m\le \frac{x}{p} \\ \frac{\gamma(m)}{m}<\frac{1}{h}}} 1 \\
&= \sum_{x^{1/u}<p\le x} \frac{x}{p}D(h) = (1 + o(1))xD(h)\log u \qquad (x \to \infty).
\end{aligned} \tag{38}$$

Gathering (37) and (38) in (36), we obtain that, in the case $u \in [1, 2]$,

$$\Psi_h(x, x^{1/u}) = (1 + o(1))x\, D(h)\, (1 - \log u) \qquad (x \to \infty). \tag{39}$$

Hence, in light of (35) and (39), we have thus proved that

$$\Psi_h(x, x^{1/u}) = (1 + o(1))x\, D(h)\, \rho(u) \qquad (x \to \infty) \tag{40}$$

holds for $0 \le u \le 2$ with

$$\rho(u) = \begin{cases} 1, & \text{if } 0 < u \le 1, \\ 1 - \log u, & \text{if } 1 \le u \le 2. \end{cases}$$

We now use induction. Assuming that (40) holds for all $u \in (0, N]$, we will now prove that it must also hold for all $u \in [N, N+1]$. For this, we will need the Buchstab identity (trivially generalized to $\Psi_h(x,y)$), that is

$$\Psi_h(x,y) = 1 + \sum_{p \le y} \sum_{\substack{mp \le x \\ P(m) < p \\ \frac{\gamma(m)}{m} < \frac{1}{h}}} 1 = 1 + \sum_{p \le y} \Psi_h\left(\frac{x}{p}, p\right). \tag{41}$$

Using (41), we have

$$\begin{aligned} \Psi_h(x, x^{1/N}) &= 1 + \sum_{p \le x^{1/N}} \Psi\left(\frac{x}{p}, p\right), \\ \Psi_h(x, x^{1/u}) &= 1 + \sum_{p \le x^{1/u}} \Psi\left(\frac{x}{p}, p\right). \end{aligned}$$

Subtracting these two equations, we obtain

$$\Psi_h(x, x^{1/u}) = \Psi_h(x, x^{1/N}) - \sum_{x^{1/u} < p \le x^{1/N}} \Psi_h\left(\frac{x}{p}, p\right). \tag{42}$$

Now, since $p > x^{1/u}$ and $u \in [N, N+1]$, we have

$$\frac{\log(x/p)}{\log p} = \frac{\log x}{\log p} - 1 < \frac{\log x}{\log(x^{1/u})} - 1 = u - 1 \le N,$$

implying that (40) holds, say with $u' = \frac{\log(x/p)}{\log p}$ instead of u, thus allowing us to replace (42) by

$$\begin{aligned} \Psi_h(x, x^{1/u}) &= (1+o(1))xD(h)\rho(N) - (1+o(1))D(h) \sum_{x^{1/u} < p \le x^{1/N}} \frac{x}{p} \rho\left(\frac{\log x}{\log p} - 1\right) \\ &= (1+o(1))xD(h)\rho(N) - (1+o(1))xD(h) \int_{x^{1/u}}^{x^{1/N}} \rho\left(\frac{\log x}{\log v} - 1\right) \frac{\mathrm{d}\,\theta(v)}{v \log v} \\ &= (1+o(1))xD(h)\left(\rho(N) - \int_N^u \frac{\rho(t-1)}{t}\, dt\right) \\ &= (1+o(1))xD(h)\rho(u) \qquad (x \to \infty), \end{aligned}$$

(where we used the prime number theorem in the form $\theta(v) = \sum_{p \le v} \log p = (1+o(1))v$ as $v \to \infty$), thus showing that (40) also holds for $u \in [N, N+1]$ and thus completing the induction argument.

4.5 The proof of Theorem 5

We first show that the bound is achieved for $n = \prod_{p \le \log x} p$. Indeed, it follows from the prime number theorem that, as $x \to \infty$,

$$\gamma_2(n) = \frac{1}{\max_{p \le \log x} p} \times \prod_{p \le \log x} p = \frac{1}{\max_{p \le \log x} p} \times e^{(1+o(1)) \log x} = (1+o(1)) \frac{x}{\log x}.$$

On the other hand, this last expression is indeed an upper bound for $\gamma_2(n)$. To prove this, first assume that $P(n) > \log(n/\log n)$. Then, in this case, as $x \to \infty$,

$$\gamma_2(n) \le \frac{n}{P(n)} \le \frac{n}{\log n - \log\log n} = \frac{n}{\log n}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right) \le (1+o(1))\frac{x}{\log x}.$$

If, on the contrary, $P(n) \le \log(n/\log n)$, we have, by the prime number theorem,

$$\gamma_2(n) < \prod_{p \le P(n)} p \le \prod_{p \le \log n - \log\log n} p = (1+o(1))\frac{n}{\log n} \le (1+o(1))\frac{x}{\log x} \qquad (x \to \infty).$$

This completes the proof of Theorem 5.

4.6 The proof of Theorem 6

Let us write each integer $n \ge 2$ as $n = st$, where s is squarefull and t is squarefree with $(s,t) = 1$. On the one hand,

$$\frac{\gamma_2(n)}{n} \ge \frac{1}{c\log n} \Longleftrightarrow \frac{sP(st)}{\gamma(s)} \le c\log n.$$

But this last inequality implies that

$$P(st) \le \frac{sP(st)}{\gamma(s)} \le c\log n \le c\log x. \tag{43}$$

Since $\gamma_2(n) \le \gamma(n) \le n/\sqrt{s}$, we get that $s \le (k\log x)^2$, which combined with (43) implies that $P(t) \le c\log x$.

Therefore, as $x \to \infty$,

$$\begin{aligned} A(x) &\le \#\{t \le x : \mu^2(t) = 1,\ P(t) \le c\log x\} \\ &\le 2^{\pi(c\log x)} = \exp\left\{(1+o(1))c\log 2\frac{\log x}{\log\log x}\right\}, \end{aligned} \tag{44}$$

where again we made use of the prime number theorem.

We will now obtain a lower bound for $B(x)$. Given any small $\delta > 0$, we have, using Lemma 12, that as $x \to \infty$,

$$\begin{aligned} B(x) &= \#\{n \le x : P(n) \le c\log n\} \\ &\ge \#\{x^{1-\delta} < n \le x : P(n) \le c\log n\} \\ &\ge \#\{x^{1-\delta} < n \le x : P(n) \le (1-\delta)c\log x\} \\ &= \Psi(x,(1-\delta)c\log x) - \Psi(x^{1-\delta},(1-\delta)c\log x) \\ &= (1+o(1))\Psi(x,(1-\delta)c\log x) \\ &= (1+o(1))\exp Z \\ &= \exp\left\{(1+o(1))\frac{\log x}{\log\log x}\left(\log(1+(1-\delta)c) + (1-\delta)c\log\left(1+\frac{1}{c(1-\delta)}\right)\right)\right\}. \end{aligned} \tag{45}$$

Since δ can be taken arbitrarily small, it follows from (45) that, as $x \to \infty$,

$$B(x) \geq \exp\left\{(1+o(1))\frac{\log x}{\log\log x}\left(\log(1+c) + c\log\left(1+\frac{1}{c}\right)\right)\right\}. \tag{46}$$

Finally, by comparing (44) with (46) and observing that for $c < \xi$, we have

$$c\log 2 < \log(1+c) + c\log\left(1+\frac{1}{c}\right),$$

the proof of Theorem 6 is complete.

4.7 The proof of Theorem 7

We first evaluate $S_1 = \#\{n \leq x : n/P(n) \leq k\}$. Writing each positive integer $n \leq x$ as $n = mp$ with $P(m) \leq p$, we have

$$\begin{aligned} S_1 &= \sum_{m \leq k} \sum_{k < p \leq x/m} 1 \\ &= \sum_{m \leq k} \sum_{p \leq x/m} 1 + O(1) \\ &= \sum_{m \leq k} \pi(x/m) + O(1) \\ &= \sum_{m \leq k} \left(\frac{x}{m\log(x/m)} + O\left(\frac{x/m}{\log^2(x/m)}\right)\right) \\ &= \frac{x}{\log x}\sum_{m=1}^{k}\frac{1}{m} + O\left(\frac{x}{\log^2 x}\right), \end{aligned}$$

where we used the prime number theorem, thus proving our first assertion.

Now let $S_2 = \#\{n \leq x : \gamma_2(n) \leq k\}$. Again, writing each positive integer $n \leq x$ as $n = mp$ with $P(m) \leq p$, we have, using Proposition 9,

$$\begin{aligned} S_2 &= \sum_{\substack{n \leq x \\ P(n)^2 \mid n \\ \gamma_2(n) \leq k}} 1 + \sum_{\substack{m \leq x \\ p \leq x/m \\ P(m) < p,\ \gamma(m) \leq k}} 1 \\ &= O\left(\frac{x}{\exp\{(1+o(1))2\sqrt{\log x \log\log x}\}}\right) + \sum_{\substack{m \leq x \\ \gamma(m) \leq k}} \sum_{P(m) < p \leq x/m} 1 \\ &= O\left(\frac{x}{\log^2 x}\right) + \sum_{\substack{m \leq x \\ \gamma(m) \leq k}} \pi(x/m) + O\left(\pi(k)\right) \\ &= \sum_{\substack{m \leq x \\ \gamma(m) \leq k}} \frac{x/m}{\log(x/m)} + O\left(\frac{x}{\log^2 x}\right) + O(\pi(k)) \end{aligned}$$

$$= \frac{x}{\log x} \sum_{\substack{m\ge 1\\ \gamma(m)\le k}} \frac{1}{m} + O\left(\frac{x}{\log^2 x}\right) + O(\pi(k))$$

$$= E(k)\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) + O(\pi(k)), \tag{47}$$

as we wanted to prove.

5 Final remarks

One could certainly extend the above results on $\gamma_2(n)$ to the more general case $\gamma_k(n)$, with $k \ge 3$, defined by $\gamma_k(n) = \gamma_{k-1}(n)/P(\gamma_{k-1}(n))$. For instance, using the above arguments, one can easily show that

$$\sum_{n\le x} \frac{1}{\gamma_k(n)} = (c_k + o(1))x\frac{(\log\log x)^{k-2}}{\log x} \qquad (x\to\infty)$$

and that

$$\max_{n\le x} \gamma_k(n) = (1+o(1))\frac{x}{\log^{k-1} x} \qquad (x\to\infty).$$

On the other hand, an interesting problem would be to investigate what kind of results one could obtain if k is replaced by a function of n, for instance, by choosing $k = \lfloor \frac{\omega(n)}{2} \rfloor$.

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