Some remarks on univalence criteria

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Abstract

The first results concerning univalence criteria are related to the univalence of an analytic function in the unit disk.

In paper [5] the author obtained a sufficient condition for the analyticity and the univalence of a family of functions defined by an integral operator, which is an extension of the univalence criteria of Becker, of Nehari and of Lewandowski.

In this note we prove that the result mentioned above also represents an extension of the univalence criteria of Ozaki and Nunokawa.

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1 Introduction

Let A be the class of functions f analytic in the unit disk $U = \{ z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = 0$ and $f'(0) = 1$.

The following sufficient conditions for univalency of an analytic function in the unit disk are well known:

Theorem 1 ([3]). *Let $f \in A$. If for all $z \in U$*

$$(1) \qquad |\{f; z\}| \leq \frac{2}{(1 - |z|^2)^2}$$

where

$$\{f; z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

then the function f is univalent in U.

Theorem 2 ([1]). *Let $f \in A$. If for all $z \in U$*

$$(2) \qquad \left(1 - |z|^2\right)\left|\frac{z f''(z)}{f'(z)}\right| \leq 1$$

then the function f is univalent in U.

Theorem 3 ([7]). *Let $f \in A$. If for all $z \in U$*

$$(3) \qquad \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1$$

then the function f is univalent in U.

Theorem 4 ([2]). *Let $f \in A$. If there exists an analytic function p with positive real part in U, $p(0) = 1$, such that the inequality*

$$(4) \qquad \left|\frac{p(z) - 1}{p(z) + 1}|z|^2 - \left(1 - |z|^2\right)\left(\frac{z p'(z)}{p(z) + 1} + \frac{z f''(z)}{f'(z)}\right)\right| < 1$$

holds true for all $z \in U$, then the function f is univalent in U.

Earlear, in paper [4], the author was obtained an univalence criteria which contains as particular cases Theorems 1, 2 and 4. Later, H. Ovesea-Tudor and S. Owa [6] given a generalization of Theorems 1, 3.

A sufficient condition for the analyticity and the univalence of a class of functions defined by an integral operator was presented by the author in paper [5]. The proofs are based on the theory of Löewner chains, essence of which is the construction of a suitable chain.

Theorem 5 ([5]). *Let α be a complex number, Re $\alpha > 0$, $f \in A$ and $g(z) = 1 + b_1 z + \ldots$, $h(z) = 1 + c_1 z + \ldots$ be analytic functions in U with $f'(z)g(z)h(z) \neq 0$ for all $z \in U$. If*

$$
(5) \qquad \left| \frac{g(z)}{h(z)} - 1 \right| < 1
$$

and

$$
(6) \quad \left| \left(\frac{g(z)}{h(z)} - 1 \right) |z|^{2\alpha} + \frac{1 - |z|^{2\alpha}}{\alpha} z \left(\frac{f''(z)}{f'(z)} - \frac{g'(z)}{g(z)} + (2\alpha + 1)\frac{h'(z)}{h(z)} \right) + \right.
$$

$$
\left. \frac{(1 - |z|^{2\alpha})^2}{\alpha |z|^{2\alpha}} z \left[(\alpha + 1)\frac{z\,(h'(z))^2}{g(z)h(z)} + \frac{zf''(z)h'(z)}{f'(z)g(z)} - \frac{zh''(z)}{g(z)} + (\alpha - 1)\frac{h'(z)}{g(z)} \right] \right| \leq 1
$$

are true for all $z \in U \setminus \{0\}$, then the function F_α,

$$
(7) \qquad F_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u)du \right)^{1/\alpha}
$$

is analytic and univalent in U, where the principal branch is intended.

Suitable choices of the functions g and h in Theorems 5 yield various types of univalence criteria.

For $g(z) \equiv 1$, $h(z) \equiv 1$ we find the following criterion of univalence :

Corollary 1 ([5]). *Let $f \in A$ and α be a complex number, Re $\alpha > 0$. If for all $z \in U \setminus \{0\}$*

$$\left| \frac{(1 - |z|^{2\alpha})}{\alpha} \cdot \frac{zf''(z)}{f'(z)} \right| \le 1$$

then the function F_α defined by (7) is analytic and univalent in U.

For $\alpha = 1$, the above corollary reduces to Becker's criterion [1].

For $h(z) \equiv 1$, $g(z) = 2/(1 + p(z))$ we get the following criterion of univalence :

Corollary 2 ([5]). *Let $f \in A$ and α be a complex number, Re $\alpha > 0$. If there exists an analytic function p with positive real part in U, $p(0) = \gamma$, such that*

$$\left| \frac{p(z) - 1}{p(z) + 1} |z|^{2\alpha} - \frac{1 - |z|^{2\alpha}}{\alpha} \left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{p(z) + 1} \right) \right| \le 1$$

for all $z \in U \setminus \{0\}$, then the function F_α defined by (7) is analytic and univalent in U.

If $\alpha = 1$, the Corollary 2 is equivalent to Lewandowski's univalence criterion [2].

Taking

$$g(z) = (f'(z))^{-1/(2\alpha)}, \qquad h(z) = (f'(z))^{-1/(2\alpha)},$$

we have the following one:

Corollary 3 ([5]). *Let $f \in A$ and α be a complex number, Re $\alpha > 0$. If for all $z \in U \setminus \{0\}$*

$$\left| \frac{(1 - |z|^{2\alpha})^2}{2\alpha^2 |z|^{2\alpha}} \left(z^2 \{f; z\} + (1 - \alpha) \frac{zf''(z)}{f'(z)} \right) \right| \le 1$$

then the function F_α defined by (7) is analytic and univalent in U.

For $\alpha = 1$, from Corollary 3 we find Nehari's univalence criterion [3].

2   Remarks

For the case when

$$(8) \qquad g(z) = \frac{z^2 f'(z) h(z)}{f^2(z)}$$

$$(9) \qquad \frac{h'(z)}{h(z)} = -\frac{1}{\alpha} \cdot \frac{f(z) - z}{f^2(z)} f'(z)$$

we obtain the following criterion of univalence found very recently by the author in [9].

Corollary 4 ([9]). *Let $f \in A$ and α be a complex number, $\mathrm{Re}\,\alpha > 0$. If the inequalities*

$$(10) \qquad \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1$$

and

$$(11) \qquad \left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) |z|^{2\alpha} + 2 \frac{1 - |z|^{2\alpha}}{\alpha} \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) + \right.$$

$$\left. \frac{(1 - |z|^{2\alpha})^2}{\alpha^2 |z|^{2\alpha}} \left[\left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) + (1 - \alpha) \left(\frac{f(z)}{z} - 1 \right) \right] \right| \leq 1$$

are true for all $z \in U \setminus \{0\}$, then the function F_α defined by (7) is analytic and univalent in U.

Proof. From (8) we have

$$\frac{g(z)}{h(z)} - 1 = \frac{z^2 f'(z)}{f^2(z)} - 1$$

and then the inequality (5) becomes (10). Taking into account (8) and (9) we get

$$z\left(\frac{f''(z)}{f'(z)} - \frac{g'(z)}{g(z)} + (2\alpha+1)\frac{h'(z)}{h(z)}\right)$$

$$= \frac{zf''(z)}{f'(z)} - z\left(\frac{2}{z} + \frac{f''(z)}{f'(z)} + \frac{h'(z)}{h(z)} - 2\frac{f'(z)}{f(z)}\right) + (2\alpha+1)\frac{h'(z)}{h(z)}$$

$$= 2\left(\frac{zf'(z)}{f(z)} - 1 + \alpha\frac{h'(z)}{h(z)}\right) = 2\left(\frac{z^2f'(z)}{f^2(z)} - 1\right)$$

and a straightforward calculation yields

$$z\left[(\alpha+1)\frac{z\left(h'(z)\right)^2}{g(z)h(z)} + \frac{zf''(z)h'(z)}{f'(z)g(z)} - \frac{zh''(z)}{g(z)} + (\alpha-1)\frac{h'(z)}{g(z)}\right]$$

$$= \frac{1}{\alpha}\left[\left(\frac{z^2f'(z)}{f^2(z)} - 1\right) + (1-\alpha)\left(\frac{f(z)}{z} - 1\right)\right]$$

So, we deduce that the inequality (6) becomes (11).

For $\alpha = 1$ from Corollary 4 we find Ozaki and Nunokawa's univalence criterion [7].

By setting $\alpha = 1$ in Theorem 5 we have $F_1(z) = f(z)$ and then Theorem 5 furnishes us a connection between the univalence criteria of Becker, of Lewandowski, of Nehari and also of Ozaki and Nunokawa.

Example 1 ([9]). Let n be a natural number, $n \geq 2$, and the function

(12)
$$f(z) = \frac{z}{1 - \frac{z^{n+1}}{n}}$$

Then f is univalent in U and $F_{\frac{n+1}{2}}$ is analytic and univalent in U, where

(13)
$$F_{\frac{n+1}{2}}(z) = \left[\frac{n+1}{2}\int_0^z u^{\frac{n-1}{2}}f'(u)du\right]^{\frac{2}{n+1}}$$

If we put $h(z) \equiv 1$ and $g(z) = zf'(z)/f(z)$ into Theorem 5, we obtain

Corollary 5 *Let $f \in A$ and α be a complex number, $Re\ \alpha > 0$. If the inequalities*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1$$

and

$$\left| \left(\frac{zf'(z)}{f(z)} - 1 \right) |z|^{2\alpha} + \frac{1 - |z|^{2\alpha}}{\alpha} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| \leq 1$$

are true for all $z \in U$, then the function F_α defined by (7) is analytic and univalent in U.

It is known that, for all $z \in U \setminus \{0\}$ and $Re\ \alpha > 0$, we have

$$\left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \leq \frac{1 - |z|^{2Re\alpha}}{Re\alpha}.$$

Also, it is easy to prove that for $Re\alpha \geq 1$ we have $|z|^{2Re\alpha} \leq |z|^2$ and

$$\frac{1 - |z|^{2Re\alpha}}{Re\alpha} \leq 1 - |z|^2.$$

From Corollary 5, for the case when $Re\alpha \geq 1$ we get a very simple and useful univalence criterion

Corollary 6 *Let $f \in A$ and α be a complex number, $Re\ \alpha \geq 1$. If the inequality*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1$$

is true for all $z \in U$, then the function F_α defined by (7) is analytic and univalent in U.

Example 2 Let α be a complex number, $Re\ \alpha \geq 1$. Then the function

$$F(z) = \left(\alpha \int_0^z u^{\alpha-1}(1+u)e^u\ du \right)^{1/\alpha}$$

is analytic and univalent in U.

To prove it, we consider the function $f \in A$, $f(z) = z \cdot e^z$ and we can apply Corollary 6 because

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = |z| < 1$$

If we set $h(z) \equiv 1$ and $g(z) = f'(z)$, from Theorem 5, we get

Corollary 7 *Let $f \in A$ and α be a complex number, $Re\ \alpha > 0$. If for all $z \in U$*

$$|f'(z) - 1| < 1\,,$$

then the function F_α defined by (7) is analytic and univalent in U.

Example 3 The function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

where $\sum_{n=2}^{\infty} n|a_n| \leq 1$ is univalent in U. For any complex number α, $Re\ \alpha > 0$, the function

$$F_\alpha(z) = z \cdot \left[1 + \sum_{n=2}^{\infty} \frac{na_n\alpha}{n+\alpha-1} z^{n-1} \right]^{1/\alpha}$$

is analytic and univalent in U.

Indeed, the condition of Corollary 7 is verified

$$|f'(z) - 1| = \left| 2a_2 z + \ldots + na_n z^{n-1} + \ldots \right| < \sum_{n=2}^{\infty} n|a_n| \leq 1.$$

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