Differential subordinations
for certain integral operators

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Abstract

Applying the Integral Existence Theorem for normalized analytic functions concerning the existence and analyticity of a general integral operator which was proven by S. S. Miller and P. T. Mocanu (J. Math. Anal. Appl. 157 (1991), 147–165), the analyticity of the functions defined by a certain integral operator is discussed. And, by making use of the properties of subordination chains and several lemmas often used in the theory of differential subordinations, some interesting subordination criteria concerning with certain integral operators are obtained.

1   Introduction

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and a complex number a, we define the subclass $\mathcal{H}[a,n]$ of $\mathcal{H}$ by

$$\mathcal{H}[a,n] = \big\{ f(z) \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \big\}.$$

Also, we define the class $\mathcal{A}_n$ of normalized analytic functions $f(z)$ as

$$\mathcal{A}_n = \big\{ f(z) \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots \big\}$$

with $\mathcal{A}_1 = \mathcal{A}$. Furthermore, a function $f(z) \in \mathcal{A}$ is said to be λ-spirallike in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left(e^{i\lambda} \frac{z f'(z)}{f(z)} \right) > 0 \qquad (z \in \mathbb{U})$$

for some real number λ with $|\lambda| < \dfrac{\pi}{2}$. We denote by $\mathcal{S}^\lambda$ the class of all such functions. And, the class of all spirallike functions defined by

$$\widehat{\mathcal{S}} = \bigcup \left\{ \mathcal{S}^\lambda : |\lambda| < \frac{\pi}{2} \right\}.$$

Specially, we note that all spirallike functions are univalent in $\mathbb{U}$.

We also introduce the familiar principle of differential subordinations between analytic functions. Let $f(z)$ and $g(z)$ be members of the class $\mathcal{H}$. Then the function $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$, written by $f(z) \prec g(z) \;\; (z \in \mathbb{U})$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1 \;\; (z \in \mathbb{U})$, and such that $f(z) = g\big(w(z)\big) \;\; (z \in \mathbb{U})$. In particular, if $g(z)$ is univalent in $\mathbb{U}$, then $f(z) \prec g(z) \;\; (z \in \mathbb{U})$ if and only

if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For the function $F(z) \in \mathcal{A}_n$, Miller and Mocanu [6] (see also [5]) proved the Integral Existence Theorem concerning with the existence and analyticity of a general integral operator of the form

$$(1) \qquad \mathbf{I}[F](z) = \left\{ \frac{\beta + \gamma}{z^\gamma \psi(z)} \int_0^z (F(t))^\alpha \varphi(t) t^{\delta-1} \, dt \right\}^{\frac{1}{\beta}},$$

where α, β, γ and δ are complex numbers, and $\varphi(z)$, $\psi(z) \in \mathcal{H}[1, n]$. The integral operator (1) was introduced by Miller, Mocanu and Reade [8].

Lemma 1. *Let α, β, γ and δ be complex numbers with $\beta \neq 0$, $\alpha+\delta = \beta+\gamma$ and $\mathrm{Re}(\alpha + \delta) > 0$. Also, let $\varphi(z)$, $\psi(z) \in \mathcal{H}[1, n]$ with $\varphi(z) \cdot \psi(z) \neq 0$ in $\mathbb{U}$. Moreover, for $F(z) \in \mathcal{A}_n$, suppose that*

$$P(z) \equiv \alpha \frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \in \mathcal{H}[\alpha + \delta, n]$$

satisfies $\mathrm{Re}\, P(z) > 0 \quad (z \in \mathbb{U})$. If $g(z) = \mathbf{I}[F](z)$ is defined by (1), then

$$g(z) \in \mathcal{A}_n, \quad \frac{g(z)}{z} \neq 0 \quad and \quad \mathrm{Re} \left\{ \beta \frac{zg'(z)}{g(z)} + \frac{z\psi'(z)}{\psi(z)} + \gamma \right\} > 0$$

for $z \in \mathbb{U}$, where all powers in (1) are principal ones.

More general form of this lemma is given by Miller and Mocanu [6, Theorem 2.5c] (see also [5, Theorem 1]). In the present paper, by taking the condition $\psi(z) \equiv 1$ in Lemma 1, we discuss the existence and analyticity of the functions defined by an integral operator

$$(2) \qquad \hat{\mathbf{I}}[F](z) = \left\{ \frac{\beta + \gamma}{z^\gamma} \int_0^z (F(t))^\alpha \varphi(t) t^{\delta-1} \, dt \right\}^{\frac{1}{\beta}}.$$

Further, by making use of the properties of subordination chains [9] (see also [6]) and several lemmas given by Miller, Mocanu and Reade [7], Miller

and Mocanu [2], [3], [4] (see also [6]) often used in the theory of differential subordinations, we deduce some subordination criteria concerning

$$f(z) \prec \hat{\mathbf{I}}[\,F\,](z) \qquad (z \in \mathbb{U})$$

for analytic functions $f(z)$ normalized by $f(0) = 0$.

We first consider a few special cases of Lemma 1. If we let $\psi(z) \equiv 1$, then we derive the following special Integral Existence Theorem.

Lemma 2. *Let α, β, γ and δ be complex numbers with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\mathrm{Re}(\alpha + \delta) > 0$. Also, for $F(z) \in \mathcal{A}_n$, $\varphi(z) \in \mathcal{H}[1,n]$ with $\varphi(z) \neq 0$ in $\mathbb{U}$, suppose that*

$$P(z) \equiv \alpha \frac{z F'(z)}{F(z)} + \frac{z \varphi'(z)}{\varphi(z)} + \delta \in \mathcal{H}[\alpha + \delta, n]$$

satisfies $\mathrm{Re}\, P(z) > 0 \ \ (z \in \mathbb{U})$. If $g(z) = \hat{\mathbf{I}}[\,F\,](z)$ is defined by (2), then

$$g(z) \in \mathcal{A}_n, \quad \frac{g(z)}{z} \neq 0 \quad and \quad \mathrm{Re}\left(\beta \frac{z g'(z)}{g(z)} + \gamma \right) > 0$$

for $z \in \mathbb{U}$, where all powers in (2) are principal ones.

This lemma provides conditions for which the function $g(z) = \hat{\mathbf{I}}[\,F\,](z)$ defined by (2) will be an analytic function.

2   Preliminaries

In order to discuss our main result, we will make use of the following several lemmas.

Lemma 3. *Let $p(z)$ be analytic in $\mathbb{U}$ with $\operatorname{Re} p(0) > 0$. If $p(z)$ satisfies*

(3) $$\operatorname{Re}\left\{ p(z) + \alpha \frac{zp'(z)}{p(z)} \right\} > 0 \qquad (z \in \mathbb{U})$$

for some real number α, then $\operatorname{Re} p(z) > 0 \;\; (z \in \mathbb{U})$.

This lemma has been proved by Miller, Mocanu and Reade [7]. Also, Miller and Mocanu [4] derived the following lemma concerning with the Briot-Bouquet differential equation.

Lemma 4. *Let β and γ be complex numbers with $\beta \neq 0$, and let $h(z)$ be analytic in $\mathbb{U}$ with $h(0) = a$. If $\operatorname{Re}\big(\beta h(z) + \gamma\big) > 0 \;\; (z \in \mathbb{U})$ with $\operatorname{Re}(\beta a + \gamma) > 0$, then the solution $q(z)$ of the Briot-Bouquet differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z),$$

with $q(0) = a$, is analytic in $\mathbb{U}$ and satisfies $\operatorname{Re}\big(\beta q(z) + \gamma\big) > 0 \;\; (z \in \mathbb{U})$.

The next lemma which comes from the general theory of the differential subordinations was given by Miller and Mocanu [3].

Lemma 5. *Let $g(z)$ be analytic and univalent on the closed unit disk $\overline{\mathbb{U}}$ except for at most one pole on $\partial\mathbb{U}$, where $\partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, $\overline{\mathbb{U}} = \mathbb{U} \cup \partial\mathbb{U}$. Also, let $a = g(0)$ and $f(z) \in \mathcal{H}[a, n]$ with $f(z) \not\equiv a$. If $f(z)$ is not subordinate to $g(z)$, then there exist points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$, and a real number m with $m \geqq n \geqq 1$ for which $f(\mathbb{U}_{r_0}) \subset g(\mathbb{U})$,*

(i) $\;f(z_0) = g(\zeta_0)$

and

(ii) $\;z_0 f'(z_0) = m\zeta_0 g'(\zeta_0)$,

where $\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

Moreover, we need the following lemma given by Miller and Mocanu [2].

Lemma 6.   *Let $p(z) \in \mathcal{H}[a, n]$ with $p(z) \not\equiv a$ and $\operatorname{Re} a > 0$. Also, let the function $\psi(r, s) : \mathbb{C}^2 \to \mathbb{C}$ satisfy*

(i)   *$\psi(r, s)$ is continuous in a domain $\mathbb{D}$ of $\mathbb{C}^2$,*

(ii)   *$(a, 0) \in \mathbb{D}$ and $\operatorname{Re} \psi(a, 0) > 0$,*

(iii) *$\operatorname{Re} \psi(\rho i, \sigma) \leqq 0$ when $(\rho i, \sigma) \in \mathbb{D}$, and*

$$\sigma \leqq -\frac{n|a - \rho i|^2}{2\operatorname{Re} a} \quad for \ real \ \rho, \ \sigma.$$

If $\big(p(z), zp'(z)\big) \in \mathbb{D}$ when $z \in \mathbb{U}$, and

$$\operatorname{Re} \psi\big(p(z), zp'(z)\big) > 0 \qquad (z \in \mathbb{U}),$$

then $\operatorname{Re} p(z) > 0 \ \ (z \in \mathbb{U})$.

In addition, we need some lemmas for subordination (or Loewner) chains. A function $L(z, t)$, $z \in \mathbb{U}$, $t \geqq 0$, is said to be a subordination chain if $L(\cdot, t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geqq 0$, $L(z, \cdot)$ is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{U}$, and $L(z, s) \prec L(z, t)$, when $0 \leqq s \leqq t$ (Pommerenke [9] or Miller and Mocanu [6]). The following lemma provides a necessary and sufficient condition for $L(z, t)$ to be a subordination chain.

Lemma 7.   *The function $L(z, t) = a_1(t)z + a_2(t)z^2 + \cdots$, with $a_1(t) \neq 0$ for $t \geqq 0$, and $\lim_{t \to \infty} |a_1(t)| = \infty$, is a subordination chain if and only if there exist constants $r \in (0, 1]$ and $M > 0$ such that*

(i)   *$L(z, t)$ is analytic in $|z| < r$ for each $t \geqq 0$, locally absolutely continuous in $t \geqq 0$ for each $|z| < r$, and satisfies*

$$|L(z, t)| \leqq M|a_1(t)|, \ \ for \ |z| < r \ and \ t \geqq 0.$$

(ii) there exists a function $p(z,t)$ analytic in $\mathbb{U}$ for all $t \in [0,\infty)$ and measurable in $[0,\infty)$ for each $z \in \mathbb{U}$, such that $\operatorname{Re} p(z,t) > 0$ for $z \in \mathbb{U}$, $t \in [0,\infty)$, and

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z}p(z,t),$$

for $|z| < r$, and for almost all $t \in [0,\infty)$.

Note that the univalency of the function $L(z,t)$ can be extended from $|z| < r$ to all of $\mathbb{U}$. This lemma is well-known as the Loewner's theorem (see [9]). In the proof of our main result, the following lemma given by Pommerenke [9] is useful to apply the slight forms of Lemma 7.

Lemma 8. *The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$, with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim\limits_{t\to\infty} |a_1(t)| = \infty$, is a subordination chain if and only if*

$$(4) \qquad \operatorname{Re}\left\{\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right\} > 0,$$

for $z \in \mathbb{U}$ and $t \geq 0$.

3   Main result

Our main theorem is contained in

Theorem 1. *Let α, β, γ and δ be complex numbers with $\alpha + \delta = \beta + \gamma$, $\operatorname{Re}\beta > 0$ and $\operatorname{Re}\gamma \geq 0$. Also, let $F(z) \in \mathcal{A}_n$, $\varphi(z) \in \mathcal{H}[1,n]$ with $\varphi(z) \neq 0$ in $\mathbb{U}$, and suppose that*

$$(5) \qquad P(z) \equiv \alpha\frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta - \gamma \in \mathcal{H}[\beta, n]$$

satisfies one of the following:

$$(i) \quad \operatorname{Re}\left\{P(z) + \frac{zP'(z)}{P(z)}\right\} > 0,$$

$$(ii) \quad \operatorname{Re} P(z) > 0, \quad when \ \gamma = 0,$$

for $z \in \mathbb{U}$. If $f(z)$ is analytic in $\mathbb{U}$ with $f(0) = 0$ and satisfies the following subordination

$$(6) \qquad \frac{f(z)}{(\beta + \gamma)^{\frac{1}{\beta}}}\left(\beta\frac{zf'(z)}{f(z)} + \gamma\right)^{\frac{1}{\beta}} \prec \left\{\big(F(z)\big)^{\alpha}\varphi(z)z^{\delta-\gamma}\right\}^{\frac{1}{\beta}} \qquad (z \in \mathbb{U}),$$

then

$$f(z) \prec \left\{\frac{\beta + \gamma}{z^{\gamma}}\int_0^z \big(F(t)\big)^{\alpha}\varphi(t)t^{\delta-1}\,dt\right\}^{\frac{1}{\beta}} \qquad (z \in \mathbb{U}).$$

Proof. From (5), we see that $\dfrac{F(z)}{z} \neq 0$ in $\mathbb{U}$. If we let

$$(7) \qquad\qquad G(z) = \left\{\big(F(z)\big)^{\alpha}\varphi(z)z^{\delta-\gamma}\right\}^{\frac{1}{\beta}},$$

then $G(z)$ can be represented by

$$G(z) = z\left(\frac{F(z)}{z}\right)^{\frac{\alpha}{\beta}}\big(\varphi(z)\big)^{\frac{1}{\beta}} = z + A_{n+1}z^{n+1} + \cdots.$$

Since the terms in the two brackets are analytic and nonzero, we conclude that $G(z) \in \mathcal{A}_n$ and $\dfrac{G(z)}{z} \neq 0$ in $\mathbb{U}$. Also, since $P(z)$ is analytic in $\mathbb{U}$ with $\operatorname{Re} P(0) = \operatorname{Re}\beta > 0$, it follows from the assumption (i) and Lemma 3 that $\operatorname{Re} P(z) > 0 \ \ (z \in \mathbb{U})$, which implies that

$$\operatorname{Re}\left\{\alpha\frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta - \gamma\right\} > 0 \qquad (z \in \mathbb{U}).$$

In particular, if $\gamma = 0$, then from the assumption (ii), it is clear that $\operatorname{Re} P(z) > 0 \ \ (z \in \mathbb{U})$.

From the above-mentioned, we see that $G(z) \in \mathcal{A}_n$ satisfies

$$(8) \quad \operatorname{Re}\left(\beta \frac{zG'(z)}{G(z)}\right) = \operatorname{Re}\left\{\alpha \frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta - \gamma\right\} > 0 \qquad (z \in \mathbb{U}).$$

Thus since $\operatorname{Re}\beta > 0$, we deduce that $G(z) \in \widehat{\mathcal{S}}$, which implies that the function $G(z)$ is univalent in $\mathbb{U}$, and hence the subordination (6) is well-defined.

Furthermore, if we set

$$(9) \qquad\qquad g(z) = \left\{\frac{\beta + \gamma}{z^\gamma} \int_0^z \left(F(t)\right)^\alpha \varphi(t) t^{\delta-1} \, dt\right\}^{\frac{1}{\beta}},$$

then by combining (7), we obtain

$$(10) \qquad \frac{g(z)}{(\beta + \gamma)^{\frac{1}{\beta}}} \left(\beta \frac{zg'(z)}{g(z)} + \gamma\right)^{\frac{1}{\beta}} = \left\{\left(F(z)\right)^\alpha \varphi(z) z^{\delta-\gamma}\right\}^{\frac{1}{\beta}} = G(z).$$

Also, since $\operatorname{Re} P(z) > 0 \ \ (z \in \mathbb{U})$, it is easy to show that

$$\operatorname{Re}\left\{\alpha \frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta\right\} > \operatorname{Re}\gamma \geqq 0 \qquad (z \in \mathbb{U}).$$

Therefore, by Lemma 2, we deduce that

$$(11) \qquad g(z) \in \mathcal{A}_n, \quad \frac{g(z)}{z} \neq 0 \quad \text{and} \quad \operatorname{Re}\left(\beta \frac{zg'(z)}{g(z)} + \gamma\right) > 0$$

for $z \in \mathbb{U}$, and hence the expression (10) is well-defined.

We now need to show that $f(z) \prec g(z) \ \ (z \in \mathbb{U})$. We will do this by using a subordination chain argument involving Lemma 8. We first consider the function

$$(12) \qquad\qquad L(z,t) = \frac{g(z)}{(\beta + \gamma)^{\frac{1}{\beta}}} \left(\beta \frac{(1+t)zg'(z)}{g(z)} + \gamma\right)^{\frac{1}{\beta}},$$

where $t \geqq 0$. From the conditions in (11), the function

$$L(z,t) = \frac{g(z)}{(\beta+\gamma)^{\frac{1}{\beta}}} \left(\beta\frac{(1+t)zg'(z)}{g(z)} + \gamma\right)^{\frac{1}{\beta}} = a_1(t)z + a_{n+1}(t)z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$ for all $t \geq 0$, and continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$. A simple calculation yields

$$\frac{\partial L(z,t)}{\partial z} = \frac{g'(z)}{(\beta+\gamma)^{\frac{1}{\beta}}} \left(\beta\frac{(1+t)zg'(z)}{g(z)} + \gamma\right)^{\frac{1}{\beta}-1}$$

$$\times \left[(1+t)\left\{(\beta-1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right)\right\} + \gamma\right]$$

and

$$\frac{\partial L(z,t)}{\partial t} = \frac{zg'(z)}{(\beta+\gamma)^{\frac{1}{\beta}}} \left(\beta\frac{(1+t)zg'(z)}{g(z)} + \gamma\right)^{\frac{1}{\beta}-1}.$$

Then, since $g(z) \in \mathcal{A}_n$, it is clear that

$$a_1(t) = \left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = \left(\frac{t\beta}{\beta+\gamma} + 1\right)^{\frac{1}{\beta}} \neq 0 \qquad (t \geqq 0),$$

and

$$\lim_{t\to\infty} |a_1(t)| = \lim_{t\to\infty} \left|\left(\frac{t\beta}{\beta+\gamma} + 1\right)^{\frac{1}{\beta}}\right| = \infty.$$

A further calculation combined with the condition $\operatorname{Re}\gamma \geqq 0$ yields

$$\operatorname{Re}\left\{\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} = \operatorname{Re}\left[(1+t)\left\{(\beta-1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right)\right\} + \gamma\right]$$

$$\geqq (1+t)\operatorname{Re}\left\{(\beta-1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right)\right\}.$$

Since $t \geqq 0$, to obtain the condition (4) of Lemma 8, we need to show that

$$(13) \qquad \operatorname{Re}\left\{(\beta-1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right)\right\} > 0 \qquad (z \in \mathbb{U}).$$

Making differentiation (10) logarithmically and multiplying β, we have

$$(14) \qquad \frac{\left(\beta \frac{zg'(z)}{g(z)}\right) \left\{(\beta-1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right)\right\}}{\beta \frac{zg'(z)}{g(z)} + \gamma} = P(z),$$

where $P(z)$ is defined by (5). Note that the equation (14) gives us that

$$(\beta-1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right) + \frac{z\left\{(\beta-1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right)\right\}'}{(\beta-1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right) + \gamma}$$

$$= P(z) + \frac{zP'(z)}{P(z)},$$

which has the form as a Briot-Bouquet differential equation. If we let

$$h(z) = P(z) + \frac{zP'(z)}{P(z)},$$

then from the assumption (i), it is clear that

$$(15) \qquad \operatorname{Re} h(z) > 0 \qquad (z \in \mathbb{U}).$$

Also, we observed that $P(z) \in \mathcal{H}[\beta, n]$ satisfies $\operatorname{Re} P(z) > 0$ $(z \in \mathbb{U})$. From these facts, it is easy to see that the function $h(z)$ is analytic in $\mathbb{U}$ with $h(0) = P(0) = \beta$, and $\operatorname{Re}(h(z) + \gamma) > 0$ $(z \in \mathbb{U})$. Therefore, by applying Lemma 4, and letting

$$(16) \qquad q(z) = (\beta-1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right),$$

we obtain the following equation

$$q(z) + \frac{zq'(z)}{q(z) + \gamma} = h(z),$$

where $q(z)$ is analytic in $\mathbb{U}$, $q(0) = \beta$ and

(17) $$\mathrm{Re}\big(q(z) + \gamma\big) > 0 \qquad (z \in \mathbb{U}).$$

Moreover, combining the above results with the condition (15), we deduce

$$\mathrm{Re}\left\{q(z) + \frac{zq'(z)}{q(z) + \gamma}\right\} = \mathrm{Re}\,\psi\big(q(z), zq'(z)\big) > 0 \qquad (z \in \mathbb{U}),$$

where $\psi(r, s) = r + \dfrac{s}{r + \gamma}$.

We now use Lemma 6 to prove that $\mathrm{Re}\,q(z) > 0$ $(z \in \mathbb{U})$. If we take $\mathbb{D} = \big(\mathbb{C}\backslash\{-\gamma\}\big) \times \mathbb{C}$, then $\psi(r, s)$ is continuous in $\mathbb{D}$. Since $q(0) = \beta$, $\mathrm{Re}\,\beta > 0$ and $\mathrm{Re}\,\gamma \geqq 0$, it is clear that $(\beta, 0) \in \mathbb{D}$ and $\mathrm{Re}\,\psi(\beta, 0) = \mathrm{Re}\,\beta > 0$. Also, from (17), we see that $\big(q(z), zq'(z)\big) \in \mathbb{D}$ when $z \in \mathbb{U}$. Hence, we only need to show that

$$\mathrm{Re}\,\psi(\rho i, \sigma) \leqq 0$$

for real ρ and σ such that $\sigma \leqq -\dfrac{n|\beta - \rho i|^2}{2\mathrm{Re}\,\beta}$. Then, a simple calculation yields that

$$\mathrm{Re}\,\psi(\rho i, \sigma) = \mathrm{Re}\left(\rho i + \frac{\sigma}{\rho i + \gamma}\right) = \mathrm{Re}\left(\frac{\sigma}{\rho i + \gamma}\right)$$
$$= \frac{\sigma \mathrm{Re}\,\gamma}{|\rho i + \gamma|^2} \leqq -\frac{n|\beta - \rho i|^2}{2\mathrm{Re}\,\beta} \cdot \frac{\mathrm{Re}\,\gamma}{|\rho i + \gamma|^2} \leqq 0.$$

Therefore, by applying Lemma 6, we obtain $\mathrm{Re}\,q(z) > 0$ $(z \in \mathbb{U})$ which is equivalent to the inequality (13). In particular, if $\gamma = 0$, then since (14) is simplified to

$$(\beta - 1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g(z)}\right) = P(z),$$

we can easily obtain the inequality (13) from the assumption (ii). From the above-mentioned, the condition (4) of Lemma 8 is satisfied. Hence by

Lemma 8, the function $L(z,t)$ given by (12) is a subordination chain, and we have $L(z,s) \prec L(z,t)$ when $0 \leqq s \leqq t$. From (10) and (12), we obtain $L(z,0) = G(z)$, and hence we must have

$$(18) \qquad\qquad L(\zeta,t) \notin G(\mathbb{U})$$

for $|\zeta| = 1$ and $t \geqq 0$.

Next, applying Lemma 5, we will show that

$$\frac{f(z)}{(\beta+\gamma)^{\frac{1}{\beta}}} \left(\beta \frac{zf'(z)}{f(z)} + \gamma \right)^{\frac{1}{\beta}} \prec G(z)$$

implies $f(z) \prec g(z)$ $(z \in \mathbb{U})$. To apply Lemma 5, we first need to discuss the univalency of the function $g(z)$ given by (9). If we define the function $p(z)$ by

$$p(z) = \beta \frac{zg'(z)}{g(z)} \qquad (z \in \mathbb{U})$$

in Lemma 3, then from (11), we see that $p(z)$ is analytic in $\mathbb{U}$ with $\operatorname{Re} p(0) = \operatorname{Re} \beta > 0$, and the condition (3) can be rewritten as the condition (13). Thus, according to Lemma 3, the condition (13) shows that $g(z)$ satisfies the following inequality

$$\operatorname{Re}\left(\beta \frac{zg'(z)}{g(z)} \right) > 0 \qquad (z \in \mathbb{U}),$$

which implies that $g(z)$ is univalent (spirallike) in $\mathbb{U}$.

We have shown that the function $g(z)$ given by (9) is univalent in $\mathbb{U}$. Here, without loss of generality, we can assume that $g(z)$ is univalent on $\overline{\mathbb{U}}$, and $g'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we can continue the remainder of the proof with the function $g(rz)$ $(0 < r < 1)$ which is univalent on $\overline{\mathbb{U}}$, and obtain our final result by letting $r \to 1^-$.

If we assume that $f(z)$ is not subordinate to $g(z)$, then by Lemma 5, there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$, and a real number $m \geqq 1$ such that $f(z_0) = g(\zeta_0)$ and $z_0 f'(z_0) = m\zeta_0 g'(\zeta_0)$. Then from (12) and (18), we have

$$\frac{f(z_0)}{(\beta+\gamma)^{\frac{1}{\beta}}} \left(\beta \frac{z_0 f'(z_0)}{f(z_0)} + \gamma \right)^{\frac{1}{\beta}} = \frac{g(\zeta_0)}{(\beta+\gamma)^{\frac{1}{\beta}}} \left(\beta \frac{m\zeta_0 g'(\zeta_0)}{g(\zeta_0)} + \gamma \right)^{\frac{1}{\beta}}$$

$$= L(\zeta_0, m-1) \notin G(\mathbb{U}),$$

where $z_0 \in \mathbb{U}$, $|\zeta_0| = 1$ and $m \geqq 1$. This contradicts the assumption (6) of the theorem, and hence we must have $f(z) \prec g(z)$. Therefore, we conclude that

$$f(z) \prec \left\{ \frac{\beta+\gamma}{z^\gamma} \int_0^z \left(F(t) \right)^\alpha \varphi(t) t^{\delta-1}\, dt \right\}^{\frac{1}{\beta}} \qquad (z \in \mathbb{U}),$$

which completes the proof of Theorem 1.

Remark 1. The result concerning the case (ii) in Theorem 1 has been proved by Kuroki and Owa [1, Theorem 3.4].

Letting $\alpha = \beta$ and $\delta = \gamma$ in Theorem 1, we obtain

Corollary 1. *Let β and γ be complex numbers with $\operatorname{Re}\beta > 0$ and $\operatorname{Re}\gamma \geqq 0$. Also, let $F(z) \in \mathcal{A}_n$, $\varphi(z) \in \mathcal{H}[1,n]$ with $\varphi(z) \neq 0$ in $\mathbb{U}$, and suppose that*

$$(19) \qquad P(z) \equiv \beta \frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} \in \mathcal{H}[\beta, n]$$

satisfies one of the following:

$$(i) \quad \operatorname{Re}\left\{ P(z) + \frac{zP'(z)}{P(z)} \right\} > 0,$$

$$(ii) \quad \operatorname{Re} P(z) > 0, \quad \text{when } \gamma = 0,$$

for $z \in \mathbb{U}$. If $f(z)$ is analytic in $\mathbb{U}$ with $f(0) = 0$ and satisfies the following subordination

$$\frac{f(z)}{(\beta + \gamma)^{\frac{1}{\beta}}} \left(\beta \frac{zf'(z)}{f(z)} + \gamma \right)^{\frac{1}{\beta}} \prec F(z)\big(\varphi(z)\big)^{\frac{1}{\beta}} \qquad (z \in \mathbb{U}),$$

then

$$f(z) \prec \left\{ \frac{\beta + \gamma}{z^{\gamma}} \int_0^z \big(F(t)\big)^{\beta} \varphi(t) t^{\gamma - 1} \, dt \right\}^{\frac{1}{\beta}} \qquad (z \in \mathbb{U}).$$

Moreover, we will give some particular case of Corollary 1. If we take $\varphi(z) \equiv 1$, then since $P(z)$ in (19) is simplified to

$$(20) \qquad\qquad P(z) = \beta \frac{zF'(z)}{F(z)},$$

we have

$$P(z) + \frac{zP'(z)}{P(z)} = (\beta - 1) \frac{zF'(z)}{F(z)} + \left(1 + \frac{zF''(z)}{F'(z)} \right).$$

Also, according to Lemma 3, the assumption (i) of Corollary 1 shows that the inequality $\operatorname{Re} P(z) > 0 \quad (z \in \mathbb{U})$. Since $P(z)$ is given by (20), and since $F(z) \in \mathcal{A}_n$, this means that $F(z)$ is univalent (spirallike) in $\mathbb{U}$. From the above-mentioned, if $F(z) \in \mathcal{A}_n$ satisfies one of the assumptions (i) and (ii) of Corollary 1 with $\varphi(z) \equiv 1$, then since $F(z)$ is univalent in $\mathbb{U}$ with $F(0) = 0$, we can deduce the following condition

$$\frac{F(z)}{z} \neq 0 \qquad (z \in \mathbb{U}),$$

and hence we see that

$$P(z) = \beta \frac{zF'(z)}{F(z)} \in \mathcal{H}[\beta, n].$$

Therefore from Corollary 1, we derive the following.

Corollary 2. *Let β and γ be complex numbers with $\operatorname{Re}\beta > 0$ and $\operatorname{Re}\gamma \geqq 0$. Also, let $F(z) \in \mathcal{A}_n$, and suppose that $F(z)$ satisfies one of the following:*

$$(i) \quad \operatorname{Re}\left\{(\beta - 1)\frac{zF'(z)}{F(z)} + \left(1 + \frac{zF''(z)}{F'(z)}\right)\right\} > 0,$$

$$(ii) \quad \operatorname{Re}\left(\beta\frac{zF'(z)}{F(z)}\right) > 0, \quad when \; \gamma = 0,$$

for $z \in \mathbb{U}$. If $f(z)$ is analytic in $\mathbb{U}$ with $f(0) = 0$ and satisfies the following subordination

$$\frac{f(z)}{(\beta + \gamma)^{\frac{1}{\beta}}}\left(\beta\frac{zf'(z)}{f(z)} + \gamma\right)^{\frac{1}{\beta}} \prec F(z) \qquad (z \in \mathbb{U}),$$

then

$$f(z) \prec \left\{\frac{\beta + \gamma}{z^\gamma}\int_0^z \left(F(t)\right)^\beta t^{\gamma - 1}\,dt\right\}^{\frac{1}{\beta}} \qquad (z \in \mathbb{U}).$$

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