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Differential subordination for classes of normalized analytic functions [1]

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Abstract

We determine the sufficient conditions for subordination for new classes of normalized analytic functions with applications in fractional calculus in complex domain.

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1 Introduction and preliminaries.

Let $\mathcal{A}_\alpha^+$ be the class of all normalized analytic functions $F(z)$ in the open disk $U := \{z \in \mathbb{C}, |z| < 1\}$, take the form

$$F(z) = z + \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}, \;\; 0 < \alpha \leq 1,$$

where $a_{0,1} = 0$, $a_{1,1} = 1$ satisfying $F(0) = 0$ and $F'(0) = 1$. And let $\mathcal{A}_\alpha^-$ be the class of all normalized analytic functions $F(z)$ in the open disk U take the form

$$F(z) = z - \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}, \;\; a_{n,\alpha} \geq 0; \;\; n = 2, 3, ...,$$

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satisfying $F(0) = 0$ and $F'(0) = 1$. With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in U. Then we say that the function f is *subordinate* to g if there exists a Schwarz function $w(z)$, analytic in U such that

$$f(z) = g(w(z)), \;\; z \in U.$$

We denote this subordination by

$$f \prec g \;\; or \;\; f(z) \prec g(z), \;\; z \in U.$$

If the function g is univalent in U the above subordination is equivalent to

$$f(0) = g(0) \;\; and \;\; f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. Assume that p, ϕ are analytic and univalent in U if p satisfies the differential superordination

$$h(z) \prec \phi(p(z)), zp'(z), z^2p''(z); z), \tag{1}$$

then p is called a solution of the differential superordination.(If f is subordinate to g, then g is called to be superordinate to f.)An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1). An univalent function q such that $p \prec q$ for all subordinants p of (1) is said to be the best subordinant.
Let $\mathcal{A}$ be the class of analytic functions of the form $f(z) = z + a_2z^2 + \dots$. Obradović and Owa [1] obtained sufficient conditions for certain normalized analytic functions $f(z) \in \mathcal{A}$ to satisfy

$$q_1(z) \prec [\frac{f(z)}{z}]^{\mu} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in U. The main object of the present work is to apply a method based on the differential subordination in order to derive sufficient conditions for functions $F \in \mathcal{A}_\alpha^+$ and $F \in \mathcal{A}_\alpha^-$ to satisfy

$$[\frac{F(z)}{z}]^{\mu} \prec q(z) \tag{2}$$

where $q(z)$ is a given univalent function in U such that $q(z) \neq 0$. Moreover, we give applications for these results in fractional calculus. We shall need the following known results.

Lemma 1 *[2] Let $q(z)$ be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that*
1. $Q(z)$ is starlike univalent in U, and
2. $\Re\frac{zh'(z)}{Q(z)} > 0$ for $z \in U$.
If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$ then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2 *[3] Let $q(z)$ be convex univalent in the unit disk U and ψ and $\gamma \in \mathbb{C}$ with $\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0$. If $p(z)$ is analytic in U and $\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z)$, then $p(z) \prec q(z)$ and q is the best dominant.*

2 Main results.

In this section, we study sufficient subordination normalized analytic functions in the classes $\mathcal{A}_\alpha^+$ and $\mathcal{A}_\alpha^-$.

Theorem 1 *Let the function $q(z)$ be univalent in the unit disk U such that $q(z) \neq 0$, $\frac{zq'(z)}{q(z)}$ is starlike univalent in U and*

(3) $$\Re\{1 + (\frac{a}{bz} + 1)(\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)})\} > 0,\ b \neq 0,\ z \neq 0,\ q'(z) \neq 0,\ z \in U.$$

If $F \in \mathcal{A}_\alpha^+$ satisfies the subordination

$$(a + bz)\frac{\mu}{z}(\frac{zF'(z)}{F(z)} - 1) \prec (a + bz)\frac{q'(z)}{q(z)},\ \ F(z) \neq 0,\ z \in U.$$

Then

$$(\frac{F(z)}{z})^\mu \prec q(z),\quad z \neq 0,\ z \in U,$$

and $q(z)$ is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) := (\frac{F(z)}{z})^\mu,\ \ z \neq 0,\quad z \in U.$$

By setting

$$\theta(\omega) := \frac{a\omega'}{\omega}\ \ and\ \ \phi(\omega) := \frac{b}{\omega},\ \ b \neq 0,$$

it can easily be observed that $\theta(\omega)$ is analytic in $\mathbb{C}-\{0\}$, $\phi(\omega)$ is analytic in $\mathbb{C}-\{0\}$ and that $\phi(\omega)\neq 0$, $\omega\in\mathbb{C}-\{0\}$. Also we obtain

$$Q(z)=zq'(z)\phi(q(z))=\frac{bzq'(z)}{q(z)} \ \ and \ \ h(z)=\theta(q(z))+Q(z)=(a+bz)\frac{q'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U,

$$\Re\{\frac{zh'(z)}{Q(z)}=\Re\{1+(\frac{a}{bz}+1)(\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)})\}>0.$$

Straightforward computation, we have

$$\begin{aligned}(a+bz)\frac{p'(z)}{p(z)}&=(a+bz)\frac{\mu}{z}(\frac{zF'(z)}{F(z)}-1)\\ &\prec(a+bz)\frac{q'(z)}{q(z)}\end{aligned}$$

Then by the assumption of the theorem we have that the assertion of the theorem follows by an application of Lemma 1.

Corollary 1 *Assume that (3) holds and q is convex univalent in U. If* $F\in\mathcal{A}_\alpha^+$ *and*

$$(a+bz)\frac{\mu}{z}(\frac{zF'(z)}{F(z)}-1)\prec\mu(a+bz)\frac{A-B}{(1+Az)(1+Bz)},$$

then

$$(\frac{F(z)}{z})^\mu\prec(\frac{1+Az}{1+Bz})^\mu,\ \ -1\leq B<A\leq 1$$

and $q(z)=(\frac{1+Az}{1+Bz})^\mu$ *is the best dominant.*

Corollary 2 *Assume that (3) holds and q is convex univalent in U. If* $F\in\mathcal{A}_\alpha^+$ *and*

$$(a+bz)\frac{\mu}{z}(\frac{zF'(z)}{F(z)}-1)\prec(a+bz)\frac{2\mu}{(1+z)(1-z)},$$

for $z\in U$, $\mu\neq 0$, *then*

$$(\frac{F(z)}{z})^\mu\prec(\frac{1+z}{1-z})^\mu$$

and $q(z)=(\frac{1+z}{1-z})^\mu$ *is the best dominant.*

Corollary 3 *Assume that (3) holds and q is convex univalent in U. If $F \in \mathcal{A}_\alpha^+$ and*

$$(a+bz)\frac{\mu}{z}(\frac{zF'(z)}{F(z)}-1) \prec \mu A(a+bz)$$

for $z \in U$, $\mu \neq 0$, then

$$(\frac{F(z)}{z})^\mu \prec e^{\mu A z}$$

and $q(z) = e^{\mu A z}$ is the best dominant.

Theorem 2 *Let the function $q(z)$ be convex univalent in the unit disk U such that $q'(z) \neq 0$ and*

$$\Re\{1+\frac{zq''(z)}{q'(z)}+\frac{1}{\gamma}\} > 0, \ \ \gamma \neq 0. \tag{4}$$

Suppose that $(\frac{F(z)}{z})^\mu$ is analytic in U. If $F \in \mathcal{A}_\alpha^-$ satisfies the subordination

$$(\frac{F(z)}{z})^\mu[1+\gamma\mu(\frac{zF'(z)}{F(z)}-1)] \prec q(z)+\gamma z q'(z), \ F(z) \neq 0.$$

Then

$$(\frac{F(z)}{z})^\mu \prec q(z), \ \ z \in U, z \neq 0$$

and $q(z)$ is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) := (\frac{F(z)}{z})^\mu, \ \ z \neq 0, , \ z \in U.$$

By setting $\psi = 1$, it can easily be observed that

$$\begin{aligned} p(z)+\gamma z p'(z) &= (\frac{F(z)}{z})^\mu[1+\gamma\mu(\frac{zF'(z)}{F(z)}-1)] \\ &\prec q(z)+\gamma z q'(z). \end{aligned}$$

Then by the assumption of the theorem we have that the assertion of the theorem follows by an application of Lemma 2.

Corollary 4 *Assume that (4) holds and q is convex univalent in U. If $F \in \mathcal{A}_\alpha^-$ and*

$$(\frac{F(z)}{z})^{\mu}[1+\gamma\mu(\frac{zF'(z)}{F(z)}-1)] \prec (\frac{1+Az}{1+Bz})^{\mu}+\mu\gamma z(A-B)\frac{(1+Az)^{\mu-1}}{(1+Bz)^{\mu+1}}$$

then

$$(\frac{F(z)}{z})^{\mu} \prec (\frac{1+Az}{1+Bz})^{\mu}, \quad -1 \le B < A \le 1$$

and $q(z)=(\frac{1+Az}{1+Bz})^{\mu}$ is the best dominant.

Corollary 5 *Assume that (4) holds and q is convex univalent in U. If $F \in \mathcal{A}_\alpha^-$ and*

$$(\frac{F(z)}{z})^{\mu}[1+\gamma\mu(\frac{zF'(z)}{F(z)}-1)] \prec [\frac{1+z}{1-z}]^{\mu}\{1+\frac{2\gamma\mu z}{1-z^2}\}$$

for $z \in U$, $\mu \neq 0$, then

$$(\frac{F(z)}{z})^{\mu} \prec (\frac{1+z}{1-z})^{\mu}$$

and $q(z)=(\frac{1+z}{1-z})^{\mu}$ is the best dominant.

Corollary 6 *Assume that (4) holds and q is convex univalent in U. If $F \in \mathcal{A}_\alpha^-$ and*

$$(\frac{F(z)}{z})^{\mu}[1+\gamma\mu(\frac{zF'(z)}{F(z)}-1)] \prec e^{\mu Az}(1+\mu\gamma Az)$$

for $z \in U$, $\mu \neq 0$, then

$$(\frac{F(z)}{z})^{\mu} \prec e^{\mu Az}$$

and $q(z)=e^{\mu Az}$ is the best dominant.

3 Applications.

In this section, we introduce some applications of section (2) containing fractional integral operators. Assume that $f(z)=\sum_{n=2}^{\infty}\varphi_n z^n$ and let us begin with the following definitions

Definition 1 *[4] The fractional integral of order α is defined, for a function f, by*

$$I_z^{\alpha}f(z) := \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta)(z-\zeta)^{\alpha-1}d\zeta; \quad \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta)>0$.

From Definition 1 and see ([5]), thus $z + I_z^\alpha f(z) \in \mathcal{A}_\alpha^+$ and $z - I_z^\alpha f(z) \in \mathcal{A}_\alpha^-$ $(\varphi_n \geq 0)$, then we have the following results

Theorem 3 *Let the assumptions of Theorem 1 hold, then*

$$(\frac{z + I_z^\alpha f(z)}{z})^\mu \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let the function $F(z)$ be defined by

$$F(z) := z + I_z^\alpha f(z), \quad z \in U,\, z \neq 0.$$

Theorem 4 *Let the assumptions of Theorem 2 hold, then*

$$(\frac{z - I_z^\alpha f(z)}{z})^\mu \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let the function $F(z)$ be defined by

$$F(z) := z - I_z^\alpha f(z), \quad z \in U,\, z \neq 0.$$

Let $F(a, b; c; z)$ be the Gauss hypergeometric function (see [6]) defined, for $z \in U$, by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)...(a+n-1), & (n \in \mathbb{N}). \end{cases}$$

We need the following definitions of fractional operators in the Saigo type fractional calculus (see [7],[8]).

Definition 2 *For $\alpha > 0$ and $\beta, \eta \in \mathbb{R}$, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by*

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}) f(\zeta) d\zeta$$

where the function $f(z)$ is analytic in a simply-connected region of the $z-$plane containing the origin, with the order

$$f(z) = O(|z|^{\epsilon})(z \to 0), \ \ \epsilon > max\{0, \beta-\eta\} - 1$$

and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$.

From Definition 2, with $\beta < 0$, we have

$$\begin{aligned} I_{0,z}^{\alpha,\beta,\eta} f(z) &= \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}) f(\zeta) d\zeta \\ &= \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n(-\eta)_n}{(\alpha)_n(1)_n} \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1}(1-\frac{\zeta}{z})^n f(\zeta) d\zeta \\ &:= \sum_{n=0}^{\infty} B_n \frac{z^{-\alpha-\beta-n}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{n+\alpha-1} f(\zeta) d\zeta \\ &= \sum_{n=0}^{\infty} B_n \frac{z^{-\beta-1}}{\Gamma(\alpha)} f(\zeta) \\ &:= \frac{\overline{B}}{\Gamma(\alpha)} \sum_{n=2}^{\infty} \varphi_n z^{n-\beta-1} \end{aligned}$$

where $\overline{B} := \sum_{n=0}^{\infty} B_n$. Denote $a_n := \frac{\overline{B}\varphi_n}{\Gamma(\alpha)}$, $\forall n = 2, 3, ...$, and let $\alpha = -\beta$ thus $z + I_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{A}_{\alpha}^{+}$ and $z - I_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{A}_{\alpha}^{-}$ ($\varphi_n \geq 0$), then we have the following results

Theorem 5 *Let the assumptions of Theorem 1 hold, then*

$$(\frac{z + I_{0,z}^{\alpha,\beta,\eta} f(z)}{z})^{\mu} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let the function $F(z)$ be defined by

$$F(z) := z + I_{0,z}^{\alpha,\beta,\eta} f(z), \quad z \in U,\, z \neq 0.$$

Theorem 6 *Let the assumptions of Theorem 2 hold, then*

$$(\frac{z - I_{0,z}^{\alpha,\beta,\eta} f(z)}{z})^{\mu} \prec q(z),$$

and $q(z)$ *is the best dominant.*

Proof. Let the function $F(z)$ be defined by

$$F(z) := z - I_{0,z}^{\alpha,\beta,\eta} f(z), \quad z \in U,\, z \neq 0.$$

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