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THE TAYLOR TOWERS FOR RATIONAL ALGEBRAIC K-THEORY AND HOCHSCHILD HOMOLOGY

MIRIAM RUTH KANTOROVITZ and RANDY MCCARTHY

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Abstract

We compute the Taylor tower for Hochschild homology as a functor from augmented commutative simplicial $\mathbb{Q}$-algebras, to chain complexes over $\mathbb{Q}$. We use this computation to obtain the layers for the Taylor tower of rational algebraic K-theory. We also show that the Hodge decomposition for rational Hochschild homology is the decomposition of the Taylor tower of the augmentation ideal functor into its homogeneous layers when evaluated at a suspension.

1. Introduction

The theory of calculus for homotopy functors between pointed spaces, which was developed by T. Goodwillie in [**G1**], [**G2**], [**G3**], has proved to be a powerful tool in algebraic topology (e.g. [**AM**] [**CC**], [**DM**] and [**M**]). The general idea of calculus is to obtain an insight about hard theories (e.g. algebraic K-theory) by using relatively easy theories ("degree n" functors). For instance, the linear approximation of the algebraic K-theory of a ring R is the topological Hochschild homology, $TH(R)$, of R (cf. [**DM**]). Thus one can study the algebraic K-theory of R via the more computable theory of $TH(R)$. Calculus can also be used to determine when two theories are equivalent by comparing their derivatives. For example, one proof that relative algebraic K-theory is rationally equal to relative topological cyclic homology along nilpotent extensions proceeds in this manner [**M**].

Another use of calculus is to derive interesting homology theories from natural functors. An example of such homology theory is the André-Quillen homology, which was described by Quillen as the "correct" homology for commutative rings [**Q**]. When the base ring k is a commutative ring of characteristic zero, André-Quillen homology over k can be viewed as the derivative of the augmentation ideal functor I, from simplicial augmented commutative k-algebras, $k\backslash\mathrm{CommAlg}/k$, to simplicial

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chain complexes over k, after adding a basepoint. This was observed by Schwede [**S**] in the world of spectra.

A modification of Goodwillie's theory to an algebraic setting was constructed by Johnson and McCarthy [**JM2**] for functors from a pointed category to an abelian category. We use this construction to compute the n-th Taylor polynomial of I, P_nI, and the homogeneous degree n layers of the Taylor tower, D_nI. The Taylor tower for I, described in Theorems (4.5)–(4.7), is

- *The layers*: D_nI is the derived functor of I^n/I^{n+1}. In particular, the derivative, D_1I, is the derived functor of I/I^2, which recovers the main result in [**S**], and D_nI computes the higher André-Quillen homology as explained in section 7.
- *The degree n Taylor polynomial*: P_nI is the derived functor of I/I^{n+1}.

The aim of this paper is to compute the Taylor tower of the rational algebraic K-theory as a functor from simplicial augmented commutative k-algebras, $k\backslash\mathrm{CommAlg}/k$, where k is a commutative ring containing $\mathbb{Q}$, to simplicial chain complexes over $\mathbb{Q}$.

Besides simple curiosity, there were two basic reasons for doing this calculation. The first was to make an approximation to the general Taylor tower of algebraic K-theory as a functor of commutative rings. The resulting spectral sequence for approximating the relative algebraic K-theory is expected to be hard (since it will involve topolological André-Quillen homology) but it may be a good tool for studying the behavior of algebraic K-theory as it deforms through commutative (instead of arbitrary) rings. Another reason to consider the Taylor tower is computational. The rational Taylor tower tends to converge only for relative nilpotent ring extensions and for these values one can simply use cyclic homology (see [**G**]). However, if one wants to study "boundary values" where the Taylor tower may still give useful information without necessarily converging (or converging for special reasons) one should use the Taylor tower itself and not an equivalent theory in the radius of convergence which may or may not provide the same boundary behavior.

The idea behind our computation is as follows. There is a natural map from (rational) algebraic K-theory to negative cyclic homology, $K_{\mathbb{Q}} \to HN$, called the Chern character. By a theorem of Goodwillie (cf. Theorem 2.3), this map is an isomorphism in the relative nilpotent case. We observe that this equivalence implies that the Chern character induces an equivalence on the derivatives at every point, $D_1K \simeq D_1HN$. We then can deduce that negative homology and rational K-theory have the same Taylor towers.

Since HN is constructed from Hochschild homology, HH, by taking homotopy fixed points under the circle action on HH, we would like to reduce the problem to a computation of the Taylor tower for Hochschild homology. However, fixed points do not behave well with respect to the construction of the Taylor polynomials. Instead, we go through cyclic homology, HC, which is the homotopy orbits of HH under the circle action.

Finally we reduce the computation of the Taylor tower for Hochschild homology to understanding the Taylor tower of the augmentation ideal functor I from $k\backslash\mathrm{CommAlg}/k$ to simplicial k-modules.

In the last section of the paper we show that using this computation and work of M. Ronco [**R**], one can interpret the Hodge decomposition for the rational Hochschild homology of a commutative ring A over k, as giving the layers of the augmentation ideal functor I from augmented commutative A-algebras to A-modules evaluated on a suspension of A. Recently, building on ideas developed here, the second author and K. Baxter have shown that this fact is true more generally. That is, the Taylor tower for any functor to rational chain complexes decomposes into a product of its homogeneous layers (D_n's) when evaluated at a suspension. The Higher Hochschild homology, as defined by T. Pirashvili [**P**], for a commutative ring is, from our point of view, the augmentation ideal functor applied to the n-fold suspension of A. Hence one immediately not only recovers Pirashvili's Hodge decomposition for Higher Hochschild homology but also obtains a description of its pieces in terms of Quillen's derived functors of Λ^k's when the sphere is odd and Quillen's derived functors of S^k's when the sphere is even.

We now briefly describe the organization of the paper. In section 2 we introduce the functors in play, namely, the forgetful functor, Hochschild homology, negative and cyclic homology and the algebraic K-theory. We also discuss a few connectivity results. In section 3 we review and give references for definitions and results from Goodwillie calculus that will be used later in the paper. Section 4 is devoted to the Taylor tower of exponential functors and in particular, we compute the Taylor tower of the forgetful functor $U : k\backslash\mathrm{CommAlg}/k \to \mathrm{Simp}(k\text{-mod})$. In section 5 we show how to view Hochschild homology as an exponential functor from $k\backslash\mathrm{CommAlg}/k$ to $HH(k)$-modules and use that to compute its Taylor polynomials. We also compute the derivative and the layers of Hochschild homology. In section 6 we compute the layers of the Taylor tower for rational algebraic K-theory as a functor of augmented commutative simplicial rings. The main main result of the paper is described in:

Theorem 6.3. *$D_nK_{\mathbb{Q}}$ is the derived functor of*

$$HH(k; \langle \mathbb{Q}[T^{n-1}] \otimes (I/I^2)^{\otimes n}\rangle_{\Sigma_n})[1] \simeq HH(k) \otimes_k \langle D_n(I)(S^1 \otimes\ -)\rangle^{hS^1}.$$

Finally, in the last section, we give a calculus interpretation of the Hodge decomposition for the rational Hochschild homology.

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2. Preliminaries

Let k be a simplicial commutative ring containing $\mathbb{Q}$. We define $k\backslash\mathrm{CommAlg}/k$ to be the category of simplicial augmented commutative k-algebras. An object A in $k\backslash\mathrm{CommAlg}/k$ is of the form $k \oplus I(A)$, where $I(A)$ is the augmentation ideal of A. Let $\mathrm{Simp}(k\text{-mod})$ be the category of simplicial k-modules and

$$I\colon\ k\backslash\mathrm{CommAlg}/k \longrightarrow \mathrm{Simp}(k\text{-mod}) \tag{1}$$

the augmentation ideal functor. We write I^n for the functor which takes $A \in k\backslash\mathrm{CommAlg}/k$ to $I(A)^n \in \mathrm{Simp}(k\text{-mod})$

There is a sequence of adjoint functors (left arrows are the left adjoint functors):

$$(*)\qquad \begin{tikzcd} k\backslash\mathrm{CommAlg}/k \arrow[r, shift left, "U"] \arrow[rr, bend left, "U"] & \mathrm{Simp}(k\text{-mod}) \arrow[l, shift left, "S"] \arrow[r, shift left, "U"] & \mathrm{Simp}(\mathrm{Sets}) \arrow[l, shift left, "{k[\ ]}"] \arrow[ll, bend left, "F"] \end{tikzcd}$$

where Simp(Sets) is the category of simplicial sets, U is the relevant forgetful functor and S is the symmetric powers functor,

$$S(M) = \bigoplus_{n\geqslant 0} S^n(M),$$

with $S^0(M) = k$ and $S^n(M)$ is the k-module $(M^{\otimes_k n})_{\Sigma_n}$, the orbits of the n-fold tensor product under the symmetric group action.

The functor $k[\ ]$ is the free functor, taking a simplicial set X to the free k-module on X, $k[X] = \bigoplus_{x\in X} kx$, and $F(X) = S \circ k[X]$ is the polynomial ring on X.

Remark 2.1. For $A \in k\backslash\mathrm{CommAlg}/k$, there is a functorial free resolution $A \xleftarrow{\simeq} L$ in $k\backslash\mathrm{CommAlg}/k$ obtained from the adjoint pair (F, U) of diagram $(*)$, where L is free in each degree. That is, $L_n \cong S(M)$, where M is a free k-module, $M \cong \bigoplus_{x\in X} kx$, for some $X \in$ Sets.

Notation 2.2. For a functor G from $k\backslash\mathrm{CommAlg}/k$ to an abelian category, we write $\widetilde{G}$ for the reduced functor, $\widetilde{G}(A) := {G(A)}\big/{G(k)}$. Note that $\widetilde{U}$ is the augmentation ideal functor I.
In this paper, unless otherwise mentioned or it is clear from the context, all tensor products are over $\mathbb{Q}$.

We are interested in the following functors from $k\backslash\mathrm{CommAlg}/k$ to chain complexes of $\mathbb{Q}$-modules, $\mathrm{Ch}(\mathbb{Q})$. We refer the reader to [**L**] and [**G**] for a detailed description of these functors.

- The Hochschild homology, $HH\colon\ k\backslash\mathrm{CommAlg}/k \to \mathrm{Ch}(\mathbb{Q})$ which takes A in $k\backslash\mathrm{CommAlg}/k$ to the Hochschild complex

  $$A \leftarrow A^{\otimes_{\mathbb{Q}} 2} \leftarrow A^{\otimes_{\mathbb{Q}} 3} \leftarrow \ \ldots$$

  and the reduced Hochschild homology

  $$\widetilde{HH}(A) \cong I(A) \leftarrow I(A^{\otimes 2}) \leftarrow \ \ldots\ .$$

  For M a simplicial A-bimodule, set

  $$HH(A, M) = M \leftarrow M \otimes A \leftarrow M \otimes A^{\otimes_{\mathbb{Q}} 2} \leftarrow M \otimes A^{\otimes_{\mathbb{Q}} 3} \leftarrow \ \ldots$$

- The negative cyclic homology, HN, and the cyclic homology, HC, as functors from $k\backslash\mathrm{CommAlg}/k$ to $\mathrm{Ch}(\mathbb{Q})$. These functors are built out of HH using the circle action on the bar complex. With S^1 acting on the bar complex by the cyclic action in each dimension, HN is equivalent to the homotopy fixed points, HH^{hS^1}, and HC is equivalent to the homotopy orbits HH_{hS^1}.

- The rational algebraic K-theory, $K_{\mathbb{Q}}$, as a functor from $k\backslash\mathrm{CommAlg}/k$ to $\mathrm{Ch}(\mathbb{Q})$, as defined in [**G**].

We will use the following theorem of Goodwillie relating $K_{\mathbb{Q}}$ to negative and cyclic homology. Let $f\colon A \to B$ be a map of simplicial rings and let

$$K_*(f)_{\mathbb{Q}} \xrightarrow{\alpha_{\mathbb{Q}}(f)} HN_*(f \otimes \mathbb{Q}) \xleftarrow{\beta_{\mathbb{Q}}(f)} HC_{*-1}(f \otimes \mathbb{Q})$$

be the natural transformations between the relative theories as described in [**G**].

Theorem 2.3. ([**G**], I.3.3-4) *With notation as above, if* $\pi_0 A \xrightarrow{\pi_0 f} \pi_0 B$ *is a surjection with nilpotent kernel then* $\alpha_{\mathbb{Q}}(f)$ *and* $\beta_{\mathbb{Q}}(f)$ *are isomorphisms.*

2.1. Hochschild homology of square zero extensions

Let $A \in k\backslash\mathrm{CommAlg}/k$ and M a simplicial A-bimodule. Let $A \ltimes M$ be the new simplicial ring whose underlying simplicial group is $A \oplus M$ with multiplication $(a, m)(a', m') = (aa', am' + ma')$. We recall ([**G**], [**M**]) that $HH(A \ltimes M)$ can be broken up into cyclic pieces

$$HH(A \ltimes M) \cong T^0(A, M) \oplus T^1(A, M) \oplus T^2(A, M) \oplus \dots ,$$

where

$$T^0(A, M) = A \leftarrow A \otimes A \leftarrow A^{\otimes 3} \leftarrow \dots = HH(A)$$

$$T^1(A, M) = M \leftarrow \begin{pmatrix} M \otimes A \\ \oplus \\ A \otimes M \end{pmatrix} \leftarrow \begin{pmatrix} M \otimes A \otimes A \\ \oplus \\ A \otimes M \otimes A \\ \oplus \\ A \otimes A \otimes M \end{pmatrix} \leftarrow \dots .$$

That is, in dimension n,

$$T^1_{[n]}(A, M) = \bigoplus_{\tau \in C_{n+1}} \tau * (M \otimes A^{\otimes n})$$

where $C_{n+1} \cong \mathbb{Z}/(n+1)\mathbb{Z}$ and $\tau \in C_{n+1}$ acts on $(M \otimes A^{\otimes n})$ by cyclic permutation. More generally, $T^{\ell}_{[n]}(A, M)$ is isomorphic to a direct sum of copies of $M^{\otimes \ell} \otimes A^{\otimes n}$ with action of the cyclic group $C_{n+\ell}$.

Note that $HH(A, M)$ sits in T^1 as a direct summand:

$$\widetilde{HH}(A, M) \cong \bigoplus_{n \geqslant 1} e * (M \otimes A^{\otimes n}),$$

where e is the trivial element of C_{n+1}.

We will use the following geometric description of T^1. Let $\mathbb{Q}[S^1]$ be the chain complex obtained from the simplicial representation Δ^1/∂ of the circle. That is,

$$\mathbb{Q}[S^1] = \mathbb{Q} \leftleftarrows \mathbb{Q} \oplus \mathbb{Q} \mathrel{\substack{\leftarrow \\ \leftarrow \\ \leftarrow}} \mathbb{Q}^{\oplus 3} \dots$$

with homology:

$$H_i(\mathbb{Q}[S^1]) = \begin{cases} \mathbb{Q} & i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

As simplicial $\mathbb{Q}$-modules, $\mathbb{Q}[S^1]$ is equivalent to

$$\mathbb{Q} \leftleftarrows \mathbb{Q}[C_2] \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \mathbb{Q}[C_3] \dots \ .$$

Using this notation, we can write

$$T^1_{[n]}(A, M) \cong \mathbb{Q}[C_{n+1}] \otimes \left(M \otimes A^{\otimes n}\right) \cong \mathbb{Q}[S^1]_{[n]} \otimes HH(A, M)_{[n]}$$

and hence

$$T^1(A, M) \simeq \mathbb{Q}[S^1] \otimes HH(A, M) \ .$$

Definition 2.4. Let $\theta\colon \mathbb{Q}[S^1] \otimes HH(k, M) \to \widetilde{HH}(k \ltimes M)$ be the map induced by sending $T^1(k, M) \cong \mathbb{Q}[S^1] \otimes HH(k, M)$ to the $T^1(k, M)$ piece of $\widetilde{HH}(k \ltimes M) \cong \bigoplus_{\ell \geqslant 1} T^\ell(k, M)$.

2.2. Connectivity results

Let R be a (simplicial) commutative ring and M a simplicial R-module. Recall (e.g. [**We**]) that there is a chain complex, $C(M)$, associated to M with $C(M)_n = M_n$ and differential $\partial_n = \sum_{i=0}^n (-1)^i d_i$, where d_i is the appropriate face map of M.

The homotopy groups of M can be defined by

$$\pi_*(M) = H_*(C(M)).$$

Definition 2.5. A simplicial R-module M is n-connected if $\pi_i(M) = 0$ for $i = 0, \dots n$.
A morphism $f : M \to N$ of simplicial R-modules is n-connected if it induces an isomorphism in the first $(n-1)$-homotopy groups and a surjection in π_n.
Note that f is ∞-connected means that f is a quasi-isomorphism.

We use the Dold-Kan correspondence to go back and forth between the categories Simp(k-mod) and $\mathrm{Ch}_{\geqslant 0}(R)$ and often we will not make a distinction between the two.

Terminology 2.6. Let $F : \mathrm{Simp}(k\text{-mod}) \to \mathrm{Simp}(k\text{-mod})$ be a functor and M an n-connected simplicial R-module. We are interested in the connectivity of $F(M)$ as a function of n. We will use the phrase "$F(M)$ is about $\phi(n)$-connected" to mean that the connectivity of $F(M)$ is $\phi(n) + c$, where c is some constant.

A standard spectral sequence argument yields the following.

Lemma 2.7. *Let M be a simplicial free R-module which is m-connected. Let N be an n-connected simplicial R-module. Then $M \otimes_R N$ is $(m + n + 1)$-connected.*

Corollary 2.8. *If M is an m-connected simplicial free A-bimodule, then $T^\ell(A, M)$ is at least ℓm connected.*

Corollary 2.9. *If M is an m-connected simplicial free A-bimodule then the map*

$$\theta\colon\ \mathbb{Q}[S^1] \otimes HH(k, M) \to \widetilde{HH}(k \ltimes M)$$

is about $2m$ connected.

Theorem 2.10. ([**W**], Proposition 1.1) *If $f\colon A \to B$ is an n-connected map of simplicial rings, $n \geqslant 1$, then the map $K(A) \to K(B)$ is $(n+1)$-connected. In particular, K preserves equivalences.*

Theorem 2.11. ([**G**], I.3) *If $f\colon\ A \to B$ is an n-connected map of simplicial rings, both flat over $\mathbb{Z}$, then $HH_*(f)$, $HN_*(f)$ and $HC_*(f)$ vanish for $* \leqslant n$.*

Lemma 2.12. *Let M be an m-connected simplicial free R-module, where R contains $\mathbb{Q}$, then $S^n_R(M)$ is $(nm + n - 1)$-connected.*

Proof. Since R contains $\mathbb{Q}$, the quotient map

$$\otimes^n_R M \to \otimes^n_R M \Big/ \Sigma_n \cong S^n_R(M)$$

splits functorially via the norm map, $\frac{1}{n!} \sum\limits_{\sigma \in \Sigma_n} \sigma$. Hence, $S^n(M)$ is a direct summand of $\otimes^n_R M$ (as simplicial R-modules). Therefore, since $\otimes^n_R M$ is $(nm+n-1)$-connected, by Lemma 2.7, so is $S^n_R(M)$. □

Definition 2.13. For $A \in k\backslash\mathrm{CommAlg}/k$, we say that A is i–connected if the map $A \to k$ is i–connected as a map of simplicial abelian groups, that is, $I(A)$ is i–connected.

Lemma 2.14. *Let L be a simplicial algebra in $k\backslash\mathrm{CommAlg}/k$ which is free in each dimension.*
If L is i-connected then the map $I(L) \to I/I^2(L)$ is about $2i$-connected.

Proof. We first note that L is a direct summand of a free simplicial algebra of the form $S(M) \cong \bigoplus\limits_{n \geqslant 0} S^n(M)$, for some free k-module M as in 2.1. Hence, we can assume that L itself is free of this form.

We have

$$I(L) = I \circ S(M) \cong M \oplus \bigoplus_{n \geqslant 2} S^n(M) \qquad \text{and}$$

$$I^2(L) = I^2 \circ S(M) \cong \bigoplus_{n \geqslant 2} S^n(M).$$

Note that since L is i-connected, the connectivity of M must be at least i. Hence, the map $I(L) \to I/I^2(L) \cong M$ has fiber isomorphic to $\bigoplus\limits_{n \geqslant 2} S^n(M)$ which is at least $(2i+1)$-connected by Lemma 2.12. □

Corollary 2.15. *Let L be a simplicial free algebra in $k\backslash\mathrm{CommAlg}/k$. Let $F\colon\ k\backslash\mathrm{CommAlg}/k \to \mathrm{Ch}(\mathbb{Q})$ be the functor HH, HN, HC, or $K_{\mathbb{Q}}$. Then $F(L) \to F\big(L/I^2(L)\big)$ is about $2i$-connected where i is the connectivity of $I(L)$.*

Proof. $L \cong k \oplus I(L) \to L/I^2(L) \cong k \oplus I/I^2(L)$ is about $2i$-connected by Lemma 2.14. Now apply Theorems 2.10 and 2.11. □

We note that from the proof of Lemma 2.14 one also has the following

Lemma 2.16. *If M is a simplicial free module with connectivity m then $I \circ S(M) \to I/I^2 \circ S(M)$ is about $2m$-connected.*

3. Goodwillie calculus

Goodwillie calculus is calculus on functors. In this section we give a brief summary of definitions and results from Goodwillie calculus in an algebraic setting which are needed for this paper. We do not attempt to give an introduction to the subject. We refer the reader to [**G1**], [**G2**] and [**G3**] for background and motivation, and to [**JM1**], [**JM2**] and [**JM3**] for further details.

3.1. Overview

Let $\mathcal{C}$ be a pointed category, that is, $\mathcal{C}$ has an object $*$ which is both initial and final, and assume $\mathcal{C}$ has finite coproducts, $\coprod$, and enough projective objects. For us, $\mathcal{C}$ is $k\backslash\mathrm{CommAlg}/k$ with coproduct $\otimes_k$ and basepoint k. Let $F : \mathcal{C} \to \mathcal{A}b$ be a functor from $\mathcal{C}$ to an abelian category $\mathcal{A}b$.

Following Tom Goodwillie's work on calculus for homotopy functors from spaces to spaces, B. Johnson and R. McCarthy ([**JM1**] [**JM2**]) defined an algebraic version of calculus on the category of functors from $\mathcal{C}$ to $\mathcal{A}b$. A Taylor tower for F (at $*$) is defined as an inverse limit of functors:

$$\ldots \xrightarrow{q_{n+1}} P_nF \xrightarrow{q_n} P_{n-1}F \xrightarrow{q_{n-1}} \ldots \xrightarrow{q_2} P_1F \xrightarrow{q_1} P_0F = F(*)$$

where, for $n > 0$, P_nF is a functor from $\mathcal{C}$ to (bounded below) chain complexes over $\mathcal{A}b$, $Ch_{\geqslant 0}(\mathcal{A}b)$, and with natural transformations $F \xrightarrow{p_n} P_nF$ such that the following diagram commutes

$$\begin{tikzcd}
 & \vdots \arrow[d, "q_{n+1}"] \\
F \arrow[ur] \arrow[r, "p_n"] \arrow[dr, "p_{n-1}"'] \arrow[ddr] & P_nF \arrow[d, "q_n"] \\
 & P_{n-1}F \arrow[d] \\
 & \vdots
\end{tikzcd}$$

The functors P_nF are functors of *degree* n in the sense explained below and the pair (P_nF, p_n) is universal with respect to degree n approximation of F.

P_nF is called the degree n Taylor polynomial for F. The *layers* of the Taylor series are the fibers $D_nF = \mathrm{fiber}(P_nF \to P_{n-1}F)$. The first layer, D_1F, is called the *derivative* of F.

Remark 3.1. If the pointed category $\mathcal{C}$ in question is the sub-category of a model category then one often wants to work up to weak equivalence of the ambient model category. However, coproducts do not necessarily preserve weak equivalences and hence, even if ones functor preserves weak equivalences, its associated n-th cross effect functors may not. If one is working with cofibrant objects, though, the weak equivalences are closed under coproducts and hence one would prefer to consider the Taylor tower only on cofibrant objects so that it remains homotopy invariant. When one can functorially replace objects by weakly equivalent cofibrant ones it is customary to do so first before applying the Taylor tower construction outlined above which we will also do in this paper.

3.2. Cross effects and degree n functors

The definition of degree n functor uses the notion of cross effects. The motivation comes from classical cross effects which were defined to study the degree of an analytic real function.

For a function $f : \mathbb{R} \to \mathbb{R}$, the n-th cross effect is a function of n variables $cr_n f : \mathbb{R}^n \to \mathbb{R}$ defined inductively as follows.

$$\begin{aligned}
\mathrm{cr}_0 f &= f(0)\\
\mathrm{cr}_1 f(x) &= f(x) - f(0)\\
\mathrm{cr}_n f(x_1,\dots,x_n) &= \mathrm{cr}_{n-1} f(x_1+x_2, x_3,\dots,x_n)\\
&\qquad - \mathrm{cr}_{n-1} f(x_1,x_3,\dots,x_n) - \mathrm{cr}_{n-1} f(x_2,x_3,\dots,x_n).
\end{aligned}$$

An analytic function f is of degree n if $\mathrm{cr}_{n+1} f = 0$. In particular, f is linear if $\mathrm{cr}_2 f = 0$.

The notion of cross effect was extended to functors of additive categories by Eilenberg and Mac Lane [**EM**].

Definition 3.2. For a functor $F : \mathcal{C} \to \mathcal{A}b$, the n-th cross effect is a functor $\mathrm{cr}_n F : \mathcal{C}^{\times n} \to \mathcal{A}b$ defined via

$$\begin{gathered}
\mathrm{cr}_0 F = F(*)\\
\mathrm{cr}_1 F(X) \oplus F(*) \cong F(X), \quad \text{that is,} \quad \mathrm{cr}_1 F = \widetilde{F}\\
\mathrm{cr}_2 F(X,Y) \cong \frac{\widetilde{F}(X \coprod Y)}{\widetilde{F}(X) \oplus \widetilde{F}(Y)}\\
\mathrm{cr}_n F(X_1,\dots,X_n) \cong \frac{\mathrm{cr}_{n-1} F(X_1 \coprod X_2, \dots X_n)}{\mathrm{cr}_{n-1} F(X_1,X_3,\dots,X_n) \oplus \mathrm{cr}_{n-1} F(X_2,X_3,\dots,X_n)}.
\end{gathered}$$

Motivated by the relationship between degree and cross effects for real functions, the following definition was made.

Definition 3.3. A functor $F : \mathcal{C} \to \mathcal{A}b$ is (strictly) degree n if $\mathrm{cr}_{n+1} F \cong 0$. If a functor is degree n then it is also degree k for $k \geqslant n$.

Example 3.4. Let R-mod be the category of modules over a commutative ring R. The n-fold tensor product, $\bigotimes^n$, the n-th exterior product, $\bigwedge^n$ and the n-th symmetric power, S^n, are (strictly) degree n functors from R-mod to R-mod.

Definition 3.5. A functor $F : \mathcal{C} \to \mathrm{Ch}(\mathcal{A}b)$ is degree n if $\mathrm{cr}_{n+1}F$ is quasi-isomorphic to zero. We say that F is *linear* if $\mathrm{cr}_2 F \simeq 0$ and *additive* if F is linear and reduced (i.e. $F(*) = 0$). In particular, if F preserves coproducts up to quasi-isomorphism then F is additive.

3.3. Cotriples

The construction of P_nF relies on the use of a cotriple arising from the adjoint pair of (3). We review some basic facts about cotriples here and refer the reader to [**We**] for further details.

Definition 3.6. A cotriple (or comonad) $(\perp,\ \epsilon,\ \delta)$ in a category $\mathcal{A}$ is a functor $\perp: \mathcal{A} \to \mathcal{A}$ together with natural transformations $\epsilon : \perp \to \mathrm{id}_{\mathcal{A}}$ and $\delta : \perp \to \perp\perp$ such that the following diagrams commute:

$$\begin{tikzcd} \perp \arrow[r, "\delta"] \arrow[d, "\delta"'] & \perp(\perp) \arrow[d, "\delta\perp"] \\ \perp(\perp) \arrow[r, "\perp\delta"] & \perp(\perp\perp) = \perp\perp(\perp) \end{tikzcd} \qquad \begin{tikzcd} & \perp \arrow[dl, "="'] \arrow[d, "\delta"] \arrow[dr, "="] & \\ \perp & \perp(\perp) \arrow[l, "\perp\epsilon"'] \arrow[r, "\epsilon\perp"] & \perp . \end{tikzcd}$$

Cotriples often arise from adjoint pairs.

Example 3.7. Let (F, U) be a pair of adjoint functors and $\perp = FU$. Let ϵ be a counit and η be a unit for the adjoint pair. Let η_U be the natural transformation that for an object B is given by $\eta_{U(B)} : U(B) \to UF(U(B))$. Then $(\perp, \epsilon, F(\eta_U))$ is a cotriple.

Cotriples yield simplicial objects in the following manner.

Definition 3.8. Let $(\perp, \epsilon, \delta)$ be a cotriple in $\mathcal{A}$ and let A be an object in $\mathcal{A}$. Then $\perp^{*+1} A$ is the following simplicial object in $\mathcal{A}$:

$$[n] \mapsto \perp^{(n+1)} A = \overbrace{\perp \cdots \perp}^{n+1 \text{ times}} A$$

$$d_i = \perp^{(i)} \epsilon \perp^{(n-i)} : \perp^{(n+1)} A \to \perp^{(n)} A$$
$$s_i = \perp^{(i)} \delta \perp^{(n-i)} : \perp^{(n+1)} A \to \perp^{(n+2)} A.$$

Observe that $\perp^{*+1}$ is augmented over $\mathrm{id}_{\mathcal{A}}$ by ϵ. In particular, if we consider $(\mathrm{id}_{\mathcal{A}}, \mathrm{id}, \mathrm{id})$ as the trivial cotriple, then ϵ gives a natural simplicial map from $\perp^{*+1}$ to id^{*+1} where id^{*+1} is the trivial simplicial $\mathcal{A}$-object.

When $\mathcal{A}$ is an abelian category, the following chain complex is associated to $\perp^{*+1} A$.

Definition 3.9. Let $(\perp, \epsilon, \delta)$ be a cotriple on an abelian category $\mathcal{A}$ and let A be an object in $\mathcal{A}$. Then $C^{\perp}_*(A)$ is the chain complex with

$$C^{\perp}_*(A) = \begin{cases} A & \text{if } n = 0, \\ \perp^n A & \text{if } n > 0 \end{cases}$$

and $\partial_n : C^{\perp}_n(A) \to C^{\perp}_{n-1}(A)$ is defined by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

Note that the chain complex $C^{\perp}_*(A)$ is the mapping cone of the composition

$$C(\perp^{*+1} A) \xrightarrow{\epsilon} C(\mathrm{id}^{*+1} A) \xrightarrow{\simeq} N(\mathrm{id}^{*+1} A) = A, \tag{2}$$

where $C(\perp^{*+1} A)$ and $C(\mathrm{id}^{*+1} A)$ are the chain complexes associated to $\perp^{*+1} A$ and $\mathrm{id}^{*+1} A$, respectively, and $N(\mathrm{id}^{*+1} A)$ is the normalized chain complex associated to $\mathrm{id}^{*+1} A$.

3.4. Cotriple construction of universal degree n approximation

Let $Func_*(\mathcal{C}, \mathcal{A}b)$ be the category of reduced functors from $\mathcal{C}$ to $\mathcal{A}b$ with natural transformations as morphisms and let $Func_*(\mathcal{C}^{\times n+1}, \mathcal{A}b)$ be the category of functors of $n+1$ variables from $\mathcal{C}$ to $\mathcal{A}b$ that are reduced in each variable separately. Let

$$\Delta^* : Func_*(\mathcal{C}^{\times n+1}, \mathcal{A}b) \to Func_*(\mathcal{C}, \mathcal{A}b)$$

be the functor obtained by composing a functor with the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C}^{\times n+1}$. That is, for $G \in Func_*(\mathcal{C}^{\times n+1}, \mathcal{A}b)$,

$$(\Delta^* G)(X) = G(\overbrace{X, \ldots, X}^{n+1 \text{ times}}).$$

We have an adjoint pair

$$Func_*(\mathcal{C}^{\times n+1}, \mathcal{A}b) \underset{\Delta^*}{\overset{\mathrm{cr}_{n+1}}{\rightleftarrows}} Func_*(\mathcal{C}, \mathcal{A}b), \tag{3}$$

where cr_{n+1} is right adjoint to Δ^*.

Definition 3.10. Let $F : \mathcal{C} \to \mathcal{A}b$ be a reduced functor, where $\mathcal{C}$ is a pointed category with finite coproducts and with enough projectives. Let $\perp_{n+1} = \Delta^* \circ \mathrm{cr}_{n+1}$ be the cotriple on $Func_*(\mathcal{C}, \mathcal{A}b)$ obtained, as in Example 3.7 from the adjoint pair $(\Delta^*, \mathrm{cr}_{n+1})$ of (3). With the notation of Definition 3.9, the *n-th Taylor polynomial of F at $*$, $P_n F$,* is defined to be the derived functor of $C^{\perp_{n+1}}_*(F)$. We define the *layers* of the Taylor series to be $D_n F = \mathrm{fiber}(P_n F \to P_{n-1} F)$ (which is algebraically naturally quasi-isomorphic to a shift of the mapping cone). The first layer, $D_1 F = P_1 F$, is called the *derivative* of F.

For functors that are not reduced, we have:

Definition 3.11. Let $F : \mathcal{C} \to \mathcal{A}b$ be any functor and $\mathcal{C}$ is as in 3.10. Then, with notation as in (2),

$$P_nF = \text{Mapping Cone } [N(\perp_{n+1}^{*+1} \widetilde{F}) \to N(\mathrm{id}^{*+1}\widetilde{F}) = \widetilde{F} \hookrightarrow F$$

and

$$D_1F = P_1(\widetilde{F}).$$

For $\mathcal{C} = k\backslash\mathrm{CommAlg}/k$, F a reduced functor and $A \in k\backslash\mathrm{CommAlg}/k$, let $L_* \overset{\simeq}{\to} A$ be a simplicial free resolution of A as in Remark 2.1. Then we think of $P_nF(A)$ as the total complex of the bi-complex

$$\begin{tikzcd}
\vdots \arrow[d] & & & \\
\mathrm{cr}^{(2)}_{n+1}F(\Delta^{(2)}L_0) \arrow[d] & \cdots \arrow[l] \arrow[d] & & \\
\mathrm{cr}_{n+1}F(\Delta L_0) \arrow[d] & \mathrm{cr}_{n+1}F(\Delta L_1) \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] & \\
F(L_0) & F(L_1) \arrow[l] & F(L_2) \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{4}$$

Theorem 3.12. *Let $F : \mathcal{C} \to \mathcal{A}b$ be a functor as in Definition 3.11. Then*

1. *P_nF is degree n.*
2. *If F is degree n then $p_n : F \to P_nF$ is a quasi-isomorphism.*
3. *The pair (P_n, p_n) is universal up to natural quasi-isomorphism with respect to degree n functors with natural transformations from F.*

3.5. Results

We will also use the following form for the derivative.

Theorem 3.13. *([**JM1**]) For $F\colon\ \mathcal{C} \to \mathrm{Ch}(k)$,*

$$D_1F(A) \simeq \varinjlim_n \Omega^n \mathrm{cr}_1 F\,(\Sigma^n A)$$

where Σ is the suspension functor in C. When $\mathcal{C}$ is $k\backslash\mathrm{CommAlg}/k$, Σ is the bar construction. The loop functor $\Omega\colon\ \mathrm{Ch}(k) \to \mathrm{Ch}(k)$ is a left shift: $\Omega(X_*) = X_*[-1]$.

As in the classical calculus, the layers of the Taylor tower can be described in terms of the derivative. The classical formula for the n-th term of the Taylor series at zero , $\frac{f^{(n)}(0)}{n!}x^n$, translates into a similar formula in Goodwillie calculus as described in the following theorem. Notice that the $n!$ is being replaced by homotopy orbits under the action of the n-th symmetric group Σ_n, and x^n is replaced by the n-tuple $\Delta X = (X, \ldots, X)$.

Theorem 3.14. ([**JM2**], 3.10) *For $X \in k\backslash\mathrm{CommAlg}/k$, $D_nF(X)$ is naturally equivalent to the homotopy orbits $(D_1^{(n)}cr_nF)_{h\Sigma_n}(\Delta X)$ where $D_1^{(n)}$ indicates taking the derivative in each of the n-variables of cr_nF separately.*

The derivative determines the Taylor tower in the following sense. For each fixed object $X \in \mathcal{C}$, we let F_X be the new functor from the category of objects over and under X in $\mathcal{C}$ to $\mathrm{Ch}(\mathcal{A}b)$ defined by: $F_X(Y) = \ker(F(Y) \to F(X))$.

Theorem 3.15. ([**JM2**], 4.12) *Let $\eta\colon F \to G$ be a natural transformation of reduced functors from $\mathcal{C}$ to $\mathrm{Ch}(\mathcal{A}b)$. If $D_1\eta_X\colon D_1F_X \to D_1G_X$ is an equivalence for all $X \in \mathcal{C}$ then $P_nF \simeq P_nG$ for all n.*

Corollary 3.16. *Let F and G be two functors from simplicial commutative algebras to $\mathrm{Ch}(\mathcal{A}b)$ and $\eta : F \to G$ a natural transformation. Suppose that there exists some fixed N and c such that for each commutative simplicial ring k and m-connected X in $k\backslash\mathrm{CommAlg}/k$ (the map $X \to k$ is m-connected) with $m \geqslant N$, the map $\tilde{\eta}_k(X) : \tilde{F}_k(X) \to \tilde{G}_k(X)$ is at least $2m - c$ connected. Then $P_n\tilde{\eta}_k : P_n\tilde{F}_k \to P_n\tilde{G}_k$ is an equivalence for all n and k.*

Remark 3.17. We note that in corollary 3.16 we could have assumed that F and G were defined on (or restricted to) the subcategory of simplicial commutative algebras containing $\mathbb{Q}$. Also, if the functors F_k and G_k preserve quasi-isomorphisms between highly connected algebras augmented over k then it suffices to consider free k–algebras X which are highly connected.

To compute the Taylor polynomial of a composite of a functor with an additive functor, we use the following property. It follows from a “chain rule” for the derivative (cf. [**JM2**]).

Lemma 3.18. ([**JM2**], 6.6) *If $G\colon \mathcal{C}' \to \mathcal{C}$ is a reduced coproduct preserving functor and $F\colon \mathcal{C} \to \mathrm{Ch}(k)$ is a functor, then*

$$P_n(F \circ G) \cong (P_nF) \circ G.$$

4. Taylor tower of the forgetful functor

As a functor from $k\backslash\mathrm{CommAlg}/k$ to $\mathrm{Ch}(\mathbb{Q})$, Hochschild homology satisfies a functional equation which allows its Taylor tower to be completely determined. In this section we give the general solution for the Taylor tower of such functors, which are called *exponential* functors in [**JM2**]. In the next section we compute the result explicitly for Hochschild homology. Let $\mathcal{C}$ be a pointed category as in Section 3.

Definition 4.1. ([**JM2**]) A reduced functor $F\colon \mathcal{C} \to \mathrm{Ch}(k)$ is *exponential* if there is a natural isomorphism

$$\mathrm{cr}_2F(A, B) \cong F(A) \otimes_k F(B).$$

The motivation for defining exponential functors this way comes from normalized exponential functions $f(x) = e^{ax} - e^{a\cdot 0} = e^{ax} - 1$. Since for such f we have

$$\mathrm{cr}_2f(x, y) = f(x + y) - f(x) - f(y) = f(x) \cdot f(y).$$

Exponential functors have the following form.

Lemma 4.2. ([**JM2**], 6.4) *Every exponential functor $F\colon \mathcal{C} \to \mathrm{Ch}(k)$ is of the form $I \circ G$ where G is a reduced coproduct preserving functor from $\mathcal{C}$ to $k\backslash\mathrm{CommAlg}/k$.*

Thus, by Lemma 3.18, to compute the Taylor tower an exponential functors, it suffices to compute the Taylor tower for the augmentation ideal functor $I : k\backslash\mathrm{CommAlg}/k \to \mathrm{Simp}(k-\mathrm{mod})$. We begin by computing the cross effects of I.

Lemma 4.3. $cr_n I \simeq I^{\otimes_k n}$.

Proof. By induction on n. When $n = 1$, $cr_1 I = I$ since I is reduced. For $n = 2$:

$$cr_2 I(A_1, A_2) \cong \frac{I(A_1 \otimes_k A_2)}{I(A_1) \oplus I(A_2)}.$$

Note that since $A_i \cong k \oplus I(A_i)$, we have

$$I(A_1 \otimes_k A_2) \cong I(A_1) \oplus I(A_2) \oplus (I(A_1) \otimes_k I(A_2)). \tag{5}$$

Hence, $cr_2 I(A_1, A_2) \cong I(A_1) \otimes_k I(A_2)$.

Let $n > 2$, then

$$\begin{aligned} cr_n I(A_1, \ldots, A_n) &\cong \frac{cr_{n-1} I(A_1 \otimes_k A_2, A_3, \ldots, A_n)}{cr_{n-1} I(A_1, A_3, \ldots, A_n) \oplus cr_{n-1} I(A_2, A_3, \ldots, A_n)}, \text{ by definition} \\ &\cong \frac{I(A_1 \otimes_k A_2) \otimes_k I(A_3) \otimes_k \ldots \otimes_k I(A_n)}{(I(A_1) \otimes_k I(A_3) \otimes_k \ldots \otimes_k I(A_n)) \oplus (I(A_2) \otimes_k \ldots \otimes_k I(A_n))}, \text{ by induction} \\ &\cong I(A_1) \otimes_k I(A_2) \otimes_k \ldots \otimes_k I(A_n), \text{ by (5)} \end{aligned}$$

□

Remark 4.4. The augmentation ideal functor I preserves weak equivalences and so we would prefer to work with the associate Taylor tower which also preserves weak equivalences (see Remark 3.1) by first replacing our simplicial ring by a weakly equivalent simplicial free one. In this way one may not actually be obtaining the Taylor tower of the original composite functor $F = I \circ G$ but one related to it. However, for the examples we will be using below, our exponential functors F have the additional property of preserving weak equivalences and taking cofibrant objects to cofibrant objects and hence no loss of information (up to weak equivalence) will occur if we use the Taylor tower of I applied to an equivalent cofibrant object.

Theorem 4.5. *$P_1 I$ is the derived functor of I/I^2.*

Proof. By Lemma 4.3, $I/I^2 = \mathrm{Coker}(cr_2 I \xrightarrow{\mu} I)$ where μ is the multiplication map. Hence, by definition 3.10 of P_n, we have a natural transformation from $P_1 I$ to I/I^2. Let $A \in k\backslash\mathrm{CommAlg}/k$ and $A \xleftarrow{\simeq} L$ a simplicial free resolution of A as in Remark 2.1. Since I preserves quasi-isomorphisms, $P_1 I$ does also and hence it suffices to show that $P_1 I(L) \simeq I/I^2(L)$. Moreover, by a standard spectral sequence argument it suffices to show that $P_1 I(L_{[n]}) \simeq I/I^2(L_{[n]})$ for each simplicial dimension

$[n]$. Recall from Section 2 that $L_{[n]} = S(M)$ for some simplicial free k-module M. Hence, it is enough to show that $P_1(I \circ S) \simeq I/I^2 \circ S$. As S is a left adjoint, S preserves coproducts and hence, by Lemma 3.18, $P_1(I/I^2 \circ S) \simeq I/I^2 \circ S$. By Lemma 2.16 we have that if M is m-connected then $(I \circ S)(M) \to (I/I^2 \circ S)(M)$ is about $2m$-connected. Hence, by Theorem 3.13,

$$P_1(I \circ S)(M) \simeq P_1(I/I^2 \circ S)(M) \simeq I/I^2 \circ S(M)$$

□

Theorem 4.6. *The n-th layer of the Taylor tower for I, D_nI, is the derived functor of I^n/I^{n+1}.*

Proof. Let $A \in k\backslash\mathrm{CommAlg}/k$ and $A \xleftarrow{\simeq} L$ a simplicial free resolution of A as in Remark 2.1. Then

$$\begin{aligned} D_nI(A) &\simeq D_nI(L) \\ &\simeq (D_1^{(n)} cr_n I(L))_{h\Sigma_n} \simeq (D_1^{(n)} cr_n I(L))_{\Sigma_n}, \text{ since } \mathbb{Q} \subset k \\ &\simeq (D_1^{(n)} I^{\otimes_k n}(L))_{\Sigma_n}, \text{ by Lemma 4.3} \\ &\simeq (D_1 I(L)^{\otimes_k n})_{\Sigma_n}, \text{ by freeness of } L \\ &\simeq ((I/I^2(L))^{\otimes_k n})_{\Sigma_n}, \text{ by Theorem 4.5} \\ &\simeq S^n(I/I^2(L)) \simeq I^n/I^{n+1}(L), \text{ since } L \text{ is levelwise free.} \end{aligned}$$

□

Theorem 4.7. *P_nI is the derived functor of I/I^{n+1}.*

Proof. The proof follows by induction using Theorems 4.5 and 4.6 and the commuting diagram of exact sequences of functors:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & D_nI \arrow[r] \arrow[d] & P_nI \arrow[r] \arrow[d] & P_{n-1}I \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & I^n/I^{n+1} \arrow[r] & I/I^{n+1} \arrow[r] & I/I^n \arrow[r] & 0. \end{tikzcd}$$

□

5. Taylor tower for Hochschild homology

We compute the Taylor tower of Hochschild homology by first viewing it as an exponential functor. To see this, note that for $A \in k\backslash\mathrm{CommAlg}/k$, the Hochschild homology, $HH(A)$, is an augmented commutative $HH(k)$-algebra. Hence the reduced Hochschild homology factors as

$$k\backslash\mathrm{CommAlg}/k \xrightarrow{HH} HH(k)\backslash\mathrm{CommAlg}/HH(k) \xrightarrow{I} \mathrm{Simp}(HH(k)\text{-mod}).$$

Lemma 5.1. *The functor*

$$HH : k\backslash\mathrm{CommAlg}/k \to HH(k)\backslash\mathrm{CommAlg}/HH(k)$$

is an additive functor and hence (by Lemma 4.2) the composite $\widetilde{HH} = I \circ HH$: $k\backslash\mathrm{CommAlg}/k \to \mathrm{Simp}(HH(k)\text{-mod})$ *is an exponential functor.*

Proof. Let $A, B \in k\backslash\mathrm{CommAlg}/k$. To show that HH is additive, we want to show that the natural map

$$HH(A) \otimes_{HH(k)} HH(B) \to HH(A \otimes_k B)$$

is an isomorphism. For this, it suffices to observe that the map from

$$A^{\otimes n} \otimes_{k^{\otimes n}} B^{\otimes n} \to (A \otimes_k B)^{\otimes n}$$

$$(a_0 \otimes \cdots \otimes a_n) \otimes_{k^{\otimes n}} (b_0 \otimes \cdots \otimes b_n) \to (a_0 \otimes_k b_0) \otimes \cdots \otimes (a_n \otimes_k b_n)$$

is an isomorphism for all n. $\square$

By the general results of Section 4 and Theorem 4.7 we can now give the following results for the Taylor polynomials of Hochschild homology

Theorem 5.2. $P_n\widetilde{HH}$ *is the derived functor of* $I/I^{n+1}(HH(-))$*, where* $HH(-)$ *is considered as a simplicial augmented* $HH(k)$*-algebra. In addition,*

$$cr_n\widetilde{HH} \cong \widetilde{HH}^{\otimes_{HH(k)} n}$$

$$D_n\widetilde{HH} \simeq \langle (D_1\widetilde{HH})^{\otimes_{HH(k)} n} \rangle_{\Sigma_n}.$$

We now wish to also compute the layers of the Taylor tower more explicitly. To do so, we express the derivate of HH more concretely.

Theorem 5.3. $D_1\widetilde{HH}$ *is the derived functor of* $\mathbb{Q}[S^1] \otimes HH(k, I/I^2)$.

Proof. We first note that $\mathbb{Q}[S^1] \otimes HH(k, I/I^2)$ is already additive since it is the composite of the following additive functors.

$$I/I^2\colon\ k\backslash\mathrm{CommAlg}/k \to \mathrm{Simp}(k\text{-mod})$$

$$HH(k, \_)\colon\ \mathrm{Simp}(k\text{-mod}) \to \mathrm{Simp}(HH(k)\text{-mod})$$

$$\mathbb{Q}[S^1] \otimes \_\colon\ \mathrm{Simp}(HH(k)\text{-mod}) \to \mathrm{Simp}(HH(k)\text{-mod})$$

Let $A \in k\backslash\mathrm{CommAlg}/k$ and $A \leftarrow L$ a simplicial free resolution of A as in 2.1. Let i be the connectivity of $I(L)$. Then, by Lemma 2.14

$$L \cong k \oplus I(L) \to L/I^2(L) \cong k \oplus I/I^2(L)$$

is about $2i$-connected. Thus, by Theorem 2.11, $\widetilde{HH}(L) \to \widetilde{HH}(L/I^2(L))$ is also about $2i$-connected. Note that $L/I^2(L) \cong k \ltimes I/I^2(L)$ as simplicial rings and hence $\widetilde{HH}(L/I^2(L))$
$\cong \widetilde{HH}(k \ltimes I/I^2(L))$. Since $I/I^2(L)$ is at least i-connected (by Lemma 2.14), then by Corollary 2.9, the map

$$\theta\colon\ \mathbb{Q}[S^1] \otimes \widetilde{HH}(k, I/I^2(L)) \to \widetilde{HH}(k \ltimes I/I^2(L))$$

is at least $2i$-connected. Finally, by Corollary 3.16 we are done. $\square$

Theorem 5.4. *$D_n\widetilde{HH}$ is the derived functor of*

$$HH(k, (\mathbb{Q}[T^n] \otimes (I/I^2)^{\otimes n})_{\Sigma_n})$$

where $\mathbb{Q}[T^n] \cong \mathbb{Q}[S^1]^{\otimes n}$ is a simplicial $\mathbb{Q}$-module which computes the homology of the n-torus, T^n, with coefficients in $\mathbb{Q}$ and Σ_n acts on the diagonal via

$$\sigma(t_1 \otimes \ldots \otimes t_n \otimes x_1 \otimes x_2 \otimes \ldots \otimes x_n) = t_{\sigma(1)} \otimes \ldots \otimes t_{\sigma(n)} \otimes x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$$

Proof.

$$\begin{aligned} D_n\widetilde{HH} &\simeq (D_1^{(n)}\widetilde{HH}^{\otimes^n_{HH(k)}})_{\Sigma_n} \text{ by Theorem 5.2} \\ &\simeq \Big(\big(\mathbb{Q}[S^1] \otimes HH(k, I/I^2)\big)^{\otimes^n_{HH(k)}}\Big)_{\Sigma_n} \text{ by Theorem 5.3} \\ &\simeq \Big(\big(HH(k, \mathbb{Q}[S^1] \otimes I/I^2)\big)^{\otimes^n_{HH(k)}}\Big)_{\Sigma_n} \text{ by linearity} \\ &\simeq \Big(HH(k, \big(\mathbb{Q}[S^1] \otimes (I/I^2)\big)\big)^{\otimes n}\big)\Big)_{\Sigma_n} \text{ by the proof of 5.1} \\ &\simeq HH(k, \big(\mathbb{Q}[T^n] \otimes (I/I^2)^{\otimes n}\big)_{\Sigma_n}). \end{aligned}$$

□

6. The Taylor tower for $K_{\mathbb{Q}}$

Theorem 6.1. *The natural transformations $K_{\mathbb{Q}} \xrightarrow{\alpha_{\mathbb{Q}}} HN \xleftarrow{\beta_{\mathbb{Q}}} HC[1]$ produce isomorphisms of the Taylor towers for the associated reduced functors from $k\backslash\mathrm{CommAlg}/k$ to $\mathrm{Ch}(\mathbb{Q})$.*

Proof. We will do only the case for the Chern character $K_{\mathbb{Q}} \xrightarrow{\alpha_{\mathbb{Q}}} HN$ as the other case is done completely analogously.

Let $A \in k\backslash\mathrm{CommAlg}/k$ be i–connected, $i > 0$, and let $A \xleftarrow{\simeq} L$ be a simplicial free resolution of A as in 2.1 (so i is also the connectivity of $I(L)$). We have

$$\begin{array}{ccc} \widetilde{K}_{\mathbb{Q}}(A) & \xrightarrow{\alpha_{\mathbb{Q}}} & \widetilde{HN}(A) \\ \uparrow{\scriptstyle\simeq} & & \uparrow{\scriptstyle\simeq} \\ \widetilde{K}_{\mathbb{Q}}(L) & \xrightarrow{\alpha_{\mathbb{Q}}} & \widetilde{HN}(L) \\ \downarrow{\scriptstyle 2i-conn} & & \downarrow{\scriptstyle 2i-conn} \\ \widetilde{K}_{\mathbb{Q}}(L/I^2(L)) & \xrightarrow{\simeq} & \widetilde{HN}(L/I^2(L)), \end{array}$$

where the bottom map is an isomorphism by Theorem 2.3 and the two maps going down are about $2i$-connected by Corollary 2.15. Using Theorem 3.13 we get $D_1\widetilde{K}_{\mathbb{Q}} \simeq D_1\widetilde{HN}$. Hence, as k was an arbitrary commutative simplicial ring containing $\mathbb{Q}$, by Corollary 3.16, the Taylor towers of $\widetilde{K}_{\mathbb{Q}}$ and $\widetilde{HN}$ are equivalent by the Chern character. □

The following theorem gives an alternative proof for the rational case of the main theorem in [**DM**], which says that D_1K is the topological Hochschild homology.

Theorem 6.2. *$D_1 K_{\mathbb{Q}}$ is the derived functor of $HH(k, I/I^2)[1]$.*

Proof. Let $A \in k\backslash\mathrm{CommAlg}/k$ and let $A \xleftarrow{\simeq} L$ be a simplicial free resolution of A as in 2.1. Then

$$\begin{aligned} D_1K_{\mathbb{Q}}(A) &\simeq D_1K_{\mathbb{Q}}(L) \\ &\simeq D_1HC[1](L) \text{ by Theorem 6.1} \\ &\simeq D_1(HH_{S^1}[1])(L) \simeq (D_1HH(L))_{hS^1}[1] \text{ since colimits commute} \\ &\simeq (\mathbb{Q}[hS^1] \otimes HH(k, I/I^2))_{hS^1}(L)[1] \text{ by Theorem 5.3} \\ &\simeq \mathbb{Q}[S^1]_{S^1} \otimes HH(k, I/I^2)(L)[1] \\ &\simeq \mathbb{Q} \otimes HH(k, I/I^2)(L)[1] \simeq HH(k, I/I^2)(L)[1] \end{aligned}$$

□

Theorem 6.3. *$D_n K_{\mathbb{Q}}$ is the derived functor of*

$$HH(k; \langle \mathbb{Q}[T^{n-1}] \otimes (I/I^2)^{\otimes n} \rangle_{\Sigma_n})[1]$$

Proof. Let $A \in k\backslash\mathrm{CommAlg}/k$ and let $A \xleftarrow{\simeq} L$ be a simplicial free resolution of A as in 2.1. Then

$$\begin{aligned} D_nK_{\mathbb{Q}}(A) &\simeq D_nK_{\mathbb{Q}}(L) \\ &\simeq D_nHC[1](L) \text{ by Theorem 6.1} \\ &\simeq D_n(HH_{S^1}[1])(L) \simeq (D_nHH)_{S^1}[1](L) \text{ since colimits commute} \\ &\simeq \left(HH(k; \langle \mathbb{Q}[T^n] \otimes (I/I^2)^{\otimes n} \rangle_{\Sigma_n}\right)_{S^1} [1] \text{ by Theorem 5.4} \\ &\simeq HH\left(k; \left(\langle \mathbb{Q}[T^n] \otimes (I/I^2)^{\otimes n} \rangle_{\Sigma_n}\right)_{S^1}\right)[1] \\ &\simeq HH\left(k; \langle \mathbb{Q}[T^n]_{S^1} \otimes (I/I^2)^{\otimes n} \rangle_{\Sigma_n}\right)[1] \text{ by switching the order of the actions} \\ &\simeq HH\left(k; \langle \mathbb{Q}[T^{n-1}] \otimes (I/I^2)^{\otimes n} \rangle_{\Sigma_n}\right)[1]. \end{aligned}$$

□

7. André-Quillen homology, Hodge decomposition and Calculus

In order to compute the Taylor tower of Hochschild homology it was natural to first consider the Taylor tower of the forgetful functor from simplicial commutative augmented k-algebras. The derivative of this was seen to be the derived functor (in the sense of Quillen) of I/I^2 which is known to be closely related to André-Quillen homology. We first recall this relationship and then use it to show that the Hodge decomposition for rational Hochschild homology is also its Taylor tower.

In this section our setup is as follows. Let k be a commutative ring. For a commutative k-algebra A, let $k\backslash\mathcal{C}/A$ be the category of simplicial commutative k-algebras over A, and let $A \xleftarrow{\simeq} P_*$ be a simplicial cofibrant resolution of A in $k\backslash\mathcal{C}/A$.

We write $\mathrm{HH}(A/k)$ for the Hochschild complex

$$\mathrm{HH}(A/k) = A \leftarrow A^{\otimes_k 2} \leftarrow A^{\otimes_k 3} \leftarrow \ \ldots$$

7.1. André-Quillen homology and Calculus

Recall (e.g. [**Q**]), that the André-Quillen homology, $\mathrm{AQ}_*(A/k)$, is defined as the homology of the cotangent complex,

$$\mathbb{L}_{A/k} = A \otimes_{P_*} \Omega_{P_*/k},$$

where $\Omega_{P/k}$ is the module of Kähler differentials of P over k. Note that $A \otimes_k A$ is augmented over A via the multiplication map $\mu : A \otimes_k A \to A$. The augmentation ideal $I(A \otimes_k A)$ is isomorphic to $\mathrm{Ker}\mu$ and $\Omega_{A/k} \cong I/I^2(A \otimes_k A)$. When P is a free k-algebra, $P = k[\mathbf{X}]$, then by the fundamental properties of the module of Kähler differentials, $\Omega_{P/k} \otimes_P A \cong I/I^2(A \otimes_k P)$. Hence, for $A \xleftarrow{\simeq} P_*$, a simplicial cofibrant resolution of A, $\mathbb{L}_{A/k} \simeq I/I^2(A \otimes_k P_*)$ To understand the André-Quillen homology from calculus point of view, consider the following composition of functors.

$$\begin{tikzcd} k\backslash\mathcal{C}/A \arrow[r, "A\otimes_k -"] \arrow[dr] & A\backslash\mathcal{C}/A \arrow[d, "U"] \\ & \mathrm{Simp}(A\text{-mod}), \end{tikzcd} \tag{6}$$

where U is the forgetful functor. To be able to apply the calculus machine, we need a category with a basepoint. The functor $A \otimes_k - : k\backslash\mathcal{C}/A \to A\backslash\mathcal{C}/A$ can be viewed as a process of “adding a basepoint” to the category $k\backslash\mathcal{C}/A$. After applying the functor $A \otimes_k -$, we have A as the designated basepoint in the category $A\backslash\mathcal{C}/A$, and calculus can be applied to the augmentation ideal functor $I = \widetilde{U}$.

When k contains $\mathbb{Q}$, then by Theorem 4.6 and the discussion above,

$$\mathbb{L}_{A/k} \simeq D_1 I(A \otimes_k -)(P_*). \tag{7}$$

Equation (7) says that the André-Quillen homology, $\mathrm{AQ}(A/k)$, can be viewed as the derivative of the augmentation ideal functor, $I : A\backslash\mathcal{C}/A \to \mathrm{Simp}(A - mod)$, evaluated at A, after adding the basepoint A. In this case, P_* is the simplicial cofibrant replacement of A in the category $k\backslash\mathcal{C}/A$.

This observation, together with Theorem 3.13 recovers the main result in [**S**].

Remark 7.1. It follows from work of Basterra, McCarthy and Mandell [**BMa**], [**BMc**], that $D_1 I(A \otimes_k -)(P_*)$ computes the Topological André-Quillen homology of A over k, $TAQ(A/k)$, and it coincides with E_∞ homology of A. When k does not contain $\mathbb{Q}$, $TAQ(A/k) \neq AQ(A/k)$.

Remark 7.2. For the purpose of analogy for the Hochschild homology, we think of $A \otimes_k A$ as “tensoring A with the zero sphere” as we explain below. Think of the zero sphere S^0 as the coproduct of two points.

$$S^0 : \qquad \bullet \coprod \bullet.$$

We write $S^0 \otimes A$ for the assignment of A for each point and taking the coproduct $A \coprod A$ in the category of k-algebras. That is

$$S^0 \otimes A := A \coprod A = A \otimes_k A.$$

7.2. Hochschild homology revisited

Following the analogy in Remark 7.2, and using the simplicial representation of the circle Δ^1/∂, write $S^1 \otimes A$ for the suspension of A, which is the simplicial k-algebra

$$S^1 \otimes A = A \leftleftarrows A \otimes_k A \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} A^{\otimes_k 3} \ldots.$$

Observe that $S^1 \otimes A$ is augmented over A with augmentation ideal $I(S^1 \otimes A) = (S^1 \otimes A)_{*>0}$. The Hochschild homology complex of A over k, $HH(A/k)$ is the composite

$$k\text{-alg} \xrightarrow{S^1 \otimes A} A \backslash \mathcal{C}/A \xrightarrow{U} \mathrm{Simp}(A\text{-mod}) \simeq \mathrm{Ch}(A). \tag{8}$$

Hodge decomposition for Hochschild homology

Let k be a commutative ring containing $\mathbb{Q}$. The Hodge decomposition of the Hochschild homology is defined via the Eulerian idempotents by Gerstenhaber and Schack [**GS**]. The Eulerian idempotents, $e_n^{(i)}$, $i = 1, \ldots n$, are mutually orthogonal idempotents in the symmetric group algebra $\mathbb{Q}[\Sigma_n]$ with $\Sigma_{i=1}^n e_n^{(i)} = 1$. Their action on the Hochschild complex commutes with the boundary operation, so the Hochschild complex $HH(A/k)$ naturally split into a direct sum of sub-complexes,

$$HH(A/k) = \bigoplus_{i \geqslant 0} HH^{(i)}(A/k),$$

where $HH^{(i)}(A/k) = e_n^{(i)} HH(A/k) \quad \text{for } n \geqslant 1.$

This decomposition is called the Hodge (or the λ-) decomposition for Hochschild homology.

When A is flat over k, the pieces of Hodge decomposition compute the (higher) André-Quillen homology.

Theorem 7.3. *(cf. [**L**] or [**R**]) Let A be a flat k-algebra then*

$$HH^{(i)}(A/k) \simeq I^i/I^{i+1}(HH(A/k)).$$

The n-th homology group, $HH_n^{(i)}(A/k)$ is isomorphic to the (higher) André-Quillen homology group $\mathrm{AQ}_{n-i}^{(i)}(A/k)$, where $\mathrm{AQ}_q^{(i)}(A/k)$ is defined by

$$\mathrm{AQ}_q^{(i)}(A/k) = H_q(\Omega^i_{P_*/k} \otimes_{P_*} A)$$

and $\Omega^i_{P/k}$ is the module of differential i-forms, $\wedge^i \Omega_{P/k}$.

Together with Theorem 4.6 we obtain

Corollary 7.4. *When A is a flat k algebra then*

$$\widetilde{HH}(A/k) \simeq \bigoplus_{i>0} D_i I(S^1 \otimes A).$$

The meaning of this corollary is that when A is flat over k, the Taylor tower for I, evaluated at the suspension of A, splits into its homogeneous layers and this decomposition coincides with the Hodge decomposition for $\widetilde{HH}(A/k) \simeq I(S^1 \otimes A)$.

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Miriam Ruth Kantorovitz `ruth@math.berkeley.edu`

Dept. of Math,
University of California,
Berkeley,
CA 94720

Randy McCarthy `randy@math.uiuc.edu`

Dept. of Math,
University of Illinois,
Urbana,
IL 61801