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Fibonacci Numbers of Generalized Zykov Sums

César Bautista-Ramos and Carlos Guillén-Galván
Facultad de Ciencias de la Computación
Benemérita Universidad Autónoma de Puebla
14 Sur y Av. San Claudio, Edif. 104C, 303
Puebla, Pue. 72570
Mexico
bautista@cs.buap.mx
cguillen@cs.buap.mx

Abstract

We show that counting independent sets in several families of graphs can be done within the framework of generalized Zykov sums by using the transfer matrix method. Then we calculate the generating function of the number of independent sets for families of generalized Zykov sums. We include many interesting particular cases (Petersen graphs, generalized Möbius ladders, carbon nanotube graphs, among others).

1 Introduction

The Fibonacci number $F(G)$ of a graph G was introduced by Prodinger and Tichy [15] and is defined as the number of independent sets of G. In combinatorial chemistry this number is also known as the Merrifield-Simmons index [11, 12]. Prodinger and Tichy calculated $F(G)$ recursively by paying attention to whether certain vertices appear or not in what they call *the usual recursion argument*. Such binary occurrence problem is formalized, in this paper, in the *transfer matrix method* [2, 6, 7, 10].

Our goal is to show how to count independent sets in graphs with some pattern structure. The structure we are dealing with is a generalization of the Zykov sum of graphs (also known as the graph join). Let us recall that the Zykov sum is the graph obtained from the disjoint union of two graphs G_1, G_2 by joining each vertex of G_1 with each vertex of G_2. This join is just a particular case of a relation set between the vertices of G_1 and G_2. In this work, we generalize Zykov sums by joining vertices only when they belong to a given relation set. We call these *open or closed Zykov sums* (the formal definitions are introduced in the next section). It is almost trivial to show that the independent sets of G_1 and G_2 give rise to a new independent set in their open or closed Zykov sum, unless their vertices belong to the relation (see Theorem 5). Such binary information, whether vertices are related or not, is stored in a matrix known as the *transfer matrix* [2, 6, 7, 10]. The transfer matrix is particularly useful for graphs that can be written as a repetitive pattern of open or closed Zykov sums. Then, the product of the transfer matrices stores information about the number of independent sets. We show that the usual matrix product gives the number of independent set in graphs that are some kind of strip, while the Hadamard product of matrices is for closing such strips in order to form some type of circular structure, like cycles or tori.

To illustrate our methods we use several families of graphs having a repetitive pattern of generalized Zykov sums which include paths, cycles, grids, cylinders, tori, Möbius ladders, and generalized Petersen graphs, among others. Additional examples are honeycomb tubes (nanotube graphs) and honeycomb torus graphs (nanotorus graphs) which are particularly interesting because they appear in parallel computing architectures [18] and nanotechnology [14].

There are other works dealing with the Fibonacci number $F(G)$ when G has a different structure from those studied here; for instance, trees [11], regular graphs [3, 17, 21], unicyclic graphs [13], graphs with small maximum degree [9], graphs with a given number of vertices and edges [4], and graphs with a given minimum degree [8]. However, these works deal with estimations for $F(G)$, in contrast to our work, which is interested mainly in exact formulas for $F(G)$ when G is an open or closed Zykov sum. Exact formulas for Fibonacci numbers of graph are also the concern of Golin, Leung, Wang, and Yong [10], Euler [7], Prodinger and Tichy [15], Warner [20], Engel [6], Calkin and Wilf [2], Burnstein, Kitaev, and Mansour [1], among others. However, there are some errors in [1], as we show in Section 5.

This paper is organized as follows: In Section 2 we introduce the basic definitions and some examples; in Section 3 we give the related theorems for counting independent sets along with some examples; in Section 4, we calculate the generating function of the Fibonacci numbers for some families of open and closed Zykov sums. Finally, in Section 5, we describe more examples, among them (almost) regular graphs [1] where we also show some counterexamples to the calculations in [1].

2 Definitions

In this paper we deal only with multigraphs [5]. However, since we are interested in the number of independent sets, the multiple edges are irrelevant, but the loops are not. Therefore, by a graph, we refer to a multigraph without multiple edges but, perhaps, with loops. For such a graph G, we denote the sets of vertices and edges of G as $V(G)$ and $E(G)$, respectively. We are mainly interested in binary relations between graphs because they describe a basic building block for patterns in families of certain graphs. These relations lead to a couple of sums of graphs: closed and open Zykov sums denoted $+_R$, $\oplus_R$ respectively. Both are defined below.

Definition 1. Let $\uplus$ be the disjoint union operator. Let $G_1, \ldots, G_n$ be a collection of graphs and $R_1, \ldots, R_{n-1}$ be a collection of relations such that each R_i is a relation set from $V(G_i)$ to $V(G_{i+1})$, $i = 1, \ldots, n-1$. Let $E_i = \{\{v, w\} \subseteq V(G_i) \uplus V(G_{i+1}) \,|\, v \in V(G_i), w \in V(G_{i+1})$ with $vR_iw\}$, $i = 1, \ldots, n-1$. We define a new graph $G_1 +_{R_1} \cdots +_{R_{n-1}} G_n$ as follows:

$$V(G_1 +_{R_1} \cdots +_{R_{n-1}} G_n) = V(G_1) \uplus \cdots \uplus V(G_n)$$

and

$$E(G_1 +_{R_1} \cdots +_{R_{n-1}} G_n) = E(G_1) \uplus \cdots \uplus E(G_n) \uplus E_1 \uplus \cdots \uplus E_{n-1}.$$

We call $G_1 +_{R_1} \cdots +_{R_{n-1}} G_n$ an *open Zykov sum*.

If we have an additional relation R_n from $V(G_n)$ to $V(G_1)$, we define an extra graph $G_1 \oplus_{R_1} \cdots \oplus_{R_{n-1}} G_n \oplus_{R_n} G_1$ as

$$V(G_1 \oplus_{R_1} \cdots \oplus_{R_{n-1}} G_n \oplus_{R_n} G_1) = V(G_1) \uplus \cdots \uplus V(G_n)$$

and

$$E(G_1 \oplus_{R_1} \cdots \oplus_{R_{n-1}} G_n \oplus_{R_n} G_1) = E(G_1) \uplus \cdots \uplus E(G_n) \uplus E_1 \uplus \cdots \uplus E_n$$

where $E_n = \{\{u, v\} : u \in V(G_n)$ and $v \in V(G_1)$ with $uR_nv\}$. We call $G_1 \oplus_{R_1} \cdots \oplus_{R_{n-1}} G_n \oplus_{R_n} G_1$ a *closed Zykov sum*.

In the following section, we present several well known families of graphs that can be generated from Zykov sums.

2.1 Examples

Let C_n be the n-cycle graph with $V(C_n) = \{0, 1, \ldots, n-1\}$ and $E(C_n) = \{\{0, 1\}, \ldots, \{n-2, n-1\}, \{n-1, 0\}\}$, where $n = 1, 2, \ldots$

Platonic graphs. The platonic graphs are made of the vertices and edges of the five platonic solids. They can be constructed from open Zykov sums as is shown in Table 1. Note that the relations used are sometimes functions, and if not, the inverse relation is a function, except in the icosahedron case, where neither $R_1^{\pm 1}$ nor $R_2^{\pm 1}$ is a function.

Hypercubes. Let Q_0 be the graph with vertex set given by the singleton set $\{v\}$ and empty set of edges. Then the hypercubes Q_n can be defined recursively by $Q_{n+1} = Q_n +_{\mathrm{Id}_n} Q_n$, $n \geq 0$ where Id_n is the identity map on $V(Q_n)$.

Paths. Let P_0 be the singleton graph, where $V(P_0) = \{v_0\}$ and $E(P_0) = \varnothing$. Let $\mathrm{Id} : \{v_0\} \to \{v_0\}$ be the identity map. Then

$$P_n = P_0 +_{\mathrm{Id}} P_0 +_{\mathrm{Id}} \cdots +_{\mathrm{Id}} P_0$$

is the path graph of length $n-1$, where P_0 appears n times.

Graph name	Zykov sum	Relation
Tetrahedral	$C_1 +_R C_3$	$R = \{(0,0),(0,1),(0,2)\}$
Octahedral	$C_3 +_R C_3$	$R = \{(0,0),(0,2),(1,1),(1,0),$ $(2,2),(2,1)\}$
Cube	$C_4 +_{\mathrm{Id}} C_4$	Id identity map on $V(C_4)$
Icosahedral	$C_3 +_{R_1} C_6 +_{R_2} C_3$	$R_1 = \{(0,0),(0,1),(0,2),(1,2),$ $(1,3),(1,4),(2,4),(2,5),(2,0)\}$ $R_2 = \{(0,0),(1,0),(1,1),(2,1),$ $(3,1),(3,2),(4,2),(5,2),(5,0)\}$
Dodecahedral	$C_5 +_{R_1} C_{10} +_{R_2^{-1}} C_5$	$R_1, R_2 : V(C_5) \to V(C_{10})$ $R_1(i) = 2i, R_2(i) = 2i+1$

Table 1: The platonic graphs as open Zykov sums. The graphs C_n are the n-cycle graphs

Grids, cylinders and tori. The grid $G_{n,m}$ is defined by $V(G_{n,m}) = \{(i,j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(G_{n,m}) = \{\{(i,j),(u,v)\} : |i-u| + |j-v| = 1\}$. Then

$$G_{n,m} = P_n +_{\mathrm{Id}} P_n +_{\mathrm{Id}} \cdots +_{\mathrm{Id}} P_n \tag{1}$$

where the path P_n appears m times and Id is the identity function on $V(P_n)$.

Now, let C_n be the n-cycle graph and Id the identity map on $V(C_n)$. The cylinder $n \times m$, as an open Zykov sum, is

$$C_{n,m} = C_n +_{\mathrm{Id}} C_n +_{\mathrm{Id}} \cdots +_{\mathrm{Id}} C_n$$

where C_n appears m times. On the other hand, the closed Zykov sum

$$T_{n,m} = C_n \oplus_{\mathrm{Id}} C_n \oplus_{\mathrm{Id}} \cdots \oplus_{\mathrm{Id}} C_n$$

is the torus $n \times m$, where C_n appears $m+1$ times. Note that in the Zykov sum $T_{n,m}$, the first and last terms have been identified by definition (see Definition 1).

3 Counting independent sets

In order to count independent sets, we are using the transfer matrix method [2, 6, 7, 10], which is based upon a perpendicularity concept. Following this idea, we propose a new inner product defined with the help of the relation set given in a Zykov sum.

Definition 2. Let $B = \{0,1\}$ and let G, H be a pair of graphs. Let $B^{V(G)}$ be the cartesian product $\prod_{v \in V(G)} B_v$, where each $B_v = B$; similarly for $B^{V(H)}$. Let R be a relation from $V(G)$ to $V(H)$. For any $\boldsymbol{a} \in B^{V(G)}, \boldsymbol{b} \in B^{V(H)}$ we define

$$\langle \boldsymbol{a} | \boldsymbol{b} \rangle_R = \sum_{\substack{v,w \\ vRw}} \pi_v(\boldsymbol{a}) \pi_w(\boldsymbol{b}) \in \mathbb{N}$$

where $\pi_v : B^{V(G)} \to B_v = B$, $\pi_w : B^{V(H)} \to B_w = B$ are the canonical projections.

Let $2^{V(G)}$ be the power set of $V(G)$. There exists a unique bijection $\mathbf{\Psi}_G : 2^{V(G)} \to B^{V(G)}$ such that

$$\pi_v \mathbf{\Psi}_G(A) = \chi_A(v), \quad \forall v \in V(G), \forall A \subset V(G) \tag{2}$$

where χ_A is the characteristic function of the set A.

The following lemma translates the concept of independent set in Zykov sums into perpendicularity.

Lemma 3. *Let G, H be a pair of graphs and R be a relation from $V(G)$ to $V(H)$. If A, B are independent sets of G, H respectively, then $A \uplus B$ is an independent set of $G +_R H$ if and only if $\langle \mathbf{\Psi}_G(A) | \mathbf{\Psi}_H(B) \rangle_R = 0$.*

Proof. The disjoint union $A \uplus B$ is an independent set of $G +_R H$ iff for any $v \in V(G), w \in V(H)$, vRw implies $\{v, w\} \not\subset A \uplus B$ iff vRw implies $\chi_A(v)\chi_B(w) = 0$ iff

$$\sum_{\substack{v,w \\ vRw}} \pi_v(\mathbf{\Psi}_G(A))\pi_w(\mathbf{\Psi}_H(B)) = 0$$

because of (2). □

The transfer matrix is defined below using the inner product relative to the Zykov sums.

Definition 4. 1. For a graph G, we denote the collection of independent sets of G with $\mathcal{I}_G$; while $F(G)$, called the *Fibonacci number of* G, stands for the cardinality of $\mathcal{I}_G$, i.e., $F(G) = |\mathcal{I}_G|$.

2. For any $z \in \mathbb{N}$ we define

$$\overline{z} = \begin{cases} 1, & \text{if } z = 0; \\ 0, & \text{otherwise.} \end{cases}$$

3. Let G, H be a pair of graphs and R be a relation from $V(G)$ to $V(H)$. The function

$$\mathbf{T}^R_{G,H} : \mathcal{I}_G \times \mathcal{I}_H \to B, \quad \mathbf{T}^R_{G,H}(A, B) = \overline{\langle \mathbf{\Psi}_G(A) | \mathbf{\Psi}_H(B) \rangle_R}$$

is called *the transfer matrix* of $G +_R H$.

Note that for any $z_1, z_2 \in \mathbb{N}$, $\overline{z_1 + z_2} = \overline{z_1}\,\overline{z_2}$. In the following Theorems 5 and 6, we show how to calculate the Fibonacci number of open and closed Zykov sums.

Theorem 5. *Let G, H be a pair of graphs and R be a relation from $V(G)$ to $V(H)$. Then*

1.

$$F(G +_R H) = \sum_{A \in \mathcal{I}_G, B \in \mathcal{I}_H} \mathbf{T}^R_{G,H}(A, B)$$

where $\mathbf{T}^R_{G,H}$ is the transfer matrix of $G +_R H$.

2. If $G = H$,

$$F(G \oplus_R G) = \sum_{A \in \mathcal{I}_G} \mathbf{T}^R_{G,G}(A, A)$$

where $\mathbf{T}^R_{G,G}$ is the transfer matrix of $G +_R G$.

Proof. 1. We have that if $J \in \mathcal{I}_{G+_RH}$ then there exist $A \in \mathcal{I}_G$ and $B \in \mathcal{I}_H$ such that $J = A \uplus B$. Now, use Lemma 3.

2. We have that $J \in \mathcal{I}_{G \oplus_R G}$ iff $J \in \mathcal{I}_G$ and $\langle \mathbf{\Psi}(J) | \mathbf{\Psi}(J) \rangle_R = 0$.

□

Theorem 6. *Let G, H, K be graphs, R be a relation from $V(G)$ to $V(H)$ and S be a relation from $V(H)$ to $V(K)$. Then,*

$$F(G +_R H +_S K) = \sum_{A \in \mathcal{I}_G, C \in \mathcal{I}_K} \sum_{B \in \mathcal{I}_H} \mathbf{T}^R_{G,H}(A, B) \mathbf{T}^S_{H,K}(B, C). \tag{3}$$

Furthermore, if $K = G$, then

$$F(G \oplus_R H \oplus_S G) = \sum_{A \in \mathcal{I}_G, B \in \mathcal{I}_H} \mathbf{T}^R_{G,H}(A, B) \mathbf{T}^S_{H,G}(B, A)\ . \tag{4}$$

Proof. We have that $J \in \mathcal{I}_{G+_RH+_SK}$ iff there exist $A \in \mathcal{I}_G, B \in \mathcal{I}_H, C \in \mathcal{I}_K$ such that $J = A \uplus B \uplus C$ and, due to Lemma 3,

$$\langle \mathbf{\Psi}_G(A) | \mathbf{\Psi}_H(B) \rangle_R + \langle \mathbf{\Psi}_H(B) | \mathbf{\Psi}_K(C) \rangle_S = 0$$

which is equivalent to

$$\overline{\langle \mathbf{\Psi}_G(A) | \mathbf{\Psi}_H(B) \rangle_R} \quad \overline{\langle \mathbf{\Psi}_H(B) | \mathbf{\Psi}_K(C) \rangle_S} = 1$$

so

$$|\mathcal{I}_{G+_RH+_SK}| = \sum_{\substack{A \in \mathcal{I}_G, B \in \mathcal{I}_H \\ C \in \mathcal{I}_K}} \overline{\langle \mathbf{\Psi}_G(A) | \mathbf{\Psi}_H(B) \rangle_R} \quad \overline{\langle \mathbf{\Psi}_H(B) | \mathbf{\Psi}_K(C) \rangle_S}$$

from which (3) follows.

Similarly $J \in \mathcal{I}_{G \oplus_R H \oplus_S G}$ iff there exist $A \in \mathcal{I}_G$ and $B \in \mathcal{I}_H$ such that $J = A \uplus B$ and $\langle \mathbf{\Psi}_G(A) | \mathbf{\Psi}_H(B) \rangle_R + \langle \mathbf{\Psi}_H(B) | \mathbf{\Psi}_G(A) \rangle_S = 0$, since Lemma 3. Thus

$$\overline{\langle \mathbf{\Psi}_G(A) | \mathbf{\Psi}_H(B) \rangle_R} \quad \overline{\langle \mathbf{\Psi}_H(B) | \mathbf{\Psi}_G(A) \rangle_S} = 1$$

which leads to

$$|\mathcal{I}_{G \oplus_R H \oplus_S G}| = \sum_{A \in \mathcal{I}_G, B \in \mathcal{I}_H} \overline{\langle \mathbf{\Psi}_G(A) | \mathbf{\Psi}_H(B) \rangle_R} \quad \overline{\langle \mathbf{\Psi}_H(B) | \mathbf{\Psi}_G(A) \rangle_S}\ .$$

□

In fact, the transfer matrix is an actual matrix indexed by the cartesian product of independent sets $\mathcal{I}_G \times \mathcal{I}_H$. We denote such matrix by

$$\mathbf{T}_{G,H}^{R} = (\mathbf{T}_{G,H}^{R}(A,B))_{A\in\mathcal{I}_G, B\in\mathcal{I}_H}.$$

Note that $\mathbf{T}_{G,H}^{R}$ depends on the inner product which is denoted as a bracket. This notation is borrowed from the Dirac notation used mainly in quantum mechanics, where it has proved of great value. So, we are going to keep using this notation when we define, for any graph G, the column matrix full of 1's indexed by the vertices of G which we denoted $|G\rangle$, while $\langle G|$ is the transpose of $|G\rangle$. Then, Theorem 6 ensures that

$$F(G+H+K) = \langle G|\mathbf{T}_{G,H}^{R}\mathbf{T}_{H,K}^{S}|K\rangle, \tag{5}$$

furthermore

$$F(G \oplus_R H \oplus_S G) = \langle G|\mathbf{T}_{G,H}^{R} * (\mathbf{T}_{H,G}^{S})^t|H\rangle \tag{6}$$

where $*$ stands for the Hadamard matrix product and the superindex t indicates matrix transposition. Let us recall that the Hadamard matrix product $A * B$ of two matrices $A = (a_{i,j})$, $B = (b_{i,j})$ of the same dimensions is given by multiplying the corresponding entries together: $A * B = (a_{i,j}b_{i,j})$. Also, note that we can write the right hand side of (6) as a standard matrix product as follows

$$F(G \oplus_R H \oplus_S G) = \operatorname{Tr}\left(\mathbf{T}_{G,H}^{R}\mathbf{T}_{H,G}^{S}\right) \tag{7}$$

where Tr denotes the matrix trace. However the formula in (6) is less difficult to calculate than (7), since the fastest known algorithms for computing the usual matrix product have complexity strictly greater than quadratic, which is the complexity of the Hadamard matrix product. Thus, the formula in (6) is useful for computer calculation.

A similar formula to (5) and (6), for a general graph, was found by Merrifield and Simmons [12, p. 209] using an exponential operator related to the annihilation and creation operators, instead of our transfer matrices and bra and ket vectors. However, said Merrifield-Simmons formula is not a convenient way to handle the Zykov sum structure.

With the aforementioned notation, the proof of Theorem 6 can be generalized in order to obtain:

Theorem 7. *1. The number of independent sets of $G_1 +_{R_1} G_2 +_{R_2} \cdots +_{R_{n-1}} G_n$ is the matrix product $\langle G_1|\mathbf{T}_{G_1,G_2}^{R_1} \cdots \mathbf{T}_{G_{n-1},G_n}^{R_{n-1}}|G_n\rangle$.*

2. The number of independent sets of $G_1 \oplus_{R_1} G_2 \oplus_{R_2} \cdots \oplus_{R_{n-1}} G_n \oplus_{R_n} G_1$ is the matrix trace $\operatorname{Tr}(\mathbf{T}_{G_1,G_2}^{R_1} \cdots \mathbf{T}_{G_{n-1},G_n}^{R_{n-1}}\mathbf{T}_{G_n,G_1}^{R_n})$.

Next, the Fibonacci numbers of two classical platonic solid graphs are calculated using Theorem 7. The remaining classical platonic solid cases are similar. Our goal here is to show that the concept of Zykov sum is an adequate framework for using the matrix transfer method for counting independent sets.

Throughout this paper, we calculate the transfer matrices after the lexicographic order in the cartesian product $\{0,1\}^{|V(G)|}$ induced by $0 < 1$, for any given graph G.

Octahedral graph. Using the decomposition of the octahedral graph as the Zykov sum given in Table 1, we get the following matrices:

$$\langle C_3| = (1,1,1,1), \quad \mathbf{T}^R_{C_3,C_3} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad |C_3\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, from Theorem 7, the Fibonacci number of the octahedral graph is $F(C_3 +_R C_3) = \langle C_3|\mathbf{T}^R_{C_3,C_3}|C_3\rangle = 10$.

Dodecahedral graph. From Table 1, we get that the dodecahedral graph G has Fibonacci number $F(G) = \langle C_5|\mathbf{T}^{R_1}_{C_5,C_{10}}\left(\mathbf{T}^{R_2}_{C_5,C_{10}}\right)^t|C_5\rangle = 5,828$, where

$$\mathbf{T}^{R_1}_{C_5,C_{10}}\left(\mathbf{T}^{R_2}_{C_5,C_{10}}\right)^t = \begin{pmatrix} 123 & 89 & 89 & 89 & 65 & 89 & 65 & 65 & 89 & 65 & 65 \\ 89 & 55 & 63 & 64 & 40 & 63 & 39 & 45 & 55 & 39 & 40 \\ 89 & 55 & 55 & 63 & 39 & 64 & 40 & 40 & 63 & 39 & 45 \\ 89 & 63 & 55 & 55 & 39 & 63 & 45 & 39 & 64 & 40 & 40 \\ 65 & 39 & 39 & 40 & 24 & 45 & 27 & 27 & 40 & 24 & 25 \\ 89 & 64 & 63 & 55 & 40 & 55 & 40 & 39 & 63 & 45 & 39 \\ 65 & 40 & 45 & 40 & 25 & 39 & 24 & 27 & 39 & 27 & 24 \\ 65 & 40 & 39 & 39 & 24 & 40 & 25 & 24 & 45 & 27 & 27 \\ 89 & 63 & 64 & 63 & 45 & 55 & 39 & 40 & 55 & 40 & 39 \\ 65 & 39 & 40 & 45 & 27 & 40 & 24 & 25 & 39 & 24 & 27 \\ 65 & 45 & 40 & 39 & 27 & 39 & 27 & 24 & 40 & 25 & 24 \end{pmatrix}.$$

4 The series of independent sets

In this section, we study the generating function of sequences of Fibonacci numbers for families of graphs determined by Zykov sums which are structures defined by repeating a fixed pattern of relations.

In the following, we are assuming that $\mathcal{G}$ is a family $(G_i)_{i\in\mathbb{N}}$ of graphs.

Definition 8. We call the infinite series

$$F_{\mathcal{G}}(x) = F(G_0) + F(G_1)x + F(G_2)x^2 + \cdots \tag{8}$$

the *Fibonacci series of* $\mathcal{G}$.

The following families, which contain repetitive patterns of open and closed Zykov sums, include many interesting cases such as grids, tori, cylinders [2, 7, 10] and so on.

Definition 9. Let $\mathcal{G} = (G_n)_{n\geq 0}$ be a family of graphs. We call $\mathcal{G}$ a family of:

1. *Strip graphs* if there exists a graph G such that $G_0 = G$, $G_1 = G +_R G$, $G_2 = G +_R G +_R G, \ldots$.

2. *Ring graphs* if there exists a graph G such that $G_0 = G$, $G_1 = G \oplus_S G$, $G_2 = G \oplus_R G \oplus_S G, G_3 = G \oplus_R G \oplus_R G \oplus_S G, \dots$. In such a case $G +_S G$ is called the *skewing* of $\mathcal{G}$.

3. *Alternating strip graphs* if there exists a graph G such that $G_0 = G$, $G_1 = G +_R G$, $G_2 = G +_R +G +_{R^{-1}} G$, $G_3 = G +_R G +_{R^{-1}} G +_R G, \dots$.

4. *Alternating ring graphs* if there exists a graph G such that $G_0 = G$, $G_1 = G \oplus_R G$, $G_2 = G \oplus_R G \oplus_{R^{-1}} G$, $G_3 = G \oplus_R G \oplus_{R^{-1}} G \oplus_R G, \dots$.

In any case G is called the *shape* of $\mathcal{G}$ and $G +_R G$ is called the *fundamental pattern* of $\mathcal{G}$.

Note that in the ring cases, the fundamental pattern is an open Zykov sum, while the family elements are closed Zykov sums.

In the following theorems, we show that the families given in Definition 9 have Fibonacci series with minor variations of the geometric series.

Theorem 10. *Let $\mathcal{G}$ be a family of graphs as in Definition 9. Let G be the shape of $\mathcal{G}$, $\mathbf{T}$ the transfer matrix of the fundamental pattern and $\mathbf{I}$ the identity matrix.*

1. *If $\mathcal{G}$ is a family of strip graphs then*

$$F_{\mathcal{G}}(x) = \langle G|(\mathbf{I} - x\mathbf{T})^{-1}|G\rangle. \tag{9}$$

2. *If $\mathcal{G}$ a family of ring graphs then*

$$F_{\mathcal{G}}(x) = F(G) + \mathrm{Tr}\left(x(\mathbf{I} - x\mathbf{T})^{-1}\mathbf{T}_1\right) \tag{10}$$

where $\mathbf{T}_1$ is the transfer matrix of the skewing of $\mathcal{G}$.

Proof.

1. From Theorem 7, we get, for any n non-negative integer, $F(G_n) = \langle G|\mathbf{T}^n|G\rangle$. So

$$F_{\mathcal{G}}(x) = \langle G|\sum_{n=0}^{\infty}\mathbf{T}^n x^n|G\rangle = \langle G|(\mathbf{I} - x\mathbf{T})^{-1}|G\rangle.$$

2. Similarly, from Theorem 7, we have $F(G_n) = \mathrm{Tr}\left(\mathbf{T}^{n-1}\mathbf{T}_1\right)$, $n \geq 1$. Then

$$\begin{aligned} F_{\mathcal{G}}(x) &= F(G) + \sum_{n=1}^{\infty}\mathrm{Tr}\left(\mathbf{T}^{n-1}\mathbf{T_1}\right)x^n \\ &= F(G) + \mathrm{Tr}\left(x(\mathbf{I} - x\mathbf{T})^{-1}\mathbf{T}_1\right) \end{aligned}$$

due to the linearity of the matrix trace.

□

Similarly, we can prove the following.

Theorem 11. *Under the notation given in Theorem 10.*

1. *If $\mathcal{G}$ is a family of alternating strip graphs then*

$$F_{\mathcal{G}}(x) = \langle G|(\mathbf{I} - x^2\mathbf{T}\mathbf{T}^t)^{-1}(\mathbf{I} + x\mathbf{T})|G\rangle. \tag{11}$$

2. *If $\mathcal{G}$ is a family of alternating ring graphs then*

$$F_{\mathcal{G}}(x) = \mathrm{Tr}\left((\mathbf{I} - x^2\mathbf{T}\mathbf{T}^t)^{-1}(\mathbf{I} + x\mathbf{T})\right).$$

5 Examples

We choose some interesting examples in order to illustrate our methods. We deal with several particular cases of strips, which include cylinders, grids, nanotubes and some kind of cylinders with Petersen graph shape as well as their closed versions as rings: tori, generalized Möbius strips, nanotori, and tori with Petersen graph shape.

Centipedes. Now, our fundamental pattern is $P_2 +_f P_2$ given in Figure 1 where the partial function $f : \{0,1\} \to \{0,1\}$ is defined just by $f(0) = 0$. Next, we take the family of strip

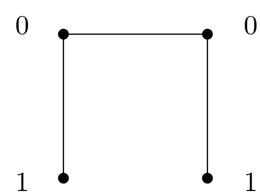

Figure 1: Fundamental pattern of the centipedes

graphs $\mathcal{G} = (G_n)_{n\geq 0}$ given by open Zykov sums $G_n = P_2 +_f \cdots +_f P_2$, where the number of P_2 is $n+1$, $n = 0, 1, \ldots$ Then, the transfer matrix of $P_2 +_f P_2$ is

$$\mathbf{T}^f_{P_2,P_2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{12}$$

Then, from (9), we get

$$F_{\mathcal{G}}(x) = \frac{3 + 2\,x}{1 - 2\,x - 2\,x^2}$$

which is the generating function, except for the first term, of the sequence A028859 in [19].

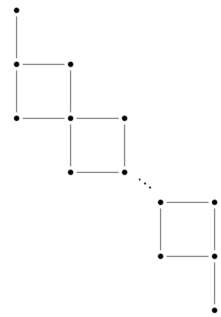

Figure 2: Strip of graphs with fundamental pattern $P_3 +_R P_3$

Diagonal grids. Let P_3 be the path graph with set of vertices $\{0, 1, 2\}$. The family $\mathcal{G}$ of strip graphs given in Figure 2 has fundamental pattern $P_3 +_R P_3$ where R is the relation on $V(P_3)$ defined by $1 R\, 0$, $2 R\, 1$ (see Figure 3).

Its transfer matrix is

$$\mathbf{T}^{R}_{P_3,P_3} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

From (9) it follows that the corresponding Fibonacci series is

$$F_{\mathcal{G}}(x) = -\frac{4\,x - 5}{3\,x^2 - 5\,x + 1}$$

which is the generating function of the sequence A188707. Similarly, we have that the family of graphs $\mathcal{R}$ given in Figure 4 is a family of ring graphs with fundamental pattern and skewing $P_3 +_R P_3$. From (10) we have that its Fibonacci series is

$$F_{\mathcal{R}}(x) = \frac{9\,x^2 - 20\,x + 5}{3\,x^2 - 5\,x + 1}\ .$$

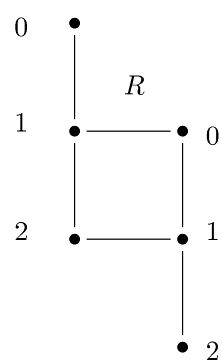

Figure 3: The fundamental pattern $P_3 +_R P_3$

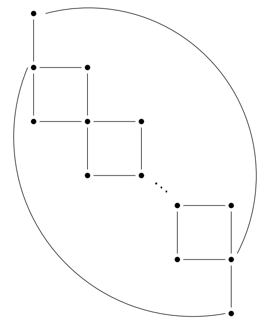

Figure 4: A generic element of the family $\mathcal{R}$

Cylinders. Let C_i be the i-cycle graph and P_n be the path graph with n vertices. Let $\mathrm{Id} : V(C_i) \to V(C_i)$ be the identity map. Then, the cylinder $C_i \times P_n$ is $G(i,n) = C_i +_{\mathrm{Id}} C_i +_{\mathrm{Id}} \cdots +_{\mathrm{Id}} C_i$, where the number of cycles is $n+1$, $n \geq 0$. Thus, the cylinders $G(i,*)$ form a family of strip graphs with fundamental pattern $C_i +_{\mathrm{Id}} C_i$ and shape C_i. From (9), for some fixed i, we can calculate the generating function $F_{G(i,*)}(x)$, as shown in Table 2.

i	$F_{G(i,*)}(x)$	Sequence
2	$-\frac{x+3}{x^2+2x-1}$	A078057
3	$-\frac{x+4}{x^2+3x-1}$	A003688
4	$-\frac{x^2-7}{x^3-x^2-5x+1}$	A051926
5	$-\frac{x^2-4x-11}{x^3-5x^2-7x+1}$	A181989
6	$-\frac{x^4-x^3-25x^2-17x+18}{x^5-2x^4-25x^3-3x^2+12x-1}$	A181961
7	$-\frac{x^4+6x^3-38x^2-16x+29}{x^5+5x^4-44x^3+8x^2+17x-1}$	A182014
8	$-\frac{x^7+6x^6-105x^5+108x^4+394x^3-163x^2-208x+47}{x^8+5x^7-109x^6+187x^5+334x^4-317x^3-65x^2+29x-1}$	A182019

Table 2: The Fibonacci series of some cylinders. These are the generating functions of the integer sequences in the third column except for the first term

Tori. Let C_i be the cycle graph of length i. Then $G(i,n) = C_i \oplus_{\mathrm{Id}} C_i \oplus_{\mathrm{Id}} \cdots \oplus_{\mathrm{Id}} C_i \oplus_{\mathrm{Id}} C_i$ where the number of cycles written is $n+1$, $n = 0, 1, \ldots$. Such graph is the torus $C_i \times C_n$. Thus, the family of ring graphs $G(i,*)$ has fundamental pattern and skewing $C_i +_{\mathrm{Id}} C_i$. Again, we can calculate $F_{G(i,*)}$ for some particular values of i with the help of (10), as shown in Table 3.

Generalized Petersen graphs. We are dealing with generalized Petersen graphs $P(i,2)$ defined by $V(P(i,2)) = \{0, 1, \ldots, 2i-1\}$ and

$$E(P(i,2)) = \big\{\{j, j+i\} \,|\, 0 \leq j \leq i-1\big\} \cup \big\{\{j,k\} \,|\, j-k \equiv 0 \pmod 2 \text{ and } i \leq j,k < 2i\big\} \cup \big\{\{j,k\} \,|\, k \equiv j+1 \pmod i \text{ and } 0 \leq j,k \leq i-1\big\}.$$

i	$F_{G(i,*)}(x)$	Sequence
3	$\frac{5\,x^2+7\,x-4}{x^3+4\,x^2+2\,x-1}$	A051928
4	$-\frac{x^5+17\,x^4-20\,x^3-72\,x^2-13\,x+7}{x^6-2\,x^5-7\,x^4+8\,x^3+15\,x^2+2\,x-1}$	A050402
5	$-\frac{11\,x^6-27\,x^5-130\,x^4+70\,x^3+220\,x^2+43\,x-11}{x^7-8\,x^6+7\,x^5+30\,x^4-10\,x^3-27\,x^2-4\,x+1}$	A182041
6	$-p(x)/q(x)$	A182052

Table 3: The Fibonacci series of several tori. In the case $i = 6$ we have $p(x) = 9\,x^{12}+67\,x^{11}-556\,x^{10}-1162\,x^9+6841\,x^8-1421\,x^7-12335\,x^6+3985\,x^5+7340\,x^4-1182\,x^3-1317\,x^2-71\,x+18$ and $q(x) = x^{13} - 4\,x^{12} - 36\,x^{11} + 119\,x^{10} + 295\,x^9 - 1032\,x^8 + 115\,x^7 + 1301\,x^6 - 360\,x^5 - 575\,x^4 + 89\,x^3 + 84\,x^2 + 4\,x - 1$

The Fibonacci number for generalized Petersen graphs $P(i, 2)$ with i an odd number was calculated by Wagner [20]. Here we calculate the Fibonacci series for the family of generalized Petersen graphs using the transfer matrix method. Let P_4 be the the path graph with vertices $V(P_4) = \{0, 1, 2, 3\}$. Let R be the relation on $V(P_4)$ defined by $0R\,0$, $3R\,3$, $2R\,1$ (see Figure 5).

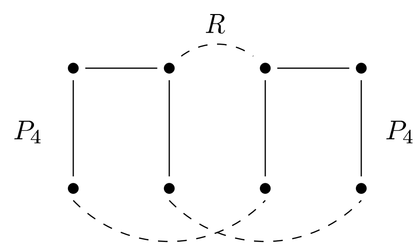

Figure 5: Fundamental pattern in the Petersen graphs $P(2i, 2)$

Then, the ring graph family with shape P_4, fundamental pattern $P_4 +_R P_4$ and skewing the same $P_4 +_R P_4$, is the family of generalized Petersen graphs $\mathcal{G} = \big(P(2i, 2)\big)_{i\geq 0}$:

$$P(2i, 2) = P_4 \oplus_R P_4 \oplus_R \cdots \oplus_R P_4 \oplus_R P_4,$$

where P_4 appears $i+1$ times, $i = 0, 1, \ldots$. Note that we included the non-standard generalized Petersen graphs $P(0, 2) = P_4$ and the multigraph $P(2, 2) = P_4 \oplus_R P_4$.

The related transfer matrix is

$$\mathbf{T}^R_{P_4,P_4} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

Then, from (10), we get

$$F_{\mathcal{G}}(x) = \frac{(6\,x^2 - 11\,x - 8)\,(2\,x^3 - 5\,x^2 - 4\,x + 1)}{4\,x^5 - 13\,x^4 + 3\,x^3 + 15\,x^2 + 3\,x - 1}$$

which is the generating function of A182054.

For generalized Petersen graphs of the type $P(2i+1,2)$, $i \geq 2$ we take the family $\mathcal{P}$ given in Figure 7, i.e.,

$$P(2i+1,2) = \bigoplus_{R_1}{}_{k=1}^{i-1} P_4 \oplus_{R_2} M \oplus_{R_3} P_4$$

where M is the graph such that $V(M) = \{0,1,2,3,4,5\}$, $E(M) = \big\{\{0,1\},\{1,2\},\{2,3\},\{2,4\},\{4,5\},\{0,5\}\big\}$ and relations $R_1 = \big\{(0,0),(3,3),(2,1)\big\}$, $R_2 = \big\{(0,0),(3,3),(2,1)\big\}$, $R_3 = \{(3,0),(5,3),(4,1)\}$.

From the Theorem 7 and proof of (10) in Theorem 10, we get

$$F_{\mathcal{P}}(x) = \sum_{j=0}^{\infty} \mathrm{Tr}(\mathbf{T}_1^j \mathbf{T}_2 \mathbf{T}_3) x^j = \mathrm{Tr}\left((\mathbf{I} - x\mathbf{T}_1)^{-1}\mathbf{T}_2\mathbf{T}_3\right)$$

where $\mathbf{T}_1 = \mathbf{T}_{P_4,P_4}^{R_1}$ is given by (13); while $\mathbf{T}_2 = \mathbf{T}_{P_4,M}^{R_2}$ and $\mathbf{T}_3 = \mathbf{T}_{M,P_4}^{R_3}$ are 8×19 and 19×8 matrices respectively. Thus,

$$F_{\mathcal{P}}(x) = -\frac{52\,x^4 - 165\,x^3 + 16\,x^2 + 207\,x + 76}{4\,x^5 - 13\,x^4 + 3\,x^3 + 15\,x^2 + 3\,x - 1}$$

which is the generating function of A182077 with a shift of one term.

Families with Petersen graph shape. Let $\mathrm{Id} : V(P(5,2)) \to V(P(5,2))$ be the identity map. We take $\mathcal{G}$ as the family of strip graphs with fundamental pattern $P(5,2) +_{\mathrm{Id}} P(5,2)$ and shape of the Petersen graph $P(5,2)$, i.e., $\mathcal{G} = (G_n)_{n\geq 0}$ where $G_n = P(5,2) +_{\mathrm{Id}} P(5,2) +_{\mathrm{Id}} \cdots +_{\mathrm{Id}} P(5,2)$ with $n+1$ copies of the Petersen graph, $n \geq 0$ (see Figure 8). Then, the transfer matrix $\mathbf{T}_{P(5,2),P(5,2)}^{\mathrm{Id}}$ is a 76×76-matrix, since $F(P(5,2)) = 76$.

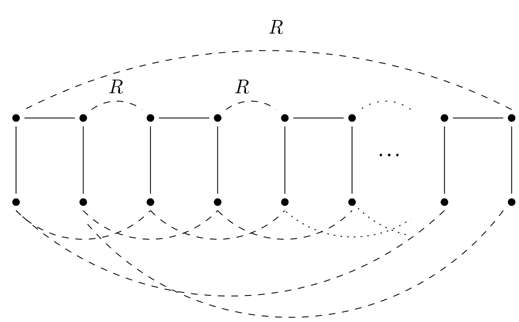

Figure 6: Generalized Petersen graph $P(2i,2)$.

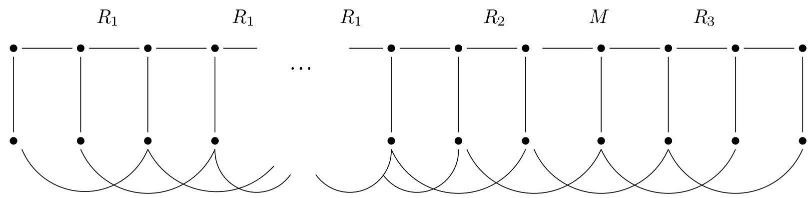

Figure 7: The family of graphs $P(2i+1,2)$

From (9) we get

$$F_{\mathcal{G}}(x) = -\frac{x^5 + 12\,x^4 - 130\,x^3 + 92\,x^2 + 237\,x + 76}{x^6 + 11\,x^5 - 137\,x^4 + 172\,x^3 + 215\,x^2 + 39\,x - 1}.$$

Similarly, for $\mathcal{G}'$ the family of ring graphs with fundamental pattern and skewing given by $P(5,2) +_{\mathrm{Id}} P(5,2)$, we get from (10), that its Fibonacci series $F_{\mathcal{G}'}(x)$ satisfies $F_{\mathcal{G}'}(x) = p(x)/q(x)$ where

$$\begin{aligned} p(x) = 59\,x^{20} + 158\,x^{19} - 17410\,x^{18} - 31425\,x^{17} + 843564\,x^{16} + 1040034\,x^{15} \\ - 10876134\,x^{14} - 9246646\,x^{13} + 52315426\,x^{12} + 29197770\,x^{11} - 101636518\,x^{10} \\ - 28773932\,x^{9} + 77606056\,x^{8} + 9105678\,x^{7} - 21502410\,x^{6} - 847682\,x^{5} + 1979331\,x^{4} \\ + 80616\,x^{3} - 50408\,x^{2} - 2203\,x + 76 \end{aligned}$$

and

$$\begin{aligned} q(x) = (x-1)\left(x^2 + 2\,x - 1\right)\left(x^5 + 5\,x^4 - 4\,x^3 - 14\,x^2 + 3\,x + 1\right) \\ \left(x^6 + 11\,x^5 - 137\,x^4 + 172\,x^3 + 215\,x^2 + 39\,x - 1\right) \\ \left(x^7 + 4\,x^6 - 52\,x^5 - 105\,x^4 + 51\,x^3 + 78\,x^2 - 10\,x - 1\right)\ . \end{aligned}$$

We obtain an additional family $\mathcal{G}''$ if, instead of closing with the relation given by the identity map, we take a rotation R by an angle of $2\pi/5$. More formally, we take $V(P(5,2)) =$

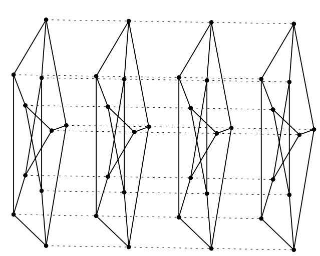

Figure 8: A strip graph with shape $P(5,2)$, the Petersen graph. The edges given by the identity map are shown by dotted lines

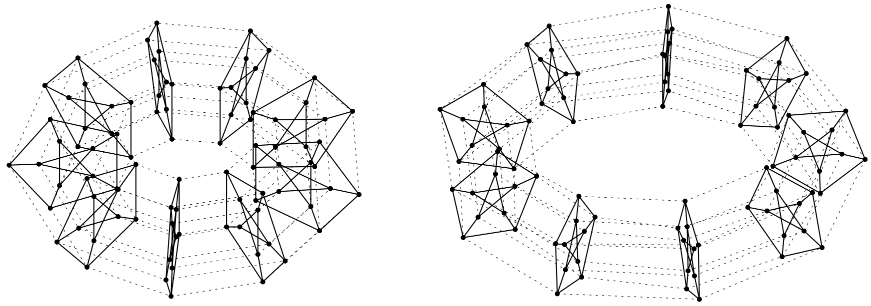

Figure 9: A pair of rings with shape $P(5,2)$, the Petersen graph. The edges given by the relation maps are shown by dotted lines. The ring on the left has relation maps the identity map; while the ring on the right has skewing given by the relation $R = \{(0,1),(1,2),(2,3),(3,4),(4,0),(5,6),(6,7),(7,8),(8,9),(9,5)\}$. The former ring has Fibonacci number 27,053,615,385,404,201. The latter has Fibonacci number 27,050,814,022,108,001.

$\{0,1,2,\ldots,9\}$ as the vertices set of the Petersen graph, and

$$E(P(5,2)) = \{\{0,1\},\{0,4\},\{0,5\},\{1,2\},\{1,6\},\{2,3\},\{2,7\},\{3,4\},\\ \{3,8\},\{4,9\},\{5,7\},\{5,8\},\{6,8\},\{6,9\},\{7,9\}\}.$$

Then, the new skewing induced by R is defined as follows (see Figure 9)

$$R = \{(0,1),(1,2),(2,3),(3,4),(4,0),(5,6),(6,7),(7,8),(8,9),(9,5)\}.$$

Then $F_{\mathcal{G}''}(x) = p_1(x)/q_1(x)$, where

$$p_1(x) = 75\,x^{15} + 1284\,x^{14} - 8828\,x^{13} - 101662\,x^{12} + 376556\,x^{11} + 1642004\,x^{10} \\ - 2174799\,x^9 - 7893320\,x^8 + 252699\,x^7 + 6559072\,x^6 + 1031350\,x^5 - 1259454\,x^4 \\ - 160398\,x^3 + 47456\,x^2 + 2305\,x - 76$$

and

$$q_1(x) = \left(x^2 + 2\,x - 1\right)\left(x^6 + 11\,x^5 - 137\,x^4 + 172\,x^3 + 215\,x^2 + 39\,x - 1\right) \\ \left(x^7 + 4\,x^6 - 52\,x^5 - 105\,x^4 + 51\,x^3 + 78\,x^2 - 10\,x - 1\right).$$

Armchair nanotube graphs. Let B_n be the graph with $2n$ vertices $\{0,1,\ldots,\ 2n-1\}$ and set of edges $\{\{0,1\},\{2,3\},\ldots,\{2n-2,2n-1\}\}$, i.e., B_n is the disjoint union of n copies of the one-length path P_2:

$$B_n = P_2 +_{\varnothing} \cdots +_{\varnothing} P_2.$$

We define a relation map $R : V(B_n) \to V(B_n)$ as $R(i) = (i-1) \bmod 2n$, $i = 0,\ldots,2n-1$. An *armchair nanotube graph* of *length* n and *breadth* k is

$$NT_{n,k} = B_n +_R B_n +_{R^{-1}} B_n +_R \cdots +_{R^{\pm 1}} B_n$$

where B_n appears $k+1$ times, $k \geq 0$ (this graph forms the structure of the (n,n) armchair carbon nanotube [14] without caps). Thomassen [16] defines a similar graph called *hexagonal cylinder circuit*, however this belongs to a different family of nanotubes: *zig-zag carbon nanotubes*.

By definition $NT_{n,*} = \left(NT_{n,k}\right)_{k\geq 0}$ is a family of alternating strip graphs with fundamental pattern $B_n +_R B_n$. By (11) and a calculation similar to those in the previous examples, we get that the Fibonacci series of the armchair nanotube graphs of length 3 is

$$F_{NT_{3,*}}(x) = -\frac{28\,x^4 + 55\,x^3 - 89\,x^2 - 29\,x + 27}{28\,x^5 + 42\,x^4 - 109\,x^3 + 17\,x^2 + 13\,x - 1}$$

which is the generating function of A182130.

Similarly, a *nanotorus graph* is the following closed Zykov sum:

$$N\tau_{n,k} = B_n \oplus_R B_n \oplus_{R^{-1}} B_n +_R \cdots \oplus_{R^{\pm 1}} B_n,$$

where, again, B_n appears $k+1$ times, $k \geq 0$. Now the family $N\tau_{3,*} = (N\tau_{3,k})_{k\geq 0}$ has Fibonacci series $F_{N\tau_{3,*}}(x) = p(x)/q(x)$, where

$$\begin{aligned} p(x) = {} & 979776\,x^{18} - 75600\,x^{17} - 12197940\,x^{16} + 5916552\,x^{15} + 35833019\,x^{14} \\ & - 19220271\,x^{13} - 44070216\,x^{12} + 23310438\,x^{11} + 26177559\,x^{10} - 13274349\,x^{9} \\ & - 7520073\,x^{8} + 3654387\,x^{7} + 940365\,x^{6} - 451464\,x^{5} - 43362\,x^{4} + 24495\,x^{3} \\ & + 25\,x^{2} - 468\,x + 27 \end{aligned}$$

and

$$\begin{aligned} q(x) = {} & (x-1)\,(x+1)\,\left(3\,x^3 - 5\,x^2 - 5\,x + 1\right)\left(36\,x^4 - x^3 - 20\,x^2 - x + 1\right) \\ & \left(36\,x^4 + x^3 - 20\,x^2 + x + 1\right)\left(28\,x^5 + 42\,x^4 - 109\,x^3 + 17\,x^2 + 13\,x - 1\right). \end{aligned}$$

The rational function $F_{N\tau_{3,*}}(x)$ is the generating function of the sequence A182141.

Generalized Möbius Ladders. Let P_n be the path graph with n vertices, $V(P_n) = \{0, \ldots, n-1\}$, σ be a permutation of $V(P_n)$ and Id be the identity map on $V(P_n)$. Then a generalized Möbius ladder $M(n,m,\sigma)$ is

$$M(n,m,\sigma) = P_n \oplus_{\mathrm{Id}} \cdots \oplus_{\mathrm{Id}} P_n \oplus_\sigma P_n$$

where P_n appears $m+1$ times, $m = 0, 1, \ldots$; for instance, if $\sigma : V(P_2) \to V(P_2)$ is the transposition $\sigma(0) = 1$ and $\sigma(1) = 0$ then $M(2,*,\sigma) = (M(2,m,\sigma))_{m\geq 0}$ is the family of usual Möbius ladders. The family of ring graphs $M(2,*,\sigma)$ has fundamental pattern $P_2 +_{\mathrm{Id}} P_2$ and skewing $P_2 +_\sigma P_2$. We have the transfer matrices,

$$\mathbf{T}^{\mathrm{Id}}_{P_2,P_2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{T}^{\sigma}_{P_2,P_2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

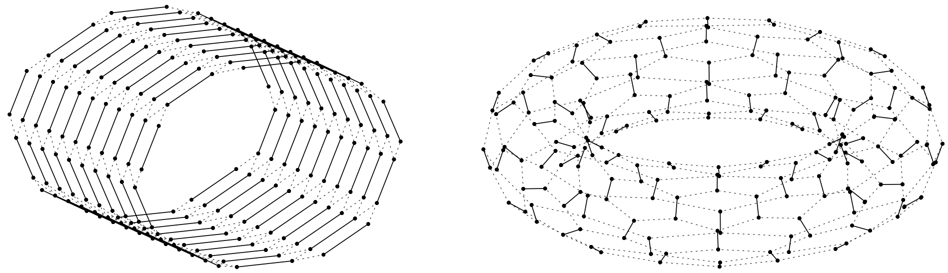

Figure 10: The nanotube graph $NT_{5,21}$ on the left and the nanotorus graph $N\tau_{5,21}$ on the right. Once again, the edges given by the relation set are shown by dotted lines. Using Theorem 7 we get that the nanotube graph $NT_{5,21}$ has Fibonacci number 14,890,453,762,710,452,477,470,450,680,772,895,445,343; while the nanotorus graph $N\tau_{5,21}$ has Fibonacci number 73,562,247,493,061,556,896,479,759,292,362,159,745

Then, from (10), we get

$$F_{M(2,*,\sigma)}(x) = \frac{2\,x^3+7\,x^2-3}{(x+1)\,(x^2+2\,x-1)}$$

which is the generating function of A182143 except for the first term.

Now, we take families $M(3,*,\sigma_i)=(M(3,m,\sigma_i))_{m\geq 0}$, $i=1,2,3$ with $\sigma_1,\sigma_2,\sigma_3$ permutations of $V(P_3)$ defined in Figure 11. Thomassen [16] calls the family of graphs $M(3,\sigma_2)$ *quadrilateral Möbius double circuits*.

We have

$$\mathbf{T}^{\sigma_1}_{P_3,P_3} = \begin{pmatrix} 1&1&1&1&1\\ 1&0&1&1&0\\ 1&1&1&0&0\\ 1&1&0&1&1\\ 1&0&0&1&0 \end{pmatrix}, \quad \mathbf{T}^{\sigma_2}_{P_3,P_3} = \begin{pmatrix} 1&1&1&1&1\\ 1&1&1&0&0\\ 1&1&0&1&1\\ 1&0&1&1&0\\ 1&0&1&0&0 \end{pmatrix},$$

$$\mathbf{T}^{\sigma_3}_{P_3,P_3} = \begin{pmatrix} 1&1&1&1&1\\ 1&1&0&1&1\\ 1&1&1&0&0\\ 1&0&1&1&0\\ 1&0&0&1&0 \end{pmatrix}, \quad \mathbf{T}^{\mathrm{Id}}_{P_3,P_3} = \begin{pmatrix} 1&1&1&1&1\\ 1&0&1&1&0\\ 1&1&0&1&1\\ 1&1&1&0&0\\ 1&0&1&0&0 \end{pmatrix}.$$

Then

$$F_{M(3,*,\sigma_1)}(x) = \frac{3\,x^5+6\,x^4-19\,x^3-30\,x^2-2\,x+5}{(x+1)\,(x^4-6\,x^2-2\,x+1)},$$

$$F_{M(3,*,\sigma_2)}(x) = \frac{2\,x^5+x^4-24\,x^3-28\,x^2-2\,x+5}{(x+1)\,(x^4-6\,x^2-2\,x+1)},$$

and

$$F_{M(3,*,\sigma_3)}(x) = \frac{4\,x^5+6\,x^4-25\,x^3-32\,x^2-x+5}{(x+1)\,(x^4-6\,x^2-2\,x+1)}.$$

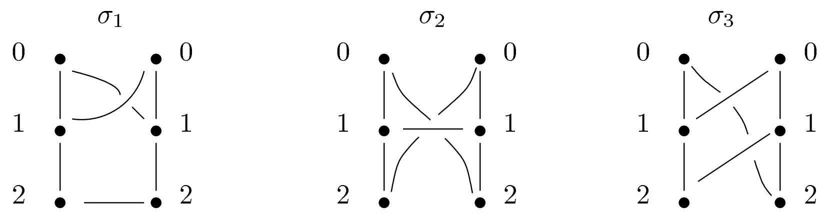

Figure 11: A subset of permutations of $V(P_3)$

Some regular and almost regular graphs. Let B_n be the disjoint union of n copies of the singleton graph K_1, i.e, $V(B_n) = \{0, \ldots, n-1\}$ and $E(B_n) = \varnothing$. Let R be the relation on $V(B_n)$ given by iRi and iRj where $j \equiv i-1 \pmod{n}$, $i = 0, \ldots n-1$. Note that R is also a relation from vertices of the n-cycle C_n to those of B_n. We define

$$G_n^k = C_n +_R B_n +_R \cdots +_R B_n$$

$$R_n^{k+1} = C_n +_R B_n +_R \cdots +_R B_n +_R C_n$$

where B_n appears $k-1$ times, $k = 1, 2, \ldots$ and $G_n^0 = R_n^0 = \varnothing$ the empty graph, $R_n^1 = C_n$. Furthermore, let C_n' be the complement graph of the n-cycle graph C_n and K_n the complete graph on n vertices with $V(K_n) = \{0, 1, \ldots, n-1\}$. Also, we define

$$K_n^k = C_n' +_R \cdots +_R C_n' +_R K_n$$

$$P_n^{k+1} = K_n +_R C_n' +_R \cdots +_R C_n' +_R K_n$$

where C_n' appears $k-1$ times, $k = 1, 2, \ldots$ and $K_n^0 = P_n^0 = \varnothing$, $P_n^1 = K_n$. The graphs G_n^k, R_n^k, K_n^k and P_n^k are called *(almost) regular graphs* class 1, class 2, class 3 and class 4, respectively by Burstein, Kitaev, and Mansour [1]. Thus, from Theorem 7, we get

$$F_{G_n^*}(x) = 1 + F(C_n)\,x + \langle C_n | \mathbf{T}^R_{C_n,B_n} (\mathbf{I} - x\,\mathbf{T}^R_{B_n,B_n})^{-1} | B_n \rangle\, x^2, \tag{14}$$

$$F_{R_n^*}(x) = 1 + F(C_n)\,x + \langle C_n | \left(x^2\,\mathbf{T}^R_{C_n,C_n} + x^3\,\mathbf{T}^R_{C_n,B_n} (\mathbf{I} - x\,\mathbf{T}^R_{B_n,B_n})^{-1} \mathbf{T}^R_{B_n,C_n} \right) | C_n \rangle, \tag{15}$$

$$F_{K_n^*}(x) = 1 + (n+1)\,x + \langle C_n' | (\mathbf{I} - x\,\mathbf{T}^R_{C_n',C_n'})^{-1} \mathbf{T}^R_{C_n',K_n} | K_n \rangle\, x^2, \tag{16}$$

$$F_{P_n^*}(x) = 1 + (n+1)\,x + \langle K_n | \left(x^2\,\mathbf{T}^R_{K_n,K_n} + x^3\,\mathbf{T}^R_{K_n,C_n'} (\mathbf{I} - x\,\mathbf{T}^R_{C_n',C_n'})^{-1} \mathbf{T}^R_{C_n',K_n} \right) | K_n \rangle. \tag{17}$$

Burstein et al [1] introduce algorithms for computing the Fibonacci series of the families G_n^*, R_n^*, K_n^*, and P_n^*. However these algorithms fail for the families R_n^*, K_n^* and P_n^*. For instance, it is easy to see that $F(R_3^3) = 32$, while 32 does not appear in the sequence A026150 which is the Fibonacci numbers sequence of the family R_3^*, according to Burstein et al. Also $F(K_4^2) = 25$, $F(P_4^3) = 77$ are counterexamples to Theorem 3.3 and Theorem 3.4 of [1], respectively. As a consequence, since 77 does not appear in A007483, Burstein et al

n	$F_{R_n^*}(x)$	$F_{K_n^*}(x)$	Sequence
3	$-\frac{4\,x^2-1}{2\,x^2-4\,x+1}$	$\frac{1}{2\,x^2-4\,x+1}$	A007070
4	$\frac{10\,x^5+41\,x^4-38\,x^3-14\,x^2+4\,x+1}{7\,x^4+15\,x^3-14\,x^2-3\,x+1}$	$\frac{3\,x^2+2\,x+1}{x^3-7\,x^2-3\,x+1}$	
5	$\frac{55\,x^6-193\,x^5-303\,x^4+149\,x^3+39\,x^2-6\,x-1}{22\,x^5-31\,x^4-69\,x^3+30\,x^2+5\,x-1}$	$\frac{4\,x^2+x+1}{x^3-7\,x^2-5\,x+1}$	
6	$\frac{516\,x^8-248\,x^7-3688\,x^6+1834\,x^5+2518\,x^4-588\,x^3-112\,x^2+10\,x+1}{108\,x^7-84\,x^6-532\,x^5+178\,x^4+280\,x^3-66\,x^2-8\,x+1}$	$\frac{5\,x^2+1}{x^3-7\,x^2-7\,x+1}$	

Table 4: Some Fibonacci series of classes 2 and 3 graphs

[1] are wrongfully relating this sequence to the class 4 graphs. The correct Fibonacci series, after the formulas (15), (16) and (17) are shown in Tables 4 and 5.

Due to Theorem 10, the family $\varnothing, C_3, C_3 +_R C_3, C_3 +_R C_3 +_R C_3, \ldots$ has Fibonacci series $(2\,x+1)/(2\,x^2+2\,x-1)$ which is the generating function of A026150; while the family $\varnothing, K_4, K_4 +_R K_4, K_4 +_R K_4 +_R K_4, \ldots$ has Fibonacci series $-(2\,x+1)/(2\,x^2+3\,x-1)$ which is the generating function of A007483.

n	$F_{P_n^*}(x)$	Sequence
3	$-\frac{(2\,x-1)\,(2\,x+1)}{2\,x^2-4\,x+1}$	A161941
4	$-\frac{8\,x^3+5\,x^2-2\,x-1}{x^3-7\,x^2-3\,x+1}$	
5	$-\frac{10\,x^3+11\,x^2-x-1}{x^3-7\,x^2-5\,x+1}$	
6	$-\frac{12\,x^3+19\,x^2-1}{x^3-7\,x^2-7\,x+1}$	

Table 5: Some Fibonacci series of the class 4 graphs. The series $F_{P_3^*}(x)$ is the generating function of two times the terms in A161941 except the first one

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