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Superordination Properties for Certain Analytic Functions [1]

H.A. Al-Kharsani, N.M. Al-Areefi

Abstract

The purpose of the present paper is to derive superordination result for functions in the class $M_{\mu}^{l,m}(\alpha, \lambda, b)$ of normalized analytic functions in the open unit disk U. A number of interesting applications of the superordination result are also considered.

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1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. We also denote by K the class of functions $f \in A$ that are convex in U.

Given two functions $f, g \in A$, where f is given by (1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{2}$$

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The Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \qquad (z \in U). \tag{3}$$

For $\alpha_j \in \mathbb{C}$ $(j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots, m)$, the generalized hypergeometric function ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$$ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n z^n}{(\beta_1)_n \cdots (\beta_m)_n n!}$$
$$(l \leq m + 1; m \in N_0 := \{0, 1, 2, \ldots\}),$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n \coloneqq \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)(\lambda+2) \cdots (\lambda+n-1) & (n \in N \coloneqq \{1, 2, 3, \ldots\}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = z {}_lF_m(\alpha_1, \alpha_l; \beta_1, \ldots, \beta_m; z). \tag{4}$$

The Dziok-Srivastava operator [4] (see also [11]) $H^{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by the Hadamrd product

$$H^{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) := h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z). \tag{5}$$

We note that the linear operator $H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ includes various other linear operators which were introduced and studied by Carlson and Shaffer [3], Hohlov [6], Ruscheweyh [10], and so on [5], [9].

Corresponding to the function $h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$, defined by (4), we introduce a function $F_\mu(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ given by

$$\begin{aligned} h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * F_\mu(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) \\ = \frac{z}{(1 - z)^\mu} \quad (z \in U, \mu > 0). \end{aligned} \tag{6}$$

Analogous to $H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$, in [2] we define the linear operator $J_\mu(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ on A as follows:

$$\begin{aligned} J_\mu(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) = F_\mu(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ (\alpha_i; \beta_j \in \mathbb{C} \setminus \overline{z}_0; i = 1, \ldots, l; j = 1, \ldots, m, \mu > 0; z \in U; f \in A). \end{aligned} \tag{7}$$

For convenience, we write

$$J_{\mu}^{l,m}(\alpha_1) := J_{\mu}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m). \tag{8}$$

This operator was defined by Cho [7] special cases were studied by Noor [8] and Alkharsani [1].

Definition 1 *Let g be analytic and univalent in U. If f is analytic in U, $f(0) = g(0)$, and $f(U) \subset g(U)$, then one says that f is subordinate to g in U, and we write $f \prec g$ or $f(z) \prec g(z)$. One also says that g is superordinate to f in U.*

Definition 2 *Let $f = \sum_{k=1}^{\infty} a_k z^k \in A$. An infinite sequence $\left\{a_k\right\}_{k=1}^{\infty}$ of complex numbers where*

$$c_k = \begin{cases} \dfrac{1}{a_k} & a_k \neq 0 \\ 0 & a_k = 0 \end{cases}$$

will be called superordinating factor if for evey $g = z + \sum_{k=2}^{\infty} b_k z^k$ in K, one has

$$f_{-1} * g \prec g \tag{9}$$

where f_{-1} is defined as follows,

$$f * f_{-1} * g \prec f * g,$$

then

$$f_{-1} = z * \sum_{k=2}^{\infty} c_k z^k$$

one also says that (11) is equivalent to

$$\sum_{k=1}^{\infty} c_k b_k z^k \prec g \qquad (z \in U; c_1 = 1),$$

or the sequence $\left\{c_k\right\}_{k=1}^{\infty}$ is a superordinating factor if and only if

$$Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \qquad (z \in U). \tag{10}$$

Definition 3 *Suppose that $f \in A$. Then the function f_{-1} is said to be a member of the class $L_{\mu}^{l,m}(\alpha,\lambda,b)$ if it satisfies*

$$\left|\frac{\lambda\mu\left(\frac{J_{\mu+1}^{l,m}(\alpha_1)f_{-1}(z)}{z}\right)+(1-\lambda\mu)\left(\frac{J_{\mu}^{l,m}(\alpha_1)f_{-1}(z)}{z}\right)-1}{\lambda\mu\left(\frac{J_{\mu+1}^{l,m}(\alpha_1)f_{-1}(z)}{z}\right)+(1-\lambda\mu)\left(\frac{J_{\mu}^{l,m}(\alpha_1)f_{-1}(z)}{z}\right)+2b(1-\alpha)-1}\right|<1 \tag{11}$$
$$(z \in U; 0 \le \alpha < 1; \lambda \ge 0; b \in \mathbb{C}\setminus\{0\}; \mu > 0),$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $L_{\mu}^{l,m}(\alpha,\lambda,b)$.

Lemma 1 *If the function f_{-1} satisfies the following conditions:*

$$\sum_{k=2}^{\infty}[1+\lambda(k-1)]C(\mu,k)|c_k| \le (1-\alpha)|b| \tag{12}$$
$$(0 \le \alpha < 1; \lambda \ge 0; b \in \mathbb{C}\setminus\{0\}; \mu > 0),$$

where

$$C(\mu,k)=\prod_{j=2}^{k}\frac{(j+\mu-2)}{(k-1)!}\psi_{k-1}, \psi_{k-1}$$
$$=\frac{(\beta_1)_{k-1}\cdots(\beta_m)_{k-1}}{(\alpha_1)_{k-1}\cdots(\alpha_l)_{k-1}}, \quad (\mu>0, k=1,2,3,\ldots), \tag{13}$$

then $f_{-1} \in L_{\mu}^{l,m}(\alpha,\lambda,b)$.

Proof. Supposes that the inequality (12) holds. Using the identity

$$z\left(J_{\mu}^{l,m}(\alpha_1)f_{-1}(z)\right)'=\mu J_{\mu+1}^{l,m}(\alpha_1)f_{-1}(z)-(\mu-1)J_{\mu}^{l,m}(\alpha_1)f_{-1}(z), \tag{14}$$

we have for $z \in U$,

$$\begin{aligned}
&\left|(1-\lambda)\frac{J_\mu^{l,m}(\alpha_1)f_{-1}(z)}{z}+\lambda(J_\mu^{l,m}(\alpha_1)f_{-1}(z))'-1\right| \\
&-\left|2b(1-\alpha)+(1-\lambda)\frac{J_\mu^{l,m}(\alpha_1)f_{-1}(z)}{z}+\lambda(J_\mu^{l,m}(\alpha_1)f_{-1}(z))'-1\right| \\
&=\left|\sum_{k=2}^{\infty}(1+\lambda(k-1)C(\mu,k)c_k z^{k-1}\right| \\
&-\left|2b(1-\alpha)+\sum_{k=2}^{\infty}(1+\lambda(k-1))C(\mu,k)c_k z^{k-1}\right| \\
&\leq \sum_{k=2}^{\infty}[1+\lambda(k-1)]C(\mu,k)|c_k||z|^{k-1} \\
&-\left\{2|b|(1-\alpha)-\sum_{k=2}^{\infty}[1+\lambda(k-1)]C(\mu,k)|c_k||z|^{k-1}\right\}
\end{aligned}$$

$$\leq 2\left\{\sum_{k=2}^{\infty}[1+\lambda(k-1)]C(\mu,k)|c_k|-|b|(1-\alpha)\right\}\leq 0, \tag{15}$$

which sows that f_{-1} belongs to $L_\mu^{l,m}(\alpha,\lambda,b)$.

Let $M_\mu^{l,m}(\alpha,\lambda,b)$ denote the class of functions f in A whose Taylor-Maclaurin coefficients a_k satisfy the condition (12).

We note that

$$M_\mu^{l,m}(\alpha,\lambda,b)\subseteq L_\mu^{l,m}(\alpha,\lambda,b). \tag{16}$$

■

Example 1 *(i) For $0\leq\alpha<1, \lambda>0, b\in\mathbb{C}\setminus\{0\}$ and $\mu>0$, the following function defined by*

$$f_0(z)=z+\frac{\mu(\lambda+1)}{2b(1-\alpha)}\psi_1 z^2\, {}_3F_2\left(1,2,1+\frac{1}{\lambda},2+\frac{1}{\lambda},\mu+1;z\right)\quad (z\in U) \tag{17}$$

is in the class $L_\mu^{l,m}(\alpha,\lambda,b)$.

(ii) For $0\leq\alpha<1,\ \lambda>0, b\in\mathbb{C}\setminus\{0\}$, and $\mu>0$, the following functions

defined by

$$f_1(z) = z \pm \frac{\mu(\lambda+1)\psi_1}{(1-\alpha)|b|} z^2 \quad (z \in U),$$

$$f_2(z) = z \pm \frac{\mu(\mu+1)(2\lambda+1)\psi_2}{(1-\alpha)|b|} z^3 \quad (z \in U),$$

$$f_3(z) = z + \mu(\lambda+1)\psi_1 z^2 \pm \frac{\mu(\mu+1)(2\lambda+1)\psi_2}{2[(1-\alpha)|b|-1]} z^3 \quad (z \in U).$$

are in the $M_\mu^{l,m}(\alpha, \lambda, b)$.

In this paper, we obtain a sharp superordination result associated with the class $M_\mu^{l,m}(\alpha, \lambda, b)$. Some applications of the main result which give important results of analytic functions are also investigated.

2 Main Theorem

Theorem 1 *Let* $f_{-1} \in M_\mu^{l,m}(\alpha, \lambda, b)$. *Then*

$$\frac{\mu(\lambda+1)\psi_1}{2[\mu(\lambda+1)\psi_1 + |b|(1-\alpha)]}(f_{-1} * g)(z) \prec g(z) \quad (z \in U) \tag{18}$$

for every function g *in* K, *and*

$$Re\ f_{-1}(z) > -\frac{\mu(\lambda+1)\psi_1 + |b|(1-\alpha)}{\mu(\lambda+1)\psi_1}. \tag{19}$$

The constant $\mu(\lambda+1)\psi_1/2[\mu(\lambda+1)\psi_1 + |b|(1-\alpha)]$ *cannot be replaced by a larger one.*

Proof. Let $f_{-1} \in M_\mu^{l,m}(\alpha, \lambda, b)$ and let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{20}$$

be any function in the class K. Then we readily have

$$\frac{\mu(\lambda+1)\psi_1}{2[\mu(\lambda+1)\psi_1 + |b|(1-\alpha)]}(f_{-1} * g)(z) = \frac{\mu(\lambda+1)\psi_1}{2[\mu(\lambda+1)\psi_1 + |b|(1-\alpha)]}\left(z + \sum_{k=2}^{\infty} b_k c_k z^k\right). \tag{21}$$

Thus, by Definition 3, the superrordination result (18) will hold true if the sequence

$$\left\{\frac{\mu(\lambda+1)c_k\psi_1}{2[\mu(\lambda+1)\psi_1+|b|(1-\alpha)]}\right\}_{k=1}^{\infty} \tag{22}$$

is a superordinating factor sequence, with $c_1 = 1$. In view of Definition 2, this is equivalent to the following inequality:

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{\mu(\lambda+1)\psi_1}{[\mu(\lambda+1)\psi_1+|b|(1-\alpha)]}c_k z^k\right\}>0, \quad (z\in U). \tag{23}$$

Now, since

$$[1+\lambda(k-1)]C(\mu,k) \qquad (\lambda\geq 0, \mu>0) \tag{24}$$

is an increasing function of K, we have

$$\begin{aligned}
&\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{\mu(\lambda+1)\psi_1}{[\mu(\lambda+1)\psi_1+|b|(1-\alpha)]}c_k z^k\right\}\\
&= \operatorname{Re}\left\{1+\frac{\mu(\lambda+1)\psi_1}{\mu(\lambda+1)\psi_1+|b|(1-\alpha)}z\right.\\
&\left.+\frac{1}{\mu(\lambda+1)\psi_1+|b|(1-\alpha)}\sum_{k=2}^{\infty}\mu(\lambda+1)\psi_1 c_k z^k\right\}\\
&> 1-\frac{\mu(\lambda+1)\psi_1}{\mu(\lambda+1)\psi_1+|b|(1-\alpha)}r\\
&-\frac{1}{\mu(\lambda+1)\psi_1+|b|(1-\alpha)}\sum_{k=2}^{\infty}(1+\lambda(k-1)C(\mu,k))|c_k|r^k\\
&> 1-\frac{\mu(\lambda+1)\psi_1}{\mu(\lambda+1)\psi_1+|b|(1-\alpha)}r\\
&-\frac{|b|(1-a)}{[\mu(\lambda+1)\psi_1+|b|(1-\alpha)]}r>0 \quad (|z|=r).
\end{aligned}$$

This proves the inequality (23), and hence also subordination result (18) asserted by Theorem 1. The inequality (19) follows from (18) by taking

$$g(z)=\frac{z}{1-z}\in K. \tag{25}$$

Next, we consider the function

$$f_1(z) = z - \frac{\mu(\lambda+1)\psi_1}{(1-\alpha)|b|} z^2 (0 \le \alpha < 1; \lambda \ge 0; b \in \mathbb{C} \setminus \{0\}, \mu > 0) \tag{26}$$

which is a member of the class $M_\mu^{l,m}(\alpha, \lambda, b)$. Then by using (18), we have

$$\frac{\mu(\lambda+1)\psi_1}{2[\mu(\lambda+1)\psi_1 + |b|(1-\alpha)]} f_{-1}(z) \prec \frac{z}{1-z} \quad (z \in U). \tag{27}$$

It can be easily verified for the function $f_1(z)$ defined by (26) that

$$\inf_{z\in U} \left\{ \mathrm{Re} \left(\frac{\mu(\lambda+1)\psi_1}{2[\mu(\lambda+1)\psi_1 + |b|(1-\alpha)]} f_{-1}(z) \right) \right\} = -\frac{1}{2} \quad (z \in U) \tag{28}$$

which completes the proof of Theorem 1. ∎

3 Some Applications

Taking $\mu = 1$ in Theorem 1, we obtain the following:

Corollary 1 *If the function f_{-1} satisfies*

$$\sum_{k=2}^{\infty} [1 + \lambda(k-1)]\psi_{k-1}|c_k| \le m \qquad (\lambda \ge 0, \quad m > 0), \tag{29}$$

then for every function g in K, one has

$$\begin{aligned} &\frac{(\lambda+1)\psi_1}{2[(\lambda+1)\psi_1 + m]} (f_{-1} * g)(z) \prec g(z) \quad (z \in U) \\ &Re\ f_{-1}(z) > -\left[1 + \frac{m}{(\lambda+1)\psi_1}\right]. \end{aligned} \tag{30}$$

The constant $\dfrac{(\lambda+1)\psi_1}{2[(\lambda+1)\psi_1 + m]}$ *cannot be replaced by larger one.*

Putting $\lambda = 0$ in Theorem 1, we have the following corollary.

Corollary 2 *If the function f_{-1} satisfies*

$$\sum_{k=2}^{\infty} C(\mu, k)|c_k| \le m, \qquad m > 0, \tag{31}$$

where $C(\mu, k)$ *is defined by (13), then for every function* g *in* K*, one has*

$$\frac{\mu\psi_1}{2[\mu\psi_1 + m]}(f_{-1} * g)(z) \prec g(z) \quad (z \in U), \quad Re\ f_{-1}(z) > -\left(1 + \frac{m}{\mu\psi_1}\right).$$

The constant $\frac{\mu\psi_1}{2[\mu\psi_1 + m]}$ *cannot be replaced by larger one.*

Next, letting $\lambda = 1$ and $\mu = 1$, in Theorem 1, we obtain the following corollary.

Corollary 3 *If the function* f_{-1} *satisfies*

$$\sum_{k=2}^{\infty} k|c_k|\psi_{k-1} \le m \qquad (m > 0), \tag{32}$$

then for every funcion g *in* K*, one has*

$$\frac{\psi_1}{2\psi_1 + m}(f_{-1} * g)(z) \prec g(z) \quad (z \in U), \quad Re\ f_{-1}(z) > -\left(1 + \frac{m}{2\psi_1}\right).$$

The constant $\frac{\psi_1}{2\psi_1 + m}$ *cannot be replaced by larger one.*

Also, by taking $\lambda = 0$ and $\mu = 1$, in Theorem 1, we have the following:

Corollary 4 *If the function* f *satisfies*

$$\sum_{k=2}^{\infty} \psi_{k-1}|c_k| \le m \quad (m > 0), \tag{33}$$

then for every function g *in* K*, one has*

$$\frac{\psi_1}{2(\psi_1 + m)}(f_{-1} * g)(z) \prec g(z) \quad (z \in U), \quad Re\ f_{-1} < -\left(1 + \frac{m}{\psi_1}\right). \tag{34}$$

The constant $\frac{\psi_1}{2(\psi_1 + m)}$ *cannot be replaced by larger one.*

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H.A. Al-Kharsani, N.M. Al-Areefi
Faculty of Science
Department of Mathematics
P.O. Box 838, Dammam 31113, Saudi Arabia
e-mail: najarifi@hotmail.com