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ON ONE FREDHOLM INTEGRAL EQUATION OF THIRD KIND

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Abstract. In the class of Hölder functions we give the necessary and sufficient condition for solvability of a Fredholm integral equation whose kernel has fixed singularity in the segment of variation of an independent variable. Finding a solution is reduced to solving a regular integral equation of second kind.

1. Introduction

The theory of linear integral equations of third kind

$$A(x)\varphi(x) = \int\limits_a^b K(x,y)\varphi(y)dy + f(x), \quad x \in [a,b] \tag{A}$$

(where $A(x)$ vanishes at some points of the segment) acquires more and more significance in applied problems of mathematical physics (theory of elasticity, transport theory, etc.) and investigations in this area are of great interest.

Immediately after the appearance of the classical theory of Fredholm integral equations of second kind Picard and Fubini initiated investigations of integral equations of the above-mentioned type.

In considering equation (A), Picard [1] supposed that $A(x)$, $K(x,y)$, and $f(x)$ are holomorphic functions with respect to the complex variables x and y in a domain containing the interval (a,b) and that $A(x)$ has simple zeros α_i $(i = 1,2,\dots,k)$ only. Ignoring the intervals $(\alpha_i - \varepsilon_i; \alpha_i + \eta_i)$, he applied Fredholm's theory to the remainder interval and proved that for $\varepsilon_i \to 0$, $\eta_i \to 0$ the limit of the solution of the resulting Fredholm equation is a solution of equation (A).

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In 1938 Friedrichs [2] performed, in the Hilbert space, spectral analysis of the operator corresponding to equation (A) under the assumption that $A(x) = x$.

In 1973 Bart and Warnock [3] investigated the problem of solvability of this equation in the class of generalized functions.

Works [4–6] generalize the above results.

In the present paper we consider integral equation (A) in the class of Hölder functions assuming that the real function $A(x)$ has simple zero in the segment of variation of an independent variable ($A(x_0) = 0$, $A'(x_0) \neq 0$, $x_0 \in (a, b)$). Along with this equation we consider, as an auxiliary one, the corresponding integral equation depending on the parameter, and using this equation, we construct a singular integral operator. The theorem on expansion of an arbitrary Hölder class function in eigen functions of the integral equation depending on both the parameter and the singular operator is proved. The application of the theorem is exemplified by solution of the original integral equation.

2. Expansion Theorem

The condition imposed on the function $A(x)$ enables us to consider instead of (A) the equation

$$x\varphi(x) = \int_{-1}^{+1} K(x, y)\varphi(y)dy + f(x), \quad x \in (-1, 1). \tag{X}$$

It is assumed that:

(a) $K(x, y)$ is the real function satisfying the Hölder condition;

(b) $f(x)$ is the real function satisfying the condition H^* [7].

By the solution of equation (X) we mean the real function of the class H^* which for every $x \in (-1, +1)$ satisfies equality (X).

Let us consider, along with equation (1), a homogeneous equation of the type

$$(x - \nu)\varphi_\nu(x) = \int_{-1}^{+1} K(x, y)\varphi_\nu(y)dy, \tag{1}$$

where ν is the complex parameter. By the solution of equation (1) is meant a complex function of the class H^*.

The equation under consideration is integral with a kernel depending analytically on the parameter ν in the plane cut along $[-1, +1]$. When $\nu \in [-1, +1]$, it reduces to an equation of third kind. Many papers (see, e.g., [8–11]) have been devoted to the investigation of such equations. Based on the results of these papers, one can directly state that characteristic numbers (values of the parameter ν for which equation (1) has a non-trivial solution) and fundamental functions (a non-trivial solution of equation (1)) possess, for $\nu \notin [-1, +1]$, the following properties:

(a) If ν_0 is the characteristic number of the kernel $K(x, y)$ of rank q, then ν_0 is likewise the characteristic number of the kernel $K(y, x)$ of the same rank.

(b) If ν_1 and ν_2 are two different characteristic numbers of the kernel $K(x, y)$, $\varphi_{\nu_1}(x)$ is the fundamental function of the kernel $K(x, y)$ corresponding to the characteristic number ν_1, and $\varphi^*_{\nu_1}(x)$ is the fundamental function of the kernel $K(y, x)$ corresponding to the characteristic number ν_2, then

$$\int_{-1}^{+1} \varphi_{\nu_1}(x)\varphi^*_{\nu_2}(x)dx = 0. \tag{2}$$

(c) The set of characteristic numbers of the kernel $K(x, y)$ is finite.

(d) If the kernel $K(x, y)$ is symmetric, then all its characteristic numbers are real.

We have

Theorem 1. *Let a homogeneous integral equation of second kind of the form*

$$M_0(t, x) = \int_{-1}^{+1} \frac{K(x, y) - K(x, t)}{y - t} M_0(t, y)dy, \quad x \in [-1, +1], \tag{3}$$

have only a trivial solution for some value of the parameter $t = t' \in [-1, +1]$. Then the number t' will not be the characteristic number of the kernels $K(x, y)$ and $K(y, x)$.

Proof. Suppose on the contrary that there exists a continuous function $\varphi^*_{t'}(x) \not\equiv 0$ such that the relation

$$(x - t')\varphi^*_{t'}(x) = \int_{-1}^{+1} K(y, x)\varphi^*_{t'}(y)dy$$

is valid. Then the equality

$$\int_{-1}^{+1} K(y, t')\varphi^*_{t'}(y)dy = 0$$

holds, and hence $\varphi^*_{t'}(x)$ is a non-trivial solution of a homogeneous integral equation of the form

$$\varphi^*_{t'}(x) = \int\limits_{-1}^{+1} \frac{K(y,x) - K(y,t')}{x - t'} \varphi^*_{t'}(y)dy,$$

for which equation (3) is the associated one having also a non-trivial solution when $t = t'$. But this contradicts the condition of the theorem. □

Consequently, if the kernel $K(x, y)$ is a function such that the homogeneous equation (3) has, for all values of the parameter $t \in [-1, +1]$, only a trivial solution, then characteristic numbers will not belong to the segment $[-1, +1]$.[1] Here we shall consider such a case, i.e., it will be assumed that the homogeneous integral equation (3) admits, for any value of the parameter $t \in [-1, +1]$, only a trivial solution. Obviously, then a nonhomogeneous integral equation of the type

$$M(t,x) = \int\limits_{-1}^{+1} \frac{K(x,y) - K(x,t)}{y - t} M(t,y)dy + K(x,t), \quad t, x \in [-1, +1], \tag{4}$$

will have a unique solution satisfying the Hölder condition with respect to t and x. Note that if

$$K(x,y) = \sum_n g_n P_n(x) P_n(y),$$

then equation (4) will have a solution as a function expressed by uniformly convergent series of the form

$$M(t,x) = \sum_n g_n P_n(x) h_n(t),$$

where $h_n(t)$ are defined from the recurrent relation

$$\begin{gathered} (n+1)h_{n+1}(t) + nh_{n-1}(t) = (2n+1)(t - g_n)h_n(t), \\ h_0(t) = 1 \quad (n = 0, 1, \dots). \end{gathered}$$

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[1] Such a property takes place, for example, in the case of a function admitting expansion into a uniformly convergent series of the type

$$K(x,y) = \sum_n g_n P_n(x) P_n(y),$$

where g_n are real numbers and $P_n(x)$ is the nth order Lagrange polynomial. Moreover, if α is an exponent, C is a Hölder constant of the function $K(x, y)$, and $\dfrac{\alpha}{2^\alpha C} > 1$, then equation (3) admits a trivial solution only.

In the class of functions H^* let us determine a singular integral operator

$$L(u)(x) = \Big(1 - \int\limits_{-1}^{+1} \frac{M(x,x')}{x'-x} dx'\Big) u(x) + \\ + \int\limits_{-1}^{+1} \frac{M(t,x)}{x-t} u(t) dt, \quad x \in (-1,+1). \tag{5}$$

Theorem 2. *The equality*

$$(x-\nu)L(u)(x) - \int\limits_{-1}^{+1} K(x,y)L(u)(y)dy = L\big((t-\nu)u(t)\big)(x) \tag{6}$$

holds.

Proof. By virtue of the fact that $M(t,x)$ satisfies equation (4), we have

$$\int\limits_{-1}^{+1} K(x,y)L(u)(y)dy = L\big((x-t)u(t)\big)(x).$$

This implies that equality (6) is valid. □

The operator L possessing the above property plays an important role in investigating the original nonhomogeneous equation. Using this operator and fundamental functions of the kernel $K(x,y)$, the solution of the original equation (X) can be expanded into a series. To prove this, we first have to establish some basic properties of the operator L.

It is obvious that the singular integral operators $L(u)$ and $L'(v)$, where

$$L'(v)(t) = \Bigg(1 - \int\limits_{-1}^{+1} \frac{M(t,x)}{x-t} dx\Bigg) v(t) + \int\limits_{-1}^{+1} \frac{M(t,x)}{x-t} v(x) dx, \quad t \in (-1,+1), \tag{7}$$

are the associated ones, which means that the equality

$$\int\limits_{-1}^{+1} v(x)L(u)(x)dx = \int\limits_{-1}^{+1} u(t)L'(v)(t)dt$$

holds.

Fundamental functions and the operators L and L' can be defined by means of the kernel $K(x,y)$. In the sequel the functions and operators defined by $K(y,x)$ will be provided with the sign *.

Theorem 3. *Fundamental functions of the kernel $K(y,x)$ are solutions of the homogeneous singular equation*

$$L'(v)(t) = 0.$$

Proof. We have

$$L'(\varphi^*_{\nu_k})(t) = \Bigg(1 - \int\limits_{-1}^{+1} \frac{M(t,x)}{x-t} dx\Bigg) \varphi^*_{\nu_k}(t) + \\ + \int\limits_{-1}^{+1} \frac{M(t,x) \int_{-1}^{+1} K(y,x) \varphi^*_{\nu_k}(y) dy}{(x-t)(x-\nu_k)} dx. \tag{8}$$

Using

$$\frac{1}{x-t} \frac{1}{x-\nu_k} = \Big(\frac{1}{x-t} - \frac{1}{x-\nu_k}\Big) \frac{1}{t-\nu_k},$$

expression (8) we write as

$$L'(\varphi^*_{\nu_k})(t) = \Bigg(1 - \int\limits_{-1}^{+1} \frac{M(t,x)}{x-t} dx\Bigg) \varphi^*_{\nu_k}(t) + \\ + \int\limits_{-1}^{+1} \frac{1}{t-\nu_k} \frac{1}{x-t} M(t,x) \int\limits_{-1}^{+1} K(y,x) \varphi^*_{\nu_k}(y) dy dx - \\ - \int\limits_{-1}^{+1} \frac{1}{t-\nu_k} \frac{1}{x-\nu_k} M(t,x) \int\limits_{-1}^{+1} K(y,x) \varphi^*_{\nu_k}(y) dy dx. \tag{9}$$

Taking into account the fact that $M(t,x)$ is the solution of equation (4) and performing transformations, we get

$$\int\limits_{-1}^{+1} \frac{1}{t-\nu_k} \frac{1}{x-\nu_k} M(t,x) \int\limits_{-1}^{+1} K(y,x) \varphi^*_{\nu_k}(y) dy dx = \\ = \int\limits_{-1}^{+1} \frac{1}{t-\nu_k} M(t,x) \varphi^*_{\nu_k}(y) dy dx = \\ = \int\limits_{-1}^{+1} \frac{1}{t-\nu_k} \varphi^*_{\nu_k}(x) \Bigg(\int\limits_{-1}^{+1} \frac{K(x,y) - K(x,t)}{y-t} M(t,y) dy + K(x,t) \Bigg) dx =$$

$$= \int_{-1}^{+1} \frac{1}{t-\nu_k} \frac{1}{y-t} M(t,y) \int_{-1}^{+1} K(x,y)\varphi^*_{\nu_k}(x)dydx =$$

$$- \int_{-1}^{+1} \frac{M(t,y)}{y-t} dy \varphi^*_{\nu_k}(t) + \varphi^*_{\nu_k}(t),$$

by virtue of which (9) implies $L'(\varphi^*_{\nu_k}) = 0$. □

Analogously, we obtain

$$L^{*\cdot}(\varphi_{\nu_k}) = 0, \quad k \in 1, 2, \dots, r. \tag{10}$$

Lemma 1. *The equalities*

$$L^{\cdot}(K(x,\cdot))(t) = M(t,x), \quad L^*(K(\cdot,x))(t) = M^*(t,x), \quad t, x \in [-1,+1],$$

hold.

Proof. Using relation (4), we have

$$L^{\cdot}(K(x,\cdot)(t) = \left(1 - \int_{-1}^{+1} \frac{M(t,y)}{y-t} dy\right) K(x,t) + \int_{-1}^{+1} \frac{M(t,y)}{y-t} K(x,y)dy =$$

$$= K(x,t) + \int_{-1}^{+1} \frac{K(x,y) - K(x,t)}{y-t} M(t,y)dy = M(t,x). \quad \square$$

The second formula is proved in a similar way.

Lemma 2. *The equality*

$$L^{*'}(M(t_0,\cdot) - K(\cdot,t_0))(t) + M^*(t,t_0) =$$
$$= L'(M^*(t,\cdot) - K(t,\cdot))(t_o) + M(t_0,t), \quad t, t_0 \in [-1,+1],$$

holds.

Proof. Using relation (4) and performing appropriate operations on the left-hand side of the above equality, we obtain

$$L^{*'}(M(t_0,\cdot) - K(\cdot,t_0))(t) + M^*(t,t_0) =$$

$$= \left(1 - \int_{-1}^{+1} \frac{M^*(t,x)}{x-t} dx\right) \int_{-1}^{+1} \frac{K(x,y) - K(x,t_0)}{y-t_0} M(t_0,y)dy +$$

$$+ \int_{-1}^{+1} \frac{M^*(t,x)}{x-t} \int_{-1}^{+1} \frac{K(x,y) - K(x,t_0)}{y-t_0} M(t_0,y)dydx + M^*(t,t_0) =$$

$$= \Bigl(1 - \int_{-1}^{+1} \frac{M(t_0,y)}{y-t_0}dy\Bigr) \int_{-1}^{+1} \frac{K(x,t_0)-K(t,t_0)}{x-t} M^*(t_0,y)dx +$$

$$+ \int_{-1}^{+1} \frac{M(t_0,y)}{y-t_0} \int_{-1}^{+1} \frac{K(x,y)-K(t,y)}{x-t} M^*(t,x)dxdy -$$

$$-\Bigl(1 - \int_{-1}^{+1} \frac{M^*(t,x)}{x-t}dx\Bigr)K(t,t_0) - \int_{-1}^{+1} \frac{M^*(t,x)}{x-t}K(x,t_0)dx +$$

$$+\Bigl(1 - \int_{-1}^{+1} \frac{M(t_0,y)}{y-t_0}dy\Bigr)K(t,t_0) + \int_{-1}^{+1} \frac{M(t_0,y)}{y-t_0}K(t,y)dy + M^*(t,t_0) =$$

$$= L'\bigl(M^*(t,\cdot) - K(t,\cdot)\bigr)(t_0) + M(t_0,t). \quad \square$$

From these lemmas there follows the equality

$$L^{*'}\bigl(M(t_0,\cdot)\bigr)(t) = L'\bigl(M^*(t,\cdot)\bigr)(t_0). \tag{11}$$

Theorem 4. *The singular integral operator* $L^{*'}$ *regularizes the operator* L, *and the equality*

$$L^{*'}\bigl(L(u)\bigr)(t) = N(t)u(t), \quad t \in (-1;+1), \tag{12}$$

holds, where

$$N(t) = \Bigl(1 - \int_{-1}^{+1} \frac{M(t,x)}{x-t}dx\Bigr)\Bigl(1 - \int_{-1}^{+1} \frac{M^*(t,x)}{x-t}dx\Bigr) + \pi^2 M^*(t,t)M(t,t).$$

Proof. Performing the operations indicated on the left-hand side of (12) and using Poincaré–Bertrand's transposition formula [7], we obtain

$$L^{*'}\bigl(L(u)\bigr)(t) = \Bigl(\Bigl(1 - \int_{-1}^{+1} \frac{M(t,x)}{x-t}dx\Bigr)\Bigl(1 - \int_{-1}^{+1} \frac{M^*(t,x)}{x-t}dx\Bigr) +$$

$$+\pi^2 M^*(t,t)M(t,t)\Bigr)u(t) +$$

$$+ \int_{-1}^{+1} \frac{u(t')}{t-t'}\Bigl(L^{*'}\bigl(M(t',\cdot)\bigr)(t) - L'\bigl(M^*(t,\cdot)\bigr)(t')\Bigr)dt'.$$

Taking into consideration equality (11), we obtain formula (12). $\square$

Lemma 3. *The inequuality $N(t) \neq 0$, $t \in [-1, +1]$, holds.*

Proof. Using equality (11), we have

$$N(t) = \Bigg(\Big(1 - \int_{-1}^{+1} \frac{M(t,x)}{x-t} dx\Big) + ixM(t,t)\Bigg) \times$$
$$\times \Bigg(\Big(1 - \int_{-1}^{+1} \frac{M^*(t,x)}{x-t} dx\Big) + i\pi M^*(t,t)\Bigg).$$

Let there exist some $t' \in [-1, +1]$ such that $N(t') = 0$. Then either

$$1 - \int_{-1}^{+1} \frac{M(t',x)}{x-t'} dx = 0 \quad \text{and} \quad M(t',t') = 0,$$

or

$$1 - \int_{-1}^{+1} \frac{M^*(t',x)}{x-t'} dx = 0 \quad \text{and} \quad M^*(t',t') = 0.$$

Therefore, owing to (4) the number $t' \in [1, +1]$ will be the fundamental value of the kernel $K(x, y)$. But this is not true. $\square$

The boundary properties of the integral operator

$$\Omega_\nu(F)(x) \equiv F(\nu, x) - \int_{-1}^{+1} \frac{K(x,y)}{y-\nu} F(\nu, y) dy, \quad x \in [-1, +1],$$

where ν is an arbitrary point on the plane, $F(\nu, x)$ is a piecewise holomorphic in ν function on the plane cut along the segment $[-1, +1]$ which satisfies the Hölder condition in x, are of great importance for further investigation of the operator L. By using the Sokhotskii–Plemelj formulas [7] for boundary values of the operator Ω_ν we get

$$\big(\Omega_t(F)(x)\big)^{\pm} \equiv F^{\pm}(t,x) - \int_{-1}^{+1} \frac{K(x,y)}{y-t} F^{\pm}(t,y) dy \mp$$
$$\mp ixK(x,t)F^{\pm}(t,t), \quad t \in (-1; +1). \tag{13}$$

Theorem 5. *Let $f_0 \in H^*$. For the singular integral equation*

$$L(u) = f_0 \tag{14}$$

to have a solution in the class H^*, it is necessary and sufficient that the function f_0 satisfy the conditions

$$\int_{-1}^{+1} f_0 \varphi^*_{\nu_k} dx = 0, \quad k = 1, 2, \dots, r. \tag{15}$$

If these conditions are fulfilled, then the solution is unique and expressed by the formula

$$u = L^{*'}(f_0). \tag{16}$$

Proof of the Necessity. Let us introduce into consideration the piecewise holomorphic function

$$\Phi(\nu, x) = \frac{1}{2\pi i} \int_{-1}^{+1} \frac{M(t,x)}{t-\nu} u(t) dt, \quad x \in [-1, +1],$$

where ν is an arbitrary point on the plane. This function possesses the following properties:

1^0. In the plane with a cut $[-1, +1]$ it is analytic with respect to the variable ν, while for the variable x it satisfies the Hölder condition.

2^0. As $\nu \to \infty$ it vanishes uniformly with respect to the variable x.

By virtue of (13) we have

$$\big(\Omega_t(\Phi)(x)\big)^+ - \big(\Omega_t(\Phi)(x)\big)^- = M(t,x)u(t) - \\ - \int_{-1}^{+1} K(x,y) \frac{M(t,y)u(t)}{y-t} dy + \int_{-1}^{+1} K(x,t) \frac{M(y,x)}{y-t} u(y) dy$$

which implies, according to (4), that

$$\big(\Omega_t(\Phi)(x)\big)^+ - \big(\Omega_t(\Phi)(x)\big)^- = K(x,t) L(u).$$

Owing to (14) the function $\Phi(\nu, x)$ will be a solution of the following boundary value problem:

$$\big(\Omega_t(\Phi)(x)\big)^+ - \big(\Omega_t(\Phi)(x)\big)^- = K(x,t) f_0(t). \quad t \in (-1, +1), \quad x \in [-1, +1].$$

Let

$$\Psi(\nu, x) = \Omega_\nu(\Phi)(x).$$

It is evident that the function $\Psi(\nu, x)$ vanishes at infinity with respect to the variable ν and it is analytic in a plane with a cut $[-1, +1]$, while with respect to the variable $x \in [-1, +1]$ it satisfies the Hölder condition; moreover,

$$\Psi^+(t, x) - \Psi^-(t, x) = K(x,t) f_0(t), \quad t \in (-1, +1), \quad x \in [-1, +1].$$

Consequently,

$$\Omega_\nu(\Phi)(x) = \frac{1}{2\pi t}\int_{-1}^{+1}\frac{K(x,t)}{t-\nu}f_0(t)dt, \quad x\in[-1,+1]. \tag{17}$$

For the integral equation (17) to define the analytic function $\Phi(\nu,x)$ in a plane with a cut $[-1,+1]$, it is necessary and sufficient that its right-hand side satisfy the conditions

$$\int_{-1}^{+1}\varphi^*_{\nu_k}(x)\int_{-1}^{+1}\frac{K(x,t)}{t-\nu_k}f_0(t)dt\,dx = 0, \quad k=1,2,\dots,r.$$

After simplification we get equality (15). The necessity is proved.

Proof of the Sufficiency. Let the function $f_0(x)\in H^*$ satisfy conditions (15). Then the solution of equation (17) is a piecewise holomorphic function vanishing at infinity. Using (13), from (17) we have

$$\Phi^+(t,x)-\Phi^-(t,x)+\int_{-1}^{+1}\frac{K(x,y)}{y-t}\big(\Phi^+(t,y)-\Phi^-(t,y)\big)dy+$$
$$+\int_{-1}^{+1}\frac{K(x,t)}{t'-t}\big(\Phi^+(t',x)-\Phi^-(t',x)\big)dt' = K(x,t)f_0(t), \tag{18}$$
$$t\in(-1,+1), \quad x\in[-1,+1].$$

Let

$$\widetilde{M}(t,x) = \Phi^+(t,x)-\Phi^-(t,x)-M(t,x)u(t),$$

where $u(t)$ is defined from the equality

$$\left(1-\int_{-1}^{+1}\frac{M(t,x)}{x-t}dx\right)u(t) = -\int_{-1}^{+1}\frac{\Phi^+(t,x)-\Phi^-(t,x)}{x-t}dx+f_0(t), \tag{19}$$
$$t\in(-1,+1).$$

(Note that the factor of $u(t)$ is different from zero. If for some $t=t_0$ this condition is violated, then equation (4) implies that the number $t_0\in(-1,+1)$ is the characteristic value of the kernel $K(x,y)$, which contradicts our assertion.) Multiplying both parts of equation (4) by $u(t)$ and subtracting from (18), we obtain

$$\widetilde{M}(t,x) = \int_{-1}^{+1}\frac{K(x,y)}{y-t}\widetilde{M}(t,y)dy, \quad t\in(-1,+1), \quad x\in[-1,+1],$$

and hence $\widetilde{M}(t,x)=0$. Thus

$$\Phi^+(t,x)-\Phi^-(t,x)=M(t,x)u(t).$$

Substituting the above-obtained equality into (19) and using again the fact that $M(t,x)$ is the solution of equation (4), we arrive at equality (14). The sufficiency is proved. □

Theorem 6. *Systems of fundamental functions* $\{\varphi_{\nu_k}\}$ *and* $\{\varphi^*_{\nu_k}\}$ *are biorthogonal.*

Proof. Owing to equality (3), it remains for us to prove that the numbers

$$N_{\nu_k}=\int\limits_{-1}^{+1}\varphi_{\nu_k}\varphi^*_{\nu_k}dx,\quad k=1,2,\dots,r,$$

are different from zero. Let us assume on the contrary that $N_{\nu_p}=0$ holds for some ν_p. Then it is obvious that φ_{ν_p} satisfyies the conditions of Theorem 5 and the integral equation

$$L(u)=\varphi_{\nu_p}$$

has a unique solution. Taking into consideration equality (10), we get $u=0$. Then $\varphi_{\nu_p}=0$, which is not true. □

From the latter two theorems there follows

Theorem 7. *Only the functions* $\varphi^*_{\nu_k}$, $k=1,2,\dots,r$, *and their linear combinations are solutions of the homogeneous singular equation*

$$L'(\nu)=0.$$

We have now come to the question which plays an important role in solving the initial equation, i.e., to the question of representing an arbitrary function of the class H^* by the characteristic functions of equation (2) and by the above-introduced operator L.

Theorem 8. *An arbitrary function* $f\in H^*$ *admits a representation of the form*

$$f(x)=\sum_1^r a_{\nu_k}\varphi_{\nu_k}(x)+L(u)(x),\quad x\in[-1,+1],\tag{20}$$

where

$$a_{\nu_k}=\frac{1}{N_{\nu_k}}\int\limits_{-1}^{+1}f(t)\varphi^*_{\nu_k}(t)dt,\ \ u(t)=\frac{1}{N(t)}L^{*'}(f)(t),$$

a_{ν_k} *and* u *being defined uniquely.*

Proof. Indeed, if such a representation is possible, then to find a_{ν_k} and u we act as follows: multiplying (20) by $\varphi^*_{\nu_k}$ and integrating both parts of the equality with respect to x, we get

$$\int_{-1}^{+1} f\varphi^*_{\nu_p} dx = \sum_1^r a_{\nu_k} \int_{-1}^{+1} \varphi_{\nu_k}\varphi^*_{\nu_p} dx + \int_{-1}^{+1} L(u)\varphi^*_{\nu_p} dx = a_{\nu_p} N_{\nu_p}.$$

To find the function u, we apply the operation $L^{*'}$ to both parts of equality (20). Then we have

$$L^{*'}(f) = \sum_1^r a_{\nu_k} L^{*'}(\varphi^*_{\nu_k}) + L^{*'}(L(u)) = Nu.$$

As for the validity of representation (20), it follows from Theorems 5 and 7. □

3. Solution of a Fredholm Equation of Third Kind

Now we shall, to the Hilbert–Schmidt approach from the theory of Fredholm integral equations of second kind and as an application (of Theorem 8), solve the equation

$$(x-\nu)\widetilde{\varphi}_\nu(x) = \int_{-1}^{+1} K(x,y)\widetilde{\varphi}_\nu(y)dy + f(x), \quad x \in (-1,+1). \tag{21}$$

Theorem 9. *If $f \in H^*$, $\nu \overline{\in} [-1,+1] \cup \{\nu_k\}$, then equation* (21) *has one and only one solution $\widetilde{\varphi}_\nu(x) \in H^*$ expressed by the formula*

$$\begin{aligned}\widetilde{\varphi}_\nu(x) = \sum_1^r \frac{1}{\nu_k - \nu}\frac{1}{N_{\nu_k}}\varphi_{\nu_k}(x)\int_{-1}^{+1} f\varphi^*_{\nu_k} dx' +\\ +L\Big(\frac{1}{t-\nu}\frac{1}{N(t)}L^{*'}(f)(t)\Big)(x).\end{aligned} \tag{22}$$

Proof. Let $\widetilde{\varphi}_\nu(x)$ be the solution of equation (21) satisfying the conditions H^*. By virtue of (20) we can write

$$\widetilde{\varphi}_\nu(x) = \sum_1^r a^{(\nu)}_{\nu_k}\varphi_{\nu_k}(x) + L\big(u^{(\nu)}(t)\big)(x).$$

Substituting this equation into (21) and using equality (6) and the fact that $\varphi_{\nu_k}(x)$ are fundamental functions, after appropriate transformations we obtain

$$f(x)=\sum_{1}^{r} a^{(\nu)}_{\nu_k}(\nu_k-\nu)\varphi_{\nu_k}(x)+L\big((\nu-t)u^{(\nu)}(t)\big)(x).$$

Using the method of finding $a^{(\nu)}_{\nu_k}$ and $u^{(\nu)}(t)$, we get

$$a^{(\nu)}_{\nu_k}=\frac{1}{\nu_k-\nu}\frac{1}{N_{\nu_k}}\int\limits_{-1}^{+1} f\varphi^*_{\nu_k}dx,\quad u^{(\nu)}(t)=\frac{1}{t-\nu}\frac{1}{N(t)}L^{*'}(f)(t).$$

Thus we have stated that if equation (21) has a solution $\widetilde{\varphi}_\nu(x)\in H^*$, then it is unique and can be expressed by formula (22).

Direct substitution shows that the function $\widetilde{\varphi}_\nu(x)$ defined by the formula (22) satisfies equation (21). □

Theorem 10. *If $\nu=\nu_p$ is the fundamental value of the kernel $K(x,y)$, then the solution of equation* (21) *exists only when the condition*

$$\int\limits_{-1}^{+1} f\varphi^*_{\nu_p}dt=0 \tag{23}$$

is fulfilled. Then equation (21) *has in the class H^* infinitely many solutions represented by the formula*

$$\widetilde{\varphi}_\nu(x)=c\varphi_{\nu_p}(x)+\sum_{k\neq p}\frac{1}{\nu_k-\nu}\frac{1}{N_{\nu_k}}\varphi_{\nu_k}(x)\int\limits_{-1}^{+1} f\varphi^*_{\nu_k}dt+$$
$$+L\Big(\frac{1}{t-\nu}\frac{1}{N(t)}L^{*\cdot}(f)(t)\Big)(x) \tag{24}$$

where c is an arbitrary constant number.

Proof. As in the previous case we get

$$a^{(\nu)}_{\nu_k}(\nu_k-\nu)=\frac{1}{N_{\nu_k}}\int\limits_{-1}^{+1} f\varphi^*_{\nu_k}dx',$$

from which it follows that equality (23) holds for $\nu=\nu_p$.

We have obtained the necessary condition which must be satisfied by the function $f(x)$ so that the nonhomogeneous equation (21) would have a solution. In that case $a^{(\nu)}_{\nu}$ becomes indefinite and the solution has form (24).

One can show that that the function $\widetilde{\varphi}_{\nu_p}(x)$ defined by formula (24) satisfies equality (21). □

Theorem 11. *For $\nu = t_0 \in (-1,+1)$ the solution of equation* (21) *exists only if the condition*

$$L^{*'}(f)(t_0) = 0 \tag{25}$$

is fulfilled. Then the unique solution $\widetilde{\varphi}_\nu(x) \in H^*$ can be written by formula (22).

Proof. As above, we get

$$(t_0 - t)u^{(t_0)}(t) = L^{*'}(f)(t),$$

whence for $t = t_0$ there follows equality (25). This in fact is the necessary condition which is satisfied by the function $f(x)$ in order that the nonhomogeneous equation (21) would hold.

We can show that the function $\widetilde{\varphi}_t(x) \in H^*$ defined by formula (22) satisfies equation (21) when condition (25) is fulfilled. □

As a particular case of Theorem 11 we obtain the theorem which answers the question we posed for equation (X).

Theorem 12. *Let $M^*(x)$ be the solution of a nonhomogeneous equation of the type*

$$M^*(x) = \int_{-1}^{+1} \frac{K(y,x) - K(0,x)}{y} M^*(y)dy + K(0,x), \quad x \in [-1,+1];$$

then the solution of equation (X) *exists in the class H^* if and only if the condition*

$$\int_{-1}^{+1} \frac{f(x) - f(0)}{x} M^*(x)dx + f(0) = 0$$

is fulfilled. Moreover, if the latter condition is fulfilled, then the solution is unique and can be expressed by the right-hand side of formula (22) *with* $\nu = 0$.

References

1. E. Picard, Les equations integrales de troisieme espace. *Ann. Sci. École Norm Sup.* **47**(1911).

2. K. O. Friedrichs, Über die Spektralzerlegung eines Integraloperators. *Math. Ann.* **115**(1938), No. 2, 249–300.

3. G. Bart and R. Warnock, Linear equations of the third kind. *SIAM J. Math. Annal.* **4**(1973), No. 4, 609–622.

4. L. D. Faddeyev, On Friedrichs model in the theory of perturbations of a continuous spectrum. (Russian) *Trudy Mat. Inst. Steklov.* **73**(1964), 292–313.

5. A. R. Khvoles, On Fredholm's integral equations of the third kind. (Russian) *Bull. Acad. Sci. Georgian SSR* (*Soobshch. Akad. Nauk Gruz. SSR*) **2**(1941), No. 5, 389–395.

6. L. G. Mikhailov, Integral equations with a homogeneous kernel of degree −1. (Russian) *Donish, Dushanbe*, 1966.

7. N. I. Muskhelishvili, Singular integral equations. (Translation from the Russian) *P. Noordhoff, Groningen*, 1953.

8. J. D. Tamarkin, On Fredholm's integral equations whose kernels are analytical in a parameter. *Ann. Math.* (2) **28**(1927), 127–152.

9. T. A. Germogenova, On completeness of the system of eigen functions of the characteristic equation in the transfer theory. (Russian) *Preprint, Inst. Appl. Math. USSR Acad. Sci.* No. 103, *Moscow*, 1976.

10. M. V. Maslennikov, The Milne problem with anisotropic scattering. (Russian) *Trudy Mat. Inst. Steklov.* **47**(1968), 3–132.

11. I. A. Feldman, On the discrete spectrum of the characteristic equation of the radiation transport theory. (Russian) *Mat. Issled.* **10**(1975), No. 1(35), 236–242.

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