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Differential subordination and superordination theorems for certain analytic functions [1]

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Abstract

Let α, β, γ and δ be complex numbers such that $\alpha \neq 0$. Define Φ on $\mathbb{D} = \mathbb{C} \setminus \{0\}$ as

$$\Phi(w, zw'; z) = w^{\delta}\left(\beta w + \alpha \frac{zw'}{w} + \gamma\right),\ z \in \mathbb{E},$$

where $\mathbb{E} = \{z : |z| < 1\}$. We find the sufficient conditions for analytic function $p,\ p(z) \neq 0$ and analytic univalent functions $q_1,\ q_1(z) \neq 0$ and $q_2,\ q_2(z) \neq 0$ in $\mathbb{E}$ such that

$$\Phi(q_1(z), zq_1'(z); z) \prec \Phi(p(z), zp'(z); z) \prec \Phi(q_2(z), zq_2'(z); z),$$

implies

$$q_1(z) \prec p(z) \prec q_2(z),$$

where q_1 and q_2 are, respectively, best subordinant and best dominant. We give applications of these results to univalent, ϕ-like and $\mathcal{P}$-valent functions and show that our results generalize and unify a number of known results.

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1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (complex plane) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$.

Let $\mathcal{A}$ be the class of functions f, analytic in $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, the classes of starlike functions of order α and convex functions of order α respectively, which are analytically defined as follows:

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in \mathbb{E} \right\}$$

and

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in \mathbb{E} \right\},$$

where α is a real number such that $0 \leq \alpha < 1$. We shall use $\mathcal{S}^*$ and $\mathcal{K}$ to denote $\mathcal{S}^*(0)$ and $\mathcal{K}(0)$, respectively, which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

For two analytic functions f and g in the open unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$.

In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$, with $(p(z), zp'(z)) \in \mathbb{C} \times \mathbb{C}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$, then the function p is said to satisfy first order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z), \ \psi(p(0), 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1). The best dominant is unique upto a rotation of $\mathbb{E}$.

Let $\pi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be analytic and univalent in a domain $\mathbb{C} \times \mathbb{C}$, p be analytic and univalent in $\mathbb{E}$, with $(p(z), zp'(z)) \in \mathbb{C} \times \mathbb{C}$ for all $z \in \mathbb{E}$. Then p is called a solution of the first order differential superordination if

$$h(z) \prec \pi(p(z), zp'(z)), \ h(0) = \pi(p(0), 0). \tag{2}$$

An analytic function q is called a subordinant of the differential superordination (2), if $q \prec p$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2), is said to be the best subordinant of (2). The best subordinant is unique up to a rotation of $\mathbb{E}$.

For any two analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$, the convolution of f and g, written as $f * g$, is defined by

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $\Re\, \phi'(0) > 0$. Then, the function $f \in \mathcal{A}$ is said to be ϕ-like in $\mathbb{E}$ if

$$\Re\, \frac{zf'(z)}{\phi(f(z))} > 0,$$

for all $z \in \mathbb{E}$. ϕ-like functions were introduced by Brickman [1]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is ϕ-like for some ϕ.

Later, Ruscheweyh [18] investigated the following general class of ϕ-like functions.

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. The function $f \in \mathcal{A}$ is called ϕ-like with respect to a univalent function $q, q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z).$$

In what follows, all the powers taken, are the principle ones.

In the present paper, we find the sufficient conditions for analytic function $p,\ p(z) \neq 0$ and analytic univalent functions $q_1,\ q_2$ with $q_1(z) \neq 0,\ q_2(z) \neq 0$ in $\mathbb{E}$ such that

$$\Phi(q_1(z), zq_1'(z); z) \prec \Phi(p(z), zp'(z); z) \prec \Phi(q_2(z), zq_2'(z); z), \tag{3}$$

implies

$$q_1(z) \prec p(z) \prec q_2(z).$$

Moreover q_1 and q_2 are, respectively, the best subordinant and the best dominant for (3) where

$$\Phi(w, zw'; z) = w^{\delta}\left(\beta w + \alpha\frac{zw'}{w} + \gamma\right),\ w \in \mathbb{D} = \mathbb{C} \setminus \{0\},\ z \in \mathbb{E}, \tag{4}$$

and α, β, γ and δ be complex numbers such that $\alpha \neq 0$. We give applications of our results to univalent, ϕ-like and $\mathcal{P}$-valent functions.

Our work is inspired by various differential operators in literature, used as criteria for starlikeness, (see ref. [3], [4], [5], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29]).

In our present work these differential operators are unified and existing results are generalized.

2 Preliminaries

We shall use the following definition and lemmas to prove our main results.

Definition 1 *([6], p.21, Definition 2.2b) We denote by Q the set of functions p that are analytic and injective on $\overline{\mathbb{E}} \setminus \mathbb{B}(p)$, where*

$$\mathbb{B}(p) = \left\{ \zeta \in \partial\mathbb{E} : \lim_{z \to \zeta} p(z) = \infty \right\},$$

and are such that $p'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{B}(p)$.

Lemma 1 *([6], p.132, Theorem 3.4 h) Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*

(i) h is convex, or

(ii) Q_1 is starlike.

In addition, assume that

(iii) $\Re \frac{zh'(z)}{Q_1(z)} > 0, \ z \in \mathbb{E}$.

If p is analytic in $\mathbb{E}$, with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p \prec q$ and q is the best dominant.

Lemma 2 *([2]) Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that* *(i)* Q_1 *is starlike in* $\mathbb{E}$ *and*

(ii) $\Re\, \frac{\theta'(q(z))}{\phi(q(z))} > 0,\ z \in \mathbb{E}.$

If $p \in \mathcal{H}[q(0), 1] \cap Q$, *with* $p(\mathbb{E}) \subset \mathbb{D}$ *and* $\theta[p(z)] + zp'(z)\phi[p(z)]$ *is univalent in* $\mathbb{E}$ *and*

$$\theta[q(z)] + zq'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)],$$

then $q \prec p$ *and* q *is the best subordinant.*

3 Main Theorems

Theorem 1 *Let* $q,\ q(z) \neq 0$, *be a univalent function in* $\mathbb{E}$ *such that*

(i) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + \frac{(\delta-1)zq'(z)}{q(z)}\right] > 0$ *and*

(ii) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + \frac{(\delta-1)zq'(z)}{q(z)} + \frac{\beta(\delta+1)q(z)}{\alpha} + \frac{\gamma\delta}{\alpha}\right] > 0.$

If the analytic function $p,\ p(z) \neq 0,\ z \in \mathbb{E}$, *satisfies the differential subordination*

$$\Phi(p(z), zp'(z); z) \prec \Phi(q(z), zq'(z); z), \tag{5}$$

where α, β, γ *and* δ *are complex numbers with* $\alpha \neq 0$ *and* Φ *is given by (4), then* $p(z) \prec q(z)$ *and* q *is the best dominant.*

Proof. Let us define the functions θ and ϕ as follows:

$$\theta(w) = (\beta w + \gamma)w^{\delta},$$

and

$$\phi(w) = \alpha w^{\delta-1}.$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0,\ w \in \mathbb{D}$.

Define the functions Q_1 and h as follows:

$$Q_1(z) = zq'(z)\phi(q(z)) = \alpha zq'(z)(q(z))^{\delta-1},$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = \Phi(q(z), zq'(z); z).$$

A little calculation yields

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + \frac{(\delta-1)zq'(z)}{q(z)},$$

and

$$\frac{zh'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + \frac{(\delta-1)zq'(z)}{q(z)} + \frac{\beta(\delta+1)q(z)}{\alpha} + \frac{\gamma\delta}{\alpha}.$$

In view of conditions (i) and (ii), we get

(1) Q_1 is starlike in $\mathbb{E}$ and

(2) $\Re \frac{zh'(z)}{Q_1(z)} > 0, \ z \in \mathbb{E}$.

Thus conditions (ii) and (iii) of Lemma 1, are satisfied.

In view of (5), we have

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof, now, follows from Lemma 1.

Theorem 2 *Let $q, \ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

(i) $\Re \left[1 + \frac{zq''(z)}{q'(z)} + \frac{(\delta-1)zq'(z)}{q(z)}\right] > 0$ *and*

(ii) $\Re \left[\frac{\beta(\delta+1)q(z)}{\alpha} + \frac{\gamma\delta}{\alpha}\right] > 0$.

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(z) \neq 0, \ z \in \mathbb{E}$, satisfies the differential superordination

$$\Phi(q(z), zq'(z); z) \prec \Phi(p(z), zp'(z); z), \tag{6}$$

where α, β, γ *and* δ *are complex numbers with* $\alpha \neq 0$, $\Phi(p(z), zp'(z); z)$ *is univalent in* $\mathbb{E}$ *and* Φ *is given by (4), then* $q(z) \prec p(z)$ *and* q *is the best subordinant.*

Proof. Let us define the functions θ and ϕ as follows:

$$\theta(w) = (\beta w + \gamma)w^{\delta},$$

and

$$\phi(w) = \alpha w^{\delta-1}.$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0,\ w \in \mathbb{D}$.

Let us define the functions Q_1 and h as follows:

$$Q_1(z) = zq'(z)\phi(q(z)) = \alpha zq'(z)(q(z))^{\delta-1},$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = \Phi(q(z), zq'(z); z).$$

A little calculation yields

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + \frac{(\delta-1)zq'(z)}{q(z)},$$

and

$$\frac{\theta'(q(z))}{\phi(q(z))} = \frac{\beta(\delta+1)q(z)}{\alpha} + \frac{\gamma\delta}{\alpha}.$$

In view of conditions (i) and (ii), we have

(1) Q_1 is starlike in $\mathbb{E}$ and

(2) $\Re \frac{\theta'(q(z))}{\phi(q(z))} > 0,\ z \in \mathbb{E}$.

Thus by (6), we obtain

$$\theta[q(z)] + zq'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)].$$

Therefore, the proof, now, follows from Lemma 2.

4 Applications to Univalent Functions

On writing $p(z) = \frac{(f*\phi)(z)}{(f*\psi)(z)}$, in Theorem 1, we have the following result.

Theorem 3 *Let q, $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ which satisfy the conditions (i) and (ii) of Theorem 1. If $f \in \mathcal{A}$ and analytic functions ϕ, ψ with $\frac{(f*\phi)(z)}{(f*\psi)(z)} \neq 0$, $z \in \mathbb{E}$, satisfy the differential subordination*

$$\Phi\left[\frac{(f*\phi)(z)}{(f*\psi)(z)}, z\left(\frac{(f*\phi)(z)}{(f*\psi)(z)}\right)'; z\right] \prec \Phi(q(z), zq'(z); z),$$

where α, β, γ and δ are complex numbers with $\alpha \neq 0$ and Φ is given by (4), then

$$\frac{(f*\phi)(z)}{(f*\psi)(z)} \prec q(z),$$

and q is the best dominant.

On writing $p(z) = \frac{(f*\phi)(z)}{(f*\psi)(z)}$, in Theorem 2, we have the following result.

Theorem 4 *Let q, $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ which satisfy the conditions (i) and (ii) of Theorem 2. If $f \in \mathcal{A}$ and analytic functions ϕ, ψ such that $\frac{(f*\phi)(z)}{(f*\psi)(z)} \in \mathcal{H}[q(0), 1] \cap Q$, with $\frac{(f*\phi)(z)}{(f*\psi)(z)} \neq 0$, $z \in \mathbb{E}$, satisfy the differential superordination*

$$\Phi(q(z), zq'(z); z) \prec \Phi\left[\frac{(f*\phi)(z)}{(f*\psi)(z)}, z\left(\frac{(f*\phi)(z)}{(f*\psi)(z)}\right)'; z\right] = h(z),$$

where α, β, γ and δ are complex numbers with $\alpha \neq 0$, h is univalent in $\mathbb{E}$ and Φ is given by (4), then

$$q(z) \prec \frac{(f*\phi)(z)}{(f*\psi)(z)},$$

and q is the best subordinant.

Remark 1 *On selecting the particular values of α, β, γ and δ in Theorem 1 and Theorem 3 and by considering the particular cases of functions ϕ and ψ*

in case of Theorem 3, we can obtain a number of known results and some of them are given below.

(i) On writing $\gamma = 1 - \beta, \delta = 1$ in Theorem 1, we obtain, Lemma 1 of [12].

(ii) On replacing $\gamma = 1$ and $\beta = 0$ in Theorem 1, we obtain, Corollary 3.2 of [21].

(iii) By taking $\alpha = \delta = 1$ and $\gamma = 0$ in Theorem 1, we obtain, Corollary 3.4 of [22].

(iv) By taking $\alpha = \delta = 2, \beta = 0$ and $\gamma = 1$ in Theorem 1, we obtain, Corollary 3.3 of [21] (see also [6], page 77).

(v) By taking $\beta = 0$ and $\gamma = \delta = 1$ in Theorem 1, we obtain, Corollary 3.4 of [21].

(vi) By taking $\alpha = 1$, $\beta = \gamma = 0$ and $\delta = -1$ in Theorem 1, we have the result of Ravichandran and Darus [15].

(vii) By taking $\alpha = \gamma = 1$, $\beta = 0$ and $\delta = \frac{1}{\lambda}$ in Theorem 1, we obtain, Lemma 1 of [13].

(viii) By taking $\phi(z) = \sum_{n=1}^{\infty} nz^n$, $\psi(z) = \sum_{n=1}^{\infty} z^n$, $\beta = \alpha$, $\gamma = 1 - \alpha$ and $\delta = 1$ in Theorem 3, we obtain the Theorem 3 of [12].

(ix) By taking $\phi(z) = \sum_{n=1}^{\infty} nz^n$, $\psi(z) = \sum_{n=1}^{\infty} z^n$, $\beta = 1$ and $\gamma = \delta = 0$ in Theorem 3, we obtain, Theorem 4.3 of [22].

(x) By taking $\phi(z) = \sum_{n=1}^{\infty} nz^n$, $\psi(z) = \sum_{n=1}^{\infty} z^n$, $\alpha = \beta = 1, \gamma = 0$ and $\delta = -1$ in Theorem 3, we obtain, Theorem 4.5 of [22].

Remark 2 *By making the selections same as in Remark 1, in Theorem 2 and Theorem 4, we can obtain the corresponding results for superordination. e.g.*

(i) For $\delta = 1$ in Theorem 2, we obtain Lemma 2.1 of [17].

(ii) On writing $\gamma = \delta = 0$ in Theorem 2, we obtain Lemma 2.4 of [17]

(iii) By taking $\phi(z) = \sum_{n=1}^{\infty} nz^n$, $\psi(z) = \sum_{n=1}^{\infty} z^n$, $\alpha = \beta$, $\gamma = 1 - \beta$ and $\delta = 1$ in Theorem 4, we obtain, Theorem 2.2 of [17].

(iv) By taking $\phi(z) = \sum_{n=1}^{\infty} nz^n$, $\psi(z) = \sum_{n=1}^{\infty} z^n$, $\beta = 1$ and $\gamma = \delta = 0$ in Theorem 4, we obtain, Theorem 2.5 of [17].

5 Applications to Multivalent Functions

Let $\mathcal{A}(\mathcal{P})$ denote the class of functions of the form $f(z) = z^{\mathcal{P}} + \sum_{k=1}^{\infty} a_{\mathcal{P}+k} z^{\mathcal{P}+k}$, $(\mathcal{P} \in \mathbb{N} = \{1, 2, 3, \cdots\})$, which are analytic and $\mathcal{P}$-valent in $\mathbb{E}$.
On writing $p(z) = \frac{1}{\mathcal{P}} \frac{zf'(z)}{f(z)})$, in Theorem 1, we have the following result.

Theorem 5 *Let q, $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, which satisfy the conditions (i) and (ii) of Theorem 1. If $f \in \mathcal{A}(\mathcal{P})$, with $\frac{1}{\mathcal{P}} \frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential subordination*

$$\Phi\left[\frac{1}{\mathcal{P}} \frac{zf'(z)}{f(z)}, z\left(\frac{1}{\mathcal{P}} \frac{zf'(z)}{f(z)}\right)'; z\right] \prec \Phi(q(z), zq'(z); z),$$

where α, β, γ and δ are complex numbers with that $\alpha \neq 0$ and Φ is given by (4), then $\frac{1}{\mathcal{P}} \frac{zf'(z)}{f(z)} \prec q(z)$ and q is the best dominant.

On writing $p(z) = \frac{1}{\mathcal{P}} \frac{zf'(z)}{f(z)}$, in Theorem 2, we have the following result.

Theorem 6 *Let q, $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, which satisfy the conditions (i) and (ii) of Theorem 2. If $f \in \mathcal{A}(\mathcal{P})$, $\frac{1}{\mathcal{P}} \frac{zf'(z)}{f(z)} \in \mathcal{H}[q(0), 1] \cap Q$, with $\frac{1}{\mathcal{P}} \frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential superordination*

$$\Phi(q(z), zq'(z); z) \prec \Phi\left[\frac{1}{\mathcal{P}} \frac{zf'(z)}{f(z)}, z\left(\frac{1}{\mathcal{P}} \frac{zf'(z)}{f(z)}\right)'; z\right] = h(z),$$

where α, β, γ and δ are complex numbers with $\alpha \neq 0$, h is univalent in $\mathbb{E}$ and Φ is given by (4), then $q(z) \prec p(z)$ and q is the best subordinant.

Remark 3 *We can obtain interesting results for $\mathcal{P}$-valent functions by selecting the particular values $\alpha,\ \beta,\ \gamma$ and δ in Theorem 5.*

e.g. For $\beta = \mathcal{P},\ \gamma = 0$ and $\delta = 0$ in Theorem 5, we obtain Theorem 1 of [25].

Also note that for the same selection in Theorem 6, we can obtain the corresponding result for superordination.

6 Applications to ϕ-like Functions

On writing $p(z) = \frac{z(f*g)'(z)}{\phi((f*g)(z))}$, in Theorem 1, we obtain the following result.

Theorem 7 *Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ which satisfy the conditions (i) and (ii) of Theorem 1. If $f,\ g \in \mathcal{A}$ such that $\frac{z(f*g)'(z)}{\phi((f*g)(z))} \neq 0,\ z \in \mathbb{E}$, satisfy the differential subordination*

$$\Phi\left[\frac{z(f*g)'(z)}{\phi((f*g)(z))}, z\left(\frac{z(f*g)'(z)}{\phi((f*g)(z))}\right)'; z\right] \prec \Phi(q(z), zq'(z); z),$$

where α, β, γ and δ are complex numbers with $\alpha \neq 0$, ϕ is an analytic function in domain containing $(f*g)(\mathbb{E})$, $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in (f*g)(\mathbb{E}) \setminus \{0\}$ and Φ is given by (4), then

$$\frac{z(f*g)'(z)}{\phi((f*g)(z))} \prec q(z),$$

and q is the best dominant.

On writing $p(z) = \frac{z(f*g)'(z)}{\phi((f*g)(z))}$, in Theorem 2, we have the following result.

Theorem 8 *Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ which satisfy the conditions (i) and (ii) of Theorem 2. If $f,\ g \in \mathcal{A}$ such that $\frac{z(f*g)'(z)}{\phi((f*g)(z))} \in$*

$\mathcal{H}[q(0),1] \cap Q$, *with* $\frac{z(f*g)'(z)}{\phi((f*g)(z))} \neq 0,\ z \in \mathbb{E}$, *satisfy the differential superordination*

$$\Phi(q(z), zq'(z); z) \prec \Phi\left[\frac{z(f*g)'(z)}{\phi((f*g)(z))}, z\left(\frac{z(f*g)'(z)}{\phi((f*g)(z))}\right)'; z\right] = h(z),$$

where α, β, γ *and* δ *are complex numbers with* $\alpha \neq 0$, h *is univalent in* $\mathbb{E}$, ϕ *is an analytic function in domain containing* $(f*g)(\mathbb{E})$, $\phi(0) = 0, \phi'(0) = 1$ *and* $\phi(w) \neq 0$ *for* $w \in (f*g)(\mathbb{E}) \setminus \{0\}$ *and* Φ *is given by (4), then*

$$q(z) \prec \frac{z(f*g)'(z)}{\phi((f*g)(z))},$$

and q *is the best subordinant.*

Remark 4 *On putting* $\gamma = 0$ *and* $\delta = 0$ *in Theorem 7, we obtain Theorem 2.1 of [19] and by the same selection in Theorem 8, we obtain Theorem 2.5 of [19].*

Remark 5 *If we select* $g(z) = \sum_{n=1}^{\infty} z^n$ *in Theorem 7 and Theorem 8, then for* $f \in \mathcal{A}$, *we have*

$$\frac{z(f*g)'(z)}{\phi(f*g)(z)} = \frac{zf'(z)}{\phi(f(z))}.$$

Now the applications of Theorem 7 and Theorem 8, can be seen by giving different values to α, β, γ *and* δ. *By doing so, we obtain the results of ([4],[13],[24]). e.g.*

(i) On writing $g(z) = \sum_{n=1}^{\infty} z^n$, $\alpha = \beta,\ \gamma = 1 - \beta$ and $\delta = 1$ in Theorem 7, we obtain, Theorem 3 of [13].

(ii) On writing $g(z) = \sum_{n=1}^{\infty} z^n$, $\alpha = \gamma = 1,\ \beta = 0$ and $\delta = \frac{1}{\lambda}$ in Theorem 7, we obtain, Theorem 4 of [13].

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