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Generalized Difference Sequence Spaces Defined by Orlicz Functions[1]

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Abstract

The idea of difference sequences was first introduced by Kizmaz [4]. In this first paper we define some generalized difference sequence combining lacunary sequences and Orlicz function. We establish some relations between these sequence space.

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1 Definitions and notations

Let l_∞, c and c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_i)$, respectively.

A sequence $x = (x_i) \in l_\infty$ is said to be almost convergent [2] if all Banach limits of $x = (x_i)$ coincide. In [2], it was shown that

$$\hat{c} = \left\{ x = (x_i) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_{i+s} \; exits, \; uniformly \; in \; s \right\}$$

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In [3,4], Maddox defined a sequence $x = (x_i)$ to strongly almost convergent to a number L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |x_{i+s} - L| = 0, \ uniformly \ in \ s$$

By a lacunary sequence $\theta = (k_r), \ r = 0, 1, 2, \ldots,$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers $h_r = (k_r - k_{r-1}) \to \infty (r \to \infty)$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al. [10] as follows:

$$N_\theta = \left\{ x = (x_i) : \ \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} |x_i - L| = 0, \ fot \ some \ L \right\}.$$

In [1], Kizmaz defined the sequence spaces $Z(\Delta) = \{x = (x_i) : \ (\Delta x_i) \in Z\}$ for $Z = l_\infty$, c and c_0, where $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$. After Et and Çolak [8] generalized the difference sequence spaces to the sequence spaces $Z(\Delta^m) = \{x = (x_i) : \ (\Delta^m x_i) \in Z\}$ for $Z = l_\infty, c$ and c_0, where $m \in N, \ \Delta^0 x = (x_i), \ \Delta x = (x_i - x_{i+1}), \ \Delta^m x = (\Delta^m x_i) = (\Delta^{m-1} x_i - \Delta^{m-1} x_{i+1})$ and so that

$$\Delta^m x_i = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{i+v}.$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing an convex with $M(0) = 0, \ M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to = \infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $T > 0$, such that $M(2u) \leq TM(u)(u \geq 0)$. The Δ_2-condition is equivalent to $M(Lu) \leq TLM(u)$, for all values of u and for $L > 1$.

An Orlicz function M can be always be represented (see[6]) in the integral form $M(x) = \int_0^x q(t)dt$, where q known as the kernel of M, is right differentiable for $t \geq 0$, $q(t) > 0$ for $t > 0$, q is non-decreasing and $q(t) \to \infty$ as $t \to \infty$.

Remark 1.1. *An Orlicz function satisfies the inequality $M(\lambda u) \leq \lambda M(u)$ for all λ with $0 < \lambda < 1$.*

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space,

$$l_M = \left\{ x = (x_i) : \sum_i M\left(\frac{|x_i|}{\rho}\right) < \infty, \; for \; some \rho > 0 \right\}.$$

The sequence space l_m with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_i M\left(\frac{|x_i|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz Sequence Space. The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

Let M be an Oriclz function and $p = (p_i)$ be any sequence of strictly positive real numbers. Güngör and Et[3] defined the following sequence spaces:

$$[\hat{c}, M, p](\Delta^m) = \left\{ x = (x_i) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[M\left(\frac{|\Delta^m x_{i+s} - L|}{\rho} \right) \right]^{p_i} = 0, \right.$$

$$\left. uniformly \; in \; s, \; for \; some \; \rho > 0 \; and \; L > 0 \right\},$$

$$[\hat{c}, M, p]_0 = \left\{ x = (x_i) : \right.$$

$$\left. \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho} \right) \right]^{p_i} = 0, \; uniformly \; in \; s, for \; some \; \rho > 0 \right\},$$

$$[\hat{c}, M, p]_\infty(\Delta^m) = \left\{ x = (x_i) : \right.$$

$$\left. \sup_{n,s} \frac{1}{n} \sum_{i=1}^{n} \left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho} \right) \right]^{p_i} < \infty, \; for \; some \; \rho > 0 \right\}.$$

The purpose of this paper is to introduce and study a concept of lacunary almost generalized Δ^m- convergence function and to examine some properties of these new sequence spaces which also generalize the well known Orlicz sequence space l_M and strongly summable sequence $[C,1,p]$, $[C,1,p]_0$ and $[C,1,p]_\infty$ [9].

In the present paper we introduce and examine the following spaces defined by Orlicz function.

Definition 1.1. *Let M be an Orlicz function and $p=(p_i)$ be any bounded sequence of strictly positive real numbers. We have*

$$[\hat{c},M,p]^\infty(\Delta^m)=\Bigg\{x=(x_i):\ \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}-L|}{\rho}\right)\right]^{p_i}=0,$$

$$uniformly\ in\ s,\ for\ some\ \rho>0\ and\ L>0\Bigg\},$$

$$[\hat{c},M,p]_0^\infty=\Bigg\{x=(x_i):$$

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right)\right]^{p_i}=0,\ uniformly\ in\ s, for\ some\ \rho>0\Bigg\},$$

$$[\hat{c},M,p]_\infty(\Delta^m)=\Bigg\{x=(x_i):$$

$$\sup_{r,s}\frac{1}{h_r}\sum_{i=1}^{n}\left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right)\right]^{p_i}<\infty,\ for\ some\ \rho>0\Bigg\}.$$

If $x=(x_i)\in[\hat{c},M,p]^\theta(\Delta^m)$, we say that $x=(x_i)$ is lacunary almost Δ^m-convergence to L with respect to Orlicz function M.

When $M(x)=x$, then we write $[\hat{c},p]^\theta(\Delta^m)$, $[\hat{c},p]_0^\theta(\Delta^m)$ for the spaces $[\hat{c},M,p]^\theta(\Delta^m)$, $[\hat{c},M,p]_0^\theta(\Delta^m)$ and $[\hat{c},M,p]_\infty^\theta(\Delta^m)$, respectively. If $p_i=1$ for all $i\in N$, then $[\hat{c},M,p]^\theta(\Delta^m)$, $[\hat{c},M,p]_0^\theta(\Delta^m)$ and $[\hat{c},M,p]_\infty^\theta(\Delta^m)$ reduce to $[\hat{c},M]^\theta(\Delta^m)$, $[\hat{c},M]_0^\theta(\Delta^m)$ and $[\hat{c},M]_\infty^\theta(\Delta^m)$, respectively.

The following inequality will be used throughout the paper,

$$|x_i + y_i|^{p_i} \le K(|x_i|^{p_i} + |y_i|^{p_i}) \tag{1.1}$$

where x_i and y_i are complex numbers, $k = \max(1, 2^{H-1})$ and $H = \sup_i p_i < \infty$.

2 Main Result

In this section we prove some results involving the sequence spaces $[\hat{c}, M, p]^{\theta}(\Delta^m)$, $[\hat{c}, M, p]_0^{\theta}(\Delta^m)$ and $[\hat{c}, M, p]_{\infty}^{\theta}(\Delta^m)$.

Theorem 2.1. *Let M be an Orlicz function and $p = (p_i)$ be a bounded sequence of strictly real numbers. Then $[\hat{c}, M, p]^{\theta}(\Delta^m)$, $[\hat{c}, M, p]_0^{\theta}(\Delta^m)$ and $[\hat{c}, M, p]_{\infty}^{\theta}(\Delta^m)$ are linear spaces over the set of complex numbers $\mathbb{C}$*

Proof. Let $x = (x_i), y = (y_i) \in [\hat{c}, M, p]_0^{\theta}(\Delta^m)$ and $\alpha, \beta \in \mathbb{C}$. Then there exists positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho_1} \right) \right]^{p_i} = 0, uniformly\ in\ s,$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{\Delta^m x_{i+s}}{\rho_2} \right) \right]^{p_i} = 0,\ uniformly\ in\ s.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing convex function, by using (1.1), we have

$$\frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{|\Delta^m(\alpha x_{i+s} + \beta y_{i+s})|}{\rho_3} \right) \right]^{p_i} =$$

$$= \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{|\alpha\Delta^m(x_{i+s})|}{\rho_3} + \frac{|\beta\Delta^m(y_{i+s})|}{\rho_3} \right) \right]^{p_i}$$

$$\le K\frac{1}{h_r} \sum_{i \in I_r} \frac{1}{2^{p_i}} \left[M\left(\frac{|\Delta^m(x_{i+s})|}{\rho_1} \right) \right]^{p_i} + K\frac{1}{h_r} \sum_{i \in I_r} \frac{1}{2^{p_i}} \left[M\left(\frac{\Delta^m(y_{i+s})}{\rho_2} \right) \right]^{p_i}$$

$$\leq +K\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{\Delta^m(x_{i+s})}{\rho_1}\right)\right]^{p_i} + K\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{\Delta^m(y_{i+s})}{\rho_2}\right)\right]^{p_i} \to 0 \ as$$

$r \to \infty$, uniformly in s.

So, $\alpha x + \beta x \in [\hat{c}, M, p]_0^\theta(\Delta^m)$. Hence $[\hat{c}, M, p]_0^\theta(\Delta^m)$ is a linear space.

The proof for the cases $[\hat{c}, M, p]^\theta(\Delta^m)$ and $[\hat{c}, M, p]_\infty^\theta(\Delta^m)$ are routine work in view of the above proof.

Theorem 2.2. *For any Orlicz function M an a bounded sequence $p = (p_i)$ of strictly positive real numbers, $[\hat{c}, M, p]_0^\theta(\Delta^m)$ is a topological linear space paranormed by*

$$h(x) = \inf\left\{\rho^{p_r/H} : \left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right)\right]^{p_i}\right)^{1/H} \leq 1,\right.$$

$$\left. r = 1, 2, \ldots, \ s = 1, 2 ... \right\}, \ \textit{where } H = \max(1, \sup_i p_i < \infty).$$

Proof. Clearly $h(x) \geq$ for all $x = (x_i) \in [\hat{c}, M]_0^\theta(\Delta^m)$. Since $M(0) = 0$, we get $h(0) = 0$. Conversely, suppose that $h(x) = 0$, then

$$\inf\left\{\rho^{p_r/H} : \left(\frac{1}{h_r}\sum_{i\in I_r}\left[\frac{|\Delta^m x_{i+s}|}{\rho}\right]^{p_i}\right)^{1/H} \leq 1, \ r = 1, 2, \ldots, s = 1, 2, \ldots\right\} = 0.$$

The implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{\Delta^m x_{i+s}}{\rho_\epsilon}\right)\right]^{p_i}\right)^{1/H} \leq 1.$$

Thus

$$\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}|}{\epsilon}\right)\right]^{p_i}\right)^{1/H} \leq \left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho_\epsilon}\right)\right]^{p_i}\right)^{1/H} \leq 1$$

for each r and s. Suppose that $x_i \neq 0$ for each $i \in N$. This implies that $\Delta^m x_{i+s} \neq 0$, for each $i, s \in N$. Let $\epsilon \to 0$, then $\frac{|\Delta^m x_{i+s}|}{\epsilon} \to \infty$. It follows that

$$\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}|}{\epsilon}\right)\right]^{p_i}\right)^{1/H} \to \infty$$

which is contradiction. Therefore, $\Delta^m x_{i+s} = 0$ for each i and s and thus $x_i = 0$ for each $i \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho_1}\right)\right]^{p_i}\right)^{1/H} \leq 1$$

and

$$\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho_2}\right)\right]^{p_i}\right)^{1/H} \leq 1$$

for each r and s. Let $\rho = \rho_1 + \rho_2$. Then, we have

$$\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m (x_{i+s}+y_{i+s})|}{\rho}\right)\right]^{p_i}\right)^{1/H}$$

$$\leq \left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m (x_{i+s})|+|\Delta^m (y_{i+s})|}{\rho_1+\rho_2}\right)\right]^{p_i}\right)^{1/H}$$

$$\leq \left(\frac{1}{h_r}\sum_{i\in I_r}\left[\frac{\rho_1}{\rho_1+\rho_2}M\left(\frac{|\Delta^m (x_{i+s})|}{\rho_1}\right)+\frac{\rho_2}{\rho_1+\rho_2}M\left(\frac{|\Delta^m (y_{i+s})|}{\rho_2}\right)\right]^{p_i}\right)^{1/H}$$

By Minkowski's inequality

$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{\Delta^m (x_{i+s})}{\rho_1}\right)\right]^{p_i}\right)^{1/H} +$$

$$+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m (y_{i+s})|}{\rho_2}\right)\right]^{p_i}\right)^{1/H} \leq 1$$

Since the ρ's are non-negative, so we have

$$h(x+y) = \inf\left\{p^{p_r/H} : \left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{\Delta^m (x_{i+s}+y_{i+s})}{\rho}\right)\right]^{p_i}\right)^{1/H} \leq 1,\right.$$

$$r=1,2,\ldots,s=1,2,\ldots\Bigg\},$$

$$\leq \inf\left\{\rho_1^{p_r/H}:\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m(x_{i+s})|}{\rho_1}\right)\right]^{p_i}\right)^{1/H}\leq 1,\right.$$

$$\left. r=1,2,\ldots,\ s=1,2,\ldots\right\}+,$$

$$+\inf\left\{\rho_2^{p_r/H}:\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m(y_{i+s})|}{\rho_2}\right)\right]^{p_i}\right)^{1/H}\leq 1,\right.$$

$$\left. r=1,2,\ldots,s=1,2,\ldots\right\}.$$

Therefore $h(x+y)\leq h(x)+h(y)$. Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition

$$h(\lambda x)=\inf\left\{\rho^{p_r/H}:\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m\lambda x_{i+s}|}{\rho}\right)\right]^{p_i}\right)^{1/H}\leq 1,\right.$$

$$\left. r=1,2,\ldots,\ s=1,2,\ldots\right\}.$$

Then,

$$h(\lambda x)=\inf\left\{(|\lambda|t)^{p_r/H}:\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}|}{t}\right)\right]^{p_i}\right)^{1/H}\leq 1,\right.$$

$$\left. r=1,2,\ldots,s=1,2,\ldots\right\}$$

where $t=\frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_r}\leq \max(1,\lambda|^{\sup p_r})$, we have

$$h(\lambda x)\leq \max(1,|\lambda|^{\sup p_r})\inf\left\{t^{p_r/H}:\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}|}{t}\right)\right]^{p_i}\right)^{1/H}\leq 1,\right.$$

$$\left. r=1,2,\ldots,s=1,2,\ldots\right\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem.

Theorem 2.3. *Let M be an Orlicz function. If $\sup_i [M(x)]^{p_i} < \infty$ for all fixed $x > 0$, then*

$$[\hat{c}, M, p]_0^\theta(\Delta^m) \subset [\hat{c}, M, p]_\infty^\theta(\Delta^m).$$

Proof. Let $x = (x_i) \in [\hat{c}, M, p]_0^\theta(\Delta^m)$. There exists some positive ρ_1 such that

$$\lim_{r \in \infty} \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho_1} \right) \right]^{p_i} = 0, \; uniformly \; in \; s.$$

Define $\rho = 2\rho_1$. Since M is non-decreasing and convex, by using (1.1), we have

$$\sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho} \right) \right]^{p_i} = \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{\Delta^m x_{i+s} - L + L}{\rho} \right) \right]^{p_i}$$

$$\leq K \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} \left[\frac{1}{2^{p_i}} M\left(\frac{|\Delta^m x_{i+s} - L|}{\rho_1} \right) \right]^{p_i} + K \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} \left[\frac{1}{2^{p_i}} M\left(\frac{|L|}{\rho_1} \right) \right]^{p_i}$$

$$\leq K \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{|\Delta^m x_{i+s} - L|}{\rho_1} \right) \right]^{p_i} + K \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{|L|}{\rho_1} \right) \right]^{p_i} < \infty.$$

Hence $x = (x_i) \in [\hat{c}, M, p]_\infty^\theta$. This completes the proof.

Theorem 2.4. *Let $0 < \inf \; p_i = h \leq p_i \leq \sup p_i = H < \infty$ and M, M_1 be Orlicz function satisfying Δ_2-condition, then we have $[\hat{c}, M_1, p]_0^\theta(\Delta^m) \subset [\hat{c}, MoM_1, p]_0^\theta(\Delta^m)$, $[\hat{c}, M_1, p]^\theta(\Delta^m) \subset [\hat{c}, MoM_1, p]^\theta(\Delta^m)$ and $[\hat{c}, M_1, p]_\infty^\theta(\Delta^m) \subset [\hat{c}, MoM_1, p]_\infty^\theta(\Delta^m)$.*

Proof. Let $x = (x_i) \in [\hat{c}, M_1, p]_0^\theta(\Delta^m)$. Then we have

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} \left[M_1\left(\frac{|\Delta^m x_{i+s} - L|}{\rho} \right) \right]^{p_i} = 0, \; uniformly \; in \; s, \; for \; some \; L.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \epsilon$ for $0 \leq t \leq \delta$. Let
$y_{i,s} = M_1\left(\frac{|\Delta^m x_{i+s} - L|}{\rho}\right)$ for all $i, s \in \mathbb{N}$. We can write

$$\frac{1}{h_r}\sum_{i\in I_r}[M(y_{i,s})]^{p_i} = \frac{1}{h_r}\sum_{\substack{i\in I_r\\ y_{i,s}\leq\delta}}[M(y_{i,s})]^{p_i} + \frac{1}{h_r}\sum_{\substack{i\in I_r\\ y_{i,s}>\delta}}[M(y_{i,s})]^{p_i}$$

By the Remark, we have
(2.1)

$$\frac{1}{h_r}\sum_{\substack{i\in I_r\\ y_{i,s}\leq\delta}}[M(y_{i,s})]^{p_i} \leq [M(1)]^H\frac{1}{h_r}\sum_{\substack{i\in I_r\\ y_{i,s}\leq\delta}}[M(y_{i,s})]^{p_i} \leq [M(2)]^H\frac{1}{h_r}\sum_{\substack{i\in I_r\\ y_{i,s}\leq\delta}}[M(y_{i,s})]^{p_i}.$$

For $y_{i,s} > \delta$

$$y_{i,s} < \frac{y_{i,s}}{\delta} < 1 + \frac{y_{i,s}}{\delta}.$$

Since M is non-decreasing and convex, it follows that

$$M(y_{i,s}) < M\left(1 + \frac{y_{i,s}}{\delta}\right) < \frac{1}{2}M(2) + \frac{1}{2}M\left(\frac{2y_{i,s}}{\delta}\right).$$

Since M satisfies Δ_2-condition, we can write

$$M(y_{i,s}) < \frac{1}{2}T\frac{y_{i,s}}{\delta}M(2) + \frac{1}{2}T\frac{y_{i,s}}{\delta}M(2) + \frac{1}{2}T\frac{y_{i,s}}{\delta}M(2) = T\frac{y_{i,s}}{\delta}M(2).$$

Hence,

$$\frac{1}{h_r}\sum_{\substack{i\in I_r\\ y_{i,s}>\delta}}[M(y_{i,s})]^{p_i} \leq \max\left(1, \left(\frac{TM(2)}{\delta}\right)^H\right)\frac{1}{h_r}\sum_{\substack{i\in I_r\\ y_{i,s}>\delta}}[(y_{i,s})]^{p_i}. \tag{2.2}$$

By (2.1) and (2.2), we have $x = (x_i) \in [\hat{c}, MoM_1, p]_0^\theta(\Delta^m)$.

Following similar arguments we can prove that $[\hat{c}, M_1, p]_0^\theta(\Delta^m) \subset [\hat{c}, MoM_1, p]_0^\theta(\Delta^m)$ and $[\hat{c}, M_1, p]_\infty^\theta(\Delta^m) \subset [\hat{c}, MoM_1, p]_\infty^\theta(\Delta^m)$. This completes the proof.

Taking $M_1(x)$ in Theorem 2.4. we have the following result.

Corollary 2.5. *Let* $0 < \inf p_i = h \leq p_i \leq \sup p_i = H < \infty$ *and* M *be an Orlicz function satisfying* Δ_2*-condition, then we have* $[\hat{c}, p]_0^\theta(\delta^m) \subset [\hat{c}, M, p]_0^\theta(\Delta^m)$ *and* $[\hat{c}, M_1, p]_\infty^\theta(\Delta^m) \subset [\hat{c}, M, p]_\infty^\theta(\Delta^m)$.

Theorem 2.6. *Let* M *be an Orlicz function. Then the following statements are equivalent:*

(a) $[\hat{c}, p]_\infty^\theta(\Delta^m) \subset [\hat{c}, M, p]_\infty^\theta(\Delta^m)$,

(b) $[\hat{c}, p]_0^\theta(\Delta^m) \subset [\hat{c}, M, p]_\infty^\theta(\Delta^m)$,

(c) $\sup_r \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{t}{\rho}\right)\right]^{p_i} < \infty \quad (t, \rho > 0)$.

Proof. (a)⇒(b): It is obvious, since $[\hat{c}, p]_0^\theta(\Delta^m) \subset [\hat{c}, p]_\infty^\theta(\Delta^m)$.

(b)⇒(c): Let $[\hat{c}, p]_0^\theta(\Delta^m) \subset [\hat{c}, M, p]_\infty^\theta(\delta^m)$. Suppose that (c) does not hold. Then for some $t, \rho > 0$

$$\sup_r \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{t}{\rho}\right)\right]^{p_i} = \infty$$

and therefore we cab find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(j)}} \sum_{i \in I_{r(j)}} \left[M\left(\frac{j^{-1}}{\rho}\right)\right]^{p_i} > j, \quad j = 1, 2, \ldots \tag{2.3}$$

Define the sequence $x = x(i)$ by

$$\Delta^m x_{i+s} = \begin{cases} j^{-1}, & i \in I_{r(j)} \\ 0, & i \notin I_{r(j)} \end{cases} \quad for \;\; all \;\; s \in \mathbb{N}.$$

Then $x = (x_i) \in [\hat{c}, p]_0^\theta(\Delta^m)$ but by (2.3) $x = (x_i) \notin [\hat{c}, M, p]_\infty^\theta(\delta^m)$, which contradicts (b).

Hence (c) must hold.

(c)⇒(a): Let (c) hold and $x = (x_i) \in [\hat{c}, p]_\infty^\theta(\Delta^m)$. Suppose that $x = (x_i) \notin [\hat{c}, M, p]_\infty^\theta(\Delta^m)$. Then

$$\sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right)\right]^{p_i} = \infty \tag{2.4}$$

Let $t=|\Delta^m x_{i+s}|$ for each i and fixed s, then by (2.4)

$$\sup_r \frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{t}{\rho}\right)\right]=\infty$$

which contradicts (c). Hence (a) must hold.

Theorem 2.7. *Let* $1\le p_i\le \sup p_i<\infty$ *and* M *be an Orlicz function. Then the following statement are equivalent:*

(a) $[\hat{c},M,p]_0^\theta(\delta^m)\subset[\hat{c},p]_0^\theta(\Delta^m)$,

(b) $[\hat{c},M,p]_0^\theta(\Delta^m)\subset[\hat{c},p]_\infty^\theta(\Delta^m)$,

(c) $\inf_r \frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{t}{\rho}\right)\right]^{p_i}>0\ \ (t,\rho>0)$.

Proof. **(a)⇒(b)**: It is obvious.

(b)⇒(c): Let (b) hold. Suppose that (c) does not hold. Then

$$\inf_r \frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{t}{\rho}\right)\right]^{p_i}=0\ \ (t,\rho>0),$$

so we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(j)}}\sum_{i\in I_{r(j)}}\left[M\left(\frac{j}{\rho}\right)\right]^{p_i}<j^{-1},\ j=1,2,... \tag{2.5}$$

Define the sequence $x=(x_i)$ by

$$\Delta^m x_{i+s}=\begin{cases} j, & i\in I_{r(j)}\\ 0, & i\notin I_{r(j)}\end{cases}\quad \text{for all}\ \ s\in\mathbb{N}$$

Thus, by (2.5), $x=(x_i)\in[\hat{c},M,p]_0^\theta(\Delta^m)$ but by (2.3) $x=(x_i)\notin[\hat{c},p]_\infty^\theta(\Delta^m)$, which contradicts (b). Hence (c) must hold.

(c)⇒(a) Let (c)hold and suppose that $x=(x_i)\in[\hat{c},M,p]_0^\theta(\Delta^m)$, i.e.,

(2.6)

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right)\right]^{p_i}=0,\ \ uniformly\ in\ s,\ forsome \rho>0.$$

Again, suppose that $x = (x_i) \notin [\hat{c}, p]_0^\theta(\Delta^m)$. Then, for some number $\epsilon > 0$ and a subinterval $I_{r(j)}$ of the set of interval I_r, we have $|\Delta^m x_{i+s}| \geq \epsilon$ for all $i \in \mathbb{N}$ and some $s \geq s_0$. Then, from the properties of the Orlicz function, we can write

$$M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right)^{p_i} \geq M\left(\frac{\epsilon}{\rho}\right)^{p_i}$$

and consequently by (2.6)

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} \left[M\left(\frac{\epsilon}{\rho}\right)\right]^{p_i} = 0,$$

which contradicts (c). Hence (a) must hold.

Finally, we consider that $p = p_i$ and $q = (q_i)$ are any bounded sequences of strictly positive real numbers. We are able to prove below theorem only under additional conditions.

Theorem 2.8. Let $0 < p_i \leq q_i$ for all $i \in \mathbb{N}$ and $\left(\frac{q_i}{p_i}\right)$ be bounded. Then,

$$[\hat{c}, M, q]^\theta(\Delta^m) \subset [\hat{c}, M, p]^\theta(\Delta^m).$$

Proof. Using the same technique of Theorem 2 of Nanda [11], it is easy to prove the theorem.

By using Theorem 2.8., it is easy to prove the following result.

Corollary 2.9. (a) If $0 < \inf p_i \leq p_i \leq 1$ for all $i \in \mathbb{N}$, then

$$[\hat{c}, M, p]^\theta(\Delta^m) \subset [\hat{c}, M]^\theta(\Delta^m).$$

(b) If $1 \leq p_i \leq \sup p_i < \infty$ for all $i \in \mathbb{N}$, then $[\hat{c}, M]^\theta(\Delta^m) \subset [\hat{c}, M, p]^\theta(\Delta^m)$.

References

[1] H. Kizmaz, *On certain sequence spaces,* Canad.Math.Bull., 24(2) (1981), 169-176.

[2] G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Mathematica, 80(1) (1948), 167-190.

[3] I.J. Maddox, *Spaces of strongly summable sequences*, Quart.J.Math., 18(1967), 345-355.

[4] I.J. Maddox, *A new type of convergence*, Math.Proc.Camb.Phil.Soc., 83(1978), 61-64.

[5] M. Güngör, M. Et, Δ^r *-strongly almost summable sequences defined by Orlicz functions*, Indian J.Pure Appl.Math., 34(8) (2003), 1141-1151.

[6] M.A. Krasnoselskii, Y.B. Rutitsky, *Convex functions and Orlicz spaces*, Groningen, Netherland, 1961.

[7] J. Lindenstrauss, L. Tzafriri, *On Orlicz sequence spaces*, Israel J.Math.10 (1971), 345-355.

[8] M.Et, and R. Çolak, *On some generalized difference sequence spaces*, Soochow J.Math., 21(4)(1995), 377-386.

[9] I.J. Maddox, *On strong almost convergence*, Math. Proc. Camb. Phil. Soc., 85(1979), 345- 350.

[10] A.R. Freedman, J.J. Sember, M. Raphael, , *Some Cesaro-type summability spaces*, Proc.London Math.Soc., 37(3) (1978), 508-520.

[11] S. Nanda, *Strongly almost summable and strongly almost convergent sequences*, Acta Math.Hung.,49(1-2)(1987), 71-76.

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