Journal of Integer Sequences, Vol. 13 (2010),
Article 10.8.2

The Euler-Seidel Matrix, Hankel Matrices and Moment Sequences

Paul Barry
School of Science
Waterford Institute of Technology
Ireland
pbarry@wit.ie

Aoife Hennessy
Department of Computing, Mathematics and Physics
Waterford Institute of Technology
Ireland
aoife.hennessy@gmail.com

Abstract

We study the Euler-Seidel matrix of certain integer sequences, using the binomial transform and Hankel matrices. For moment sequences, we give an integral representation of the Euler-Seidel matrix. Links are drawn to Riordan arrays, orthogonal polynomials, and Christoffel-Darboux expressions.

1 Introduction

The purpose of this note is to investigate the close relationship that exists between the Euler-Seidel matrix [3, 4, 5, 6, 10] of an integer sequence, and the Hankel matrix [9] of that sequence. We do so in the context of sequences that have integral moment representations, though many of the results are valid in a more general context. While partly expository in nature, the note assumes a certain familiarity with generating functions, both ordinary and exponential, orthogonal polynomials [2, 8, 16] and Riordan arrays [12, 15] (again, both ordinary, where we use the notation (g, f), and exponential, where we use the notation $[g, f]$). Many interesting examples of sequences and Riordan arrays can be found in Neil Sloane's

On-Line Encyclopedia of Integer Sequences, [13, 14]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix $\mathbf{B}$ is A007318.

The *Euler-Seidel matrix* of a sequence $(a_n)_{n\geq 0}$, which we will denote by $\mathbf{E} = \mathbf{E}_a$, is defined to be the rectangular array $(a_{n,k})_{n,k\geq 0}$ determined by the recurrence $a_{0,k} = a_k$ $(k \geq 0)$ and

$$a_{n,k} = a_{n,k-1} + a_{n+1,k-1} \quad (n \geq 0, k \geq 1). \tag{1}$$

The sequence $(a_{0,k})$, the first row of the matrix, is usually called the *initial sequence*, while the sequence $(a_{n,0})$, first column of the matrix, is called the *final sequence*. They are related by the binomial transform (or Euler transform, after Euler, who first proved this [7]). We recall that the binomial transform of a sequence a_n has general term $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. Thus the first row and column of the matrix are determined from Eq. (1) as follows:

$$a_{n,0} = \sum_{k=0}^{n} \binom{n}{k} a_{0,k}, \tag{2}$$

$$a_{0,n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_{k,0}. \tag{3}$$

In general, we have

$$a_{n,k} = \sum_{i=0}^{n} \binom{n}{i} a_{i+k,0} = \sum_{i=0}^{n} \binom{n}{i} a_{i+k}. \tag{4}$$

Example 1. We take $a_{0,n} = a_n = C_n = \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers. Thus the initial sequence, or first row, is the Catalan numbers, while the final sequence, or first column, will be the binomial transform of the Catalan numbers. We obtain the following matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 5 & 14 & 42 & \cdots \\ 2 & 3 & 7 & 19 & 56 & 174 & \cdots \\ 5 & 10 & 26 & 75 & 230 & 735 & \cdots \\ 15 & 36 & 101 & 305 & 965 & 3155 & \cdots \\ 51 & 137 & 406 & 1270 & 4120 & 13726 & \cdots \\ 188 & 543 & 1676 & 5390 & 17846 & 60398 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We now remark that the Catalan numbers C_n A000108 have the following moment representation:

$$C_n = \frac{1}{2\pi} \int_0^4 x^n \frac{\sqrt{x(4-x)}}{x}\, dx. \tag{5}$$

For the rest of this note, we shall assume that all sequences discussed have a moment representation of the form

$$a_n = \int_{\mathbf{R}} x^n d\mu_a$$

for a suitable measure $d\mu_a$. The reader is directed to the Appendix for the link between the generating function of a sequence and the corresponding measure, when it exits.

We give some examples of such sequences.

Example 2. The aerated Catalan numbers.

We have seen (see Eq. (5)) that the Catalan numbers are a moment sequence. Similarly, the aerated Catalan numbers

$$1, 0, 1, 0, 2, 0, 5, 0, 14, \ldots$$

can by represented by

$$C_{\frac{n}{2}} \frac{1+(-1)^n}{2} = \frac{1}{2\pi} \int_{-2}^{2} x^n \sqrt{4-x^2}\, dx.$$

This is the famous semi-circle distribution, of importance in random matrix theory.

Example 3. The factorial numbers $n!$.

We have the well-known integral representation of $n!$ A000142

$$n! = \int_0^\infty x^n e^{-x}\, dx.$$

Example 4. The aerated double factorials.

We recall that the double factorials A001147 are given by

$$(2n-1)!! = \prod_{k=1}^{n} (2k-1) = \frac{(2n)!}{n!\, 2^n}.$$

The aerated double factorials, which begin

$$1, 0, 3, 0, 5, 0, 15, 0, 105, 0, 945, \ldots$$

have integral representation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n\, e^{-\frac{x^2}{2}}\, dx.$$

The aerated double factorial numbers have exponential generating function $e^{\frac{x^2}{2}}$. (Note that the numbers $2^n \cdot n!$ A000165 are also called double factorials).

2 The Euler-Seidel and Hankel matrix for moment sequences

We recall that for a sequence $(a_n)_{n \ge 0}$, its Hankel matrix is the matrix $\mathbf{H} = \mathbf{H}_a$ with general term a_{n+k}. Note that if a_n has o.g.f. $A(x)$ then the bivariate generating function of $\mathbf{H}_a$ is given by

$$\frac{xA(x) - yA(y)}{x-y}.$$

If a_n has an exponential generating function $G(x)$, then the n-th row (and n-th column) of $\mathbf{H}_a$ has exponential generating function given by

$$\frac{d^n}{dx^n} G(x).$$

Example 5. We have seen that the aerated double factorial numbers have e.g.f. $e^{\frac{x^2}{2}}$. Thus the n-th row of the Hankel matrix associated to them has e.g.f.

$$\frac{d^n}{dx^n}e^{\frac{x^2}{2}} = e^{\frac{x^2}{2}}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2}(2k-1)!!x^{n-2k}.$$

(We note that written as

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2}(2k-1)!! = e^{-\frac{x^2}{x}}\frac{d^n}{dx^n}e^{\frac{x^2}{x}},$$

this is a statement about scaled Hermite polynomials.)

Note that if

$$a_n = \int x^n d\mu_a$$

then

$$a_{n+k} = \int x^{n+k} d\mu_a = \int x^n x^k d\mu_a.$$

The binomial matrix is the matrix $\mathbf{B}$ with general term $\binom{n}{k}$. The binomial transform of a sequence a_n is the sequence with general term

$$b_n = \sum_{k=0}^{n}\binom{n}{k}a_k.$$

In this case, the sequence b_n has o.g.f. given by

$$\frac{1}{1-x}A\left(\frac{x}{1-x}\right).$$

The sequence $(b_n)_{n\geq 0}$ can be viewed as

$$\mathbf{B}\cdot(a_n)^t.$$

Note that we have

$$\begin{aligned} b_n &= \sum_{k=0}^{n}\binom{n}{k}a_k \\ &= \sum_{k=0}^{n}\binom{n}{k}\int x^k\,d\mu_a \\ &= \int\sum_{k=0}^{n}\binom{n}{k}x^k\,d\mu_a \\ &= \int(1+x)^n\,d\mu_a. \end{aligned}$$

In similar fashion, we have

$$a_n = \int(x-1)^n\,d\mu_b.$$

Proposition 6. *We have*

$$\mathbf{E}_a = \mathbf{B}\mathbf{H}_a. \tag{6}$$

Proof. We have

$$\mathbf{B}\mathbf{H}_a = \left(\binom{n}{k}\right) \cdot \left(a_{n+k}\right).$$

The result follows from Eq. (4). □

We now let

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k,$$

the binomial transform of a_n. In the sequel, we shall be interested in the product $\mathbf{B}^{-1}\mathbf{H}_b$.

Example 7. Taking $b_n = \sum_{k=0}^{n} \binom{n}{k} C_k$ A0007317, the binomial transform of the Catalan numbers C_n, we obtain

$$\mathbf{H}_b = \begin{pmatrix} 1 & 2 & 5 & 15 & 51 & 188 & \dots \\ 2 & 5 & 15 & 51 & 188 & 731 & \dots \\ 5 & 15 & 51 & 188 & 731 & 2950 & \dots \\ 15 & 51 & 188 & 731 & 2950 & 12235 & \dots \\ 51 & 188 & 731 & 2950 & 12235 & 51822 & \dots \\ 188 & 731 & 2950 & 12235 & 51822 & 223191 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Multiplying by $\mathbf{B}^{-1}$, we obtain

$$\mathbf{B}^{-1}\mathbf{H}_b = \begin{pmatrix} 1 & 2 & 5 & 15 & 51 & 188 & \dots \\ 1 & 3 & 10 & 36 & 137 & 543 & \dots \\ 2 & 7 & 26 & 101 & 406 & 1676 & \dots \\ 5 & 19 & 75 & 305 & 1270 & 5390 & \dots \\ 14 & 56 & 230 & 965 & 4120 & 17846 & \dots \\ 42 & 174 & 735 & 3155 & 13726 & 60398 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is the transpose of the Euler-Seidel matrix for C_n.

This result is general. In order to prove this, we will use the follow lemma.

Lemma 8.

$$x^n(1+x)^k = \sum_{i=0}^{k} \binom{k}{i} x^{i+n} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \sum_{i=0}^{j+k} \binom{j+k}{i} x^i. \tag{7}$$

Proof. Since $(1+x)^k = \sum_{i=0}^k \binom{k}{i} x^i$ by the binomial theorem, we immediately have

$$x^n(1+x)^k = x^n \sum_{i=0}^{k} \binom{k}{i} x^i = \sum_{i=0}^{k} \binom{k}{i} x^{i+n}.$$

But also, we have

$$\begin{aligned}
\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \sum_{i=0}^{j+k} \binom{j+k}{i} x^i &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (1+x)^{j+k} \\
&= (1+x)^k \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (1+x)^j \\
&= (1+x)^k x^n.
\end{aligned}$$

□

Proposition 9. *The Euler-Seidel matrix of the sequence $(a_n)_{n\geq 0}$ is equal to the transpose of the matrix given by $\mathbf{B}^{-1}\mathbf{H}_b$, where $\mathbf{H}_b$ is the Hankel matrix of the binomial transform*

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$$

of the initial sequence a_n. That is,

$$\mathbf{E}_a^t = \mathbf{B}^{-1}\mathbf{H}_b. \tag{8}$$

Proof. The general element of

$$\mathbf{B}^{-1}\mathbf{H} = \left((-1)^{n-k} \binom{n}{k} \right) \cdot (b_{n+k})$$

is given by

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} b_{j+k}.$$

Now

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} b_{j+k} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \sum_{i=0}^{j+k} \binom{j+k}{i} a_i.$$

We thus wish to show that

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \sum_{i=0}^{j+k} \binom{j+k}{i} a_i = \sum_{i=0}^{k} \binom{k}{i} a_{i+n}.$$

To this end, we assume that

$$a_n = \int x^n d\mu_a,$$

and observe that

$$\begin{aligned}\sum_{i=0}^{k}\binom{k}{i}a_{i+n} &= \sum_{i=0}^{k}\binom{k}{i}\int x^{i+n}d\mu_a \\ &= \int\sum_{i=0}^{k}\binom{k}{i}x^{i+n}\,d\mu_a \\ &= \int\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{j+k}\binom{j+k}{i}x^i\,d\mu_a \\ &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{j+k}\binom{j+k}{i}a_i.\end{aligned}$$

The result now follows from Eq. (4). □

Corollary 10.

$$\mathbf{E}_a = \left(\int x^k(1+x)^n d\mu_a\right). \tag{9}$$

We are interested in characterising the main diagonal of the Euler-Seidel matrix of a_n, which by the above is the same as the main diagonal of $\mathbf{B}^{-1}\mathbf{H}_b$, where $\mathbf{H}$ is the Hankel matrix of b_n, the binomial transform of a_n.

Note that the diagonal is given by

$$a_{n,n} = \sum_{i=0}^{n}\binom{n}{i}a_{n+i}.$$

Example 11. We have seen that the diagonal of the Euler-Seidel matrix for the Catalan numbers C_n begins

$$1, 3, 26, 305, 4120, 60398, 934064, \ldots$$

By the above, the general term of this sequence is

$$d_n = \sum_{i=0}^{n}\binom{n}{i}C_{n+i}.$$

Now consider the moment representation of the Catalan numbers given by

$$C_n = \int x^n d\mu = \frac{1}{2\pi}\int_0^4 x^n\frac{\sqrt{x(4-x)}}{x}dx.$$

We claim that

$$d_n = \int (x(1+x))^n d\mu = \frac{1}{2\pi}\int_0^4 (x(1+x))^n\frac{\sqrt{x(4-x)}}{x}dx.$$

This follows from the result above, or directly, since

$$\begin{aligned}\int (x(1+x))^n d\mu &= = \int (x+x^2)^n d\mu \\ &= \int \sum_{i=0}^n \binom{n}{i} x^{2i} x^{n-i} d\mu \\ &= \sum_{i=0}^n \binom{n}{i} \int x^{n+i} d\mu \\ &= \sum_{i=0}^n \binom{n}{i} C_{n+i}.\end{aligned}$$

Note that by the change of variable $y = x(1+x)$ we obtain in this case the alternative moment representation for d_n given by

$$d_n = \frac{1}{2\pi} \int_0^{20} y^n \frac{\sqrt{2}(1+\sqrt{1+4y})\sqrt{5\sqrt{1+4y}-2y-5}}{4y\sqrt{1+4y}}\, dy.$$

The above method of proof is easily generalised. Thus we have

Proposition 12. *Let a_n be a sequence which can be represented as the sequence of moments of a measure:*

$$a_n = \int x^n d\mu_a.$$

Then the elements d_n of the main diagonal of the Euler-Seidel matrix have moment representation given by

$$d_n = \int (x(1+x))^n d\mu_a.$$

3 Examples: the Fibonacci and Jacobsthal cases

Example 13. We first of all look at the Fibonacci numbers, $F_{n+1} = F(n+1)$ A000045. It is well known that the binomial transform of $F(n+1)$ is $F(2n+1)$ A122367:

$$F(2n+1) = \sum_{k=0}^n \binom{n}{k} F(k+1).$$

Letting $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1}{\phi} = \frac{1-\sqrt{5}}{2}$, we have

$$\begin{aligned}F_{n+1} &= \frac{1}{\sqrt{5}}(\phi^{n+1} - \bar{\phi}^{n+1}) \\ &= \frac{1}{\sqrt{5}}(\phi\phi^n - \bar{\phi}\bar{\phi}^n) \\ &= \frac{1}{\sqrt{5}} \int_{\mathbf{R}} x^n (\phi\delta_\phi - \bar{\phi}\delta_{\bar{\phi}})\, dx.\end{aligned}$$

Then the general element of the Euler-Seidel matrix for F_{n+1} is given by

$$\begin{aligned}\frac{1}{\sqrt{5}}\int_{\mathbf{R}} x^k(1+x)^n(\phi\delta_\phi - \bar{\phi}\delta_{\bar{\phi}})\,dx &= \frac{\phi\cdot\phi^k(1+\phi)^n - \bar{\phi}\cdot\bar{\phi}^k(1+\bar{\phi})^n}{\sqrt{5}}\\ &= \frac{\phi^{k+1}(1+\phi)^n - \bar{\phi}^{k+1}(1+\bar{\phi})^n}{\sqrt{5}},\end{aligned}$$

where δ_a represents the Dirac delta "function" at a:

$$\int_{\mathbf{R}} f(x)\delta_a dx = <\delta_a, f> = f(a).$$

This gives us

$$\mathbf{E}_{F_{n+1}} = \begin{pmatrix} 1 & 1 & 2 & 3 & 5 & 8 & \dots \\ 2 & 3 & 5 & 8 & 13 & 21 & \dots \\ 5 & 8 & 13 & 21 & 34 & 55 & \dots \\ 13 & 21 & 34 & 55 & 89 & 144 & \dots \\ 34 & 55 & 89 & 144 & 233 & 377 & \dots \\ 89 & 144 & 233 & 377 & 610 & 987 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The diagonal of this matrix begins

$$1, 3, 13, 55, 233, \dots$$

This is F_{3n+1} A033887, which by the above can be written

$$F_{3n+1} = \frac{\phi^{n+1}(1+\phi)^n - \bar{\phi}^{n+1}(1+\bar{\phi})^n}{\sqrt{5}}.$$

Example 14. Our next example is based on the Jacobsthal numbers J_n A001045, where

$$J_n = \frac{2^n}{3} - \frac{(-1)^n}{3}.$$

We have

$$J_{n+1} = \frac{2\cdot 2^n}{3} + \frac{(-1)^n}{3} = \frac{1}{3}\int_{\mathbf{R}} x^n\,(2\delta_2 + \delta_{-1})\,dx.$$

Then the Euler-Seidel matrix for J_{n+1} has general element given by

$$\begin{aligned}\int x^k(1+x)^n\,(2\delta_2 + \delta_{-1})\,dx &= \frac{2\cdot 2^k(1+2)^n + (-1)^k(1+(-1))^n}{3}\\ &= \frac{2^{k+1}3^n + (-1)^k\,0^n}{3}.\end{aligned}$$

We have

$$\mathbf{E}_{J_{n+1}} = \begin{pmatrix} 1 & 1 & 3 & 5 & 11 & 21 & \dots \\ 2 & 4 & 8 & 16 & 32 & 64 & \dots \\ 6 & 12 & 24 & 48 & 96 & 192 & \dots \\ 18 & 36 & 72 & 144 & 288 & 576 & \dots \\ 54 & 108 & 216 & 432 & 864 & 1728 & \dots \\ 162 & 324 & 648 & 1296 & 2592 & 5184 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The diagonal sums, which begin

$$1, 4, 24, 144, 864, \ldots$$

are A067411, with general term

$$\frac{2^{n+1}\, 3^n + (-1)^n\, 0^n}{3} = \frac{2 \cdot 6^n + 0^n}{3}.$$

4 Hankel matrices, Riordan arrays and orthogonal polynomials

From the last section, we have

$$\mathbf{E}_a = \mathbf{B}\mathbf{H}_a$$

and

$$\mathbf{E}_a^t = \mathbf{B}^{-1}\mathbf{H}_b$$

where b_n is the binomial transform of a_n. The second equation shows us that

$$\begin{aligned}\mathbf{E}_a &= (\mathbf{B}^{-1}\mathbf{H}_b)^t \\ &= \mathbf{H}_b^t(\mathbf{B}^{-1})^t \\ &= \mathbf{H}_b(\mathbf{B}^t)^{-1},\end{aligned}$$

since $\mathbf{H}_b$ is symmetric. Thus we obtain

$$\mathbf{B}\mathbf{H}_a = \mathbf{H}_b(\mathbf{B}^t)^{-1},$$

which implies that

$$\mathbf{H}_b = \mathbf{B}\mathbf{H}_a\mathbf{B}^t. \tag{10}$$

Since $\det(\mathbf{B}) = 1$, we deduce the well-known result that the Hankel transform of b_n is equal to that of a_n. We can also use this result to relate the LDU decomposition of $\mathbf{H}_b$ [11, 17] to that of $\mathbf{H}_a$. Thus we have

$$\begin{aligned}\mathbf{H}_b &= \mathbf{B}\mathbf{H}_a\mathbf{B}^t \\ &= \mathbf{B} \cdot \mathbf{L}_a\mathbf{D}_a\mathbf{L}_a^t \cdot \mathbf{B}^t \\ &= (\mathbf{B}\mathbf{L}_a)\mathbf{D}_a(\mathbf{B}\mathbf{L}_a)^t.\end{aligned}$$

One consequence of this is that the coefficient triangle of the polynomials orthogonal with respect to $d\mu_b$ is given by

$$\mathbf{L}_a^{-1}\mathbf{B}^{-1},$$

where $\mathbf{L}_a^{-1}$ is the coefficient array of the polynomials orthogonal with respect to $d\mu_a$.

Example 15. We take the example of the Catalan numbers $a_n = C_n$ and their binomial transform $b_n = \sum_{k=0}^{n} C_k$. It is well known that the Hankel transform of C_n is the all 1's sequence, which implies that $\mathbf{D}_a$ is the identity matrix. Thus in this case,

$$\mathbf{H}_a = \mathbf{L_a}{\mathbf{L_a}}^t$$

where

$$\mathbf{L}_a = \mathbf{L}_{C_n} = (c(x), xc(x)^2) \qquad \text{(A039599)}$$

with

$$\mathbf{L}_a^{-1} = \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right) \qquad \text{(A129818)}$$

with general term $(-1)^{n-k}\binom{n+k}{2k}$, where we have used the notation of Riordan arrays. The polynomials

$$P_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} x^k$$

are thus a family of polynomials orthogonal on $[0,4]$ with respect to the density function $\frac{1}{2\pi}\frac{\sqrt{x(4-x)}}{x}$ [18]. It is known that the bi-variate generating function of the inverse of the n-th principal minor of $\mathbf{H}_a$ is given by the Christoffel-Darboux quotient

$$\frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{x-y}.$$

We deduce that the orthogonal polynomials defined by b_n have coefficient matrix

$$\begin{aligned} \mathbf{L}_b^{-1} &= \mathbf{L}_a^{-1}\mathbf{B}^{-1} \\ &= \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right) \cdot \left(\frac{1}{1+x}, \frac{x}{1+x}\right) \\ &= \left(\frac{1+x}{1+3x+x^2}, \frac{x}{1+3x+x^2}\right). \end{aligned}$$

It turns out that these polynomials $Q_n(x)$ are given simply by

$$Q_n(x) = P_n(x-1).$$

Thus $\mathbf{H}_b^{-1}$ has n-th principal minor generated by

$$\frac{Q_{n+1}(x)Q_n(y) - Q_{n+1}(y)Q_n(x)}{x-y}.$$

In similar manner, we can deduce that the Euler-Seidel matrix $\mathbf{E}_a = \mathbf{E}_{C_n}$ is such that the n-th principal minor of $\mathbf{E}_a^{-1}$ is generated by

$$\frac{P_{n+1}(x)P_n(y-1) - P_{n+1}(y-1)P_n(x)}{x-y} = \frac{P_{n+1}(x)Q_n(y) - Q_{n+1}(y)P_n(x)}{x-y}.$$

Example 16. For the aerated double factorials, we have

$$\mathbf{H}_a = \mathbf{L}_a \mathbf{D}_a \mathbf{L}_a^t$$

where

$$\mathbf{L}_a = [e^{\frac{x^2}{2}}, x], \quad \mathbf{D}_a = \text{diag}(n!).$$

The associated orthogonal polynomials (which are scaled Hermite polynomials) have coefficient matrix

$$\mathbf{L}_a^{-1} = [e^{-\frac{x^2}{2}}, x],$$

and we have

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (2k-1)!!(-1)^k x^{n-2k},$$

or equivalently,

$$P_n(x) = \sum_{k=0}^{n} Bessel^* \left(\frac{n+k}{2}, k \right) (-1)^{\frac{n-k}{2}} \frac{1+(-1)^{n-k}}{2} x^k,$$

where

$$Bessel^*(n,k) = \frac{(2n-k)!}{k!(n-k)!2^{n-k}} = \binom{n+k}{2k}(2k-1)!!,$$

(see [1]). With

$$Q_n(x) = P_n(x-1)$$

we again have that the Euler-Seidel matrix $\mathbf{E}_a$ is such the n-th principal minor of $\mathbf{E}_a^{-1}$ is generated by

$$\frac{P_{n+1}(x)P_n(y-1) - P_{n+1}(y-1)P_n(x)}{x-y} = \frac{P_{n+1}(x)Q_n(y) - Q_{n+1}(y)P_n(x)}{x-y}.$$

5 Appendix: The Stieltjes transform of a measure

The *Stieltjes transform* of a measure μ on $\mathbb{R}$ is a function G_μ defined on $\mathbb{C} \setminus \mathbb{R}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} \mu(t).$$

If f is a bounded continuous function on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} f(x)\mu(x) = -\lim_{y \to 0^+} \int_{\mathbb{R}} f(x) \Im G_\mu(x+iy) dx.$$

If μ has compact support, then G_μ is holomorphic at infinity and for large z,

$$G_\mu(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}},$$

where $a_n = \int_{\mathbb{R}} t^n \mu(t)$ are the moments of the measure. If $\mu(t) = d\psi(t) = \psi'(t)dt$ then

$$\psi(t) - \psi(t_0) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_{t_0}^{t} \Im G_\mu(x+iy)dx.$$

If now $g(x)$ is the generating function of a sequence a_n, with $g(x) = \sum_{n=0}^{\infty} a_n x^n$, then we can define

$$G(z) = \frac{1}{z} g\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}.$$

By this means, under the right circumstances we can retrieve the density function for the measure that defines the elements a_n as moments.

Example 17. We let $g(z) = \frac{1-\sqrt{1-4z}}{2z}$ be the g.f. of the Catalan numbers. Then

$$G(z) = \frac{1}{z} g\left(\frac{1}{z}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{x-4}{x}}\right).$$

Then

$$\Im G_\mu(x+iy) = -\frac{\sqrt{2}\sqrt{\sqrt{x^2+y^2}\sqrt{x^2-8x+y^2+16} - x^2 + 4x - y^2}}{4\sqrt{x^2+y^2}},$$

and so we obtain

$$\begin{aligned} \psi'(x) &= -\frac{1}{\pi} \lim_{y \to 0^+} \left\{ -\frac{\sqrt{2}\sqrt{\sqrt{x^2+y^2}\sqrt{x^2-8x+y^2+16} - x^2 + 4x - y^2}}{4\sqrt{x^2+y^2}} \right\} \\ &= \frac{1}{2\pi} \frac{\sqrt{x(4-x)}}{x}. \end{aligned}$$

6 Acknowledgements

The author would like to thank the anonymous referees for their careful reading and cogent suggestions which have hopefully led to clearer paper.

References

[1] P. Barry, On a family of generalized Pascal triangles defined by exponential Riordan arrays, *J. Integer Seq.* **10** (2007), Article 7.3.5.

[2] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[3] A. Dil and V. Kurt, Applications of Euler-Seidel matrices, *Adv. Stud. Contemp. Math. (Kyungshang)*, **18** (2009), 219–233.

[4] A. Dil and V. Kurt, Investigating exponential and geometric polynomials with Euler-Seidel algorithm, http://arxiv.org/abs/0908.2585.

[5] A. Dil, V. Kurt, and M. Cenkci, Algorithms for Bernoulli and allied polynomials, *J. Integer Seq.* **10** (2007), Article 07.5.4.

[6] D. Dumont, Matrices d'Euler-Seidel, *Sém. Lothar. Combin.*, B05c (1981). Available at http://www.emis.de/journals/SLC/opapers/s05dumont.html.

[7] L. Euler, De transformatione serierum, *Opera Omnia* ser. 1, Vol. 10, Teubner, 1913.

[8] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Clarendon Press, Oxford, 2003.

[9] J. W. Layman, The Hankel transform and some of its properties, *J. Integer Seq.*, **4**, (2001), Article 01.1.5.

[10] I. Mezö and A. Dil, Euler-Seidel method for certain combinatorial numbers and a new characterization of Fibonacci sequence, *Cent. Eur. J. Math.*, **7** (2009), 310–321.

[11] P. Peart and W.-J. Woan, Generating functions via Hankel and Stieltjes matrices, *J. Integer Seq.*, **3** (2000), Article 00.2.1.

[12] L. W. Shapiro, S. Getu, W-J. Woan and L. C. Woodson, The Riordan Group, *Discr. Appl. Math.* **34** (1991), 229–239.

[13] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://www.research.att.com/∼njas/sequences/, 2010.

[14] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, *Notices Amer. Math. Soc.*, **50** (2003), 912–915.

[15] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* **132** (1994), 267–290.

[16] G. Szegö, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc., 1975.

[17] W.-J. Woan, Hankel matrices and lattice paths, *J. Integer Seq.*, **4** (2001), Article 01.1.2.

[18] E. Weisstein, Morgan-Voyce Polynomials, MathWorld, http://mathworld.wolfram.com/Morgan-VoycePolynomials.html, 2009.

2010 *Mathematics Subject Classification*: Primary 11B83; Secondary 11B39, 11C20, 15B05, 15B36, 42C05.

Keywords: Euler-Seidel matrix, Hankel matrix, integer sequence, moment sequence, orthogonal polynomial, Riordan array.

(Concerned with sequences A000045, A000108, A000142, A000165, A000731, A001045, A001147, A007317, A007318, A033887, A039599, A067411, A122367, and A129818.)

Received April 27 2010; revised version received July 21 2010. Published in *Journal of Integer Sequences*, August 2 2010.

Return to Journal of Integer Sequences home page.