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On a subclass of analytic functions with negative coefficient associated with convolution structure [1]

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Abstract

The main object of this paper is to study the subclass $SC(\gamma, \lambda, \beta)$ of analytic univalent functions with negative coefficients in unit disc U $= \{z : |z| < 1\}$. Further coefficient estimates, distortion theorem and integral operators for this class are also obtained. We also discuss radii of convexity and closure properties for functions belonging to the class $SC(\gamma, \lambda, \beta)$.

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1 Introduction

Let $\mathcal{A}$ denote the class of the functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

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which are analytic in the unit disk $U = \{z : |z| < 1\}$.

A function $f \in \mathcal{A}$ is said to belong to the class A of *Starlike* functions of order α $(0 \leq \alpha < 1)$, if it is satisfies, for $z \in U$, the conditions

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha. \tag{2}$$

We denote this class by $S^*(\alpha)$. Further, $f \in \mathcal{A}$ is said to be convex function of order α in U, if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad z \in U, \tag{3}$$

for some α $(0 \leq \alpha < 1)$. We denote this class $K(\alpha)$.
Let T denote subclass of A, consisting functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0. \tag{4}$$

The function

$$S_\alpha(z) = z(1-z)^{-2(1-\alpha)}, \ \ \alpha(0 \leq \alpha \leq 1) \tag{5}$$

is the familiar extremal function for the class $S^*(\alpha)$, setting

$$C(\alpha, k) = \frac{\prod_{i=2}^{k} (i - 2\alpha)}{(k-1)!}, k \geq 2, \tag{6}$$

using (5) and (6) we can write

$$\mathrm{S}_\alpha(z) = z + \sum_{k=2}^{\infty} C(\alpha, k) z^k. \tag{7}$$

Clearly, $C(\alpha, k)$ is a decreasing function in α, and that

$$\lim_{k\to\infty} C(\alpha, k) = \begin{cases} \infty, \alpha < 1/2 \\ 1, \alpha = 1/2 \\ 0, \alpha > 1/2. \end{cases} \tag{8}$$

If we now define g(z) as

$$\text{g}(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{9}$$

then the Hadamard product (or convolution) of two analytic functions f(z) and g(z), where f(z) , g(z) is given by equations (1) and (9) respectively, is defined as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \tag{10}$$

For a function f(z) in $\mathcal{A}$, we can define the differential operator D^n, introduced by Salagean [9] as

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z) = z + \sum_{k=2}^{\infty} \text{k}\text{a}_k z^k$$

$$D^2 f(z) = D(Df(z)) = z + \sum_{k=2}^{\infty} \text{k}^2 \text{a}_k z^k$$

$$D^n f(z) = D(D)^{n-1} f(z) = z + \sum_{k=2}^{\infty} \text{k}^{\text{n}} \text{a}_k z^k \tag{11}$$

We also define a subclass of $\mathcal{A}$ consisting of functions f(z), denoted by $SC(\gamma, \lambda, \beta)$ which satisfy the following condition

$$Re\left[1 + \frac{1}{\gamma}\left(\frac{z\left[\lambda z(D^n f * S_\alpha)'(z) + (1-\lambda)(D^n f * S_\alpha)'(z)\right]}{\lambda z(D^n f * S_\alpha)'(z) + (1-\lambda)(D^n f * S_\alpha)(z)} - 1\right)\right] > \beta, \tag{12}$$
$$(0 \leq \lambda \leq 1, 0 \leq \beta < 1; \gamma \in C, z \in U).$$

Special case of class.

(a) When $\lambda = 0$, and $\alpha = 1/2$ then our class reduces in class of starlike functions of order β.

(b) When $\lambda = 0$, then our class reduces in class of starlike functions of complex order γ.

(c) When $\alpha = 1/2$ then this class reduces in class defined and studied by Altintas, Irmak, Owa and Srivastava [5].

2 Coefficient estimates.

Theorem 1 *Let the function $f(z)\varepsilon A$ is in the class $SC(\gamma, \lambda, \beta)$, if and only if*

$$\sum_{k=2}^{\infty} k^n \left[\lambda k + 1 - \lambda\right] \left[k - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, \mathrm{k})\mathrm{a}_{\mathrm{k}} \leq \gamma\,(1{-}\beta) \tag{13}$$

Proof. Assume that the inequality (13) holds true, then

$$\left|\frac{1}{\gamma}\left(\frac{z\ [\lambda z(D^n f * S_\alpha)'(z) + (1-\lambda)(D^n f * S_\alpha)'(z)]}{\lambda z(D^n f * S_\alpha)'(z) + (1-\lambda)(D^n f * S_\alpha)(z)} - 1\right)\right|$$
$$= \left|\frac{1}{\gamma}\left(\frac{\sum\limits_{k=2}^{\infty} \mathrm{k}^{\mathrm{n}} \left[\lambda k + 1 - \lambda\right](1-k)\mathrm{C}(\alpha, k)\mathrm{a}_{\mathrm{k}}\mathrm{z}^{\mathrm{k}-1}}{1 - \sum\limits_{k=2}^{\infty} \mathrm{k}^{\mathrm{n}} \left[\lambda k + 1 - \lambda\right] \mathrm{C}(\alpha, k)\mathrm{a}_{\mathrm{k}}\mathrm{z}^{\mathrm{k}-1}}\right)\right|$$
$$\leq (1 - \beta).$$

Hence, by using the maximum modulus principle, $f(z) \in SC(\gamma, \lambda, \beta)$. Conversely, assume that the function f(z) defined by (1) is in the class $SC(\gamma, \lambda, \beta)$. Then we will have

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{z\,[\lambda\, z(D^n f * S_\alpha)'(z) + (1-\lambda)(D^n f * S_\alpha)'(z)]}{\lambda z(D^n f * S_\alpha)'(z) + (1-\lambda)(D^n f * S_\alpha)(z)} - 1\right)\right\} > \beta,$$

$$Re\left[1 + \frac{1}{\gamma}\left\{\frac{\sum\limits_{k=2}^{\infty} \mathrm{k}^{\mathrm{n}} \left[\lambda k + 1 - \lambda\right](1-k)\mathrm{C}(\alpha, k)\mathrm{a}_{\mathrm{k}}\mathrm{z}^{\mathrm{k}}}{z - \sum\limits_{k=2}^{\infty} \mathrm{k}^{\mathrm{n}} \left[\lambda k + 1 - \lambda\right] \mathrm{C}(\alpha, k)\mathrm{a}_{\mathrm{k}}\mathrm{z}^{\mathrm{k}}}\right\}\right] > \beta,$$

$$Re\left[1+\frac{1}{\gamma}\left\{\frac{\sum\limits_{k=2}^{\infty}\mathrm{k^n}\left[\lambda k+1-\lambda\right](1-k)\mathrm{C}(\alpha,k)\mathrm{a_k z^{k-1}}}{1-\sum\limits_{k=2}^{\infty}\mathrm{k^n}\left[\lambda k+1-\lambda\right]\mathrm{C}(\alpha,k)\mathrm{a_k z^{k-1}}}\right\}\right]>\beta,$$

and now when $z\to 1^-$, we obtain

$$\frac{\sum\limits_{k=2}^{\infty}\mathrm{k^n}\left[\lambda k+1-\lambda\right](1-k)\mathrm{C}(\alpha,k)\mathrm{a_k}}{1-\sum\limits_{k=2}^{\infty}\mathrm{k^n}\left[\lambda k+1-\lambda\right]\mathrm{C}(\alpha,k)\mathrm{a_k}}\geq\gamma(\beta-1)$$

and finally,

$$\sum_{k=2}^{\infty}\mathrm{k^n}\left[\lambda k+1-\lambda\right]\left[\mathrm{k}-1-\gamma(\beta-1)\right]\mathrm{C}(\alpha,k)\mathrm{a_k}\leq\gamma(1-\beta).$$

Corollary 1 *Let the function f(z) defined by (1) be in the class SC*(γ,λ,β)*. Then*

$$a_k\leq\frac{\gamma(1-\beta)}{\mathrm{k^n}\left[\lambda k+1-\lambda\right]\left[\mathrm{k}-1-\gamma(\beta-1)\right]\mathrm{C}(\alpha,k)},\quad(\mathrm{k}\geq 2)\tag{14}$$

and the equality is attained for the function f(z) given by

$$f(z)=\mathrm{z}-\frac{\gamma(1-\beta)}{\mathrm{k^n}\left[\lambda k+1-\lambda\right]\left[\mathrm{k}-1-\gamma(\beta-1)\right]\mathrm{C}(\alpha,k)}\mathrm{z^k}.\tag{15}$$

3 Distortion Theorem.

Theorem 2 *Let the function f(z) be in class SC*(γ,λ,β) *then for* $0\leq|z|=r$

$$\mathrm{r}-\frac{\gamma(1-\beta)}{\mathrm{k^n}\left[\lambda k+1-\lambda\right]\left[\mathrm{k}-1-\gamma(\beta-1)\right]\mathrm{C}(\alpha,k)}\mathrm{r^k}\leq\ |\mathrm{f(z)}|\tag{16}$$

$$\leq\mathrm{r}+\frac{\gamma(1-\beta)}{\mathrm{k^n}\left[\lambda k+1-\lambda\right]\left[\mathrm{k}-1-\gamma(\beta-1)\right]\mathrm{C}(\alpha,k)}\mathrm{r^k}.\tag{17}$$

Proof. From equation (15), easy to find that

$$|\mathrm{z}| - \frac{\gamma(1-\beta)}{\mathrm{k^n}\,[\lambda k+1-\lambda]\ [\mathrm{k}-1-\gamma(\beta-1)]\,\mathrm{C}(\alpha,k)}\,|\mathrm{z}|^{\mathrm{k}} \le |\mathrm{f(z)}|$$

$$\le |\mathrm{z}| + \frac{\gamma(1-\beta)}{\mathrm{k^n}\,[\lambda k+1-\lambda]\ [\mathrm{k}-1-\gamma(\beta-1)]\,\mathrm{C}(\alpha,k)}\,|\mathrm{z}|^{\mathrm{k}}$$

Now using the fact that $|z| = r < 1$, we obtain the desired result.

Corollary 2 *If the function f(z) is in the class $SC(\gamma,\lambda,\beta)$ then f(z) is included in a disc with centre at the origin and radius r, where*

$$\mathrm{r} = 1 + \frac{\gamma(1-\beta)}{\mathrm{k^n}\,[\lambda k+1-\lambda]\ [\mathrm{k}-1-\gamma(\beta-1)]\,\mathrm{C}(\alpha,k)} \tag{18}$$

Theorem 3 *Let the function f(z) be in the class $SC(\gamma,\lambda,\beta)$, then*

$$1 - \frac{\gamma(1-\beta)}{\mathrm{k^{n-1}}\,[\lambda k+1-\lambda]\ [\mathrm{k}-1-\gamma(\beta-1)]\,\mathrm{C}(\alpha,k)}\mathrm{r^{k-1}} \le |\mathrm{f(z)}|$$

$$\le 1 + \frac{\gamma(1-\beta)}{\mathrm{k^{n-1}}\,[\lambda k+1-\lambda]\ [\mathrm{k}-1-\gamma(\beta-1)]\,\mathrm{C}(\alpha,k)}\mathrm{r^{k-1}}$$

Where equality holds for the function f(z) given by (15).

$$1 - \frac{k\gamma(1-\beta)}{\mathrm{k^n}\,[\lambda k+1-\lambda]\ [\mathrm{k}-1-\gamma(\beta-1)]\,\mathrm{C}(\alpha,k)}\,|\mathrm{z}|^{\mathrm{k-1}} \le |\mathrm{f(z)}|$$

$$\le 1 + \frac{k\gamma(1-\beta)}{\mathrm{k^n}\,[\lambda k+1-\lambda]\ [\mathrm{k}-1-\gamma(\beta-1)]\,\mathrm{C}(\alpha,k)}\,|\mathrm{z}|^{\mathrm{k-1}}$$

Again using the fact that $|z| = r$, we obtain the desired result.

4 Integral Operators

Theorem 4 *Let the function f(z) defined by (1) be in the class $SC(\gamma,\lambda,\beta)$ and let c be a real number such that $c > -1$. Then F(z), defined by*

$$\mathrm{F(z)} = \frac{\mathrm{c}+1}{\mathrm{z^c}} \int_0^z t^{c-1} f(t)dt \tag{19}$$

also belongs to the class $SC(\gamma,\lambda,\beta)$.

Proof. From the representation of F(z), it is obtained that

$$\mathrm{F(z)} = z - \sum_{k=2}^{\infty} b_k z^k, \tag{20}$$

where $b_k = \left(\frac{c+1}{k+c}\right) a^k$

Therefore

$$\sum_{k=2}^{\infty} \mathrm{k^n} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k)\mathrm{b_k}$$

$$= \sum_{k=2}^{\infty} \mathrm{k^n} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k) \left(\frac{c+1}{k+c}\right) \mathrm{a_k}$$

$$\leq \sum_{k=2}^{\infty} \mathrm{k^n} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k)\mathrm{a_k}$$

$$\leq \gamma(\beta - 1),$$

since f(z) belongs to $SC(\gamma, \lambda, \beta)$ so by virtue of Theorem 1, F(z) is also element of $SC(\gamma, \lambda, \beta)$.

Theorem 5 *Let the function*

$$F(z) = z - \sum_{k=2}^{\infty} a_k z^k, \mathrm{a_k} \geq 0$$

be in the class $SC(\gamma, \lambda, \beta)$ and is defined by equation (19). Now if $c > -1$, then F(z) is univalent in $|z| < R^*$, where

$$R^* = \inf \left\{ \frac{\mathrm{k^{n-1}} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k)(\mathrm{c}+1)}{(c+\mathrm{k})\gamma(1-\beta)} \right\}^{\frac{1}{\mathrm{k}-1}}, \mathrm{k} \geq 2 \tag{21}$$

The result is sharp.

Proof. From (19) we have

$$f(z) = \frac{z^{1-c} \left(z^c F(z)\right)'}{c+1} = z - \sum_{k=2}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k.$$

In order to obtain the required result, it is sufficient to prove that

$$\left|f'(z)-1\right|<1 \; for \; |z|<R^{*}.$$

Now since

$$(22) \qquad \left|\mathrm{f}'(\mathrm{z})-1\right| = \left|-\sum_{k=2}^{\infty} \mathrm{k}\left(\frac{c+k}{c+1}\right) a_k z^{k-1}\right|$$

$$\leq \sum_{k=2}^{\infty} \mathrm{k}\left(\frac{c+k}{c+1}\right) a_k \, |z|^{k-1}$$

$$\leq \sum_{k=2}^{\infty} \mathrm{k}\left(\frac{c+k}{c+1}\right) a_k \, |z|^{k-1}$$

$$\leq \sum_{k=2}^{\infty} \mathrm{k}\left(\frac{c+k}{c+1}\right) a_k \, |z|^{k-1} \leq 1$$

But from Theorem 1, we know that

$$(23) \qquad \sum_{\mathrm{k}=2}^{\infty} \frac{\mathrm{k}^{\mathrm{n}}\left[\lambda k+1-\lambda\right]\left[\mathrm{k}-1-\gamma(\beta-1)\right]\mathrm{C}(\alpha,k)\mathrm{a_k}}{\gamma(1-\beta)} < 1$$

From equation (22) and (23) we have

$$\mathrm{k}\left(\frac{c+k}{c+1}\right)|z|^{k-1} \leq \frac{\mathrm{k}^{\mathrm{n}}\left[\lambda k+1-\lambda\right]\left[\mathrm{k}-1-\gamma(\beta-1)\right]\mathrm{C}(\alpha,k)}{\gamma(1-\beta)}$$

or

$$(24) \qquad |z| \leq \left\{\frac{\mathrm{k}^{\mathrm{n-1}}\left[\lambda k+1-\lambda\right]\left[\mathrm{k}-1-\gamma(\beta-1)\right]\mathrm{C}(\alpha,k)(\mathrm{c}+1)}{(c+k)\gamma(1-\beta)}\right\}^{\frac{1}{\mathrm{k}-1}}, (\mathrm{k}\geq 2),$$

we obtain the desired result. The result is sharp for the function

$$\mathrm{f(z)} = z - \frac{(c+k)\gamma(1-\beta)}{\mathrm{k}^{\mathrm{n}}\left[\lambda k+1-\lambda\right]\left[\mathrm{k}-1-\gamma(\beta-1)\right](c+k)\mathrm{C}(\alpha,k)}\mathrm{z}^{\mathrm{k}}, (\mathrm{k}\geq 2).$$

5 Radius of Convexity

Theorem 6 *If f(z) given by (1) is in class $SC(\gamma, \lambda, \beta)$ then f(z) is convex in $|z| < Rp$, where*

$$\mathrm{R_p} = \inf \left\{ \frac{\mathrm{k}^{\mathrm{n}-2} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k) a_{\mathrm{k}}}{\gamma(1-\beta)} \right\}^{\frac{1}{(\mathrm{k}-1)}} \tag{25}$$

The result is sharp.

Proof. In order to establish the required result it is sufficient to show that

$$\left| \frac{zf'(z)}{f'(z)} \right| < 1, \quad |z| < R_p$$

in view of (1), we have

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{\sum\limits_{k=2}^{\infty} k(k-1) a_k \left|\mathrm{z}\right|^{k-1}}{1 - \sum\limits_{k=2}^{\infty} k a_k \left|\mathrm{z}\right|^{k-1}}$$

Hence, we obtain

$$\sum_{k=2}^{\infty} k^2 a_k \left|\mathrm{z}\right|^{k-1} \leq 1 \tag{26}$$

but from Theorem 1, we know that

$$\sum_{\mathrm{k}=2}^{\infty} \frac{\mathrm{k}^{\mathrm{n}} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k) \mathrm{a_k}}{\gamma(1-\beta)} < 1 \tag{27}$$

and thus from (26) and (27) we have

$$k^2 \left|\mathrm{z}\right|^{k-1} \leq \frac{\mathrm{k}^{\mathrm{n}} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k)}{\gamma(1-\beta)}$$

or

$$|\mathrm{z}| \leq \left\{ \frac{\mathrm{k}^{\mathrm{n}-2} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k)}{\gamma(1-\beta)} \right\}^{\frac{1}{(k-1)}}$$

Hence, f(z) is convex in $|z| < R_p$. The result is sharp and is given by (25).

6 Closure Theorem

Theorem 7 *Let the function $f_j(z), (j = 1, 2, , m)$ be defined by*

$$f_j(z) = z - \sum_{k=2}^{\infty} \mathrm{a_{kj}} \mathrm{z^k} \quad (a_{kj} > 0) \tag{28}$$

for $z \in U$, be in the class $SC(\gamma, \lambda, \beta)$ then the function h(z) defined by

$$h(z) = z - \sum_{k=2}^{\infty} \mathrm{b_k} \mathrm{z^k}$$

also belongs to the class $SC(\gamma, \lambda, \beta)$, where

$$b_k = \frac{1}{m} \sum_{j=1}^{m} \mathrm{a_{kj}}$$

Proof. Since $f_j(z)$ $SC(\gamma, \lambda, \beta)$, it follows from Theorem 1, that

$$\sum_{k=2}^{\infty} \mathrm{k^n} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k)\mathrm{a_{kj}} < \gamma(1 - \beta), \quad (j = 1, 2, , m).$$

Therefore,

$$\begin{aligned} &\sum_{k=2}^{\infty} \mathrm{k^n} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k)\mathrm{b_k} \\ &= \sum_{k=2}^{\infty} \mathrm{k^n} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k) \left(\frac{1}{m} \sum_{j=1}^{m} \mathrm{a_{kj}}\right) \\ &= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=2}^{\infty} \mathrm{k^n} \left[\lambda k + 1 - \lambda\right] \left[\mathrm{k} - 1 - \gamma(\beta - 1)\right] \mathrm{C}(\alpha, k)\mathrm{a_{kj}}\right) \\ &< \gamma(1 - \beta). \end{aligned}$$

Hence by Theorem 1, $h(z) \in SC(\gamma, \lambda, \beta)$ also.

Theorem 8 *The class $SC(\gamma, \lambda, \beta)$ is closed under linear combination.*

Proof. *Employing same techniques used by Silverman [14] with the aid of Theorem 8, it can be easily proved.*

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