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NECESSARY AND SUFFICIENT CONDITIONS FOR WEIGHTED ORLICZ CLASS INEQUALITIES FOR MAXIMAL FUNCTIONS AND SINGULAR INTEGRALS. I

A. GOGATISHVILI AND V. KOKILASHVILI

ABSTRACT. Criteria of various weak and strong type weighted inequalities are established for singular integrals and maximal functions defined on homogeneous type spaces in the Orlicz classes.

§ 1. INTRODUCTION

This paper is devoted to the study of one-weighted inequalities in Orlicz classes for singular integrals

$$\mathcal{K}f(x)=\int_X k(x,y)f(y)\,d\mu$$

and maximal functions

$$\mathcal{M}f(x)=\sup_{B\ni x}\frac{1}{\mu B}\int_B |f(y)|\,d\mu$$

defined on a homogeneous type space X. Necessary and sufficient conditions are found that must be satisfied by the Orlicz class generating function φ and by the weight function w so that either of the following two inequalities be fulfilled:

$$\int\limits_{\{x:|\mathcal{K}f(x)|>\lambda\}} w(x)\,d\mu\le c\int_X \varphi\big(cf(x)\big)w(x)\,d\mu, \tag{1.1}$$

$$\int_X \varphi\big(\mathcal{K}f(x)\big)w(x)\,d\mu\le c\int_X \varphi\big(f(x)\big)w(x)\,d\mu. \tag{1.2}$$

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A similar problem is solved for inequalities of the form

$$\int_X \varphi\big(\mathcal{K}f(x)w(x)\big)\,d\mu \le c\int_X \varphi\big(f(x)w(x)\big)\,d\mu, \tag{1.3}$$

$$\int_X \varphi\big(\mathcal{M}f(x)w(x)\big)\,d\mu \le c\int_X \varphi\big(f(x)w(x)\big)\,d\mu \tag{1.4}$$

and inequalities of some other forms.

The class of weight functions providing the validity of a one-weighted inequality in L^p $(1 < p < \infty)$ for the Hilbert transform is given in [1]. Sufficient conditions for analogous inequalities for Calderon–Zygmund singular integrals in R^n are established in [2], [3], [4]. In the multidimensional case it is a well-known fact that for Riesz transforms the Muchenhoupt condition $\mathcal{A}_p$ completely describes the class of weight functions guaranteeing the validity of a one-weighted inequality in Lebesgue space L^p $(p > 1)$ [5]. Some other cases are also described in [3].

In the nonweighted case $(w \equiv 1)$ the criteria for the function φ in inequalities of form (1) and (2) are given in [6], [7] for Riesz transforms. Reference [6] also contains a study of the vector-valued case. These results are surveyed in the monograph [8].

The one-weighted problem for Hilbert transforms in Orlicz spaces was solved in [9] (see also [10]) under the a priori assumptions that the Young function φ and its complementary function satisfied the condition Δ_2. Earlier, an analogous problem for classical Hardy–Littlewood–Wiener functions was solved in [11] (see also [28]). Subsequently, an attempt was made in [12], [13] at solving a one-weighted problem for Hilbert transforms under less restrictive assumptions, but the conditions indicated in the said papers for the pair (φ, w) turned out difficult to survey.

The nonweighted L^p-theory for singular integrals given on homogeneous type spaces is developed in [14]–[20]. For analogous singular integrals the one-weighted problem in Lebesgue spaces is solved in [21].

The merit of this paper in solving the one-weighted problem for singular integrals given on homogeneous type spaces should be viewed in several directions. Firstly, the criteria obtained for the pair (φ, w) are as simple as possible; secondly, the investigation of general singular integrals enables one to take into consideration the previously known results for classical singular integrals such as multidimensional singular integrals in R^n, Hilbert transforms for odd and even functions and singular integrals with the Cauchy kernel on regular curves. We thereby generalize the results of many authors stated, for example, in [1], [2], [3], [12], [13], [22], [23], [24], [25]. Simultaneously, we derive a solution of the one-weighted problem in Orlicz classes also in the case of Hilbert transforms for even functions. The case of odd functions was investigated in [13], but the condition ensuing from our general theorem is easier to verify.

In conclusion, we would like to note that the solution of a one-weighted problem in Orlicz classes for maximal functions given on homogeneous type spaces was obtained by us in our recent paper [24]. In this paper an analogous problem is solved for other possible kinds of weighted inequalities.

The paper is organized as follows:

§2 is auxiliary. It contains the description of the class of quasiconvex functions and some properties of functions satisfying the condition Δ_2. §3 deals with the criteria of weak type multiweighted inequalities for maximal functions given on homogeneous type spaces. In §4 we prove the weighted inequalities for analogs in homogeneous type spaces of the Marcinkiewicz integral that arise naturally in the theory of singular integrals. §5 is devoted to the investigation of singular integrals with kernels of the Calderon–Zygmund kernel type. Sufficient conditions are established for weight functions as well as for functions generating Orlicz classes ensuring the validity of weak type weighted inequalities. In §6 we distinguish a class of kernels for which the sufficient conditions found in §5 prove to be the necessary ones as well. Based on the above-mentioned results, in §7 we develop the criteria for various strong type weighted inequalities both for maximal functions and for singular integrals. §8 presents concrete examples of classical singular integrals for which the criteria of weighted estimates are obtained from the formulated general theorems. The appendix descusses analogous problems for vector-valued maximal functions and singular integrals.

§ 2. On Some Classes of Functions

This section is auxiliary. It begins by describing the class of quasiconvex functions.

In what follows the symbol Φ will be used to denote the set of all functions $\varphi : \mathbb{R}^1 \to \mathbb{R}^1$ which are nonnegative, even, and increasing on $(0,\infty)$ such that $\varphi(0+) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$.

To accomplish our task we shall need the following basic definitions of quasiconvex functions:

A function ω is called a Young function on $[0,\infty)$ if $\omega(0) = 0$, $\omega(\infty) = \infty$ and it is not identically zero on $(0,\infty)$; at some point $t > 0$ it may have a jump up to ∞ but in that case it must be left continuous at t (see [7]).

A function φ is called quasiconvex if there exist a Young function ω and a constant $c > 1$ such that $\omega(t) \le \varphi(t) \le \omega(ct)$, $t \ge 0$.

To each quasiconvex function φ we can put into correspondence its complementary function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(t) = \sup_{s\ge 0} \big(st - \varphi(s)\big). \tag{2.1}$$

The subadditivity of the supremum readily implies that $\widetilde{\varphi}$ is always a

Young function and $\tilde{\tilde{\varphi}} \le \varphi$. Equality (2.1) holds if φ is itself a Young function. If $\varphi_1 \le \varphi_2$, then $\tilde{\varphi}_2 \le \tilde{\varphi}_1$, and if $\varphi_1(t) = a\psi(bt)$, then

$$\tilde{\varphi}_1(t) = a\tilde{\varphi}\Big(\frac{t}{ab}\Big).$$

This and (2.1) imply

$$\tilde{\omega}\Big(\frac{t}{c}\Big) \le \tilde{\varphi}(t) \le \tilde{\omega}(t).$$

Now from the definition of $\tilde{\varphi}$ we obtain the Young inequality

$$st \le \varphi(s) + \tilde{\varphi}(t), \quad s, t \ge 0.$$

By definition, φ satisfies the global condition Δ_2 ($\varphi \in \Delta_2$) if there is $c > 0$ such that

$$\varphi(2t) \le 2\varphi(t), \quad t > 0.$$

If $\psi \in \Delta_2$, then there exist constants p and c such that $p > 1$, $c > 1$ and

$$t_2^{-p}\varphi(t_2) \le c t_1^{-p}\varphi(t_1) \tag{2.2}$$

for $0 < t_1 < t_2$ (see [8], Lemma 1.3.2).

Lemma 2.1 ([8], [24]). *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*

(i) *φ is quasiconvex;*

(ii) *there is a constant $c_1 > 0$ such that*

$$t_1^{-1}\varphi(t_1) \le c_1 t_2^{-1}\varphi(c_1 t_2) \tag{2.3}$$

for each t_1 and t_2, $0 < t_1 < t_2$ ($t^{-1}\varphi(t)$ quasiincreases);

(iii) *there is a constant $c_2 > 0$ such that*

$$\varphi(t) \le \tilde{\tilde{\varphi}}(c_2 t), \quad t > 0; \tag{2.4}$$

(iv) *there are positive ε and c_3 such that*

$$\tilde{\varphi}\Big(\varepsilon\,\frac{\varphi(t)}{t}\Big) \le c_3\varphi(t), \quad t > 0; \tag{2.5}$$

(v) *there is a constant $c_4 > 0$ such that*

$$\varphi\Big(\frac{1}{\mu B}\int_B f(y)\,d\mu\Big) \le \frac{c_4}{\mu B}\int_B \varphi\big(c_4 f(y)\big)\,d\mu. \tag{2.6}$$

Remark 1. If the function $t^{-1}\varphi(t)$ quasidecreases, then we have the inverse of Jensen's inequality and therefore the inverse of (2.6).

Lemma 2.2 ([24]). *For a quasiconvex φ we have*

$$\varepsilon\varphi(t) \le \varphi(c\varepsilon t), \quad t > 0, \ \varepsilon > 1, \tag{2.7}$$

and

$$\varphi(\gamma t) \le \gamma\varphi(ct), \quad t > 0, \ \gamma > 1, \tag{2.8}$$

where the constants c do not depend on t, ε, and γ.

Lemma 2.3 ([24]). *Let $\varphi \in \Phi$ and φ be quasiconvex. Then there is a constant $\delta > 0$ such that for an arbitrary $t > 0$ we have*

$$\widetilde{\varphi}\Big(\delta\,\frac{\varphi(t)}{t}\Big) \le \varphi(t) \le \widetilde{\varphi}\Big(2\,\frac{\varphi(t)}{t}\Big) \tag{2.9}$$

and

$$\varphi\Big(\delta\,\frac{\widetilde{\varphi}(t)}{t}\Big) \le \widetilde{\varphi}(t) \le \varphi\Big(2\,\frac{\widetilde{\varphi}(t)}{t}\Big). \tag{2.10}$$

Lemma 2.4 ([8], [24]). *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*

(i) *the function φ^α is quasiconvex for some α, $0 < \alpha < 1$;*
(ii) *the function φ is quasiconvex and $\widetilde{\varphi} \in \Delta_2$;*
(iii) *there is a constant $\alpha > 1$ such that*

$$\varphi(at) \ge 2a\varphi(t), \quad t > 0; \tag{2.11}$$

(iv) *there is a constant $c > 0$ such that*

$$\int_0^t \frac{\varphi(s)}{s^2}\,ds \le c\,\frac{\varphi(ct)}{t}, \quad t > 0. \tag{2.12}$$

Lemma 2.5. *Given a quasiconvex function φ and a number p, $1 < p < \infty$, the following two conditions are equivalent:*

(i) *the function $t^{-p}\varphi(t)$ quasiincreases;*
(ii) *the function $t^{-p'}\widetilde{\varphi}(t)$ quasidecreases.*

Here and in what follows p' will always denote $\frac{p}{p-1}$.

Proof. Let $t^{-p}\varphi(t)$ quasiincrease. Then there exists a constant $c > 0$ such that

$$t_1^{-p}\varphi(t_1) \le t_2^{-p}(ct_2)$$

for arbitrary t_1, t_2 provided that $0 < t_1 \le t_2$. As a result, we have

$$\widetilde{\varphi}(t_2) = \sup_s \big(st_2 - \varphi(s)\big) = \sup_s \bigg(c\Big(\frac{t_2}{t_1}\Big)^{p'-1} st_2 - \varphi\Big(c\Big(\frac{t_2}{t_1}\Big)^{p'-1} s\Big)\bigg) \le$$
$$\le \sup_s \bigg(c\Big(\frac{t_2}{t_1}\Big)^{p'} st_1 - \Big(\frac{t_2}{t_1}\Big)^{(p'-1)p}\varphi(s)\bigg) = \Big(\frac{t_2}{t_1}\Big)^{p'}\widetilde{\varphi}(ct_1).$$

The obtained inequality means that the function $t^{-p'}\widetilde{\varphi}(t)$ quasidecreases.

Now let the function $t^{-p'}\widetilde{\varphi}(t)$ quasidecrease. Then there exists a constant $c>0$ such that

$$t_2^{-p'}\widetilde{\varphi}(t_2)\le t_1^{-p'}\widetilde{\varphi}(ct_1),\quad 0<t_1\le t_2.$$

From this inequality we obtain

$$\begin{aligned}\widetilde{\widetilde{\varphi}}(t_1)=\sup_s\big(st_1-\widetilde{\varphi}(s)\big)=\sup_s\bigg(c\Big(\frac{t_1}{t_2}\Big)^{p-1}st_1-\widetilde{\varphi}\Big(c\Big(\frac{t_1}{t_2}\Big)^{p-1}s\Big)\bigg)\le\\ \le\sup_s\bigg(c\Big(\frac{t_1}{t_2}\Big)^{p}st_2-\Big(\frac{t_1}{t_2}\Big)^{p}\widetilde{\varphi}(s)\bigg)=\Big(\frac{t_1}{t_2}\Big)^{p}\widetilde{\widetilde{\varphi}}(ct_2).\end{aligned}$$

Next, since $\widetilde{\widetilde{\varphi}}\sim\varphi$ (see Lemma 2.1), we have

$$\frac{\varphi(t_1)}{t_1^p}\le\frac{\widetilde{\widetilde{\varphi}}(ct_1)}{t_1^p}\le\frac{\widetilde{\widetilde{\varphi}}(c^2t_2)}{t_2^p}\le\frac{\varphi(c^2t_2)}{t_2^p},\ \ t_1^{-p}\varphi(t_1)\le t_1^{-p}\,\widetilde{\widetilde{\varphi}}(ct_1)\le\cdots,$$

i.e., the function $t^{-p}\varphi(t)$ quasiincreases. $\square$

Definition 2.1. For any quasiconvex function φ we define a number $p(\varphi)$ as

$$\frac{1}{p(\varphi)}=\inf\big\{\beta\ :\ \varphi^\beta\ \text{ is quasiconvex}\big\}.$$

Lemma 2.6. *The inequality*

$$\frac{1}{p(\varphi)}+\frac{1}{p(\widetilde{\varphi})}\ge 1 \tag{2.13}$$

holds for every quasiconvex function φ.

Proof. By the definition of number $p(\varphi)$ we have $p(\varphi)=\sup\{p:\varphi^{\frac{1}{p}}$ is quasiconvex$\}$. Further by Lemma 2.1 $p(\varphi)=\sup\{p:t^{-p}\varphi(t)$ quasiincrease$\}$. Therefore $t^{-(p(\varphi)-\varepsilon)}\varphi(t)$ quasiincreases, which by Lemma 2.5 is equivalent to the fact that the function $t^{-(p(\varphi)-\varepsilon)'}\widetilde{\varphi}(t)$ quasidecreases. Thus $p(\widetilde{\varphi})\le(p(\varphi)-\varepsilon)'$. Hence we have $p(\varphi)-\varepsilon\le(p(\widetilde{\varphi}))'$ from which we conclude that (2.13) holds since ε is an arbitrary number. $\square$

§ 3. Criteria of Multiweighted Weak Type Inequalities for Maximal Functions Defined on Homogeneous Type Spaces

Let (X,d,μ) be a homogeneous type space. It is a metric space with a complete measure μ such that the class of compactly supported continuous functions is dense in the space $L^1(X,\mu)$. It is also assumed that there is a nonnegative real-valued function $d:X\times X\to\mathbb{R}^1$ satisfying the following conditions:

(i) $d(x,x)=0$ for all $x\in X$;

(ii) $d(x,y) > 0$ for all $x \neq y$ in X;

(iii) there is a constant a_0 such that $d(x,y) \leq a_0 d(y,x)$ for all x,y in X;

(iv) there is a constant a_1 such that $d(x,y) \leq a_1(d(x,z) + d(z,y))$ for all x,y,z in X;

(v) for each neighborhood V of x in X there is an $r > 0$ such that the ball $B(x,r) = \{y \in X : d(x,y) < r\}$ is contained in V;

(vi) the balls $B(x,r)$ are measurable for all x and $r > 0$;

(vii) there is a constant b such that $\mu B(x,2r) \leq b\mu B(x,r)$ for all $x \in X$ and $r > 0$.

For the definitions of homogeneous type spaces see [16],[17],[26].

An almost everywhere positive locally μ-summable function $w : X \to \mathbb{R}^1$ will be called a weight function. For an arbitrary μ-measurable set E we shall assume

$$wE = \int_E w(x)\, d\mu.$$

Definition 3.1. The weight function $w \in \mathcal{A}_p(X)$ $(1 \leq p < \infty)$ if

$$\sup_B \left(\frac{1}{\mu B}\int_B w(x)\, d\mu\right)\left(\frac{1}{\mu B}\int_B \big(w(x)\big)^{-1/(p-1)} d\mu\right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset X$ and

$$\frac{1}{\mu B}\int_B w(x)\, d\mu \leq c \operatorname*{inf\,ess}_{y\in B} w(y) \quad \text{for } \ p = 1.$$

In the latter inequality c does not depend on B. The above conditions are the analogs of the well-known Muckenhoupt's conditions.

Let us recall the basic properties of classes $\mathcal{A}_p$ (see [20],[26]).

Proposition 3.1. *If $w \in \mathcal{A}_p$ for some $p \in [1,\infty)$, then $w \in \mathcal{A}_s$ for all $s \in [p,\infty)$ and there is an $\varepsilon > 0$ such that $w \in \mathcal{A}_{p-\varepsilon}$.*

Definition 3.2. The weight function w belongs to $\mathcal{A}_\infty(X)$ if to each $\varepsilon \in (0,1)$ there corresponds $\delta \in (0,1)$ such that if $B \subset X$ is a ball and E is any measurable set of B, then $\mu E < \delta\mu B$ implies $wE < \varepsilon wB$.

On account of the well-known properties of classes $\mathcal{A}_p$ we have

$$\mathcal{A}_\infty(X) = \bigcup_{p\geq 1} \mathcal{A}_p(X).$$

Given locally integrable real functions f on X, we define the maximal function by

$$\mathcal{M}f(x) = \sup(\mu B)^{-1}\int_B |f(y)|\, d\mu, \quad x \in X,$$

where the supremum is taken over all balls B containing x.

Proposition 3.2 ([20]). *For $1 < p < \infty$ the following conditions are equivalent:*

(i) *there exists a positive constant c such that for an arbitrary measurable f we have*

$$\int_X \big(\mathcal{M}f(x)\big)^p w(x) d\mu \le c \int_X |f(x)|^p w(x)\, d\mu;$$

(ii) $w \in \mathcal{A}_p$.

Proposition 3.3 ([20], [26]). *The following conditions are equivalent:*

(i) *the operator $\mathcal{M} : f \to \mathcal{M}f$ is of weak type with respect to the measure w, i.e., there exists a positive constant c such that*

$$w\big\{x : \mathcal{M}f(x) > \lambda\big\} \le \frac{c}{\lambda} \int_X |f(x)| w(x)\, d\mu$$

for all $\lambda > 0$ and measurable $f : X \to \mathbb{R}^1$;

(ii) $w \in \mathcal{A}_1(X)$.

The above propositions are proved in a standard manner using the covering lemma to be formulated below and the well-known Marcinkiewicz's interpolation theorem.

Proposition 3.4 ([26], Lemma 2). *Let $\mathcal{F}$ be a family $\{B(x, r)\}$ of balls with bounded radii. Then there is a countable subfamily $\{B(x_i, r_i)\}$ consisting of pairwise disjoint balls such that each ball in $\mathcal{F}$ is contained in one of the balls $B(x_i, ar_i)$ where $a = 3a_1^2 + 2a_0a_1$. The constants a_0, a_1 are from the definition of the space (X, d, μ).*

Further for any $B = B(x, r)$ and $a > 0$ let aB denote the ball $B(x, ar)$.

Now let us prove the criteria of a weak type multiweighted inequality for the maximal functions defined on homogeneous type spaces.

In what follows the notation $(f)_B = \frac{1}{\mu B} \int_B f(x)\, d\mu$ will be used.

Theorem 3.1. *Let $\varphi \in \Phi$ and w_i $(i = 1, 2, 3, 4)$ be weight functions. Then the following conditions are equivalent:*

(i) *the inequality*

$$\int\limits_{\{x:\mathcal{M}f(x)>\lambda)\}} \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \le c_1 \int\limits_X \varphi\big(c_1 f(x) w_3(x)\big) w_4(x)\, d\mu \quad (3.1)$$

with a constant c_1 independent of f and $\lambda > 0$ is valid;

(ii) *there is a constant $c_2 > 0$ such that the inequality*

$$\int_B \varphi\big((f)_B w_1(x)\big) w_2(x)\, d\mu \le c_2 \int_B \varphi\big(c_2 f(x) w_3(x)\big) w_4(x)\, d\mu \qquad (3.2)$$

is fulfilled for any nonnegative μ-measurable function $f : X \to \mathbb{R}^1$ and for any ball B;

(iii) *there are positive constants ε and c_3 such that we have the inequality*

$$\int_B \widetilde{\varphi}\left(\varepsilon \frac{\int_B \varphi(\lambda w_1(y)) w_2(y) d\mu}{\lambda \mu B w_3(x) w_4(x)}\right) w_4(x) d\mu \leq c_3 \int_B \varphi\big(\lambda w_i(x)\big) w_2(x) d\mu \quad (3.3)$$

for any $\lambda > 0$ and an arbitrary ball B.

Proof. The implication (i)$\Rightarrow$(ii) is a consequence of the obvious inclusion $B \subset \{x : \mathcal{M}(2f\chi_B)(x) > |f|_B\}$.

Let us prove the implication (ii)$\Rightarrow$(iii). First we shall show that the condition (ii) implies that φ is quasiconvex.

Take k such that the set $E = \{x : k^{-1} \leq w_1(x),\ k^{-1} \leq w_2(x),\ w_3(x) \leq k,\ w_4(x) \leq k\}$ has a positive measure and assume that $x_0 \in E$ is a density point such that it is not an atom. Therefore there is a constant $r_0 > 0$ such that

$$\mu B(x_0, r) \cap E \geq \frac{1}{2}\, \mu B(x_0, r)$$

for all r, $0 \leq r \leq r_0$. Assume further that for fixed s and t, $0 \leq s \leq t$, we have

$$r_1 = \inf\big\{r > 0 \ :\ \mu B(x_0, r) > \frac{s}{t}\, \mu B(x_0, r)\big\}.$$

It is evident that

$$\mu B(x_0, r_1) \leq b \mu B\big(x_0, \frac{r_1}{2}\big) \leq \frac{bs}{t}\, \mu B(x_0, r_0) \leq b \mu B(x_0, 2r_1) \leq b^2 \mu B(x_0, r_1),$$

where the constant b is from the doubling condition.

Now let us consider the function

$$f(x) = 2bkt\chi_{B(x_0, r_1) \cap E}(x).$$

We can readily see that

$$\big(f\big)_{B(x_0, r_0)} \geq sk.$$

Putting f in (ii), by the above-obtained chain of inequalities we have

$$\varphi(s) \leq c_2 kb\, \frac{s}{t}\, \varphi(2bc_2k^2t).$$

Therefore, on account of Lemma 2.1, we conclude that φ is quasiconvex.

Let B be a fixed ball and $\lambda > 0$. Given $k \in \mathbb{N}$, put

$$B_k = \big\{x \in B \ :\ w_3(x) w_4(x) > \frac{1}{k}\big\}$$

and

$$g(x) = \left(\frac{\int_B \varphi(\lambda w_1(y)) w_2(y) dy}{\lambda \mu B w_3(x) w_4(x)}\right)^{-1} \widetilde{\varphi}\left(\varepsilon\, \frac{\int_B \varphi(\lambda w_1(y)) w_2(y) d\mu}{\lambda \mu B w_3(x) w_4(x)}\right) \chi_{B_k(x)}$$

with ε to be specified later.

By our notation we have

$$I = \int_{B_k} \widetilde{\varphi}\left(\frac{\varepsilon \int_B \varphi(\lambda w_1(y)) w_2(y) d\mu}{\lambda \mu B w_3(x) w_4(x)}\right) w_4(x)\, d\mu =$$
$$= \frac{1}{\lambda \mu B} \int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \int_B \frac{g(x)}{w_3(x)}\, d\mu.$$

If the condition

$$\frac{1}{\mu B} \int_B \frac{g(x)}{w_3(x)}\, d\mu < \lambda$$

is satisfied, for the ball B and λ we have

$$I \leq \int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \tag{3.4}$$

Let now

$$\frac{1}{\mu B} \int_B \frac{g(x)}{w_3(x)}\, d\mu > \lambda.$$

From (ii) for the function $f(x) = cg(x)(w_3(x))^{-1}$ where the constant c is from condition (2.7), we derive the estimates

$$I \leq \int_B \varphi\left(\frac{c}{\mu B}\Big(\int_B \frac{g(t)}{w_3(t)}\, d\mu\Big) w_1(x)\right) w_2(x)\, d\mu \leq$$
$$\leq c_2 \int_B \varphi\big(c_2 c g(x)\big) w_4(x)\, d\mu. \tag{3.5}$$

From (3.4) and (3.5) we conclude that

$$I \leq \int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu + c_2 \int_B \varphi\big(c_2 c g(x)\big) w_4(x)\, d\mu.$$

Choosing ε so small that $\delta^{-1} c_2 c^2 \varepsilon < 1$, where the constant δ is from inequality (2.10), by the definition of g, (2.8), and (2.9) (see Lemmas 2.2 and 2.3) we derive the estimate

$$I \leq \int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu + \frac{c_2^2 c^2 \varepsilon}{\delta}\, I. \tag{3.6}$$

Now we want to show that I is finite for a sufficiently small ε. If $\lim_{t\to\infty} t^{-1}\varphi(t) = \infty$, then $\widetilde{\varphi}$ is finite everywhere and thus

$$I \leq \widetilde{\varphi}\left(\varepsilon k \frac{\int_B \varphi(\lambda w_1(x)) w_2(x) d\mu}{\lambda \mu B}\right) w_4 B < \infty. \tag{3.7}$$

If the function $t^{-1}\varphi(t)$ is bounded, then the condition (ii) implies

$$\int_B \varphi\big((f)_{B} w_1(x)\big) w_2(x)\, d\mu \le c_4 \int_B |f(x)| w_3(x) w_4(x)\, d\mu. \tag{3.8}$$

If in (3.8) we put $f(x) = \lambda\mu B(\mu E)^{-1}\chi_{E}(x)$, where E is an arbitrary measurable subset of B, then

$$\frac{1}{\mu B}\int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \le \frac{c_4\lambda}{\mu E}\int_E w_3(x) w_4(x)\, d\mu$$

which yields the estimate

$$\frac{\int_B \varphi(\lambda w_1(y)) w_2(y)\, d\mu}{\lambda\mu B w_3(x) w_4(x)} \le c_4$$

almost everywhere on B, where c_4 is independent of B and λ. Thus we conclude that $I \le \widetilde{\varphi}(\varepsilon c_4) w_4 B$. Choosing ε so small that $\widetilde{\varphi}(\varepsilon c_4) < \infty$, we see that I is finite.

Now inequality (3.6) implies

$$\begin{aligned}\int_{B_k} \widetilde{\varphi}\left(\varepsilon\, \frac{\int_B \varphi(\lambda w_1(y)) w_2(y)\, d\mu}{\lambda\mu B w_3(x) w_4(x)}\right) w_4(x)\, d\mu \le \\ \le \frac{\delta}{\delta - c_2^2 c^2\varepsilon}\int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu.\end{aligned}$$

Passing in the latter inequality to the limit as $k \to \infty$, we derive the desired inequality (3.3).

Now let us prove the implication (iii)$\Rightarrow$(i). For each natural number n we put

$$\mathcal{M}^n f(x) = \sup \frac{1}{\mu B}\int_B |f(y)|\, d\mu,$$

where the supremum is taken over all balls B in X, containing x and $\operatorname{rad} B \le n$.

First it will be shown that (iii) implies (i) if $\mathcal{M}f$ is replaced by $\mathcal{M}^n f$ and the constant c_1 is independent of n. Once this has been done, the result follows by letting n tend to infinity.

Assume B to be any ball such that

$$\lambda \le \frac{1}{\mu B}\int_B |f(y)|\, d\mu. \tag{3.9}$$

Let a be the constant from Covering Lemma 3.4. By the doubling condition there exists a constant c_5 such that $\mu aB \le c_5\mu B$ for any ball B in X.

Let the constants c_3 and ε be from (3.3). Applying the Young inequality, (3.9), and (3.3), we obtain the estimates

$$\int_{aB}\varphi\big(\lambda w_1(x)\big)w_2(x)\,d\mu \le \frac{1}{\lambda\mu B}\int_B |f(x)|\,d\mu = \frac{c_5}{\lambda\mu aB}\int_B |f(x)|\,d\mu =$$
$$= \frac{1}{2c_3}\int_B \frac{2c_3c_5}{\varepsilon}\,|f(x)|w_3(x)\,\varepsilon\,\frac{\int_{aB}\varphi(\lambda w_1(y))w_2(y)d\mu}{\lambda\mu aBw_3(x)w_4(x)}\,w_4(x)\,d\mu \le$$
$$\le \frac{1}{2c_3}\int_B \varphi\Big(\frac{2c_3c_5}{\varepsilon}\,f(x)w_3(x)\Big)w_4(x)\,d\mu +$$
$$+\frac{1}{2c_3}\int_B \widetilde{\varphi}\left(\varepsilon\frac{\int_{aB}\varphi(\lambda w_1(y))w_2(y)d\mu}{\lambda\mu aBw_3(x)w_4(x)}\right)\,w_4(x)\,d\mu \le$$
$$\le \frac{1}{2}\,c_1\int_B \varphi\big(c_1f(x)w_3(x)\big)w_4(x)\,d\mu + \frac{1}{2}\int_{aB}\varphi\big(\lambda w_1(x)\big)w_2(x)\,d\mu,$$

where $c_1 = \sup(\frac{1}{c_3}, \frac{2c_3c_5}{\varepsilon})$.

Therefore

$$\int_{aB}\varphi\big(\lambda w_1(x)\big)w_2(x)\,d\mu \le c_1\int_B \varphi\big(c_1f(x)w_3(x)\big)w_4(x)\,d\mu. \tag{3.10}$$

For any point $x \in \{x : \mathcal{M}^n f(x) > \lambda\}$ there exists a ball $B_x = B(y,r) \ni x$, $0 < r \le n$, such that

$$\int_{B_x} |f(y)|\,d\mu > \lambda\mu B_x.$$

Let now $\{B_i\}$ be the sequence of pairwise disjoint balls corresponding to the family $\mathcal{F} = \{B_x\}$ and existing on account of Covering Lemma 3.4. It is obvious that

$$\big\{x\ :\ \mathcal{M}^n f(x) > \lambda\big\} \subseteq \bigcup_{B\in\mathcal{F}} B \subset \bigcup_i aB_i$$

and by (3.10)

$$\int_{aB_i}\varphi\big(\lambda w_1(x)\big)w_2(x)\,d\mu \le c_1\int_B \varphi\big(c_1f(x)w_3(x)\big)w_4(x)\,d\mu.$$

Consequently

$$\int\limits_{\{x:\mathcal{M}^n f(x)>\lambda\}} \varphi\big(\lambda w_1(x)\big)w_2(x)\,d\mu \le \sum_i \int_{aB_i}\varphi\big(\lambda w_1(x)\big)w_2(x)\,d\mu \le$$
$$\le c_1\sum_i\int_{B_i}\varphi\big(c_1f(x)w_3(x)\big)w_4(x)\,d\mu \le c_1\int_X \varphi\big(c_1f(x)w_3(x)\big)w_4(x)\,d\mu. \qquad \square$$

Remark 2. While proving Theorem 3.1, it was shown that condition (3.2) and hence either of conditions (3.1) and (3.3) guarantees the quasiconvexity of the function φ.

Below we shall consider concrete cases where criteria of different kinds of one-weighted inequalities have a quite simple form.

Theorem 3.2. *Let $\varphi \in \Phi$. Then the following statements are valid:*

(i) *for $w_1 \equiv w_3 \equiv 1$ and $w_2 \equiv w_4 \equiv w$ each of conditions* (3.1), (3.2), *and* (3.3) *is equivalent to the fact that φ is quasiconvex and $w \in \mathcal{A}_{p(\varphi)}$;*

(ii) *for $w_1 \equiv w_3 \equiv w$ and $w_2 \equiv w_4 \equiv 1$ each of conditions* (3.1), (3.2), *and* (3.3) *is equivalent to the fact that φ is quasiconvex, $w \in \mathcal{A}_{p(\varphi)}$, and $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$;*

(iii) *for $w_2 \equiv w_4 \equiv w$ and $w_1 \equiv w_3 \equiv \frac{1}{w}$ each of conditions* (3.1), (3.2), *and* (3.3) *is equivalent to the fact that φ is quasiconvex and $w \in \mathcal{A}_{p(\widetilde{\varphi})}$.*

To prove Theorem 3.2 we must first prove some auxiliary statements.

Proposition 3.5. *Let $\varphi \in \Phi$. The following statements are valid:*

(i) *if $\varphi \in \Delta_2$ and condition* (3.3) *is fulfilled for $w_1 \equiv w_3$ and $w_2 \equiv w_4$, then the function $\varphi(\lambda w_1) w_2 \in \mathcal{A}_\infty$ uniformly with respect to λ;*

(ii) *if $\widetilde{\varphi} \in \Delta_2$ and* (3.3) *holds for $w_3 \equiv w_1$ and $w_4 \equiv w_2$, then the function $\widetilde{\varphi}(\frac{\lambda}{w_1 w_2}) w_2 \in \mathcal{A}_\infty$ uniformly with respect to λ.*

Proof. (i) Let B be an arbitrary ball in X and E be its arbitrary μ-measurable subset. Applying the Young inequality and condition (3.3), we obtain

$$\begin{aligned}
&\int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu = \\
&= \frac{1}{2c_4} \int_E \frac{\int_B \varphi(\lambda w_1(y)) w_2(y) d\mu}{\lambda \mu B w_1(x) w_2(x)}\, 2c_4 \frac{\mu B}{\mu E}\, \lambda w_1(x) w_2(x)\, d\mu \le \\
&\le \frac{1}{2c_4} \int_E \widetilde{\varphi}\left(\frac{\int_B \varphi(\lambda w_1(y)) w_2(y) d\mu}{\lambda \mu B w_1(x) w_2(x)} \right) w_2(x)\, d\mu + \\
&+ \frac{1}{2c_4} \int_E \varphi\Big(2c_4 \frac{\mu B}{\mu E}\, w_1(x)\Big) w_2(x)\, d\mu \le \\
&\le \frac{1}{2} \int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu + \frac{1}{2c_4} \int_E \varphi\Big(2c_4 \frac{\mu B}{\mu E}\, w_1(x)\Big) w_2(x)\, d\mu
\end{aligned}$$

and hence

$$\int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \le \frac{1}{c_4} \int_E \varphi\Big(2c_4 \frac{\mu B}{\mu E}\, w_1(x)\Big) w_2(x)\, d\mu. \tag{3.11}$$

Since, by assumption, $\varphi \in \Delta_2$, there exist a $p > 1$ and a constant $c_5 > 0$ such that $\varphi(at) \le c_5 a^p \varphi(t)$ for any $a > 1$ and (see §2).

Therefore from (3.11) we derive

$$\int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \le c_6 \Big(\frac{\mu B}{\mu E}\Big)^p \int_E \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu.$$

Since the constant c_6 is independent of λ, the latter inequality implies that $\varphi(\lambda w_1)w_2 \in \mathcal{A}_\infty$ uniformly with respect to λ.

(ii) Assume that $\widetilde{\varphi} \in \Delta_2$. Then the reasoning will be as above.

Put

$$\frac{\int_B \varphi(\lambda w_1(x))w_2(x)\,d\mu}{\lambda\mu B} = t. \tag{3.12}$$

Condition (3.3) can be rewritten in the form

$$\int_B \widetilde{\varphi}\Big(\frac{t}{w_1(x)w_2(x)}\Big)\,d\mu \le c_7\lambda t\mu B.$$

Hence it follows that

$$\int_B \varphi\left(\frac{1}{c_7}\frac{w_1(x)}{\mu B\cdot t}\int_B \widetilde{\varphi}\Big(\frac{t}{w_1(y)w_2(y)}\Big)w_2(y)\,d\mu\right)w_2(x)\,d\mu \le$$
$$\le \int_B \varphi\big(\lambda w_1(x)\big)w_2(x)\,d\mu = \lambda t\mu B.$$

The Young inequality, equality (3.12), and the condition Δ_2 give us

$$\lambda t\mu B \le c_8\int_B \widetilde{\varphi}\Big(\frac{t}{w_1(x)w_2(x)}\Big)w_2(x)\,d\mu.$$

The latter two estimates yield

$$\int_B \varphi\left(\frac{1}{c_7}\frac{w_1(x)}{\mu B\cdot t}\int_B \widetilde{\varphi}\Big(\frac{t}{w_1(y)w_2(y)}\Big)w_2(y)\,d\mu\right)w_2(x)\,d\mu \le$$
$$\le c_8\int_B \widetilde{\varphi}\Big(\frac{t}{w_1(y)w_2(y)}\Big)w_2(x)\,d\mu. \tag{3.13}$$

Following (i), from (3.13) and the property $\widetilde{\widetilde{\varphi}}\sim\varphi$ for quasiconvex functions we conclude that $\widetilde{\varphi}(\frac{t}{w_1w_2})w_2 \in \mathcal{A}_\infty$ uniformly with respect to t. $\square$

Lemma 3.1. *If* $1 \le p_1 < p_2 < \infty$, $\rho_1 \in \mathcal{A}_{p_1}$, $\rho_2 \in \mathcal{A}_{p_2}$, *then* $\rho_1^\theta\rho_2^{1-\theta} \in \mathcal{A}_p$ *where* $p = \theta p_1 + (1-\theta)p_2$, $o < \theta < 1$.

It is easy to accomplish the proof using the Hölder inequality and the definition of the class $\mathcal{A}_p$.

Our further discussion will essentially be based on the result we established earlier (see from [24], Proposition 2.4).

Propostion A [24]. *Let* $\varphi \in \Phi$. *Then the following conditions are equivalent:*

(i) *there is* $c_1 > 0$ *such that the inequality*

$$\varphi(\lambda)w\big\{x \in X\ :\ \mathcal{M}f(x) > \lambda\big\} \le c_1\int_X \varphi\big(c_1 f(x)\big)w(x)\,d\mu$$

is fulfilled for any $\lambda > 0$ and locally summable function $f : X \to \mathbb{R}^1$;

(ii) *there are positive constant ε and c_2 such that the inequality*

$$\int_B \widetilde{\varphi}\Big(\varepsilon \frac{\varphi(\lambda)}{\lambda} \frac{wB}{\mu B w(x)}\Big) w(x)\, d\mu \le c_2 \varphi(\lambda) wB$$

is fulfilled for any ball B and positive number λ;

(iii) *φ is quasiconvex and $w \in \mathcal{A}_{p(\varphi)}$.*

Proposition 3.6. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*

(i) *there exists $\varepsilon > 0$ and $c_{10} > 0$ such that for any $\lambda > 0$ and ball $B \subset X$ we have*

$$\int_B \widetilde{\varphi}\left(\varepsilon \frac{\int_B \varphi(\lambda w(y))\, d\mu}{\lambda \mu B w(x)}\right) d\mu \le c_{10} \int_B \varphi\big(\lambda w(x)\big)\, d\mu; \tag{3.14}$$

(ii) *φ is quasiconvex, $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$, and $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$.*

Proof. First we shall show that the implication (i)$\Rightarrow$(ii) holds. The quasiconvexity of φ is obtained from Remark 3.1 to Theorem 3.1, since (3.14) is the particular case of (3.3) for $w_1 \equiv w_3 \equiv w$ and $w_2 \equiv w_4 \equiv 1$. Therefore $p(\varphi) \ge 1$.

Let us show that $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$. First we assume that $p(\varphi) = 1$. By (2.8) and (2.6) there exists a number $\varepsilon_1 > 0$ such that

$$\varphi\Big(\varepsilon_1 \frac{1}{\mu B} \int_B \lambda w(x)\, d\mu\Big) \le \frac{1}{\mu B} \int_B \varphi\big(\lambda w(x)\big)\, d\mu.$$

Next, recalling that φ is quasiconvex and applying Lemma 2.2, from (3.14) we obtain

$$\frac{1}{\mu B} \int_B \widetilde{\varphi}\left(\varepsilon \frac{\varphi(\varepsilon_1 \lambda \frac{wB}{\mu B})}{\lambda w(x)}\right) d\mu \le \varphi\Big(\varepsilon_1 \lambda \frac{wB}{\mu B}\Big).$$

If in this inequality we insert $\lambda = 2s^{-1}\widetilde{\varphi}(s)\mu B(wB)^{-1}$, then by (2.7) and the right-hand side of (2.10) we shall have

$$\frac{1}{\mu B} \int_B \widetilde{\varphi}\Big(\varepsilon_2 s \frac{wB}{\mu B w(x)}\Big) d\mu \le c_{11} \widetilde{\varphi}(s)$$

for some $\varepsilon_2 > 0$ not depending on the ball B.

Putting $\widetilde{\varphi}_1(t) = t\widetilde{\varphi}(t)$, we rewrite the latter inequality as

$$\int_B \widetilde{\varphi}_1\Big(\varepsilon_2 \frac{swB}{\mu B w(x)}\Big) w(x)\, d\mu \le c_{11} \widetilde{\varphi}(s) wB,$$

which by Proposition 2.5 from [24] implies that $w \in \mathcal{A}_{p(\varphi_1)}$. Since, by assumption, $p(\varphi) = 1$, applying Lemma 2.4 we have $\widetilde{\varphi} \notin \Delta_2$, i.e., $\widetilde{\varphi}_1 \notin \Delta_2$, and again applying Lemma 2.4 we find that $p(\varphi_1) = 1$ and therefore $w \in \mathcal{A}_1$.

Let now $p(\varphi) > 1$. Then by Lemma 2.4 $\widetilde{\varphi} \in \Delta_2$ and thus from Proposition 3.5 we conclude that $\widetilde{\varphi}(\frac{t}{w}) \in \mathcal{A}_\infty$ uniformly with respect to t. Due to the inverse Hölder inequality (see [6]) there exist $\delta > 0$ and $c_{12} > 0$ such that

$$\left(\frac{1}{\mu B}\int_B \widetilde{\varphi}^{1+\delta}\Big(\frac{t}{w(x)}\Big)\,d\mu\right)^{\frac{1}{1+\delta}} \le c_{12}\frac{1}{\mu B}\int_B \widetilde{\varphi}\Big(\frac{t}{w(x)}\Big)\,d\mu$$

for an arbitrary ball B and number $t > 0$.

Thus from (3.14) we can obtain

$$\begin{aligned}\frac{1}{\mu B}\int_B \widetilde{\varphi}^{1+\delta}\left(\varepsilon\,\frac{\int_B \varphi(\lambda w(y))d\mu}{\lambda \mu B w(x)}\right)\,d\mu \le \\ \le c_{13}\left(\frac{1}{\mu B}\int_B \varphi\big(\lambda w(x)\big)\,d\mu\right)^{1+\delta}.\end{aligned} \tag{3.15}$$

Further, by the definition of the number $p(\varphi)$ the function $\varphi(t^{\frac{1}{p(\varphi)-\varepsilon}})$ is quasiconvex for all sufficiently small $\varepsilon > 0$. Let this number be chosen so that

$$\frac{(p(\varphi)-\varepsilon)'}{1+\delta} < p'(\varphi).$$

By Lemma 2.1 there exists a $\varepsilon_3 > 0$ such that

$$\varphi\left(c_3\lambda\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\Big)^{\frac{1}{p(\varphi)-\varepsilon}}\right) \le \frac{1}{\mu B}\int_B \varphi\big(\lambda w(x)\big)\,d\mu.$$

Hence from (3.15) we have

$$\begin{aligned}\frac{1}{\mu B}\int_B \widetilde{\varphi}^{1+\delta}\left(\varepsilon\,\frac{1}{\lambda w(x)}\,\varphi\left(c_3\lambda\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu\Big)^{\frac{1}{p(\varphi)-\varepsilon}}\right)\right)\,d\mu \le \\ \le c_{14}\varphi^{1+\delta}\left(\varepsilon_3\lambda\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu\Big)^{\frac{1}{p(\varphi)-\varepsilon}}\right).\end{aligned}$$

If in this inequality we insert

$$\lambda = 2\,\frac{\widetilde{\varphi}(s^{\frac{1}{p(\varphi)-\varepsilon}})}{s^{\frac{1}{p(\varphi)-\varepsilon}}}\,\left(\frac{\mu B}{\varepsilon\int_B w^{p(\varphi)-\varepsilon}(x)d\mu}\right)^{\frac{1}{p(\varphi)-\varepsilon}}$$

and apply Lemmas 2.2 and 2.3, then we shall find

$$\frac{1}{\mu B}\int_B \widetilde{\varphi}^{1+\delta}\left(\varepsilon_4\,\frac{s^{\frac{1}{p(\varphi)-\varepsilon}}}{w(x)}\,\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu\Big)\right) \le c_{15}\widetilde{\varphi}^{1+\delta}\big(s^{\frac{1}{p(\varphi)-\varepsilon}}\big).$$

Putting $\widetilde{\psi}_1(t) = \widetilde{\varphi}^{1+\delta}(t^{\frac{1}{p(\varphi)-\varepsilon}}) \cdot t$, we can rewrite the latter inequality as

$$\int_B \widetilde{\psi}_1\Big(\frac{\varepsilon_4 s}{\mu B w^{p(\varphi)-\varepsilon}(x)} \int_B w^{p(\varphi)-\varepsilon}(y)\,d\mu\Big) w^{p(\varphi)-\varepsilon}(x)\,d\mu \leq$$
$$\leq c_{15}\widetilde{\psi}_(s) \int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu.$$

Hence by virtue of Proposition A we conclude that $w^{p(\varphi)-\varepsilon} \in \mathcal{A}_{p(\psi_1)}$. Now let us show that $p(\psi_1) \leq p(\varphi) - \varepsilon$. Let the number ε_5 be chosen so that

$$\frac{(p(\varphi)-\varepsilon)'}{1+\delta} = \big(p(\varphi)+\varepsilon_5\big)'.$$

By the definition of $p(\varphi)$ the function $\frac{\varphi(t)}{t^{p(\varphi)+\varepsilon_5}}$ is not quasiincreasing and therefore by Lemma 2.5 the function $t^{-p(\varphi)+\varepsilon_5)'}\widetilde{\varphi}(t)$ cannot be decreasing. This means that the function

$$t^{-\frac{(1+\delta)(p(\varphi)+\varepsilon_5)'}{p(\varphi)-\varepsilon}-1}\widetilde{\psi}(t)$$

is not quasiincreasing. Thus by Lemma 2.5 the function $t^{-(p(\varphi)-\varepsilon)}\psi_1(t)$ cannot quasiincrease. Hence it follows that $p(\psi_1) \leq p(\varphi) - \varepsilon$. On the other hand, as shown above, $w^{p(\varphi)-\varepsilon} \in \mathcal{A}_{p(\psi_1)}$. We finally conclude that $w^{p(\varphi)-\varepsilon} \in \mathcal{A}_{p(\varphi)-\varepsilon}$ for $p(\varphi) > 1$ and $w \in \mathcal{A}_1$ for $p(\varphi) = 1$.

Using a reasoning similar to the above one and taking $\widetilde{\varphi}$ instead of φ, one can prove that $w^{-(p(\widetilde{\varphi})-\varepsilon)} \in \mathcal{A}_{p(\widetilde{\varphi})-\varepsilon}$ for $p(\widetilde{\varphi}) > 1$ and $w^{-1} \in \mathcal{A}_1$ for $p(\widetilde{\varphi}) = 1$.

Now we shall show that $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$ for $p(\varphi) > 1$. Let first $p(\widetilde{\varphi}) = 1$. As shown above, then $w^{p(\varphi)-\varepsilon} \in \mathcal{A}_{p(\varphi)-\varepsilon}$ and $w^{-1} \in \mathcal{A}_1$. The function w^{-1} satisfies the inverse Hölder inequality by which we have

$$\Big(\frac{1}{\mu B}\int_B w^{-(1+\delta)}(x)\,d\mu\Big)^{\frac{1}{1+\delta}} \Big(\frac{1}{\mu B}\int_B w^{(1+\delta)'}(x)\,d\mu\Big)^{\frac{1}{(1+\delta)'}} \leq$$
$$\leq c_{17}\Big(\frac{1}{\mu B}\int_B w^{-1}(x)\,d\mu\Big)\Big(\frac{1}{\mu B}\int_B w^{(1+\delta)'}(x)\,d\mu\Big) \leq c_{18}.$$

Therefore $w^{(1+\delta)'} \in \mathcal{A}_{(1+\delta)'}$. Choose the number δ so small that $p(\varphi) < (1+\delta)'$. Thus by the latter inclusion, the condition $w^{p(\varphi)-\varepsilon} \in \mathcal{A}_{p(\varphi)-\varepsilon}$, and Lemma 3.1 we conclude that $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$. If $p(\widetilde{\varphi}) > 1$, then due to the above-said $w^{-(p(\widetilde{\varphi})-\varepsilon)} \in \mathcal{A}_{p(\widetilde{\varphi})-\varepsilon}$. Hence we conclude that $w^{(p(\widetilde{\varphi})-\varepsilon)'} \in \mathcal{A}_{(p(\varphi)-\varepsilon)'}$.

Further by Lemma 2.6 we have $p(\varphi) \leq p'(\widetilde{\varphi}) \leq (p(\widetilde{\varphi})-\varepsilon)'$. Again from Lemma 3.1 we conclude that $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$. In a similar way we show that $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$.

Finally, let us show that (ii)$\Rightarrow$(i). By the definition of the number $p(\widetilde{\varphi})$ the function $\widetilde{\varphi}(\frac{1}{t^{p(\widetilde{\varphi})-\varepsilon}})$ is quasiconvex and therefore by Lemma 2.5 $t^{-(p(\widetilde{\varphi})-\varepsilon)'}\varphi(t)$ quasidecreases. Using Remark 2.1 and the definition of the class $\mathcal{A}_p$, we obtain

$$\frac{1}{\mu B}\int_B \varphi\big(\lambda w(x)\big)\,d\mu \le \varphi\Big(c_{18}\Big(\frac{1}{\mu B}\int_B \big(\lambda w(x)\big)^{(p(\widetilde{\varphi})-\varepsilon)'}d\mu\Big)^{\frac{1}{(p(\widetilde{\varphi})-\varepsilon)'}}\Big) \le$$
$$\le \varphi\Big(c_{18}\Big(\frac{1}{\mu B}\int_B \big(\lambda w(x)\big)^{-(p(\widetilde{\varphi})-\varepsilon)}d\mu\Big)^{\frac{1}{p(\widetilde{\varphi})-\varepsilon}}\Big) \le$$
$$\le \varphi\Big(c_{19}\lambda\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu\Big)^{\frac{1}{(p(\varphi)-\varepsilon}}\Big).$$

Hence we conclude

$$\int_B \widetilde{\varphi}\Big(\frac{\varepsilon}{\lambda w(x)}\,\frac{1}{\mu B}\int_B \varphi\big(\lambda w(y)\big)\,d\mu\Big)\,d\mu \le$$
$$\le \int_B \widetilde{\varphi}\Big(\frac{\varepsilon}{\lambda w(x)}\,\varphi\Big(c_{19}\lambda\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(y)\,d\mu\Big)\Big)^{\frac{1}{p(\varphi)-\varepsilon}}\Big)d\mu.$$

Let

$$E = \Big\{x \in X \ :\ c_{19}\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(y)\,d\mu\Big)^{\frac{1}{p(\varphi)-\varepsilon}} \ge w(x)\Big\}.$$

By Lemma 2.3 we derive the estimate

$$\int_{B\setminus E} \widetilde{\varphi}\Big(\frac{\varepsilon}{\lambda w(x)}\,\varphi\Big(c_{19}\lambda\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(y)\,d\mu\Big)^{\frac{1}{p(\varphi)-\varepsilon}}\Big)\Big)d\mu \le$$
$$\le \int_{B\setminus E} \widetilde{\varphi}\Big(\varepsilon\,\frac{\varphi(c_{20}\lambda w(x))}{\lambda w(x)}\Big)\,d\mu \le c_{21}\int_B \varphi\big(\lambda w(x)\big)\,d\mu. \tag{3.16}$$

Since the function $\widetilde{\varphi}(t)t^{-(p(\varphi)-\varepsilon)'}$ quasidecreases, there exists a constant $c_{22} > 0$ such that $\widetilde{\varphi}(at) \le c_{22}a^{(p(\varphi)-\varepsilon)'}\widetilde{\varphi}(t)$ for an arbitrary $a > 1$.

Thus we can derive the estimates

$$\int_E \widetilde{\varphi}\left(\frac{\varepsilon}{\lambda w(x)}\,\varphi\Big(c_{19}\lambda\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu\Big)^{\frac{1}{p(\varphi)-\varepsilon}}\Big)\right)\,d\mu \le$$
$$\le c_{23}\widetilde{\varphi}\left(cc_{24}\,\frac{\varphi\left(c_{25}\lambda\left(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu\right)^{\frac{1}{p(\varphi)-\varepsilon}}\right)}{\lambda c_{25}\left(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu\right)}\right)\times$$
$$\times\int_E \left(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(y)\,d\mu\right)^{\frac{(p(\varphi)-\varepsilon)'}{p(\varphi)-\varepsilon}} w^{-(p(\varphi)-\varepsilon)'}(x)\,d\mu \le$$

$$\leq \varphi\left(c\lambda\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu\Big)^{\frac{1}{p(\varphi)-\varepsilon}}\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu\Big)^{\frac{(p(\varphi)-\varepsilon)'}{p(\varphi)-\varepsilon}}\times\right.$$

$$\left.\times\int_B w^{-(p(\varphi)-\varepsilon)'}(x)\,d\mu\right)\leq\varphi\left(c\Big(\frac{1}{\mu B}\int_B w^{p(\varphi)-\varepsilon}(x)\,d\mu\Big)^{\frac{1}{p(\varphi)-\varepsilon}}\right)\mu B\leq$$

$$\leq c\int_B \varphi\big(\lambda w(x)\big)\,d\mu.$$

From the above estimates and (3.16) we conclude that (i) is valid. □

Proof of Theorem 3.2. The validity of statements (i) and (iii) follows from Propositions 2.4 and 2.7 in [24], while that of (ii) from Proposition 3.6. □

Since the particular cases of Theorem 3.2 are very interesting, we shall formulate them as separate theorems.

Theorem 3.3. *Let $\varphi\in\Phi$. Then the following conditions are equivalent:*
(i) *the inequality*

$$\varphi(\lambda)\,w\big\{x\in X:\mathcal{M}f(x)>\lambda\big\}\leq c_1\int_X\varphi\big(cf(x)\big)w(x)\,d\mu$$

holds, where the constant c_1 is independent of $\lambda>0$ and f;
(ii) *φ is quasiconvex and $w\in\mathcal{A}_{p(\varphi)}$.*

Theorem 3.4. *Let $\varphi\in\Phi$. The following statements are equivalent:*
(i) *the inequality*

$$\int\limits_{\{x\in X:\mathcal{M}f(x)>\lambda\}}\varphi\big(\lambda w(x)\big)\,d\mu\leq c_2\int\limits_X\varphi\big(c_2f(x)w(x)\big)\,d\mu$$

holds, where the constant c_2 is independent of $\lambda>0$ and f;
(ii) *φ is quasiconvex, $w^{p(\varphi)}\in\mathcal{A}_{p(\varphi)}$, and $w^{-p(\widetilde{\varphi})}\in\mathcal{A}_{p(\widetilde{\varphi})}$.*

Theorem 3.5. *Let $\varphi\in\Phi$. Then the following conditions are equivalent:*
(i) *the inequality*

$$\int\limits_{\{x\in X:\mathcal{M}f(x)>\lambda\}}\varphi\Big(\frac{\lambda}{w(x)}\Big)\,d\mu\leq c_3\int\limits_X\varphi\Big(c_3\,\frac{f(x)}{w(x)}\Big)w(x)\,d\mu$$

holds, where the constant c_3 is independent of f and $\lambda>0$;
(ii) *φ is quasiconvex and $w\in\mathcal{A}_{p(\widetilde{\varphi})}$.*

Theorem 3.3 was proved by us earlier in [24] (see Proposition 2.4). The other two theorems are new.

§ 4. Weighted Inequalities for the Marcinkiewicz Integral

We shall investigate the Marcinkiewicz integral in weighted Orlicz classes. This integral plays an important role in the theory of singular integrals. Conceptually, the investigation of this paragraph is close to [27]. We introduce some analogues of the Marcinkiewicz integral in homogeneous type spaces and generalize the results of [24] for weighted Orlicz classes. The results obtained will further be used in investigating weighted problems for singular integrals defined on homogeneous type spaces.

In what follows a nonnegative function $h : [0, \infty) \times [0, \infty) \to \mathbb{R}^1$ will be assumed to be such that for arbitrary $t \geq 0$ and $x \in X$ the function

$$\frac{h(t, s)}{\mu B(x, t + s)}$$

nondecreases and there exists a positive constant c such that

$$\int_0^\infty \frac{h(t, s)}{t + s} \frac{\mu B(x, s)}{\mu B(x, t + s)} \, ds < \infty \tag{4.1}$$

for an arbitrary $t > 0$.

Fix a measurable nonnegative function $F : X \to \mathbb{R}^1$. The expression

$$\mathcal{I} f(x) = \int_X \frac{h(F(y), d(y, x))}{\mu B(y, d(y, x) + F(y))} f(y) \, d\mu$$

will be called the generalized Marcinkiewicz integral of a nonnegative measurable function $f : X \to \mathbb{R}^1$.

Theorem 4.1. *There exists a positive constant c such that the inequality*

$$\int_X \mathcal{I} f(x) g(x) \, d\mu \leq c \int_X f(x) \mathcal{M} g(x) \, d\mu \tag{4.2}$$

holds for arbitrary nonnegative measurable functions $f : X \to \mathbb{R}^1$ and $g : X \to \mathbb{R}^1$.

We omit the proof because it repeats that of the corresponging theorem from [27].

Theorem 4.2. *Let $\varphi \in \Phi$ and be a quasiconvex function satisfying the condition Δ_2. If $w \in \mathcal{A}_{p(\varphi)}$, then there exists some constant $c > 0$ such that the inequality*

$$\int_X \varphi(\mathcal{I} f)(x) w(x) \, d\mu \leq c \int_X \varphi\big(f(x)\big) w(x) \, d\mu \tag{4.3}$$

holds for any measurable $f : X \to \mathbb{R}^1$.

The proof of this theorem is based on

Theorem A [24]. *Let $\varphi \in \Phi$. Then the following statements are equivalent:*

(i) *there is a constant $c_1 > 0$ such that the inequality*

$$\int_X \varphi\Big(\frac{\mathcal{M}f(x)}{w(x)}\Big) w(x)\,d\mu \le c_1 \int_X \varphi\Big(c_1 \frac{f(x)}{w(x)}\Big) w(x)\,d\mu$$

holds for all μ-measurable $f: X \to \mathbb{R}^1$;

(ii) *φ^α is quasiconvex for some $\alpha \in (0,1)$ and $w \in \mathcal{A}_{p(\widetilde{\varphi})}$.*

Proof of Theorem 4.2. By Theorem 4.1 and the Young inequality we obtain the estimates

$$\int_X \varphi\big(\mathcal{I}f(x)\big) w(x)\,d\mu = \int_X \frac{\varphi(\mathcal{I}f(x))}{\mathcal{I}f(x)}\,\mathcal{I}f(x) w(x)\,d\mu \le$$
$$\le x \int_X \Big(\frac{\varphi(\mathcal{I}f(x))}{\mathcal{I}f(x)}\,w(x)\Big) f(x)\,d\mu \le \frac{1}{2c}\int_X \varphi\Big(\frac{2c}{\varepsilon}\,f(x)\Big) w(x)\,d\mu +$$
$$+\frac{1}{2c}\int_X \widetilde{\varphi}\left(\frac{\varepsilon\mathcal{M}\left(\frac{\varphi(\mathcal{I}f(x))}{\mathcal{I}f(x)}\,w(x)\right)}{w(x)}\right) w(x)\,d\mu \tag{4.4}$$

with ε to be specified later.

Since $\varphi \in \Delta_2$, the function $\widetilde{\varphi}$ is quasiconvex for some $\alpha \in (0,1)$ and, using the condition $w \in \mathcal{A}_{p(\varphi)}$ from Theorem A, we can estimate the second term as follows:

$$I = \frac{1}{2c}\int_X \widetilde{\varphi}\left(\frac{\varepsilon\mathcal{M}\left(\frac{\varphi(\mathcal{I}f(x))}{\mathcal{I}f(x)}\,w(x)\right)}{w(x)}\right) w(x)\,d\mu \le$$
$$\le \frac{1}{2}\int_X \widetilde{\varphi}\left(\varepsilon c\frac{\varphi(\mathcal{I}f(x))}{\mathcal{I}f(x)}\right) w(x)\,d\mu.$$

Choosing $\varepsilon c < 1$ appropriately, by Lemma 2.3 we have

$$I \le \frac{1}{2}\int_X \varphi\big(\mathcal{I}f(x)\big) w(x)\,d\mu. \tag{4.5}$$

From (4.4) and (4.5) we obtain (4.3). $\square$

Theorems 4.1 and 4.2 lead to some corollaries which will further be used in investigating weighted problems for singular integrals defined on homogeneous type spaces.

First we introduce one lemma which is an analogue of Whitney's covering lemma for homogeneous type spaces.

Lemma 4.1 ([16]). *Let $\Omega \subseteq X$ be an open bounded set and $c \geq 1$. Then there exists a sequence of balls $B_j = B(x_j, r_j)$ such that*

(i) $\Omega = \bigcup\limits_{j=1}^{\infty} \overline{B}_j$, $\overline{B}_j = B(x_j c r_j)$;

(ii) *there exists a positive number $M = M(c, b, a_0, a_1)$ such that*

$$\sum_{j=1}^{\infty} \chi_{\overline{B}_j}(x) \leq M;$$

(iii) $(X \backslash \Omega) \cap \overset{=}{B}_j \leq \varnothing$ *for each j, where* $\overset{=}{B}_j = B(x_j, 3ca_1 r_j)$.

Let $h(t,s) = \omega(t/(t+s))$, where $\omega : [0,1] \to \mathbb{R}^1$ is a nondecreasing function with the condition $\omega(0) = 0$, and

$$\int_0^1 \frac{\omega(s)}{s}\, ds < \infty, \tag{4.6}$$

Then condition (4.1) will be fulfilled.

Let Ω be an open bounded set and $F(y) = \operatorname{dist}(y, X \backslash \Omega)$. By Lemma 4.1 for $c > 2a_0^1 a_1$ there exists a sequence of balls $B(x_j, r_j)$ with the above-mentioned conditions.

Under our assumptions, for $f \geq 0$, $\operatorname{supp} f \subset \Omega$ we easily obtain

$$\mathcal{I} f(x) \sim \sum_{j=1}^{\infty} \omega\Big(\frac{r_j}{r_j + d(x, x_j)}\Big) \frac{\int_{B_j} f(y)\, d\mu}{\mu B(x_j, d(x_j, x_j + r_j))}.$$

In what follows it will be assumed that $f(x) = \chi_{\bigcup\limits_j B_j}$ and the corresponding Marcinkiewicz integral will be denoted by $\mathcal{I}_\omega$.

Theorems 4.1 and 4.2 yield the following corollaries.

Corollary 4.1. *By condition (4.6) for an arbitrary nonnegative μ-measurable function $g : X \to \mathbb{R}^1$ we have*

$$\int\limits_{X \backslash \bigcup\limits_j B_j} \mathcal{I}_\omega(x) g(x)\, d\mu \leq c \int\limits_{\bigcup\limits_j B_j} \mathcal{M} g(x)\, d\mu,$$

where the constant c does not depend on g and $\{B_j\}$.

Corollary 4.2. *Let $1 \leq p < \infty$ and $w \in \mathcal{A}_p$. Then the inequality*

$$\int\limits_{X \backslash \bigcup\limits_j B_j} \mathcal{I}_\omega^p(x) g(x)\, d\mu \leq c \int\limits_{\bigcup\limits_j B_j} w(x)\, d\mu$$

holds, where the constant c does not depend on $\{B_j\}$.

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Authors' address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Z. Rukhadze St., Tbilisi 380093
Republic of Georgia