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Generalized q-Taylor's series and applications [1]

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Abstract

A generalized q-Taylor's formula in fractional q-calculus is established and used in deriving certain q-generating functions for the basic hypergeometric functions and basic Fox's H-function.

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1 Introduction

In the theory of q-series [3], the q-shifted factorial for a real (or complex) number a is defined by

$$(1)\qquad (a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \qquad (n \in \mathbf{N};\ |q|<1).$$

Also, the q-analogue of $(x \pm y)^n$ ([8]) is given by

$$(2)\quad (x \pm y)^{(n)} = (x \pm y)_n = x^n(\mp y/x;q)_n = x^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2}\,(\pm y/x)^k$$

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$$(n \in \mathbf{N}; \ \ |q| < 1),$$

where the q-binomial coefficient is defined by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{(q^{-n};q)_k}{(q;q)_k}(-q^n)^k q^{-k(k-1)/2}. \tag{3}$$

For a bounded sequence of real (or complex) numbers $\{A_n\}$, let $f(x) = \sum_{n=-\infty}^{\infty} A_n x^n$, then ([4]; see also [2, p. 502])

$$f\left[(x \pm y)\right] = \sum_{n=-\infty}^{\infty} A_n x^n (\mp y/x;q)_n. \tag{4}$$

The q-gamma function (cf. [3]) is defined by

(5)

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty (1-q)^{a-1}} = \frac{(q;q)_{a-1}}{(1-q)^{a-1}} \quad (a \neq 0,-1,-2,\cdots; \ \ |q|<1),$$

and in terms of (2) and (5), the Riemann-Liouville fractional q-differential operator of a function $f(x)$ is defined by ([1])

$$D^{\mu}_{x,q}\left\{f(x)\right\} = \frac{1}{\Gamma_q(-\mu)} \int_0^x (x-tq)_{-\mu-1} f(t)d(t;q) \tag{6}$$

$$(\Re(\mu) < 0; |q| < 1).$$

In particular, for $f(x) = x^p$, (6) gives

$$D^{\mu}_{x,q}\left\{x^p\right\} = \frac{\Gamma_q(1+p)}{\Gamma_q(1+p-\mu)} \ x^{p-\mu} \quad (\Re(p) > -1; \ \Re(\mu) < 0). \tag{7}$$

The generalized basic hypergeometric series (cf. Slater [11]) is given by

$${}_r\Phi_s \left[\begin{array}{lc} a_1,\cdots,a_r & ; \\ & q,x \\ b_1,\cdots,b_s & ; \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n} \ x^n \ , \tag{8}$$

where for convergence, $|q| < 1$ ($|x| < 1$ if $r = s+1$; and for any x: if $r \leq s$).

Saxena *et al* [9] introduced a basic analogue of the H-function in terms of the Mellin-Barnes type basic contour integral in the following manner:

$$H_{A,B}^{m_1,n_1}\left[x;q\left|\begin{array}{l}(a_1,\alpha_1),\cdots,(a_A,\alpha_A)\\(b_1,\beta_1),\cdots,(b_B,\beta_B)\end{array}\right.\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j s})\prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j s})\pi x^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j s})\prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j s})\, G(q^{1-s}) sin\pi s}ds, \tag{9}$$

where

$$G(q^\alpha)=\prod_{n=0}^{\infty}\left\{(1-q^{\alpha+n})\right\}^{-1}=\frac{1}{(q^\alpha;q)_\infty}, \tag{10}$$

and $0\le m_1\le B$; $0\le n_1\le A$; α_j and β_j are all positive integers. The contour C is a line parallel to $\Re(\omega s)=0$, with indentations, if necessary, in such a manner that all the poles of $G(q^{b_j-\beta_j s})$ $(1\le j\le m_1)$ are to its right, and those of $G(q^{1-a_j+\alpha_j s})$ $(1\le j\le n_1)$ are to the left of C. The basic integral converges if $\Re\left[s\ log(x)-logsin\pi s\right]<0$, for large values of $|s|$ on the contour C, that is if $\left|\left\{arg(x)-\omega_2\ \omega_1^{-1}log\,|x|\right\}\right|<\pi$, where $|q|<1$, $logq=-\omega=-(\omega_1+i\omega_2)$, ω_1 and ω_2 being real.

For $\alpha_j=\beta_i=1$ $(j=1,\cdots,A;\ i=1,\cdots,B)$, (9) reduces to the q-analogue of the Meijer's G-function [9] defined by

$$G_{A,B}^{m_1,n_1}\left[x;q\left|\begin{array}{l}a_1,\cdots,a_A\\b_1,\cdots,b_B\end{array}\right.\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-s})\prod_{j=1}^{n_1} G(q^{1-a_j+s})\pi x^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+s})\prod_{j=n_1+1}^{A} G(q^{a_j-s})\, G(q^{1-s}) sin\pi s}ds, \tag{11}$$

where $0\le m_1\le B$; $0\le n_1\le A$ and $\Re\left[s\ log(x)-logsin\pi s\right]<0$.

The object of this paper is to derive a generalized q-Taylor's formula in fractional q-calculus using Riemann-Liouville fractional q-differential operator (6). The usefulness of the main result is exhibited by deriving certain q-generating functions for the basic hypergeometric function ${}_r\Phi_s(.)$ and for the basic analogue of the Fox's H-function.

2 Main result

In this section, we prove the following theorem which may be regarded as a generalization of the q-Taylor's formula.

Theorem 1 *Let η be an arbitrary complex number and $\Re(p) > -1$, then*

$$(x+t)_p \; f[(x+tq^p)] = \sum_{n=-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2} \; t^{n+\eta}}{\Gamma_q(n+\eta+1)} D_{x,q}^{n+\eta} \left\{ x^p f(x) \right\}, \tag{12}$$

valid for all t where $|t/x| < 1$, $|tq^p/x| < 1$ and $|q| < 1$.

Proof. Making use of (4) in conjunction with (2), the left-hand side of (12) (say L) gives

$$L = \sum_{m=0}^{\infty} A_m x^{p+m} (-t/x; q)_p (-tq^p/x; q)_m$$

$$= \sum_{m=0}^{\infty} A_m x^{p+m} (-t/x; q)_{p+m}. \tag{13}$$

On the other hand, the right-hand side (say R) of (12) leads to

$$R = \sum_{n=-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2} \; t^{n+\eta}}{\Gamma_q(n+\eta+1)} D_{x,q}^{n+\eta} \left\{ \sum_{m=0}^{\infty} A_m x^{p+m} \right\}.$$

Using the fractional q-derivative formula (6), the right-hand side of (12) becomes

$$R = \sum_{n=-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2} \; (t/x)^{n+\eta}}{\Gamma_q(n+\eta+1)} \sum_{m=0}^{\infty} A_m \frac{\Gamma_q(p+m+1)}{\Gamma_q(p+m+1-n-\eta)} x^{p+m}. \tag{14}$$

On interchanging the order of summations and carring out elementary simplifications, we get

$$R = \frac{(1-q)^{-\eta}}{\Gamma_q(\eta+1)} \sum_{m=0}^{\infty} A_m x^{p+m} (t/x)^{\eta} \sum_{n=-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2} \; (t/x)^n}{(q^{1+\eta}; q)_n (q^{p+m+1}; q)_{-n-\eta}}, \tag{15}$$

which in view of the q-identities [3, pp. 233-234]:

$$(a;q)_{-n} = \frac{(-q/a)^n}{(q/a;q)_n} q^{n(n-1)/2}, \qquad (a;q)_{n+k} = (a;q)_n (aq^n;q)_k$$

yields

$$R = \frac{(1-q)^{-\eta}}{\Gamma_q(\eta+1)} \sum_{m=0}^{\infty} A_m x^{p+m} (-tq^{p+m}/x)^{\eta} (q^{-p-m};q)_{\eta} \sum_{n=-\infty}^{\infty} \frac{(q^{\eta-p-m};q)_n \; (-tq^{p+m}/x)^n}{(q^{1+\eta};q)_n}. \tag{16}$$

Applying the Ramanujan's summation formula (cf. [3, II.29, p. 239]), viz.

$${}_1\psi_1(a;b;q,z) = \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n}\; z^n = \frac{(q;q)_\infty (b/a;q)_\infty (az;q)_\infty (q/az;q)_\infty}{(b;q)_\infty (q/a;q)_\infty (z;q)_\infty (b/az;q)_\infty}, \tag{17}$$

we find that (16) reduces to

$$R = \frac{(1-q)^{-\eta}}{\Gamma_q(\eta+1)} \sum_{m=0}^{\infty} A_m x^{p+m} (-tq^{p+m}/x)^{\eta} (q^{-p-m};q)_{\eta} \frac{(q;q)_\infty (q^{1+m+p};q)_\infty (-tq^{\eta}/x;q)_\infty (-q^{1-\eta}x/t;q)_\infty}{(q^{1+\eta};q)_\infty (q^{1+m+p-\eta};q)_\infty (-tq^{m+p}/x;q)_\infty (-qx/t;q)_\infty} \tag{18}$$

which implies that

$$R = \sum_{m=0}^{\infty} A_m x^{p+m} (-t/x;q)_{p+m} = L. \tag{19}$$

This completes the proof of the theorem.

It may be observed that a generalized Taylor's formula involving the Riemann-Liouville type operator was obtained earlier by Raina [6, p. 81, eqn. (2.1)]. If we set $\eta = 0$ in the above theorem, we get the following corollary (giving a simple form of q-Taylor's formula).

Corollary 1 *If* $\Re(p) > -1$, *then*

$$(x+t)_p \; f[(x+tq^p)] = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} \; t^n}{\Gamma_q(n+1)} D_{x,q}^n \left\{ x^p f(x) \right\}, \tag{20}$$

valid for all t *where* $|t/x| < 1$, $|tq^p/x| < 1$ *and* $|q| < 1$.

A similar type of q-Taylor's formula was also given by Jackson [5].

3 Applications of the main result

The generalized fractional q-Taylor's formula established in the previous section would find many applications giving q-generating functions and series summation for the basic functions.

To illustrate the applications, we first apply formula (12) to obtain the series summation (or q-generating function) for the basic hypergeometric function ${}_r\Phi_s(\cdots)$, defined by (8).

Let us set

$$f(x) = {}_r\Phi_s\left[\begin{array}{lr} a_1, \cdots, a_r & ; \\ & q, \rho x \\ b_1, \cdots, b_s & ; \end{array}\right]$$

in (12), then we get

$$(21)\quad (x+t)_p\, {}_r\Phi_s\left[\begin{array}{lr} a_1, \cdots, a_r & ; \\ & q, \rho(x+tq^p) \\ b_1, \cdots, b_s & ; \end{array}\right] = \sum_{n=-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2}\; t^{n+\eta}}{\Gamma_q(n+\eta+1)}$$

$$D_{x,q}^{n+\eta}\left\{x^p\; {}_r\Phi_s\left[\begin{array}{lr} a_1, \cdots, a_r & ; \\ & q, \rho x \\ b_1, \cdots, b_s & ; \end{array}\right]\right\}.$$

Using the result (due to Yadav and Purohit [12]):

$$(22)\qquad D_{x,q}^{\lambda}\left\{x^p {}_r\Phi_s\left[\begin{array}{lr} a_1, \cdots, a_r & ; \\ & q, \rho x \\ b_1, \cdots, b_s & ; \end{array}\right]\right\} = \frac{\Gamma_q(p+1)}{\Gamma_q(p+1-\lambda)}\; x^{p-\lambda}$$

$${}_{r+1}\Phi_{s+1}\left[\begin{array}{lr} a_1, \cdots, a_r, q^{p+1} & ; \\ & q, \rho x \\ b_1, \cdots, b_s, q^{p+1-\lambda} & ; \end{array}\right],$$

valid for all values of λ, the series relation (21) leads to

$$(23)\quad (x+t)_p\, {}_r\Phi_s\left[\begin{array}{lr} a_1, \cdots, a_r & ; \\ & q, \rho(x+tq^p) \\ b_1, \cdots, b_s & ; \end{array}\right] = \sum_{n=-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2}\; t^{n+\eta}}{\Gamma_q(n+\eta+1)}$$

$$\frac{\Gamma_q(p+1)}{\Gamma_q(p+1-n-\eta)}\; x^{p-n-\eta}\; {}_{r+1}\Phi_{s+1}\left[\begin{array}{ll} a_1,\cdots,a_r,q^{p+1} & ; \\ & q,\rho x \\ b_1,\cdots,b_s,q^{p+1-n-\eta} & ; \end{array}\right].$$

On replacing t by $-xt$ in (23), we arrive at the following q-generating function.

$$(t;q)_p\; {}_{r+1}\Phi_s\left[\begin{array}{ll} a_1,\cdots,a_r,tq^p & ; \\ & q,\rho x \\ b_1,\cdots,b_s & ; \end{array}\right] = \sum_{n=-\infty}^{\infty}\frac{(q^{-p};q)_{n+\eta}\,(tq^p)^{n+\eta}}{(q;q)_{n+\eta}} \tag{24}$$

$${}_{r+1}\Phi_{s+1}\left[\begin{array}{ll} a_1,\cdots,a_r,q^{p+1} & ; \\ & q,\rho x \\ b_1,\cdots,b_s,q^{p+1-n-\eta} & ; \end{array}\right],$$

provided that both the sides exist.
For $\eta = 0$, (21) yields the q-generating function

$$(t;q)_p\; {}_{r+1}\Phi_s\left[\begin{array}{ll} a_1,\cdots,a_r,tq^p & ; \\ & q,\rho x \\ b_1,\cdots,b_s & ; \end{array}\right] = \sum_{n=0}^{\infty}\frac{(q^{-p};q)_n\,(tq^p)^n}{(q;q)_n} \tag{25}$$

$${}_{r+1}\Phi_{s+1}\left[\begin{array}{ll} a_1,\cdots,a_r,q^{p+1} & ; \\ & q,\rho x \\ b_1,\cdots,b_s,q^{p+1-n} & ; \end{array}\right].$$

Further, if we put $r = s = 0$, then (25) yields the following series summation:
(26)

$$(t;q)_p\; {}_1\Phi_0\left[\begin{array}{ll} tq^p & ; \\ & q,\rho x \\ - & ; \end{array}\right] = \sum_{n=0}^{\infty}\frac{(q^{-p};q)_n\,(tq^p)^n}{(q;q)_n}\; {}_1\Phi_1\left[\begin{array}{ll} q^{p+1} & ; \\ & q,\rho x \\ q^{p+1-n} & ; \end{array}\right].$$

The q-extensions of the Fox's H-function and Meijer's G-function defined, respectively by (9) and (11) in terms of the Mellin-Barne's type of basic integrals possess the advantage that a number of q-special functions (including the basic hypergeometric functions) happen to be the particular cases of these functions. For various basic special functions which are deducible from basic analogue of Fox's H-function or Meijer's G-function, one may refer to the paper of Saxena *et al* [10]. We apply q-Taylor's formula (12) to obtain a series summation (or q-generating function) for the basic Fox's H-function.

Let us choose

$$f(x) = H_{A,B}^{m_1,n_1}\left[\rho x; q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right]$$

in (12), then using the fractional q-derivative formula for H-function of Yadav and Purohit [13], we arrive at the following result:

(27)

$$(x+t)_p\ H_{A,B}^{m_1,n_1}\left[\rho(x+tq^p); q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] = \sum_{n=-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2}\ x^p\ (t/x)^{n+\eta}}{\Gamma_q(n+\eta+1)(1-q)^{n+\eta}}$$

$$H_{A+1,B+1}^{m_1,n_1+1}\left[\rho x; q \,\middle|\, \begin{matrix}(-p,1),(a,\alpha)\\(b,\beta),(n+\eta-p,1)\end{matrix}\right],$$

where η is an arbitrary complex number, $0 \le m_1 \le B$; $0 \le n_1 \le A$ and the H-function satisfies the existence conditions as stated with (9).

A generalized Taylor's formula involving Weyl type fractional derivatives was also used (see Raina [7]) to derive generating function relationship for the Fox's H-function.

For $\alpha_j = \beta_i = 1$ $(j = 1, \cdots, A;\ i = 1, \cdots, B)$, the result (27) reduces to a q-generating function for the basic analogue of G-function given by

$$(28) \qquad (x+t)_p\ G_{A,B}^{m_1,n_1}\left[\rho(x+tq^p); q \,\middle|\, \begin{matrix}a_1,\cdots,a_A\\b_1,\cdots,b_B\end{matrix}\right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2}\ x^p\ (t/x)^{n+\eta}}{\Gamma_q(n+\eta+1)(1-q)^{n+\eta}}\, G_{A+1,B+1}^{m_1,n_1+1}\left[\rho x; q \,\middle|\, \begin{matrix}-p,a_1,\cdots,a_A\\b_1,\cdots,b_B,n+\eta-p\end{matrix}\right].$$

We conclude this paper by remarking that several series summations and generating functions to various basic (or q-analogue) special functions can be deduced from the results (24) and (27).

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