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# Starlikeness of analytic maps satisfying a differential inequality [1]

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**Abstract**

In the present note, the authors present a criterion for starlikeness of analytic maps satisfying a differential inequality in the open unit disc $\mathbb{E} = \{z : |z| < 1\}$ and claim that their result unifies a number of previously known results in this direction.

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## 1 Introduction

Let $\mathcal{A}$ be the class of functions $f$, analytic in $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Denote by $S^*(\alpha)$, the class of starlike functions of order $\alpha$, which is analytically defined as follows:

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in \mathbb{E} \right\},$$

where $\alpha$ is a real number such that $0 \leq \alpha < 1$.

We write $S^* = S^*(0)$. Therefore $S^*$ is the class of univalent starlike functions (w.r.t. the origin).

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Obtaining different criteria for starlikeness of an analytic function has always been a subject of interest e.g. Miller, Mocanu and Reade [5] studied the class of $\alpha$-convex functions and proved that if a function $f \in \mathcal{A}$ satisfies the differential inequality

$$\Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0,\ z \in \mathbb{E},$$

where $\alpha$ is any real number, then $f$ is starlike in $\mathbb{E}$. Lewandowski et al.[2] proved that for a function $f \in \mathcal{A}$, the differential inequality

$$\Re\left[\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f'(z)}\right] > 0,\ z \in \mathbb{E},$$

ensures membership for $f$ in the class $S^*$. For more such results, we refer the reader to [3], [6], [7] and [8].

In the present paper, we generalize these sufficient conditions and obtain an interesting criterion for starlikeness. In Section 4, we show that some well-known results follow as corollaries to our result.

## 2 Preliminaries

We shall need the following lemma of Miller and Mocanu [4] to prove our result.

**Lemma 1** *Let $\Omega$ be a set in the complex plane $\mathbb{C}$ and let $\psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$. For $u = u_1 + iu_2,\ v = v_1 + iv_2$, assume that $\psi$ satisfies the condition $\psi(iu_2, v_1; z) \notin \Omega$, for all $u_2, v_1 \in \mathbb{R}$, with $v_1 \le -(1+u_2^2)/2$ and for all $z \in \mathbb{E}$. If the function $p$, $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, is analytic in $\mathbb{E}$ and if $\psi(p(z), zp'(z); z) \in \Omega$, then $\Re\, p(z) > 0$ in $\mathbb{E}$.*

## 3 Main Theorem

**Theorem 2** *Let $\alpha$, $\alpha \ge 0$, $\lambda, 0 \le \lambda < 1$, and $\beta, 0 \le \beta \le 1$, be given real numbers.*

**(i)** *For $1/2 \le \lambda < 1$, if a function $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ in $\mathbb{E}$, satisfies*

(1)
$$\Re\left[\frac{zf'(z)}{f(z)}\left(1 + \frac{\alpha z f''(z)}{f'(z)}\right) + \alpha\beta\left(1 - \frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > M(\alpha,\beta,\lambda),$$

*then* $f \in S^*(\lambda)$.

**(ii)** *For* $0 \leq \lambda < 1/2$, *let a function* $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ *in* $\mathbb{E}$, *satisfy*

**(a)**

(2)
$$\Re\left[\frac{zf'(z)}{f(z)}\left(1+\frac{\alpha zf''(z)}{f'(z)}\right)+\alpha\beta\left(1-\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > M(\alpha,\beta,\lambda),$$

*whenever*

(3)
$$\beta(2\lambda-1-3\lambda^3+2\lambda^4)+(3-2\lambda)\lambda^3 \geq 0,$$

*and*

**(b)**

(4)
$$\Re\left[\frac{zf'(z)}{f(z)}\left(1+\frac{\alpha zf''(z)}{f'(z)}\right)+\alpha\beta\left(1-\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > N(\alpha,\beta,\lambda),$$

*whenever*

(5)
$$\beta(2\lambda-1-3\lambda^3+2\lambda^4)+(3-2\lambda)\lambda^3 \leq 0.$$

*Then* $f \in S^*(\lambda)$. *Here*

(6)
$$M(\alpha,\beta,\lambda) = [1-\alpha(1-\beta)]\lambda+\alpha(1-\beta)\lambda^2-\frac{\alpha(1-\beta)(1-\lambda)}{2}-\frac{\alpha\beta(1-\lambda)}{2\lambda},$$

*and*

$$N(\alpha,\beta,\lambda) = [1-\alpha(1-\beta)]\lambda+\alpha(1-\beta)\lambda^2-\frac{\alpha(1-\beta)(1-\lambda)}{2}$$

(7)
$$-\frac{\alpha}{2(1-\lambda)}[2\sqrt{\beta\lambda(1-2\lambda)(1-\beta)(3-2\lambda)}+\beta\lambda-\lambda^2(1-\beta)(3-2\lambda)].$$

**Proof.** Define a function $p$ by

(8)
$$\frac{zf'(z)}{f(z)} = \lambda+(1-\lambda)p(z).$$

Then $p$ is analytic in $\mathbb{E}$ and $p(0) = 1$. A simple calculation yields

$$\frac{zf'(z)}{f(z)}\left(1+\frac{\alpha zf''(z)}{f'(z)}\right)+\alpha\beta\left(1-\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$=(1-\alpha+\alpha\beta)[\lambda+(1-\lambda)p(z)]+\alpha(1-\beta)[\lambda+(1-\lambda)p(z)]^2$$
$$+\alpha(1-\beta)(1-\lambda)zp'(z)+\alpha\beta\frac{(1-\lambda)zp'(z)}{\lambda+(1-\lambda)p(z)}$$
$$=\psi(p(z),zp'(z);z) \tag{9}$$

where,

$$\psi(u,v;z)=(1-\alpha+\alpha\beta)[\lambda+(1-\lambda)u]+\alpha(1-\beta)[\lambda+(1-\lambda)u]^2$$
$$+\alpha(1-\beta)(1-\lambda)v+\alpha\beta\frac{(1-\lambda)v}{\lambda+(1-\lambda)u}$$

Let $u=u_1+iu_2, v=v_1+iv_2$, where $u_1,u_2,v_1,v_2$ are all real with $v_1\le -(1+u_2^2)/2$. Then, we have

$$\begin{aligned}
&\Re\,\psi(iu_2,v_1;z)\\
&= (1-\alpha+\alpha\beta)\lambda+\alpha(1-\beta)[\lambda^2-(1-\lambda)^2u_2^2]+\\
&\qquad +\alpha(1-\beta)(1-\lambda)v_1+\alpha\beta\frac{\lambda(1-\lambda)v_1}{\lambda^2+(1-\lambda)^2u_2^2}\\
&\le (1-\alpha+\alpha\beta)\lambda+\alpha(1-\beta)[\lambda^2-(1-\lambda)^2u_2^2]+\\
&\qquad -\frac{\alpha(1-\beta)(1-\lambda)(1+u_2^2)}{2}-\alpha\beta\frac{\lambda(1-\lambda)(1+u_2^2)}{2(\lambda^2+(1-\lambda)^2u_2^2)}\\
&= (1-\alpha+\alpha\beta)\lambda+\alpha(1-\beta)\lambda^2-\frac{\alpha(1-\beta)(1-\lambda)}{2}-\\
&\qquad -\alpha(1-\beta)(1-\lambda)\left(\frac{3}{2}-\lambda\right)u_2^2-\alpha\beta\frac{\lambda(1-\lambda)(1+u_2^2)}{2(\lambda^2+(1-\lambda)^2u_2^2)}\\
&= (1-\alpha+\alpha\beta)\lambda+\alpha(1-\beta)\lambda^2-\frac{\alpha(1-\beta)(1-\lambda)}{2}-\\
&\qquad -\alpha(1-\beta)(1-\lambda)\left(\frac{3}{2}-\lambda\right)t-\alpha\beta\frac{\lambda(1-\lambda)(1+t)}{2(\lambda^2+(1-\lambda)^2t)}\\
&= \phi(t)\quad\text{(say)},\quad\text{where } u_2^2=t\\
&\le \max\phi(t). \qquad (10)
\end{aligned}$$

Writing

$$(1-\alpha+\alpha\beta)\lambda+\alpha(1-\beta)\lambda^2-\frac{\alpha(1-\beta)(1-\lambda)}{2}=a,$$

$$(1-\beta)(1-\lambda)\left(\frac{3}{2}-\lambda\right)=b$$

and $\frac{\lambda}{1-\lambda}=c$, we have

$$\phi(t)=a-\alpha bt-\frac{\alpha\beta c}{2}\left(\frac{1+t}{c^2+t}\right).$$

Clearly, $\phi(t)$ is continuous at $t=0$. A simple calculation gives

$$\phi'(t)=-\alpha b-\frac{\alpha\beta c}{2}\left(\frac{c^2-1}{(c^2+t)^2}\right).$$

**Case (i).** When $1/2\leq\lambda<1$, then $c=\frac{\lambda}{1-\lambda}\geq 1$. Since $\alpha\geq 0,\ 0\leq\beta\leq 1$, therefore, $b>0$. Hence, $\phi'(t)\leq 0$ which implies that $\phi$ is a decreasing function of t $(\geq 0)$. Thus

$$\begin{aligned}\max\phi(t) &= \phi(0)\\ &= M(\alpha,\beta,\lambda).\end{aligned}\qquad(11)$$

Let

$$\Omega=\{w:\ \Re\, w>M(\alpha,\beta,\lambda)\}.$$

Then from (1) and (9), we have $\psi(p(z),zp'(z);z)\in\Omega$ for all $z\in\mathbb{E}$, but $\psi(iu_2,v_1;z)\notin\Omega$, in view of (10) and (11). Therefore, by Lemma 1 and (8), we conclude that $f\in S^*(\lambda)$.

**Case (ii).** When $0\leq\lambda<1/2$, we get $c=\frac{\lambda}{1-\lambda}<1$. Now, $\phi'(t)=0$ implies

$$-\alpha b-\frac{\alpha\beta c}{2}\left(\frac{c^2-1}{(c^2+t)^2}\right)=0$$

which gives

$$t=-c^2\pm\sqrt{\frac{\beta c(1-c^2)}{2b}}.$$

Writing $-c^2-\sqrt{\frac{\beta c(1-c^2)}{2b}}=t_1$ and $-c^2+\sqrt{\frac{\beta c(1-c^2)}{2b}}=t_2$, we observe that $t_1<0$ and also, $t_1<t_2$.

**Subcase (i).** When $t_2 < 0$, i.e. when (3) holds true. In that case $t_1$ and $t_2$ both are negative. (Here, $t$ is positive.) It can be easily verified that

$$\phi'(t) = -\frac{\alpha}{2(c^2+t)^2}(t-t_1)(t-t_2) < 0.$$

Thus $\phi$ is a decreasing function of $t$ and again

$$\begin{aligned} \max\phi(t) &= \phi(0) \\ &= M(\alpha,\beta,\lambda). \end{aligned} \tag{12}$$

Proceeding as in case (i), we obtain the required result.

**Subcase (ii).** When $t_2 > 0$, i.e. when (5) holds true. In that case, $\phi$ is a increasing function of $t, t \geq 0$, and therefore,

$$\begin{aligned} \max\phi(t) &= \phi(t_2) \\ &= N(\alpha,\beta,\lambda). \end{aligned} \tag{13}$$

Let

$$\Omega = \{w : \Re\, w > N(\alpha,\beta,\lambda)\}.$$

Then from (4) and (9), we have $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{E}$, but $\psi(iu_2, v_1; z) \notin \Omega$, in view of (10) and (13). Result now follows by Lemma 1.

## 4 Applications to Univalent Functions

In this section, we apply Theorem 2 and obtain certain well-known criteria for starlikeness of an analytic function.

Writing $\beta = 1$ in Theorem 2, we obtain the following result of Fukui [1] for the class of $\alpha$-convex functions.

**Corollary 3** *Let $\alpha$, $\alpha \geq 0$ be a given real number. For all $z \in \mathbb{E}$, let a function $f \in \mathcal{A}$ satisfy*

$$\Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \begin{cases} > \lambda - \frac{\alpha\lambda}{2(1-\lambda)}, & 0 \leq \lambda < 1/2, \\ > \lambda - \frac{\alpha(1-\lambda)}{2\lambda}, & 1/2 \leq \lambda < 1. \end{cases}$$

*Then $f \in S^*(\lambda)$.*

The case, when we write $\beta = 0$ in Theorem 2, gives the following result of Ravichandran et al. [9].

**Corollary 4** *Let $\alpha$, $\alpha \geq 0$ be a given real number. For a real number $\lambda, 0 \leq \lambda < 1$, and for all $z \in \mathbb{E}$, let a function $f \in \mathcal{A}$ satisfy*

$$\Re\left[\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}\right] > \alpha\lambda\left(\lambda - \frac{1}{2}\right) + \lambda - \frac{\alpha}{2}.$$

*Then $f \in S^*(\lambda)$.*

Setting $\beta = 0$ and $\lambda = \alpha/2, 0 < \alpha < 2$ in Theorem 2, we obtain the following result of Li and Owa [3].

**Corollary 5** *If a function $f \in \mathcal{A}$ satisfies*

$$\Re\left[\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}\right] > -\frac{\alpha^2}{4}(1-\alpha), \quad 0 < \alpha < 2,$$

*then $f \in S^*(\alpha/2)$.*

Setting $\beta = \alpha = 1$ in Theorem 2, we obtain the following result.

**Corollary 6** *For all $z \in \mathbb{E}$, let $f$ in $\mathcal{A}$ satisfy the condition*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \begin{cases} > \lambda - \frac{\lambda}{2(1-\lambda)}, \; 0 \leq \lambda < 1/2, \\ > \lambda - \frac{(1-\lambda)}{2\lambda}, \; 1/2 \leq \lambda \leq 1. \end{cases}$$

*Then $f \in S^*(\lambda)$.*

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