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Common fixed point theorems for subcompatible D-maps of integral type [1]

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Abstract

Some common fixed point theorems for two pairs of subcompatible single and multivalued D-maps in metric spaces are obtained extending some results of single-valued maps of Jungck and Rhoades [9].

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1 Introduction

To generalize commuting maps, Sessa [10] introduced the notion of weakly commuting maps.

Later on, Jungck generalized commuting and weakly commuting maps, first to compatible maps [6] and then to weakly compatible maps [7].

And in 1998, the same author with Rhoades [8] extended the concept of weakly compatible maps to the setting of single and multivalued maps by giving the notion of subcompatible maps.

Recently in 2008, Al-Thagafi and Shahzad [2] introduced the concept of occasionally weakly compatible maps (owc) which is a proper generalization of nontrivial weakly compatible maps which do have a coincidence point.

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2 Preliminaries

Throughout this paper $\mathcal{X}$ stands for a metric with the metric d and $B(\mathcal{X})$ denotes the family of all nonempty, bounded subsets of $\mathcal{X}$. Define for all A, B in $B(\mathcal{X})$

$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}.$$

If $A = \{a\}$, we write $\delta(A,B) = \delta(a,B)$ and $\delta(A,B) = d(a,b)$ if $A = \{a\}$ and $B = \{b\}$. For all A, B, C in $B(\mathcal{X})$, the definition of δ yields the following properties:

$$\begin{aligned}
\delta(A,B) &= \delta(B,A) \geq 0,\\
\delta(A,B) &\leq \delta(A,C) + \delta(C,B),\\
\delta(A,A) &= diam A,\\
\delta(A,B) &= 0 \Leftrightarrow A = B = \{a\}.
\end{aligned}$$

Definition 1 *([4]) A sequence $\{A_n\}$ of nonempty subsets of $\mathcal{X}$ is said to be convergent towards a subset A of $\mathcal{X}$ if,*
(i) each point a of A is a limit of a convergent sequence $\{a_n\}$, where $a_n \in A_n$ for $n \in \mathbb{N}$,
(ii) for arbitrary $\epsilon > 0$, there is an integer m such that $n > m$, $A_n \subseteq A_\epsilon$. $A_\epsilon = \{x \in \mathcal{X} : \exists a \in A, a$ depending on x and $d(x,a) < \epsilon\}$. A is then said to be the limit of the sequence $\{A_n\}$.

Lemma 1 *([4]) Let $\{A_n\}$, $\{B_n\}$ be sequences in $B(\mathcal{X})$ converging respectively to A and B in $B(\mathcal{X})$, then the sequence of numbers $\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.*

Lemma 2 *([5]) Let $\{A_n\}$ be a sequence in $B(\mathcal{X})$ and y be a point in $\mathcal{X}$ such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(\mathcal{X})$.*

Definition 2 *([10]) Self-maps f and g of a metric space $(\mathcal{X}, d)$ are said to be weakly commuting if, for all $x \in \mathcal{X}$*

$$d(fgx, gfx) \leq d(gx, fx).$$

Definition 3 *([6]) Self-maps f and g of a metric space $(\mathcal{X}, d)$ are called compatible if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ *is a sequence in* $\mathcal{X}$ *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ *for some* $t \in \mathcal{X}$.

Definition 4 *([7]) Two maps* $f,\, g : \mathcal{X} \to \mathcal{X}$ *are said to be weakly compatible if they commute at their coincidence points.*

Definition 5 *([8]) Maps* $f : \mathcal{X} \to \mathcal{X}$ *and* $F : \mathcal{X} \to B(\mathcal{X})$ *are said to be subcompatible if they commute at coincidence points; that is,*

$$\{t \in \mathcal{X}/Ft = \{ft\}\} \subseteq \{t \in \mathcal{X}/Fft = fFt\}.$$

Definition 6 *([2]) Two self-maps* f *and* g *of a set* $\mathcal{X}$ *are owc if and only if there is a point* $t \in \mathcal{X}$ *which is a coincidence point of* f *and* g *at which* f *and* g *commute.*

In their paper [3], Djoudi and Khemis gave the notion of D-maps which extended the notion of property (E.A) given by Aamri and El Moutawakil [1].

Definition 7 *([3]) Maps* $f : \mathcal{X} \to \mathcal{X}$ *and* $F : \mathcal{X} \to B(\mathcal{X})$ *are said to be* D*-maps iff there exists a sequence* $\{x_n\}$ *in* $\mathcal{X}$ *such that for some* $t \in \mathcal{X}$

$$\lim_{n\to\infty} fx_n = t \text{ and } \lim_{n\to\infty} Fx_n = \{t\}.$$

Our objective here is to prove some common fixed point theorems for two pairs of subcompatible single and multivalued D-maps satisfying contractive condition of integral type in metric spaces. These results extend the results of Jungck and Rhoades [9].

For our main results we need the following:
Let Ψ be the set of all continuous maps $\psi : \mathbb{R}^+ \to \mathbb{R}$ such that
(ψ_1) : for all u, v in $\mathbb{R}^+$, if
$(\psi_a) : \psi(u, v, v, u, u + v, 0) \leq 0$ or
$(\psi_b) : \psi(u, v, u, v, 0, u + v) \leq 0$
we have $u \leq v$
$(\psi_2) : \varphi(u, u, 0, 0, u, u) > 0$ for all $u > 0$,
next, let Φ be the set of all maps $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that φ is Lebesgue-integrable which is summable nonnegative and satisfies $\int_0^\epsilon \varphi(t)dt > 0$ for each $\epsilon > 0$,
and let $\mathcal{F}$ be the set of all continuous maps $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that $F(t) = 0$ iff $t = 0$.

3 Main results

Theorem 1 *Let $(\mathcal{X}, d)$ be a metric space and let f, $g : \mathcal{X} \to \mathcal{X}$; F, $G : \mathcal{X} \to B(\mathcal{X})$ be single and multivalued maps, respectively. Suppose that*
(1) f and g are surjective,

$$(2) \qquad \psi\left(\int_0^{\delta(Fx,Gy)} \varphi(t)dt, \int_0^{d(fx,gy)} \varphi(t)dt, \int_0^{\delta(fx,Fx)} \varphi(t)dt\right., \\ \left.\int_0^{\delta(gy,Gy)} \varphi(t)dt, \int_0^{\delta(fx,Gy)} \varphi(t)dt, \int_0^{\delta(gy,Fx)} \varphi(t)dt\right) \leq 0$$

for all x, y in $\mathcal{X}$, where $\psi \in \Psi$ and $\varphi \in \Phi$. If either
(3) f and F are subcompatible D-maps; g and G are subcompatible, or
(3′) g and G are subcompatible D-maps; f and F are subcompatible.
Then, f, g, F and G have a unique common fixed point $t \in \mathcal{X}$ such that $Ft = Gt = \{ft\} = \{gt\} = \{t\}$.

Proof. Suppose that f and F are D-maps, then, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in \mathcal{X}$. By vertue of condition (1) there are two points u and v in $\mathcal{X}$ such that $t = fu = gv$.
We show that $Gv = \{gv\}$. Indeed, by inequality (2) we have

$$\psi\left(\int_0^{\delta(Fx_n,Gv)} \varphi(t)dt, \int_0^{d(fx_n,gv)} \varphi(t)dt, \int_0^{\delta(fx_n,Fx_n)} \varphi(t)dt\right., \\ \left.\int_0^{\delta(gv,Gv)} \varphi(t)dt, \int_0^{\delta(fx_n,Gv)} \varphi(t)dt, \int_0^{\delta(gv,Fx_n)} \varphi(t)dt\right) \leq 0.$$

Since ψ is continuous, we get at infinity

$$\psi\left(\int_0^{\delta(gv,Gv)} \varphi(t)dt, 0, 0, \int_0^{\delta(gv,Gv)} \varphi(t)dt, \int_0^{\delta(gv,Gv)} \varphi(t)dt, 0\right) \leq 0$$

which from (ψ_a), gives $\int_0^{\delta(gv,Gv)} \varphi(t)dt \leq 0$, and hence $\delta(gv, Gv) = 0$, which implies that $Gv = \{gv\} = \{t\}$. Since the pair (g, G) is subcompatible, then, $Ggv = gGv$; i.e., $Gt = \{gt\}$.

We claim that $Gt = \{gt\} = \{t\}$. If not, then condition (2) implies that

$$\psi\Bigg(\int_0^{\delta(Fx_n,Gt)}\varphi(t)dt, \int_0^{d(fx_n,gt)}\varphi(t)dt, \int_0^{\delta(fx_n,Fx_n)}\varphi(t)dt\ ,$$
$$\int_0^{\delta(gt,Gt)}\varphi(t)dt, \int_0^{\delta(fx_n,Gt)}\varphi(t)dt, \int_0^{\delta(gt,Fx_n)}\varphi(t)dt\Bigg) \leq 0.$$

At infinity we get

$$\psi\left(\int_0^{d(t,gt)}\varphi(t)dt, \int_0^{d(t,gt)}\varphi(t)dt, 0, 0, \int_0^{d(t,gt)}\varphi(t)dt, \int_0^{d(gt,t)}\varphi(t)dt\right) \leq 0$$

which contradicts (ψ_2). Thus, $\int_0^{d(t,gt)}\varphi(t)dt = 0$, which implies that $\{gt\} = \{t\} = Gt$.
Next, we show that $Fu = \{fu\} = \{t\}$. Suppose not. Then inequality (2) gives

$$\psi\Bigg(\int_0^{\delta(Fu,Gt)}\varphi(t)dt, \int_0^{d(fu,gt)}\varphi(t)dt, \int_0^{\delta(fu,Fu)}\varphi(t)dt\ ,$$
$$\int_0^{\delta(gt,Gt)}\varphi(t)dt, \int_0^{\delta(fu,Gt)}\varphi(t)dt, \int_0^{\delta(gt,Fu)}\varphi(t)dt\Bigg) \leq 0;$$

that is,

$$\psi\left(\int_0^{\delta(Fu,t)}\varphi(t)dt, 0, \int_0^{\delta(t,Fu)}\varphi(t)dt, 0, 0, \int_0^{\delta(t,Fu)}\varphi(t)dt\right) \leq 0$$

which implies by (ψ_b) that $\int_0^{\delta(Fu,t)}\varphi(t)dt \leq 0$ and hence $Fu = \{t\} = \{fu\}$.
Since f and F are subcompatible, then, $Ffu = fFu$; i.e., $Ft = \{ft\}$.
Then, the use of (2) gives

$$\psi\Bigg(\int_0^{\delta(Ft,Gt)}\varphi(t)dt, \int_0^{d(ft,gt)}\varphi(t)dt, \int_0^{\delta(ft,Ft)}\varphi(t)dt\ ,$$
$$\int_0^{\delta(gt,Gt)}\varphi(t)dt, \int_0^{\delta(ft,Gt)}\varphi(t)dt, \int_0^{\delta(gt,Ft)}\varphi(t)dt\Bigg) \leq 0;$$

i.e.,

$$\psi\left(\int_0^{d(ft,t)}\varphi(t)dt, \int_0^{d(ft,t)}\varphi(t)dt, 0, 0, \int_0^{d(ft,t)}\varphi(t)dt, \int_0^{d(t,ft)}\varphi(t)dt\right) \leq 0$$

contradicts (ψ_2). Hence, $\{ft\} = \{t\} = Ft$. Therefore t is a common fixed point of maps f, g, F and G.
Now, suppose that there exists another common fixed point t' such that $t' \neq t$. Then, using inequality (2) we obtain

$$\psi\left(\int_0^{\delta(Ft,Gt')}\varphi(t)dt, \int_0^{d(ft,gt')}\varphi(t)dt, \int_0^{\delta(ft,Ft)}\varphi(t)dt\right.,$$
$$\left.\int_0^{\delta(gt',Gt')}\varphi(t)dt, \int_0^{\delta(ft,Gt')}\varphi(t)dt, \int_0^{\delta(gt',Ft)}\varphi(t)dt\right)$$
$$= \psi\left(\int_0^{d(t,t')}\varphi(t)dt, \int_0^{d(t,t')}\varphi(t)dt, 0, 0, \int_0^{d(t,t')}\varphi(t)dt, \int_0^{d(t,t')}\varphi(t)dt\right)$$
$$\leq 0$$

which contradicts (ψ_2). Thus, $t' = t$.
The proof is similar by replacing (3) with (3′).

If we let in Theorem 1, $f = g$ and $F = G$, then, we get the next corollary.

Corollary 1 *Let $(\mathcal{X}, d)$ be a metric space and let $f : \mathcal{X} \to \mathcal{X}$; $F : \mathcal{X} \to B(\mathcal{X})$ be a single and a multivalued map, respectively. If*
(1) f is surjective,

$$(2) \qquad \psi\left(\int_0^{\delta(Fx,Fy)}\varphi(t)dt, \int_0^{d(fx,fy)}\varphi(t)dt, \int_0^{\delta(fx,Fx)}\varphi(t)dt\right.,$$
$$\left.\int_0^{\delta(fy,Fy)}\varphi(t)dt, \int_0^{\delta(fx,Fy)}\varphi(t)dt, \int_0^{\delta(fy,Fx)}\varphi(t)dt\right) \leq 0$$

for all x, y in $\mathcal{X}$, where $\psi \in \Psi$ and $\varphi \in \Phi$,
(3) f and F are subcompatible D-maps.
Then, f and F have a unique common fixed point $t \in \mathcal{X}$ such that $Ft = \{ft\} = \{t\}$.

Now, if we put in Theorem 1, $f = g$, then, we obtain the following result.

Corollary 2 *Let $(\mathcal{X}, d)$ be a metric space and let $f : \mathcal{X} \to \mathcal{X}$; F, $G : \mathcal{X} \to B(\mathcal{X})$ be maps satisfying the conditions*
(1) f is surjective,

$$(2)\qquad \psi\left(\int_0^{\delta(Fx,Gy)}\varphi(t)dt, \int_0^{d(fx,fy)}\varphi(t)dt, \int_0^{\delta(fx,Fx)}\varphi(t)dt\,,\right.$$
$$\left.\int_0^{\delta(fy,Gy)}\varphi(t)dt, \int_0^{\delta(fx,Gy)}\varphi(t)dt, \int_0^{\delta(fy,Fx)}\varphi(t)dt\right)\leq 0$$

for all x, y in $\mathcal{X}$, where $\psi\in\Psi$ and $\varphi\in\Phi$. If either
(3) *f and F are subcompatible D-maps; f and G are subcompatible, or*
(3′) *f and G are subcompatible D-maps; f and F are subcompatible.*
Then, f, F and G have a unique common fixed point $t\in\mathcal{X}$ such that $Ft = Gt = \{ft\} = \{t\}$.

Using recurrence on n, we obtain the following result.

Theorem 2 *Let $(\mathcal{X}, d)$ be a metric space and let f, $g : \mathcal{X}\to\mathcal{X}$; $F_n : \mathcal{X}\to B(\mathcal{X})$, $n = 1, 2, \ldots$ be maps such that*
(1) *f and g are surjective,*

$$(2)\qquad \psi\left(\int_0^{\delta(F_nx,F_{n+1}y)}\varphi(t)dt, \int_0^{d(fx,gy)}\varphi(t)dt, \int_0^{\delta(fx,F_nx)}\varphi(t)dt\,,\right.$$
$$\left.\int_0^{\delta(gy,F_{n+1}y)}\varphi(t)dt, \int_0^{\delta(fx,F_{n+1}y)}\varphi(t)dt, \int_0^{\delta(gy,F_nx)}\varphi(t)dt\right)\leq 0$$

for all x, y in $\mathcal{X}$, where $\psi\in\Psi$ and $\varphi\in\Phi$. If either
(3) *f and F_n are subcompatible D-maps; g and F_{n+1} are subcompatible, or*
(3′) *g and F_{n+1} are subcompatible D-maps; f and F_n are subcompatible.*
Then, there exists a unique point $t\in\mathcal{X}$ such that $F_nt = \{ft\} = \{gt\} = \{t\}$.

Now, we prove our second main theorem.

Theorem 3 *Let $(\mathcal{X}, d)$ be a metric space and let f, $g : \mathcal{X}\to\mathcal{X}$; F, $G : \mathcal{X}\to B(\mathcal{X})$ be single and multivalued maps, respectively. Suppose that*
(*a*) *$F(\mathcal{X})\subseteq g(\mathcal{X})$ and $G(\mathcal{X})\subseteq f(\mathcal{X})$,*

$$(b)\qquad \psi\left(\int_0^{F(\delta(Fx,Gy))}\varphi(t)dt, \int_0^{F(d(fx,gy))}\varphi(t)dt, \int_0^{F(\delta(fx,Fx))}\varphi(t)dt\,,\right.$$
$$\left.\int_0^{F(\delta(gy,Gy))}\varphi(t)dt, \int_0^{F(\delta(fx,Gy))}\varphi(t)dt, \int_0^{F(\delta(gy,Fx))}\varphi(t)dt\right)\leq 0$$

for all x, y in $\mathcal{X}$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $\digamma \in \mathcal{F}$. If either
(c) f and F are subcompatible D-maps; g and G are subcompatible and $F(\mathcal{X})$ is closed, or
(c′) g and G are subcompatible D-maps; f and F are subcompatible and $G(\mathcal{X})$ is closed.
Then, f, g, F and G have a unique common fixed point $t \in \mathcal{X}$ such that $Ft = Gt = \{ft\} = \{gt\} = \{t\}$.

Proof. Suppose that g and G are D-maps, then, there is a sequence $\{y_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} gy_n = t$ and $\lim_{n\to\infty} Gy_n = \{t\}$ for some $t \in \mathcal{X}$. Since $G(\mathcal{X})$ is closed and $G(\mathcal{X}) \subseteq f(\mathcal{X})$, then, there exists a point $u \in \mathcal{X}$ such that $fu = t$. First, we claim that $Fu = \{fu\} = \{t\}$. If not, then, from (b),

$$\psi\Bigg(\int_0^{\digamma(\delta(Fu,Gy_n))}\varphi(t)dt, \int_0^{\digamma(d(fu,gy_n))}\varphi(t)dt, \int_0^{\digamma(\delta(fu,Fu))}\varphi(t)dt\ ,$$
$$\int_0^{\digamma(\delta(gy_n,Gy_n))}\varphi(t)dt, \int_0^{\digamma(\delta(fu,Gy_n))}\varphi(t)dt, \int_0^{\digamma(\delta(gy_n,Fu))}\varphi(t)dt\Bigg) \leq 0.$$

Since ψ and $\digamma$ are continuous, at infinity we get

$$\psi\Bigg(\int_0^{\digamma(\delta(Fu,fu))}\varphi(t)dt, 0, \int_0^{\digamma(\delta(fu,Fu))}\varphi(t)dt, 0, 0, \int_0^{\digamma(\delta(fu,Fu))}\varphi(t)dt\Bigg) \leq 0$$

which from (ψ_b) gives $\int_0^{\digamma(\delta(Fu,fu))}\varphi(t)dt \leq 0$ and therefore $\digamma(\delta(Fu, fu)) = 0$ which implies that $Fu = \{fu\} = \{t\}$. Since f and F are subcompatible, then, $Ffu = fFu$; i.e., $Ft = \{ft\}$.
Suppose that $ft \neq t$, then, from inequality (b),

$$\psi\Bigg(\int_0^{\digamma(\delta(Ft,Gy_n))}\varphi(t)dt, \int_0^{\digamma(d(ft,gy_n))}\varphi(t)dt, \int_0^{\digamma(\delta(ft,Ft))}\varphi(t)dt\ ,$$
$$\int_0^{\digamma(\delta(gy_n,Gy_n))}\varphi(t)dt, \int_0^{\digamma(\delta(ft,Gy_n))}\varphi(t)dt, \int_0^{\digamma(\delta(gy_n,Ft))}\varphi(t)dt\Bigg) \leq 0.$$

At infinity we obtain

$$\psi\Bigg(\int_0^{\digamma(d(ft,t))}\varphi(t)dt, \int_0^{\digamma(d(ft,t))}\varphi(t)dt, 0, 0,$$
$$\int_0^{\digamma(d(ft,t))}\varphi(t)dt, \int_0^{\digamma(d(t,ft))}\varphi(t)dt\Bigg) \leq 0$$

which contradicts (ψ_2). Therefore $\int_0^{F(d(ft,t))}\varphi(t)dt = 0$ which implies that $F(d(ft,t)) = 0$; i.e., $\{ft\} = \{t\} = Ft$.
Since $F(\mathcal{X}) \subseteq g(\mathcal{X})$, there exists an element $v \in \mathcal{X}$ such that $gv = t$. We claim that $Gv = \{gv\} = \{t\}$. If not, then, using condition (b) we have

$$\psi\left(\int_0^{F(\delta(Ft,Gv))}\varphi(t)dt, \int_0^{F(d(ft,gv))}\varphi(t)dt, \int_0^{F(\delta(ft,Ft))}\varphi(t)dt\right.,$$
$$\left.\int_0^{F(\delta(gv,Gv))}\varphi(t)dt, \int_0^{F(\delta(ft,Gv))}\varphi(t)dt, \int_0^{F(\delta(gv,Ft))}\varphi(t)dt\right)$$
$$= \psi\left(\int_0^{F(\delta(t,Gv))}\varphi(t)dt, 0, 0, \int_0^{F(\delta(t,Gv))}\varphi(t)dt, \int_0^{F(\delta(t,Gv))}\varphi(t)dt, 0\right) \leq 0$$

which from (ψ_a) gives $\int_0^{F(\delta(t,Gv))}\varphi(t)dt = 0$ and hence $F(\delta(t,Gv)) = 0$ which implies that $Gv = \{t\} = \{gv\}$. Since the pair (G,g) is subcompatible, then, $Ggv = gGv$; i.e., $Gt = \{gt\}$.
Suppose that $gt \neq t$. Then, by (b) we have

$$\psi\left(\int_0^{F(\delta(Ft,Gt))}\varphi(t)dt, \int_0^{F(d(ft,gt))}\varphi(t)dt, \int_0^{F(\delta(ft,Ft))}\varphi(t)dt\right.,$$
$$\left.\int_0^{F(\delta(gt,Gt))}\varphi(t)dt, \int_0^{F(\delta(ft,Gt))}\varphi(t)dt, \int_0^{F(\delta(gt,Ft))}\varphi(t)dt\right)$$
$$= \psi\left(\int_0^{F(d(t,gt))}\varphi(t)dt, \int_0^{F(d(t,gt))}\varphi(t)dt, 0, 0,\right.$$
$$\left.\int_0^{F(d(t,gt))}\varphi(t)dt, \int_0^{F(d(gt,t))}\varphi(t)dt\right) \leq 0$$

contradicts (ψ_2). Therefore $\int_0^{F(d(t,gt))}\varphi(t)dt = 0$ which implies that $F(d(t,gt)) = 0$; i.e., $\{gt\} = \{t\} = Gt$, and t is a common fixed point of f, g, F and G.
The uniqueness of the common fixed point follows easily from condition (b).
The proof is thus completed.
The proof is similar by replacing (c') with (c).

Corollary 3 *Let $(\mathcal{X},d)$ be a metric space and let $f : \mathcal{X} \to \mathcal{X}$; $F : \mathcal{X} \to B(\mathcal{X})$ be a single and a multivalued map, respectively. Suppose that*
(a) $F(\mathcal{X}) \subseteq f(\mathcal{X})$,

$$(b) \qquad \psi\left(\int_0^{\digamma(\delta(Fx,Fy))}\varphi(t)dt, \int_0^{\digamma(d(fx,fy))}\varphi(t)dt, \int_0^{\digamma(\delta(fx,Fx))}\varphi(t)dt\ ,\right.$$
$$\left.\int_0^{\digamma(\delta(fy,Fy))}\varphi(t)dt, \int_0^{\digamma(\delta(fx,Fy))}\varphi(t)dt, \int_0^{\digamma(\delta(fy,Fx))}\varphi(t)dt\right) \leq 0$$

for all x, y in $\mathcal{X}$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $\digamma \in \mathcal{F}$. If f and F are subcompatible D-maps and $F(\mathcal{X})$ is closed, then, f and F have a unique common fixed point $t \in \mathcal{X}$ such that $Ft = \{ft\} = \{t\}$.

Corollary 4 *Let $(\mathcal{X}, d)$ be a metric space and let $f : \mathcal{X} \to \mathcal{X}$; F, $G : \mathcal{X} \to B(\mathcal{X})$ be maps. If*
(a) $F(\mathcal{X}) \subseteq f(\mathcal{X})$ and $G(\mathcal{X}) \subseteq f(\mathcal{X})$,

$$(b) \qquad \psi\left(\int_0^{\digamma(\delta(Fx,Gy))}\varphi(t)dt, \int_0^{\digamma(d(fx,fy))}\varphi(t)dt, \int_0^{\digamma(\delta(fx,Fx))}\varphi(t)dt\ ,\right.$$
$$\left.\int_0^{\digamma(\delta(fy,Gy))}\varphi(t)dt, \int_0^{\digamma(\delta(fx,Gy))}\varphi(t)dt, \int_0^{\digamma(\delta(fy,Fx))}\varphi(t)dt\right) \leq 0$$

for all x, y in $\mathcal{X}$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $\digamma \in \mathcal{F}$. If either
(c) f and F are subcompatible D-maps; f and G are subcompatible and $F(\mathcal{X})$ is closed, or
(c′) f and G are subcompatible D-maps; f and F are subcompatible and $G(\mathcal{X})$ is closed.
Then, there is a unique point $t \in \mathcal{X}$ such that $Ft = Gt = \{ft\} = \{t\}$.

By recurrence on n, we get the next result.

Theorem 4 *Let $(\mathcal{X}, d)$ be a metric space and let f, $g : \mathcal{X} \to \mathcal{X}$; $F_n : \mathcal{X} \to B(\mathcal{X})$ be single and multivalued maps, respectively. Suppose that*
(a) $F_n(\mathcal{X}) \subseteq g(\mathcal{X})$ and $F_{n+1}(\mathcal{X}) \subseteq f(\mathcal{X})$,

$$(b) \qquad \psi\left(\int_0^{\digamma(\delta(F_nx,F_{n+1}y))}\varphi(t)dt, \int_0^{\digamma(d(fx,gy))}\varphi(t)dt, \int_0^{\digamma(\delta(fx,F_nx))}\varphi(t)dt\ ,\right.$$
$$\left.\int_0^{\digamma(\delta(gy,F_{n+1}y))}\varphi(t)dt, \int_0^{\digamma(\delta(fx,F_{n+1}y))}\varphi(t)dt, \int_0^{\digamma(\delta(gy,F_nx))}\varphi(t)dt\right) \leq 0$$

for all x, y in $\mathcal{X}$, where $\psi \in \Psi$, $\varphi \in \Phi$, $F \in \mathcal{F}$ and $n \in \mathbb{N}^* = \{1, 2, \ldots\}$. If either
(c) f and F_n are subcompatible D-maps; g and F_{n+1} are subcompatible and $F_n(\mathcal{X})$ is closed, or
(c′) g and F_{n+1} are subcompatible D-maps; f and F_n are subcompatible and $F_{n+1}(\mathcal{X})$ is closed.
Then, there exists a unique point t in $\mathcal{X}$ such that $F_n t = \{ft\} = \{gt\} = \{t\}$.

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