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pr - Homeomorphisms On Quotient Spaces [1]

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Abstract

This paper is aimed to introduce pr - homeomorphisms a new weaker form of g-homeomorphisms. Further the notion of pr^* - homeomorphisms is defined. Different characterizations of the introduced concept are found to develop a good insight into the spaces. Some properties of pr - homeomorphisms and pr^* - homeomorphisms from quotient space to other spaces are obtained.

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1 Introduction

Crossley and Hildebrand [2] studied semi-homeomorphisms which are generalizations of homeomorphisms. Maki et al [6] introduced the notions of generalized homeomorphisms and gc - homeomorphisms. In this paper we introduce a new classes of homeomorphisms namely pr - homeomorphisms and pr^* - homeomorphisms which are weaker than g - homeomorphisms. Further we investigate the notions of pr - homeomorphisms and pr^* - homeomorphisms on quotient spaces. Some properties of them with pr - compactness are also studied.Throughout the paper X, Y and Z denotes the topological spaces (X, τ), (Y, σ) and (Z, μ).

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2 Preliminaries

Definition 1 *A subset A of* (X, τ) *is called*
(i) a preclosed set [7] if $cl(int(A)) \subset A$.
(ii) a regular open set [11] if $A = int(cl(A))$ *and a regular closed set if* $A = cl(int(A))$.
(iii) a regular semiopen set [4] if there exit a regular open set U *such that* $U \subset A \subset cl(U)$*.The family of all regular semiopen sets of* X *is denoted by* $RSO(X)$.
(iv) pr - closed set [8] if $pcl(A) \subset U$ *whenever* $A \subset U$ *and* U *is regular semiopen in* (X, τ)*. The family of all pr - closed subsets of the space* (X, τ) *is denoted by* $PRC(X, \tau)$.

Definition 2 *Let* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *be a map.* f *is said to be*
(i) pr - continuous [8] if $f^{-1}(V)$ *is pr - closed in* X *for every closed set* V *of* Y.
(ii) pr - irresolute [8] if the inverse image of every pr - closed set in Y *is pr - closed in* X.

3 Characterizations Of *pr* - Homeomorphisms On Quotient Spaces

Definition 3 *A bijection* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *is called pr - homeomorphism if* f *is both pr - continuous and pr - open.*

Definition 4 *A map* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *is pr-closed if the image* $f(A)$ *is pr-closed in* Y *for every closed set* A *in* X.

Definition 5 *A map* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *is pr*-closed map if the image* $f(A)$ *of every pr-closed set* A *in* X *is pr - closed in* Y.

Definition 6 *A map* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *is pr-regular semiclosed if the image of every preclosed set in* X *is regular semiclosed in* Y.

Definition 7 *A function* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *is called regular semi pr-closed if the inverse image of every preclosed set in* Y *is regular semiclosed in* X.

Proposition 1 *If a map* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *is injective pr-regular semi-closed and regular semi pr - closed, then* $RSO(X, \tau) = \tau$.

Proof : Let A be closed in (X,τ). As f is pr - regular semiclosed, $f(A)$ is regular semiclosed in (Y,σ). Since $f(A)$ is preclosed and f is injective regular semi pr - closed $f^{-1}(f(A)) = A$ is regular semiclosed in X. Thus every closed set is regular semiclosed. Hence $RSO(X,\tau) = \tau$.

Proposition 2 *Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be regular semi continuous and pr-closed. Then for every pr - closed set $A \subset X$, $f(A)$ is pr - closed in Y.*
Proof : Let A be pr - closed in X and $f(A) \subset R$ where R is regular semiopen in Y. Then $A \subset f^{-1}(R)$. Since A is pr - closed, $pcl(A) \subset f^{-1}(R)$ that is $f(pcl(A)) \subset R$. Since f is pr - closed, $pcl(f(pcl(A))) \subset R$ and so $pcl(f(A)) \subset R$. Thus $f(A)$ is pr - closed.

Corollary 1 *If $f : (X,\tau) \longrightarrow (Y,\sigma)$ is a pr-regular semiclosed and regular semi pr-closed map then f is pr-irresolute and pr^*-closed.*
Proof : The proof is obvious.

Proposition 3 *(i) Suppose the canonical projection $p : (X,\tau) \longrightarrow (X^*,\tau^*)$ is pr-closed and $RSO(X,\tau) = \tau$. If for a subset F of (X^*,τ^*) the inverse $p^{-1}(F)$ is pr-closed in (X,τ), then the set F is pr-closed in (X^*,τ^*).*
(ii) Suppose that p is injective pr-regular semiclosed and regular semi pr-closed. Then a subset F is pr-closed in (X^*,τ^*) if and only if the inverse image $p^{-1}(F)$ is pr-closed in (X,τ).
Proof : (i) Since p is continuous and $RSO(X,\tau) = \tau$ it is regular semi continuous. Also it is pr- closed. Thus the image $p(p^{-1}(F)) = F$ of the pr-closed set $p^{-1}(F)$ is pr-closed in (X,τ) by Proposition 2 .
(ii) (Necessity) Suppose F is pr-closed in (X^*,τ^*). By Corollary 1 $p^{-1}(F)$ is pr-closed in (X,τ). (Sufficiency) Suppose $p^{-1}(F)$ is pr-closed in (X,τ). By Proposition 1 $RSO(X,\tau) = \tau$. By Corollary 1, p is pr^*-closed and thus pr-closed. By (i), F is pr-closed in (X^*,τ^*).

Remark 1 *Suppose p is injective pr-regular semiclosed and regular semi pr-closed . Then a subset V is pr-open in (X^*,τ^*) if and only if the inverse image $p^{-1}(V)$ is pr-open in (X,τ).*

Remark 2 *Given any partition X^* of X, there is exactly one equivalence relation on X from which it is derived. Suppose a map $f : (X,\tau) \longrightarrow (Y,\sigma)$ satisfies the condition that if xRy for x, $y \in X$, then $f(x) = f(y)$. Then the induced map $f^{\perp} : (X^*,\tau^*) \longrightarrow (Y,\sigma)$ is well defined by $f^{\perp}([x]) = f(x)$ for every $x \in X$ where $[x]$ is the equivalence class of x or the set containing x.*

Theorem 1 *Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a map satisfying the condition if xRy for $x,\ y \in X$, then $f(x) = f(y)$. Suppose that the canonical projection $p : (X,\tau) \longrightarrow (X^*,\tau^*)$ is a pr-closed map and $RSO(X,\tau) = \tau$. If f is pr-continuous (resp. pr-irresolute) then the induced map $f^{\perp} : (X^*,\tau^*) \longrightarrow (Y,\sigma)$ is pr-continuous (resp. pr-irresolute).*
Proof : Let V be a closed set (resp. pr-closed set) in (Y,σ) . Since $p^{-1}((f^{\perp})^{-1}(V)) = f^{-1}(V)$ and f is pr-continuous (resp. pr-irresolute) the set $p^{-1}((f^{\perp})^{-1}(V))$ is pr-closed in (X,τ). $(f^{\perp})^{-1}(V)$ is pr-closed in (X^*,τ^*) by Proposition 3 (i). That is $f^{\perp}$ is pr-continuous (resp. pr-irresolute).

Theorem 2 *Suppose that $p : (X,\tau) \longrightarrow (X^*,\tau^*)$ is injective pr-regular semi-closed and regular semi pr-closed , then the following statements are equivalent.*
(i) f is pr-continuous(resp. pr-irresolute).
(ii) The induced map $f^{\perp} : (X^*,\tau^*) \longrightarrow (Y,\sigma)$ is pr-continuous (resp. pr-irresolute).
Proof : (i) $\Rightarrow$ (ii)
Let U be closed(pr−closed) in (Y, σ). As f is pr-continuous, $f^{-1}(U)$ is pr-closed in (X,τ). By definition of $f^{\perp}$, $f^{-1}(U) = p^{-1}((f^{\perp})^{-1}(V))$. So $p^{-1}((f^{\perp})^{-1}(U))$ is pr-closed in (X,τ). By Corollary 1 and Proposition 3 (i) , $(f^{\perp})^{-1}(U)$ is pr-closed in (X^*,τ^*) and hence $f^{\perp}$ is pr-continuous (resp. pr-irresolute).
(ii) $\Rightarrow$ (i)
Let U be closed(pr −closed) in (Y, σ). By hypothesis $(f^{\perp})^{-1}(U)$ is pr-closed in (X^*,τ^*). By Proposition 3(ii), $p^{-1}((f^{\perp})^{-1}(V))$ is pr-closed in (X,τ). By definition of $f^{\perp}, p^{-1}((f^{\perp})^{-1}(V)) = f^{-1}(V)$. Thus f is pr-continuous (resp. pr-irresolute).

Theorem 3 *Suppose $p : (X,\tau) \longrightarrow (X^*,\tau^*)$ is a injective pr-regular semi-closed and regular semi pr-closed map. If $f : (X,\tau) \longrightarrow (Y,\sigma)$ is pr-continuous, onto and pr-closed, and satisfies the condition $(\Theta)xRy$ for $x,y \in X$ if and only if $f(x) = f(y)$ [R is a relation associated with the partition X^* of X]. Then the induced map $f^{\perp} : (X^*,\tau^*) \longrightarrow (Y,\sigma)$ is a pr-homeomorphism.*
Proof: Consider $f^{\perp} : (X^*,\tau^*) \longrightarrow (Y,\sigma)$. Suppose $f^{\perp}([x]) = f^{\perp}([y])$, then $f(x) = f(y)$. By (Θ), [x] = [y]. Hence $f^{\perp}$ is one to one. Let $y \in Y$, since f is onto there exists an $x \in X$ such that $f(x) = y$. Thus $f^{\perp}([x]) = y$ and so $f^{\perp}$ is onto. Since f is pr-continuous, $f^{\perp}$ is pr-continuous by Theorem 1. Let U be closed in (X^*,τ^*). As p is regular semi pr-closed, $p^{-1}(U)$ is regular semiclosed and thus closed in (X,τ). As f is pr-closed $f(p^{-1}(U))$ is pr-closed in (Y,σ). That is, $f^{\perp}(U) = f(p^{-1}(U))$ is pr-closed in (Y,σ). Thus $f^{\perp}$ is pr-closed. Hence the induced map $f^{\perp}$: $(X^*,\tau^*) \longrightarrow (Y,\sigma)$ is a pr-homeomorphism.

Remark 3 *Consider the partition Y^* of Y. Let B be an equivalence relation on (Y, σ) associated with the partition Y^*. Let $p_1 : (Y, \sigma) \longrightarrow (Y^*, \sigma^*)$ be the quotient map. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be the map satisfying the condition ($\Theta\Theta$) $if\ x\ R\ y\ for\ x,\ y \in X$ then $f(x)Bf(y)$.Then the induced map $f_* : (X^*, \tau^*) \longrightarrow (Y^*, \sigma^*)$ is well defined by $f_*([x]) = p_1(f(x))$ for every $[x] \in X^*$.*

Proposition 4 *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a bijective pr-continuous map, then the following are equivalent*
(i) f is a pr-open map.
(ii) f is a pr-homeomorphism.
(iii) f is a pr-closed map.
Proof: The proof is immediate.

Definition 8 *A bijection $f : (X, \tau) \longrightarrow (Y, \sigma)$ is pr^*-homeomorphism if f is pr-irresolute and its inverse f^{-1} is also pr-irresolute.*

Theorem 4 *Suppose that p: $(X, \tau) \longrightarrow (X^*, \tau^*)$ is pr-regular semiclosed and regular semi pr-closed and $p_1 : (Y, \sigma) \longrightarrow (Y^*, \sigma^*)$ is regular semi continuous and pr-closed. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a pr-continuous map (resp. pr-irresolute map) satisfying the condition xRy if and only if $f(x)Bf(y)$ for all x,y $\in X$.*
(i) The induced map f_*: $(X^*, \tau^*) \longrightarrow (Y^*, \sigma^*)$ is pr-continuous (resp. pr-irresolute).
(ii) If there exists a pr-continuous (resp. pr-irresolute) map $k : (Y, \sigma) \longrightarrow (X, \tau)$ such that $f_* \circ p \circ k = p_1$ and the converse of $(\Theta\Theta)$ holds, then f_* is a pr-homeomorphism (resp. pr^*-homeomorphism).
Proof: (i) Let $g = p_1 \circ f : (X, \tau) \longrightarrow (Y^*, \sigma^*)$ then xRy for x,y $\in X$ implies $f(x)Bf(y)$ and so $[f(x)] = [f(y)]$. But $g(x) = p_1(f(x)) = [f(x)]$ and $g(y) = p_1(f(y)) = [f(y)]$. Thus $g(x) = g(y)$. Also the induced map $g^\perp$: $(X^*, \tau^*) \longrightarrow (Y^*, \sigma^*)$ defined by $g^\perp([x]) = g(x)$ is well defined. $g^\perp = f_*$. Since $g^\perp([x]) = g(x) = p_1(f(x)) = f_*([x])$ for every $[x] \in X^*$. Now g is pr-continuous (resp. pr-irresolute). Since p_1 is continuous and f is pr-continuous. By Theorem 1, $g^\perp$ is pr-continuous (resp. pr-irresolute). That is, f_* is pr-continuous (resp. pr-irresolute).
(ii) From (i) and hypothesis, follows that f_* is pr-continuous (resp. pr-irresolute) bijection. Let F be closed (resp. pr-closed) set of (X^*, τ^*). Then $f_*(F) = p_1(p \circ k)^{-1}(F)$ holds and $p \circ k$ is pr-continuous, $(p \circ k)^{-1}(F)$ is pr-closed in Y. Since p_1 is regular semi continuous and pr-closed, $f_*(F)$ is pr-closed and hence $f_*(resp.(f_*)^{-1})$ is pr-closed (resp. pr-irresolute). Therefore by Proposition 4 (resp. Definition 8) f_* is a pr-homeomorphism (resp. pr^*-

homeomorphism).
Some properties of pr-compactness is investigated here.

Definition 9 *A collection* $\{A_i : i \in \Lambda\}$ *of pr-open sets in a topological space* X *is called a pr-open cover of a subset* S *if* $S \subset \cup\{A_i : i \in \Lambda\}$ *holds.*

Definition 10 *A topological space* (X, τ) *is pr-compact if every pr-open cover of* X *has a finite subcover.*

Definition 11 *A subset* S *of a topological* X *is said to be pr-compact relative to* X*, if for every collection* $\{A_i : i \in \Lambda\}$ *of pr-open subsets of* X *such that* $S \subset \cup\{A_i : i \in \Lambda\}$ *there exists a finite subset* Λ_o *of* Λ *such that* $S \subset \cup\{A_i : i \in \Lambda_o\}$.

Proposition 5 *Suppose that the canonical projection* $p : (X, \tau) \longrightarrow (X^*, \tau^*)$ *is pr-regular semiclosed and regular semi pr-closed. If* (X, τ) *is pr-compact, then* (X^*, τ^*) *is pr-compact.*
Proof : *Let* $\{A_i : i \in \Lambda\}$ *be any pr-open covering of* (X^*, τ^*)*. That is,* $X^* = \cup\{A_i : i \in \Lambda\}$ *where each* A_i *is a pr-open set of* (X^*, τ^*)*. Now by Remark 1, the family* $\left\{p^{-1}(A_i) : i \in \Lambda\right\}$ *is a open covering of* (X, τ)*. Since* X *is pr-compact there exists a finite subcovering say* $\left\{p^{-1}(A_i) : i = 1, 2, ..., n\right\}$ *such that* $X = \cup\left\{p^{-1}(A_i) : i = 1, 2, ..., n\right\}$.
Now $X^* = p(X) = p(\cup\left\{p^{-1}(A_i) : i = 1, 2, ..., n\right\})$
$= \cup\left\{p(p^{-1}(A_i) : i = 1, 2, ..., n)\right\}$
$= \cup\{A_i : i = 1, 2, ..., n\}$*. Hence* $\{A_i : i = 1, 2, ..., n\}$ *forms a finite subcovering of* X^* *and thus* X^* *is pr-compact.*

Proposition 6 *Let* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *be a pr-continuous map and let* H *be a pr-compact set relative to* (X, τ)*, then* $f(H)$ *is compact in* (Y, σ).
Proof : *Let* $\{A_i : i \in \Lambda\}$ *be a collection of open subsets of* (Y, σ) *such that* $f(H) \subset \cup\{A_i : i \in \Lambda\}$*. Then* $H \subset f^{-1}(\cup\{A_i : i \in \Lambda\}) = \cup\left\{f^{-1}(A_i) : i \in \Lambda\right\}$*. Since* f *is pr-continuous,* $\left\{f^{-1}(A_i) : i \in \Lambda\right\}$ *is a covering of* H *by pr-open sets in* X*. Since* H *is pr-compact, there exists a finite subcovering say*
$\left\{f^{-1}(A_1), f^{-1}(A_2), ..., f^{-1}(A_n)\right\}$.
Then $H \subset \cup\left\{f^{-1}(A_i) : i = 1, 2, ..., n\right\}$ *and so*
$f(H) \subset f(\cup\left\{f^{-1}(A_i) : i = 1, 2, ..., n\right\}) = \cup\left\{f(f^{-1}(A_i)) : i = 1, 2, ..., n\right\} \subset \cup\{A_i : i = 1, 2, ..., n\}$*. Therefore* $\{A_1, A_2, ..., A_n\}$ *is a finite subcovering of* $f(H)$ *and thus* $f(H)$ *is compact in* Y.

Proposition 7 *A pr-closed subset of pr-compact space* X *is pr-compact relative to* X.

Proof : Let A be pr-closed subset of a pr-compact space X, then $X - A$ is pr-open. Let Ω be a pr-open cover for A. Then $\{\Omega, X - A\}$ is a pr-open cover for X. Since X is pr-compact, it has a finite subcover, say $\{P_1, P_2, ..., P_n\} = \Omega_1$. If $X - A \notin \Omega$ then $Omega_1$ is a finite subcover of A. If $X - A \in \Omega_1$, then $\Omega_1 - (X - A)$ is a subcover of A. Hence A is pr-compact relative to X.

Theorem 5 *Suppose that (X, τ) is pr-compact, (Y, σ) is Hausdroff and map $p : (X, \tau) \longrightarrow (X^*, \tau^*)$ is pr-regular semiclosed and regular semi pr-closed. If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a pr-continuous (resp. pr-irresolute) and onto map satisfying (Θ) and the converse of $(\Theta\Theta)$, then the induced map $f^{\perp} : (X^*, \tau^*) \longrightarrow (Y, \sigma)$ is a pr-homeomorphism (resp. pr*-homeomorphism).*
Proof: By hypothesis and Theorem 1, the induced map $f^{\perp} : (X^*, \tau^*) \longrightarrow (Y, \sigma)$ is pr-continuous (resp. pr-irresolute) and bijective. Let F be closed (resp. pr-closed) in (X^*, τ^*). By Proposition 5, (X^*, τ^*) is pr-compact. Since F is pr-closed in (X^*, τ^*), it is pr-compact relative to (X^*, τ^*) by Proposition 7. By Proposition 6, we have $f^{\perp}(F)$ is compact in (Y, σ). As (Y, σ) is Hausdroff, $f^{\perp}(F)$ is closed and thus pr-closed and so $(f^{\perp})^{-1}$ is pr-continuous (resp. pr-irresolute).

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