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MODULAR PROPERTIES OF THETA-FUNCTIONS AND REPRESENTATION OF NUMBERS BY POSITIVE QUADRATIC FORMS

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Abstract. By means of the theory of modular forms the formulas for a number of representations of positive integers by two positive quaternary quadratic forms of steps 36 and 60 and by all positive diagonal quadratic forms with seven variables of step 8 are obtain.

Let $r(n;f)$ denote a number of representations of a positive integer n by a positive definite quadratic form f with a number of variables s. It is well known that, for the case $s>4$, $r(n;f)$ can be represented as

$$r(n;f)=\rho(n;f)+\nu(n;f),$$

where $\rho(n;f)$ is a "singular series" and $\nu(n;f)$ is a Fourier coefficient of cusp form. This can be expressed in terms of the theory of modular forms by stating that

$$\vartheta(\tau;f)=E(\tau;f)+X(\tau),$$

where $E(\tau;f)$ is the Eisenstein series and $X(\tau)$ is a cusp form.

In his work [1] Malyshev formulated the following problem: to define the Eisenstein series and to develop a full theory of singular series for arbitrary $s\geq 2$. For $s\geq 3$, its solution follows from Ramanathan's results [2].

In the present paper we work out a full solution of this problem. Moreover, convenient formulas are obtained for calculating values of the function $\rho(n;f)$.

Thus, if the genus of the quadratic form f contains one class, then according to Siegel's theorem ([2], [3], [4]), $\vartheta(\tau;f)=E(\tau;f)$ and in that case the problem for obtaining "exact" formulas for $r(n;f)$ is solved completely. If the genus contains more than one class, then it is necessary to find a cusp form $X(\tau)$.

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A large number of papers is devoted to finding such formulas. Cusp forms in these works are constructed in the form of linear combinations of products of simple theta-functions with characteristics or their derivatives (see, e.g., [5]), products of Jacobi theta-functions or their derivatives (see, e.g., [6]), and theta-functions with spherical polynomials (see, e.g., [7]). All these functions are certain particular cases of linear combinations of the so-called generalized theta-functions with characteristics defined below by the formula (1). In the present paper, using modular properties of these functions, we have obtained by the unique method the exact formulas for a number of representations of numbers by positive quadratic forms both with an even (forms of such a kind were considered earlier) and an odd number of variables. Moreover, it is shown that using [8], one can reduce cumbersome calculations for obtaining formulas for a number of representations of numbers by the quadratic forms considered earlier and to obtain new formulas.

§ 1. Let $f=\frac{1}{2}x'Ax$ be a positive definite qadratic form, let A be an integral matrix with even diagonal elements, and the vector column $x\in\mathbb{Z}^s$, $s\in N$, $s\geq 2$. We call $u\in\mathbb{Z}^s$ a special vector with respect to the form f if $Au\equiv 0 \pmod N$, where N is a step of the form f. Moreover, let $P_\nu=P_\nu(x)$ be a spherical function of the νth order (ν is a positive integer) corresponding to the form f (see [9], p. 454). Then the generalized theta-function with characteristics we define as follows:

$$\vartheta_{gh}(\tau;p_\nu,f)\sum_{x\equiv g \pmod N}(-1)^{\frac{h'A(x-g)}{N^2}}p_\nu(x)e^{\frac{\pi i\tau x'Ax}{N^2}};\tag{1}$$

here and below, g and h are the special vectors with respect to the form f.

In the sequel, use will be made of the following lemmas (see Lemmas 1 and 4 in [8]).

Lemma 1. *Let k be an arbitrary integral vector and l be a special vector with respect to the form f. Then the following equalities hold:*

$$\vartheta_{g+Nk,h}(\tau;p_\nu,f)=(-1)^{\frac{h'Ak}{N}}\vartheta_{gh}(\tau;p_\nu,f),$$
$$\vartheta_{g,h+2l}(\tau;p_\nu,f)=\vartheta_{gh}(\tau;p_\nu,f).$$

Lemma 2. *Let $F(\tau)$ be an entire modular form of the type $(-r,N,v(L))$ and let there exist an integer l for which $(v(L))^l=1$. Then the function $F(\tau)$ is identically equal to zero if in its expansion in powers of $Q=e^{2\pi i\tau}$ the coefficients c_n equal 0 for $n\leq(r/12)N\prod_{p|N}(1+p^{-1})$.*

In the main theorem below we formulate modular properties of linear combinations of functions (1).

Theorem 1. *Let* $f_1 = f_1(x) = \frac{1}{2}x'A_1x, \dots, f_j = f_j(x) = \frac{1}{2}x'A_jx$ *be positive definite quadratic forms with a number of variables* s, *let* $P_\nu^{(k)} = P_\nu^{(k)}(x)$ *be the coresponding spherical functions of order* ν, Δ_k *be the determinant of the matrix* A_k, N_k *the step of the form* f_k $(k = 1, \dots, j)$, Δ *the determinant of the matrix* A *of some positive definite quadratic form* $\frac{1}{2}x'Ax$ *with a number of variables* $s + 2\nu$ *and the step* N.

Next, let $g^{(k)}$ *and* $h^{(k)}$ *be special vectors with respect to* f_k; *moreover, given* $2 \nmid \frac{N}{N_k}$, *let* h_k *be the vector with even components* $(k = 1, \dots, j)$;

$$L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N);$$

$$\begin{aligned} v(L) &= \left(i^{\frac{1}{2}\eta(\gamma)(\operatorname{sgn}\delta - 1)}\right)^{s+2\nu} \left(i^{\left(\frac{|\delta|-1}{2}\right)^2}\right)^{s+2\nu} \left(\frac{2(\operatorname{sgn}\delta)\beta\Delta}{|\delta|}\right) \quad \textit{for}\ \ 2 \nmid s, \\ &= (\operatorname{sgn}\delta)^{\frac{s}{2}+\nu} \left(\frac{(-1)^{\frac{s}{2}+\nu}\Delta}{|\delta|}\right) \quad \textit{for}\ \ 2 \mid s; \end{aligned} \tag{2}$$

$$\begin{aligned} \eta(\gamma) &= 1 \quad \textit{for} \quad \gamma \geq 0, \\ &= -1 \quad \textit{for} \quad \gamma < 0; \end{aligned}$$

$\left(\frac{(-1)^{\frac{s}{2}+\nu}\Delta}{|\Delta|}\right)$ *is the Kronecker symbol,* $\left(\frac{2(\operatorname{sgn}\delta)\beta\Delta}{|\delta|}\right)$ *is the Jacobi symbol.*

Then the function

$$X(\tau) = \sum_{k=1}^{j} B_k \vartheta_{g^{(k)}h^{(k)}}(\tau; P_\nu^{(k)}, f_k) \tag{3}$$

for arbitrary complex numbers B_k *is an entire modular form of the type* $(-(\frac{s}{2}+\nu), N, v(L))$ *if and only if the conditions*

$$N_k | N, \quad N_k^2 | f_k(g^{(k)}), \quad 4N_k \Big| \frac{N}{N_k} f_k(h^{(k)})$$

are fulfilled, and for all α *and* δ *such that* $\alpha\delta \equiv 1 \pmod N$ *we have*

$$\sum_{k=1}^{j} B_k \vartheta_{\alpha g^{(k)}, h^{(k)}}(\tau; p_\nu^{(k)}, f_k)(\operatorname{sgn}\delta)^\nu \left(\frac{(-1)^{[\frac{1}{2}]}\Delta_k}{|\delta|}\right) =$$
$$= \left(\frac{(-1)^{[\frac{s+2\nu}{2}]}\Delta}{|\delta|}\right) \sum_{k=1}^{j} B_k \vartheta_{g^{(k)}h^{(k)}}(\tau; p_\nu^{(k)}, f_k).$$

This theorem has been proved in [8] for even h_k. It can easily be adjusted to the case $2|\frac{N}{N_k}h_k$. From this theorem we obtain the following two theorems which are analogues of Theorems 4 and 2 from [8].

Theorem 2. *If all the conditions of Theorem 1 are fulfilled and either $\nu > 0$, or $\nu = 0$ and all the $g^{(k)}$ vectors are nonzero, then the function (3) is a cusp form of the type $(-(\frac{1}{2}+\nu), N, v(L))$.*

Theorem 3. *Let f be an integral positive quadratic form with a number of variables s and let Δ be a determinant of the form f. Then the function $\vartheta(\tau; f)$ defined by the formula*

$$\vartheta(\tau; f) = 1 + \sum_{n=1}^{\infty} r(n; f) e^{2\pi i \tau f} \quad (\operatorname{Im}\tau > 0) \tag{4}$$

is the entire modular form of the type $(-\frac{s}{2}, N, \nu(L))$, where $\nu(L)$ are defined by the formulas (2) for $\nu = 0$.

From the results of [2], [3], [4] and [10] we obtain

Theorem 4. *Let f be a positive quadratic form with a number of variables s and let Δ be its determinant. Then the function $E(\tau, z; f)$, determined for $\operatorname{Re} z \geq 2 - \frac{s}{2}$ and $\operatorname{Im}\tau > 0$ by the formula*

$$E(\tau, z; f) = 1 + \frac{e^{\frac{\pi i s}{4}}}{2^{\frac{s}{2}} \Delta^{\frac{s}{2}}} \sum_{q=1}^{\infty} \sum_{\substack{H=-\infty \\ (H,q)=1}}^{\infty} \frac{S(fh, q)}{q^{\frac{s}{2}} (q\tau - H)^{\frac{s}{2}} |q\tau - H|^{z}},$$

where $S(fh, q)$ is the Gaussian sum, can be continued analytically into the neighborhood of the point $z = 0$. Further, having defined the Eisenstein series $E(\tau; f)$ by the formulas

$$\begin{aligned} E(\tau; f) &= \frac{1}{2} E(\tau, z; f)\big|_{z=0} \ \textit{for } s = 2 \\ &= E(\tau, z; f)\big|_{z=0} \ \textit{for } s > 2, \end{aligned}$$

we have

$$\begin{aligned} E(\tau; f) &= 1 + \frac{1}{2} \sum_{n=1}^{\infty} \rho(n; f) e^{2\pi i \tau n} \quad \textit{for} \quad s = 2, \\ &= 1 + \sum_{n=1}^{\infty} \rho(n; f) e^{2\pi i \tau n} \quad \textit{for} \quad s > 2; \end{aligned} \tag{5}$$

here $\rho(n; f)$ is a singular series which is calculated as follows:

(1) *If* $2 \mid s$, $v = \prod_{\substack{p|n \\ p\nmid 2\Delta}} p^w$, $\Delta = r^2\omega$ *(*ω *is a square-free number), then*

$$\rho(n;f) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})\Delta^{1/2}} n^{\frac{s}{2}-1}\chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{\substack{p|r \\ p>2}} \Big(1 - \Big(\frac{(-1)^{\frac{s}{2}}\omega}{p}\Big)p^{-\frac{s}{2}}\Big)^{-1} \times$$

$$\times \mathcal{L}^{-1}\Big(\frac{s}{2}; (-1)^{\frac{1}{2}}\omega\Big) \sum_{k|v} \Big(\frac{(-1)^{\frac{s}{2}}\Delta}{k}\Big)k^{1-\frac{s}{2}}.$$

(2) *If* $2 \nmid s$, $\Delta n = 2^{\alpha+\gamma}v_1v_2 = r^2\omega$, $2^\alpha \| n$, $2^\gamma \| \Delta$, $p^l \| \Delta$, $p^w \| n$ $(p > 2)$, $v_1 = \prod_{\substack{p|n \\ p\nmid 2\Delta}} p^w = r_1^2\omega_1$, $v_2 = \prod_{\substack{p|\Delta n \\ p|\Delta,\, p>2}} p^{w+l} = r_2^2\omega_2$ *(*$\omega, \omega_1, \omega_2$ *are square-free numbers), then*

$$\rho(n;f) = \frac{(s-1)!\, r_1^{2-s} n^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})2^{s-2}\pi^{\frac{s}{2}-1}|B_{s-1}|\Delta^{\frac{1}{2}}}\chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \times$$

$$\times \prod_{p|2\Delta}(1-p^{1-s})^{-1}\mathcal{L}\Big(\frac{s-1}{2}; (-1)^{\frac{s-1}{2}}\omega\Big) \prod_{\substack{p|r_2 \\ p>2}} \Big(1 - \Big(\frac{(-1)^{\frac{s-1}{2}}\omega}{p}\Big)p^{\frac{1-s}{2}}\Big) \times$$

$$\times \sum_{k|r_1} k^{s-2} \prod_{p|k} \Big(1 - \Big(\frac{(-1)^{\frac{s-1}{2}}\omega}{p}\Big)p^{\frac{1-s}{2}}\Big).$$

The values of χ_2 *and* χ_p *are given in* [11] *(formulas* (9)–(13), *p.* 66*);*

$$\mathcal{L}(k; (-1)^k\omega) = \sum_{\substack{l=1 \\ 2\nmid l}}^{\infty} \Big(\frac{(-1)^k\omega}{l}\Big)\frac{1}{l^k} = \prod_{\substack{p \\ p>2}} \Big(1 - \Big(\frac{(-1)^k\omega}{p}\Big)p^{-k}\Big)^{-1};$$

B_{s-1} *are Bernoulli's numbers.*

§ 2. In this section we will obtain exact formulas for a number of representations of numbers by quaternary quadratic forms:

$$x_1^2 + x_2^2 + x_3^2 + 15x_4^2 \quad \text{and} \quad 2x_1^2 + 2x_1x_2 + 5x_2^2 + 2x_3^2 + 2x_3x_4 + 5x_3^2.$$

The first form has been considered by Lomadze in [5] who, for the construction of cusp form $X(\tau)$, used products of simple theta-functions with characteristics and of their derivatives (some particular cases of the function (1)) and therefore he had to use modular forms of step 240 instead of 60.

Theorem 5. *Let*

$$f = x_1^2 + x_2^2 + x_3^2 + 15x_4^2, \quad f_1 = 3x_1^2 + 15x_2^2,$$
$$f_2 = 4x_1^2 + 2x_1x_2 + 4x_2^2, \quad g^{(1)} = \begin{pmatrix} 20 \\ 20 \end{pmatrix}, \quad h^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$g^{(2)} = \begin{pmatrix} 15 \\ 0 \end{pmatrix}, \quad h^{(2)} = \begin{pmatrix} 0 \\ 15 \end{pmatrix}, \quad p_1^{(1)} = x_2, \quad p_1^{(2)} = x_1 + x_2.$$

Then the equality

$$\vartheta(\tau; f) = E(\tau; f) + \frac{4}{15}\vartheta_{g^{(1)}h^{(1)}}(\tau; p^{(1)}, f_1) + \frac{1}{10}\vartheta_{g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2) \quad (6)$$

holds, where the functions

$$\vartheta(\tau; f), \quad \vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1), \quad \vartheta_{g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2)$$

are defined by formulas (4) *and* (1)*, while the function* $E(\tau; f)$ *by formula* (5)*.*

Proof. By Theorem 3, the function $\vartheta(\tau; f)$ belongs to the space of entire modular forms of the type $(-2, 60, v(L))$, where $v(L)$ is the corresponding multiplier system. Then, according to Siegel's theorem (see [2]), $E(\tau; f)$ also belongs to this space. Using Lemma 1, we can check that the function

$$X(\tau) = \frac{4}{15}\vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) + \frac{1}{10}\vartheta_{g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2)$$

satisfies all the conditions of Theorem 1.

Indeed, f_1 is the binary form of step 60 and f_2 is the binary form of step 30 ($N_1 = 60$, $N_2 = 30$), $g^{(1)} = \begin{pmatrix} 20 \\ 20 \end{pmatrix}$, and $h^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are special vectors with respect to the form $f_1 = 3x_1^2 + 15x_2^2$, and $g^{(2)} = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$ and $h^{(2)} = \begin{pmatrix} 0 \\ 15 \end{pmatrix}$ are special vectors with respect to the form $f_2 = 4x_1^2 + 2x_1x_2 + 4x_2^2$. $2 \mid h^{(1)}$, $2 \mid \frac{N}{N_2}h^{(2)}$, since $N_1 = N = 60$, $N_2 = 30$; but $60 \mid N$, $30 \mid N$, $60^2 \mid f_1(g^{(1)})$, $30^2 \mid f_2(g^{(2)})$, $240 \mid f_1(h^{(n)})$, $120 \mid 2f_2(h^{(2)})$.

If $\alpha\delta \equiv 1 \pmod{60}$, then either

$$\alpha \equiv \delta \equiv 1 \pmod{3} \quad \text{or} \quad \alpha \equiv \delta \equiv -1 \pmod{3}.$$

Because of Lemma 1,

$$\vartheta_{\alpha g^{(1)}, h^{(1)}}(\tau; p_1^{(1)}, f_1) = \begin{cases} \vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) & \text{for} \quad \alpha \equiv \delta \equiv 1 \pmod{3}, \\ \vartheta_{-g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) & \text{for} \quad \alpha \equiv \delta \equiv -1 \pmod{3}. \end{cases}$$

Due to (1) we have

$$\vartheta_{g^{(1)}h^{(1)}}(\tau; p_1, f_1) = \sum_{x \equiv g^{(1)} \pmod{60}} x_2 e^{2\pi i\tau \frac{3x_1^2+15x_2^2}{60^2}}$$

$$= -\sum_{x \equiv -g^{(1)} \pmod{60}} x_2 e^{2\pi i\tau \frac{3x_1^2+15x_2^2}{60^2}} = -\vartheta_{-g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1).$$

Thus

$$\vartheta_{\alpha g^{(1)},h^{(1)}}(\tau; p_1^{(1)}, f_1) = \begin{cases} \vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) & \text{for } \alpha \equiv \delta \equiv 1 \pmod 3, \\ \vartheta_{-g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) & \text{for } \alpha \equiv \delta \equiv -1 \pmod 3. \end{cases} \quad (7)$$

We have

$$\operatorname{sgn}\delta\Big(\frac{-\Delta_1}{|\delta|}\Big) = \operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big)\Big(\frac{5}{|\delta|}\Big), \quad \Big(\frac{(-1)^2\Delta}{|\delta|}\Big) = \Big(\frac{-1}{|\delta|}\Big)\Big(\frac{|\delta|}{3}\Big)\Big(\frac{5}{|\delta|}\Big). \quad (8)$$

Furthermore, we have

$$\Big(\frac{|\delta|}{3}\Big) = \begin{cases} \operatorname{sgn}\delta & \text{for } \delta \equiv 1 \pmod 3, \\ -\operatorname{sgn}\delta & \text{for } \delta \equiv -1 \pmod 3. \end{cases}$$

We can easily verify that formulas (7) and (8) imply

$$\vartheta_{\alpha g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1)\operatorname{sgn}\delta\Big(\frac{-\Delta_1}{|\delta|}\Big) = \Big(\frac{\Delta}{|\delta|}\Big)\vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1^{(1)}). \quad (9)$$

Analogously, we get

$$\vartheta_{\alpha g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2)\operatorname{sgn}\delta\Big(\frac{-\Delta_2}{|\delta|}\Big) = \Big(\frac{\Delta}{|\delta|}\Big)\vartheta_{\alpha g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2). \quad (10)$$

Consequently, according to (9) and (10), the function

$$X(\tau) = \frac{4}{15}\vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) + \frac{1}{10}\vartheta_{g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2) \quad (11)$$

satisfies the conditions of Theorem 1 and, due to Theorem 2, belongs to the space of cusp forms of the type $(-2, 60, v(L))$.

Thus, owing to Lemma 2, the function

$$\Psi(\tau) = \vartheta(\tau; f) - E(\tau; f) - X(\tau),$$

where $X(\tau)$ is defined by (11), will be identically zero if all coefficients for $Q^n (n \le 24)$ are zero in its expansion in powers of $Q = e^{2\pi i\tau}$.

Next, let $n = 2^\alpha 3^{\beta_1} 5^{\beta_2} u, (u, 30) = 1$. Then by Theorem 4,

$$E(\tau; f) = 1 + \sum_{n=1}^{\infty} \rho(n; f) Q^n \quad \big(Q = e^{2\pi i\tau}\big), \quad (12)$$

where

$$\rho(n;f)=\frac{1}{12}\Big(2^{\alpha+1}+(-1)^{\beta_1}\Big(\frac{-1}{u}\Big)\Big)\Big(3^{\beta_1+1}-(-1)^{\alpha+\beta_2}\Big(\frac{u}{3}\Big)\Big)\times$$
$$\times\Big(5^{\beta_2+1}+(-1)^{\alpha+\beta_1+\beta_2}\Big(\frac{u}{5}\Big)\Big)\sum_{d_1d_2=u}\Big(\frac{15}{d_1}\Big)d_2. \tag{13}$$

Having calculated the values $\rho(n;f)$ for all $n\le 24$ by formula (13), we obtain because of (12):

$$E(\tau;f)=1+3Q+\frac{20}{3}Q^2+\frac{8}{3}Q^3+9Q^4+24Q^5+15Q^6+$$
$$+\frac{16}{3}Q^7+\frac{68}{3}Q^8+39Q^9+\frac{65}{3}Q^{10}+24Q^{11}+\frac{56}{3}Q^{12}+24Q^{13}+$$
$$+48Q^{14}+\frac{65}{3}Q^{15}+33Q^{16}+72Q^{17}+\frac{140}{3}Q^{18}+18Q^{19}+$$
$$+72Q^{20}+96Q^{21}+24Q^{22}+\frac{88}{3}Q^{23}+75Q^{24}+\dots\,. \tag{14}$$

Formulas (4) and (1) yield

$$\vartheta(\tau;f)=1+6Q+12Q^2+8Q^3+6Q^4+24Q^5+24Q^6+$$
$$+12Q^8+30Q^9+24Q^{10}+24Q^{11}+8Q^{12}+24Q^{13}+$$
$$+480Q^{14}+2Q^{15}+18Q^{16}+72Q^{17}+52Q^{18}+36Q^{19}+$$
$$+72Q^{20}+96Q^{21}+24Q^{22}+24Q^{23}+84Q^{24}+\dots\,. \tag{15}$$

$$\frac{4}{15}\vartheta_{g^{(1)}h^{(1)}}(\tau;p_1^{(1)},f_1)=\frac{16}{3}(Q^2+Q^3-Q^7-2Q^8+Q^{10}-$$
$$-2Q^{12}-2Q^{15}+Q^{18}-Q^{23}+4Q^{27}+\dots), \tag{16}$$

$$\frac{1}{10}\vartheta_{g^{(2)}h^{(2)}}(\tau;p_1^{(2)},f_2)=\frac{3}{2}(2Q-2Q^4-6Q^6-6Q^9-$$
$$-2Q^{10}-6Q^{15}-10Q^{16}+12Q^{19}+6Q^{24}+\dots). \tag{17}$$

Taking into account (14)–(17), we can easily verify that all coefficients for $Q^n (n\le 24)$ in the expansion of the function $\psi(\tau)$ in powers of Q are zero. Thus identity (6) is proved. □

Theorem 6. *Let* $f=x_1^2+x_2^2+x_3^2+15x_4^2$, $n=2^\alpha 3^{\beta_1}5^{\beta_2}u$, $(u,30)=1$. *Then*

$$r(n;f)=\frac{1}{6}\Big(3^{\beta_1+1}-(-1)^{\beta_2}\Big(\frac{u}{3}\Big)\Big)\times$$
$$\times\Big(5^{\beta_2+1}+(-1)^{\beta_1+\beta_2}\Big(\frac{u}{5}\Big)\Big)\sum_{d_1d_2=u}\Big(\frac{15}{d_1}\Big)d_2+$$

$$+\frac{3}{2}\sum_{\substack{n=x_1^2+x_1x_2+4x_2^2\\ 2\nmid x_1}}(-1)^{\frac{x_1-1}{2}}(x_1+2x_2) \quad \textit{for} \quad n\equiv 1 \pmod 4,$$

$$=\frac{1}{12}\Big(2^{\alpha+1}+(-1)^{\beta_1}\Big(\frac{-1}{u}\Big)\Big)\Big(3^{\beta_1+1}-(-1)^{\alpha+\beta_2}\Big(\frac{u}{3}\Big)\Big)\times$$
$$\times\Big(5^{\beta_2+1}+(-1)^{\alpha+\beta_1+\beta_2}\Big(\frac{u}{5}\Big)\Big)\sum_{d_1d_2=u}\Big(\frac{15}{d_1}\Big)d_2+$$
$$+\frac{3}{2}\sum_{\substack{n=x_1^2+x_1x_2+4x_2^2\\ 2\nmid x_1}}(-1)^{\frac{x_1-1}{2}}(x_1+2x_2)+$$
$$+\frac{16}{3}\sum_{\substack{3n=x_1^2+5x_2^2\\ x_1\equiv x_2\equiv 1 \pmod 3}}x_2 \quad \textit{otherwise.}$$

Proof. Equating coefficients of the same powers Q in both parts of identity (6), we get

$$r(n;f)=\rho(n;f)+\frac{16}{3}\nu_1(n)+\frac{3}{2}\nu_2(n), \tag{18}$$

where $\nu_1(n)$, $\nu_2(n)$ denote respectively the coefficients for Q in the expansions of the functions

$$\frac{1}{20}\vartheta_{g^{(1)}h^{(1)}}(\tau;p_1^{(1)},f_1), \quad \frac{1}{15}\vartheta_{g^{(2)}h^{(2)}}(\tau;p_1^{(2)},f_2)$$

in powers of Q.

From (1) we have

$$\frac{1}{20}\vartheta_{g^{(1)}h^{(1)}}(\tau;p_1^{(1)},f_1)=\sum_{x_1,x_2=-\infty}(3x_2+1)e^{\frac{2\pi i\tau((3x_1+1)^2+5(3x_2+1)^2)}{3}},$$

i.e.,

$$\nu_1(n)=\sum_{\substack{3n=x_1^2+5x_2^2\\ x_1\equiv x_2\equiv 1 \pmod 3}}x_2. \tag{19}$$

It follows from (4) that

$$\frac{1}{15}\vartheta_{g^{(2)}h^{(2)}}(\tau;p_1^{(2)},f_2)=$$
$$=\sum_{x_1,x_2=-\infty}(-1)^{x_1}(2x_1+1+2x_2)e^{\frac{2\pi i\tau[(2x_1+1)^2+(2x_1+1)x_2+4x_2^2]}{1}},$$

i.e.,

$$\nu_2(n)=\sum_{\substack{n=x_1^2+x_1x_2+4x_2^2\\ 2\nmid x_1}}(-1)^{\frac{x_1-1}{2}}(x_1+2x_2). \tag{20}$$

From formulas (18), (13), (19), (20) we obtain the desired exspression for $r(n;f)$. □

Theorem 7. *Let* $f=2x_1^2+2x_1x_2+5x_2^2+2x_3^2+2x_3x_4+5x_4^2$, $f_1=3x_1^2+9x_2^2$, $g=\begin{pmatrix}18\\6\end{pmatrix}$, $h=\begin{pmatrix}0\\0\end{pmatrix}$, $p_1=x_2$. *Then*

$$\vartheta(\tau;f)=E(\tau;f)-\frac{1}{9}\vartheta_{gh}(\tau;p_1,f_1),$$

where the functions $\vartheta(\tau;f)$, $E(\tau;f)$ *and* $\vartheta_{gh}(\tau;p_1,f_1)$ *are defined respectively by the formulas* (4), (5) *and* (1).

Proof. Let $n=2^\alpha 3^\beta u$, $(u,6)=1$. Then by Theorem 4, $E(\tau;f)=1+\sum_{n=1}^{\infty}\rho(n;f)Q^n$ $\left(Q=e^{2\pi i\tau}\right)$, where

$$\begin{aligned}\rho(n;f)&=12(3^{\beta-1}-1)\sum_{\mu|u}\mu \quad \text{for} \quad \alpha>0,\ \ \beta>0,\\ &=4(3^{\beta-1}-1)\sum_{\mu|u}\mu \quad \text{for} \quad \alpha=0,\ \ \beta>0,\\ &=4\sum_{\mu|u}\mu \quad \text{for} \quad \alpha>0,\ \ \beta>0,\\ &=\frac{4}{3}\sum_{\mu|u}\mu \quad \text{for} \quad (n,6)=1.\end{aligned} \tag{21}$$

Formulas (5) and (21) imply

$$\begin{gathered}E(\tau;f)=1+\frac{4}{3}Q+\dots,\quad \vartheta(\tau;f)=1+4Q^2+\dots,\\ -\frac{1}{9}\vartheta_{9h}(\tau;p_1,f_1)=-\frac{4}{3}Q+\dots\ .\end{gathered} \tag{22}$$

By Theorem 3, the function $\vartheta(r;f)$ belongs to the space of entire modular forms of the type $(-2,36,1)$. Then by Siegel's theorem (see [2]), $E(\tau;f)$ also belongs to this space. Using Lemma 1, we can easily verify that the function $\vartheta_{gh}(\tau;p_1,f_1)$ satisfies all the conditions of Theorem 1. Therefore by Theorem 2, it belongs to the space of cusp forms of the type $(-2,36,1)$. It is well known that this space is one-dimensional (see [12]). Therefore from (22) we obtain the above assertion. □

From Theorem 7 immediately follows

Theorem 8. *Let* $n = 2^\alpha 3^\beta u$, $(u,6)=1$, $f = 2x_1^2 + 2x_1x_2 + 5x_2^2 + 2x_3^2 + 2x_3x_4 + 5x_4^2$. *Then*

$$\begin{aligned} r(n;f) &= 12(3^{\beta-1}-1)\sum_{\mu|u}\mu \quad \textit{for} \quad \alpha>0,\ \beta>0,\\ &= 4(3^{\beta-1}-1)\sum_{\mu|u}\mu \quad \textit{for} \quad \alpha=0,\ \beta>0,\\ &= 4\sum_{\mu|u}\mu \quad \textit{for} \quad \alpha>0,\ \beta=0,\\ &= \frac{4}{3}\sum_{\mu|u}\mu - \frac{2}{3}\sum_{\substack{4n=3x_1^2+x_2^2\\ x_1\equiv 1 \pmod 2\\ x_2\equiv 1 \pmod 6}} x_2 \quad \textit{for} \quad (n,6)=1. \end{aligned}$$

Remark to Theorem 8. Let

$$\nu(n) = \frac{1}{2}\sum_{\substack{4n=3x_1^2+x_2^2\\ x_1\equiv 1 \pmod 2\\ x_2\equiv 1 \pmod 6}} x_2.$$

It can be easily shown that

$$\nu(n) = \frac{1}{2}\sum_{\substack{4n=3x_1^2+x_2^2\\ x_1\equiv 1 \pmod 2\\ x_2\equiv 1 \pmod 6}} (x_1+x_2).$$

Further, arguing as in [12] (p. 233), we can easily show that
(1) $\nu(n_1n_2) = \nu(n_1)\nu(n_2)$ if $(n_1,n_2)=1$;
(2) $\nu(p^\beta) = \sum_{0\le k<\frac{\beta}{2}} p^k Tr(\pi^{\beta-2k}(p)) + \delta\Big(\frac{\beta}{2}\Big)p^{\frac{\beta}{2}}$,
where $\pi(p)$ is the Frobenius endomorphism of a curve $y^2 = x^3+1$ reduced in modulo p, $\delta(r)$ is equal to one or to zero according to whether the number r is an integer or not. In particular, if $n=p$ is a prime number, then

$$\nu(p) = -\sum_{x=0}^{p-1}\Big(\frac{x^3+1}{p}\Big),$$

where $\big(\frac{x^3+1}{p}\big)$ is the Legendre symbol.

§ 3. In this section we obtain formulas for a number of representations of numbers by quadratic forms with seven variables

$$f = 2\sum_{j=1}^{s} x_j^2 + \sum_{j=s+1}^{7} x_j^2 \quad (1 \le s \le 6). \tag{23}$$

The cases $s = 0$ and $s = 1$ are considered earlier (see, e.g., [13], Vol. II, pp. 305, 309, 335 and Vol. III, p. 237). In these cases the corresponding forms belong to one-class genera. The case $s = 3$ was considered in [6].

Theorem 9. *Let f be of the kind* (23), $f_1 = 2x_1^2 + 2x_2^2 + x_3^2$, $f_2 = x_1^2 + x_2^2 + 2x_3^2$, $g^{(1)} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$, $h^{(1)} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$, $p_2 = x_1x_2$, $g^{(2)} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$, $h^{(2)} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$. *Then the following equality holds,*

$$\vartheta(\tau; f) = E(\tau; f) + X(\tau; f), \tag{24}$$

where

$$\begin{aligned} X(\tau; f) &= \frac{1}{32}\vartheta_{g^{(1)}h^{(1)}}(\tau; p_2, f_1) \quad \textit{for} \quad s = 2, 4, \\ &= \frac{1}{32(s-1)}\vartheta_{g^{(2)}h^{(2)}}(\tau; p_2, f_2) \quad \textit{for} \quad s = 3, 5, \\ &= 0 \quad \textit{otherwise}. \end{aligned} \tag{25}$$

Proof. By Theorem 3, the functions $\vartheta(\tau; f)$ belong to two different spaces of modular forms $(-\frac{7}{2}, 8, v(L))$, where $v(L)$ is a system of multipliers corresponding to the form f. This system is the same for all s with the same evenness. Then, according to Siegel's theorem (see [1]), the functions $E(\tau; f)$ also belong to appropriate spaces of modular forms. It can be easily verified that functions (25) satisfy all the conditions of Theorem 1 and, by Theorem 2, they belong to two different spaces of cusp forms depending on s.

Let $n = 2^\alpha u$ $(2 \nmid u,\ \alpha \ge 0)$, $2^s n = r_s^2\omega_s$, $u = r^2\omega$ $(s = 1, 2, \dots, 6)$ and let ω and ω_s be square-free numbers. Then by Theorem 4 we have

$$E(\tau; f) = 1 + \sum_{n=1}^{\infty} \rho(n; f) Q^n \quad (Q = e^{2\pi i\tau}), \tag{26}$$

where

$$\rho(n; f) = 2^{\frac{5\alpha}{2}+9-\frac{s}{2}}\pi^{-3}\omega^{\frac{5}{2}}\mathcal{L}(3; -\omega_s)\chi_2 \sum_{\mu|r}\mu^5 \prod_{p|\mu}\Big(1 - \Big(\frac{-\omega_s}{p}\Big)p^{-3}\Big).$$

By Lemma 27 from [14] we have

$$\mathcal{L}(3;-1) = \frac{\pi^3}{32}, \qquad \mathcal{L}(3;-2) = \frac{3\pi^3}{64\sqrt{2}}; \tag{27}$$

$$\begin{aligned}
\mathcal{L}(3;-\omega) &= \frac{\pi^3}{16\omega^{\frac{5}{2}}}\Big\{\sum_{1\le h\le \frac{\omega}{4}}(\omega^2-16h^2)\Big(\frac{h}{\omega}\Big) + 3\omega^2 \sum_{\frac{\omega}{4}<h<\frac{\omega}{2}}\Big(\frac{h}{\omega}\Big) + \\
&\quad + 16\sum_{\frac{\omega}{4}<h\le\frac{\omega}{2}} h(h-\omega)\Big(\frac{h}{\omega}\Big)\Big\}, \text{ if } \omega\equiv 1 \pmod 4,\ \omega>1,\\
&= \frac{\pi^3}{2\omega^{\frac{5}{2}}}\sum_{1\le h\le\frac{\omega}{2}} h(\omega-2h)\Big(\frac{h}{\omega}\Big), \text{ if } \omega\equiv 3 \pmod 4,\\
&= \frac{\pi^3}{32\omega^{\frac{5}{2}}}\Big\{\sum_{1\le h\le\frac{\omega}{16}}(3\omega^2-256h^2)\Big(\frac{h}{\frac{1}{2}\omega}\Big) + \\
&\quad + 4\omega\sum_{\frac{\omega}{16}<h<\frac{3\omega}{16}}(\omega-8h)\Big(\frac{h}{\frac{1}{2}\omega}\Big) + 13\omega^2\sum_{\frac{3\omega}{16}<h\le\frac{\omega}{4}}\Big(\frac{h}{\frac{1}{2}\omega}\Big) - \\
&\quad - 128\sum_{\frac{3\omega}{16}<h\le\frac{\omega}{4}} h(\omega-2h)\Big(\frac{h}{\frac{1}{2}\omega}\Big)\Big\}, \text{ if } \omega\equiv 2 \pmod 8,\ \ \omega>2,\\
&= \frac{\pi^3}{32\omega^{\frac{5}{2}}}\Big\{32\omega\sum_{1\le h\le\frac{\omega}{16}} h\Big(\frac{h}{\frac{1}{2}\omega}\Big) - \omega^2\sum_{\frac{\omega}{16}<h<\frac{3\omega}{16}}\Big(\frac{h}{\frac{1}{2}\omega}\Big) + \\
&\quad + 64\sum_{\frac{\omega}{16}<h\le\frac{3\omega}{16}} h(\omega-4h)\Big(\frac{h}{\frac{1}{2}\omega}\Big) + 8\omega\sum_{\frac{3\omega}{16}<h\le\frac{\omega}{4}}(\omega-4h)\Big(\frac{h}{\frac{1}{2}\omega}\Big)\Big\},\\
&\quad \text{if } \omega\equiv 6 \pmod 8.
\end{aligned} \tag{28}$$

Using formulas (33) of [9], after calculation of values χ_2, we obtain

$$\begin{aligned}
\chi_2 &= 1 \quad \text{for} \quad 2\nmid s,\ \alpha=0, \text{ or for } 2|s,\ \alpha=0,\ u\equiv 1 \pmod 4,\\
&\quad \text{or } 2|s,\ \alpha=1,\\
&= 1+(-1)^{\frac{u^2-1}{6}}2^{\frac{s}{2}-5}, \quad \text{for} \quad 2|s,\ \alpha=0,\ u\equiv 3 \pmod 4,\\
&= 1+\frac{2^{\frac{s}{2}-3}(1-2^{-\frac{5\alpha}{2}}\cdot 63)}{31} \quad \text{for} \quad 2|s,\ 2|\alpha,\ u\equiv 1 \pmod 4,\\
&= 1+\frac{2^{\frac{s}{2}-3}(1-2^{-\frac{5\alpha}{2}}+(-1)^{\frac{u^2-1}{8}}2^{-\frac{5\alpha}{2}-2}\cdot 31)}{31} \text{ for } 2|s,\\
&\quad 2|\alpha,\ u\equiv 3 \pmod 4\\
&= 1+\frac{2^{\frac{s}{2}-3}(1-2^{-\frac{5\alpha}{2}+\frac{5}{2}}\cdot 63)}{31} \text{ for } 2|s,\ 2\nmid\alpha,\ \alpha>1,
\end{aligned}$$

$$= 1 + \frac{2^{\frac{s}{2}-\frac{1}{2}}(1 - 2^{-\frac{5\alpha}{2}} \cdot 63)}{31} \quad \text{for } 2\nmid s,\ 2|\alpha,\ \alpha > 0,$$
$$= 1 + \frac{2^{\frac{s}{2}-\frac{1}{2}}(1 - 2^{-\frac{5\alpha}{2}-\frac{5}{2}} \cdot 63)}{31} \quad \text{for } 2\nmid s,\ 2\nmid \alpha,\ u \equiv 1 \pmod 4,$$
$$= 1 + \frac{2^{\frac{s}{2}-\frac{1}{2}}(1 - 2^{-\frac{5\alpha}{2}-\frac{5}{2}} + (-1)^{\frac{u^2-1}{8}} 2^{-\frac{5\alpha}{2}-\frac{9}{2}} \cdot 31)}{31}$$
$$\text{for } 2\nmid s,\ 2\nmid \alpha,\ u \equiv 3 \pmod 4. \tag{29}$$

By (1) we have

$$\vartheta_{g^{(1)}h^{(1)}}(\tau; p_2, f_1) = 16 \sum_{x_1,x_2,x_3=-\infty}^{\infty} (-1)^{x_1+x_2+x_3}(2x_1+1)(2x_2+1) \times$$
$$\times\, e^{\frac{2\pi i\tau[(2x_1+1)^2+(2x_2+1)^2+2x_3^2]}{2}}, \tag{30}$$
$$\vartheta_{g^{(2)}h^{(2)}}(\tau; p_2, f_2) = 16 \sum_{x_1,x_2,x_3=-\infty}^{\infty} (-1)^{x_1+x_2}(2x_1+1)(2x_2+1) \times$$
$$\times\, e^{\frac{2\pi i\tau[(2x_1+1)^2+(2x_2+1)^2+2(2x_3+1)^2]}{4}}. \tag{31}$$

Taking then into account (26)–(31) and arguing as in the proof Theorem 5, we obtain the above assertion. □

From Theorem 9 we have

Theorem 10. *Let* $n = 2^\alpha u$ $(2\nmid u,\ \alpha \ge 0)$, $2^s n = r_1^2\omega_s$, $u = r^2\omega$, *and let* ω *and* ω_s *be square-free numbers,* $s = 1, 2, \dots, 6$,

$$f = 2\sum_{j=1}^{s} x_j^2 + \sum_{j=s+1}^{7} x_j^2.$$

Then

$$r(n; f) = 2^{\frac{5\alpha}{2}-\frac{s}{2}+9}\omega^{\frac{5}{2}}\pi^{-3}\mathcal{L}(3; -\omega_s)\chi_2 \times$$
$$\times \sum_{\mu|r} \mu^5 \prod_{p|\mu}\Big(1 - \Big(\frac{-\omega_s}{p}\Big)p^{-3}\Big) + \nu(n; f),$$

where

$$\nu(n;f) = 0 \quad for \quad s = 1,6,$$
$$= \frac{1}{2} \sum_{\substack{2n=x_1^2+x_2^2+2x_3^2 \\ 2\nmid x_1,\ 2\nmid x_2}} (-1)^{\frac{x_1x_2-1}{2}+x_3} x_1x_2 \quad for \quad s = 2,4,$$
$$= \frac{1}{2s-2} \sum_{\substack{4n=x_1^2+x_2^2+2x_3^2 \\ 2\nmid x_1,\ 2\nmid x_2,\ 2\nmid x_3}} (-1)^{\frac{x_1x_2-1}{2}} x_1x_2 \quad for \quad s = 3,5.$$

The values $\mathcal{L}(3;-\omega_3)$ *and* χ_2 *can be calculated by formulas* (27)–(29).

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