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On Certain Subclasses of Meromorphic Functions With Positive Coefficients [1]

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Abstract

In this paper, we obtain coefficient estimates, distortion theorem, radii of starlikeness and convexity for the class $\sum_p^*(A,B,\beta)$ of meromorphic functions with positive coefficients. Several interesting results involving Hadamard product of functions belonging to the classes $\sum_p^*(\alpha,\beta)$ and $\sum_p^*(A,B,\beta)$ are also derived.

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1 Introduction

Let $\sum_p$ denote the class of functions of the form

$$(1.1)\qquad f(z)=\frac{1}{z}+\sum_{n=p}^{\infty}a_n\,z^n\ \ (\ p\in N^*=\{2i-1:i\in N=\{1,2,3,....\}\})$$

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which are analytic in the punctured disc $U^* = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue one there.

A function $f(z) \in \sum_p$ is said to be meromorphically starlike of order α if it satisfies

$$\operatorname{Re}\left\{-\frac{z\,f'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

for some $\alpha (0 \le \alpha < 1)$ and for all $z \in U^*$.

Further a function $f(z) \in \sum_p$ is said to be meromorphically convex of order α if it satisfies

$$\operatorname{Re}\left\{-(1 + \frac{z\,f''(z)}{f'(z)})\right\} > \alpha \tag{1.3}$$

for some $\alpha (0 \le \alpha < 1)$ and for all $z \in U^*$.

Some subclasses of $\sum_1$ when $p = 1$ were recently introduced and studied by Pommerenke [8], Miller [6], Mogra. et al. [7], Cho [3], Cho et al. [4] and Aouf ([1] and [2]) .

Let $\sum_p^*$ be the subclass of $\sum_p$ consisting of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n\, z^n \quad (a_n \ge 0,\, p \in N^*). \tag{1.4}$$

We denote by $\sum_p^*(A, B)$ the class of functions $f(z) \in \sum_p^*$, that satisfy the condition

$$-z^{p+1}\, f^{(p)}(z) \prec p!\, \frac{1 + Az}{1 + Bz} \quad (z \in U^*), \tag{1.5}$$

where $\prec$ denotes subordination, $-1 \le A < B \le 1$ and $-1 \le A < 0$.

It is easy to see that (1.5) is equivalent to

$$\left| \frac{z^{p+1} f^{(p)}(z) + p!}{B\, z^{p+1} f^{(p)}(z) + p!\, A} \right| < 1 \ \ (z \in U^*). \tag{1.6}$$

Definition 1 *A function $f(z) \in \sum_p^*$ is said to be a member of the class $\sum_p^*(A, B, \beta)$ if it satisfies*

$$\left| \frac{z^{p+1} f^{(p)}(z) + p!}{B\, z^{p+1} f^{(p)}(z) + p!\, A} \right| < \beta \tag{1.7}$$

for some $\beta(0 < \beta \leq 1), -1 \leq A < B \leq 1$, $-1 \leq A < 0$ and for all $z \in U^*$.

We note that:

(i) $\sum_p^*(-1, 1, \beta) = T_p(\beta)$(Kim et al. [5]);

(ii) $\sum_1^*(-A, -B, 1) = \sum_d(A, B)$(Cho [3]);

(iii) $\sum_1^*(2\alpha\gamma - 1, 2\gamma - 1, \beta) = \sum_d(\alpha, \beta, \gamma)(0 \leq \alpha < 1, 0 \leq \gamma \leq \frac{1}{2},$ $0 < \beta \leq 1)$ (Cho et al. [4]);

(iv) $\sum_p^*(\frac{2\alpha}{p!} - 1, 1, \beta) = \sum_p^*(\alpha, \beta)$

$$= \left\{ f(z) \in \sum_p^* : \left| \frac{z^{p+1} f^{(p)}(z) + p!}{z^{p+1} f^{(p)}(z) + 2\alpha - p!} \right| < \beta, 0 \leq \alpha < p!, 0 < \beta \leq 1 \right\};$$

(v) $\sum_p^* \left(\frac{2\alpha}{p!} - 1, 1, 1 \right) = \sum_p^*(\alpha)$

$$= \left\{ f(z) \in \sum_p^* : \ \mathrm{Re} \left\{ -z^{p+1} f^{(p)}(z) \right\} > \alpha \, , 0 \leq \alpha < p!, \, z \in U^* \right\}.$$

In this paper, we obtain coefficient estimates, distortion theorem and radii of starlikeness and convexity for the class $\sum_p^*(A, B, \beta)$ of meromorphic functions with positive coefficients. Several interesting results involving Hadamard product of functions belonging to the classes $\sum_p^*(\alpha, \beta)$ and $\sum_p^*(A, B, \beta)$ are also derived.

2 Coefficient estimates

The following theorem gives a necessary and sufficient condition for a function to be in the class $\sum_p^*(A, B, \beta)$.

Theorem 1 *A function* $f(z) \in \sum_p^*$ *is in the class* $\sum_p^*(A, B, \beta)$*if and only if*

$$\sum_{n=p}^{\infty} \binom{n}{p} a_n \leq \frac{(B-A)\beta}{(1+\beta B)}, \tag{2.1}$$

where

$$\binom{n}{p} = \frac{n!}{p!(n-p)!} .$$

Proof. Suppose (2.1) holds for all admissible values of A, B and β. Then we have

$$\left|z^{p+1} f^{(p)}(z) + p!\right| - \beta \left|B\, z^{p+1} f^{(p)}(z) + p!\, A\right| \tag{2.2}$$

$$= \left|\sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n\, z^{n+1}\right| - \beta \left|p!(B-A) - B \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n\; z^{n+1}\right|$$

$$\leq \sum_{n=p}^{\infty} \frac{n!}{(n-p)!}(1+\beta B) a_n\, |z|^{n+1} - p!(B-A)\beta\ .$$

Since the above inequality holds for all $r = |z|, 0 < r < 1$, letting $r \to 1$, we have

$$\sum_{n=p}^{\infty} \frac{n!}{(n-p)!}(1+\beta B) a_n - p!(B-A)\beta \leq 0, \tag{2.3}$$

which shows that $f(z) \in \sum_p^*(A, B, \beta)$.

Conversely, if $f(z) \in \sum_p^*(A, B, \beta)$, then

$$(2.4) \qquad \left| \frac{z^{p+1} f^{(p)}(z) + p!}{B\, z^{p+1} f^{(p)}(z) + p!\, A} \right|$$

$$= \left| \frac{\sum\limits_{n=p}^{\infty} \frac{n!}{(n-p)!}\, a_n\, z^{n+1}}{p!(B-A) - B \sum\limits_{n=p}^{\infty} \frac{n!}{(n-p)!}\, a_n\, z^{n+1}} \right| < \beta \quad (z \in U^*).$$

Since $\mathrm{Re}(z) \leq |z|$ for all z, (2.4) gives

$$(2.5) \qquad \mathrm{Re} \left\{ \frac{\sum\limits_{n=p}^{\infty} \frac{n!}{(n-p)!}\, a_n\, z^{n+1}}{p!(B-A) - B \sum\limits_{n=p}^{\infty} \frac{n!}{(n-p)!}\, a_n\, z^{n+1}} \right\} < \beta \quad (z \in U^*).$$

Choose values of z on the real axis so that $z^{p+1} f^{(p)}(z)$ is real. Upon clearing the denominator in (2.5) and letting $z \to 1^-$, we have

$$(2.6) \qquad \sum_{n=p}^{\infty} \frac{n!}{(n-p)!}\, a_n \;\leq \frac{p!(B-A)\beta}{(1+\beta B)} \;.$$

Hence the result follows.

Corollary 1 *If the function* $f(z) = \frac{1}{z} + \sum\limits_{n=p}^{\infty} a_n\, z^n (a_n \geq 0)$ *is in the class* $\sum_p^*(A, B, \beta)$*, then*

$$(2.7) \qquad a_n \leq \frac{(B-A)\beta}{\binom{n}{p}(1+\beta B)} \quad (n \geq p;\, p \in N^*).$$

The result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{(B-A)\beta}{\binom{n}{p}(1+\beta B)}\;z^{n}\quad(n\geq p;\;p\in N^{*}). \tag{2.8}$$

Putting $p=1$ in Theorem 1, we have

Corollary 2 *$f(z)\in\sum_1^*(A,B,\beta)$ if and only if*

$$\sum_{n=1}^{\infty}na_n\leq\frac{(B-A)\beta}{(1+\beta B)}\;. \tag{2.9}$$

Putting $A=\dfrac{2\alpha}{p!}-1(0\leq\alpha<p!)$ and $B=1$ in Theorem 1, we obtain the following necessary and sufficient condition for functions in $\sum_p^*(\alpha,\beta)$.

Corollary 3 *Let a function $f(z)$ defined by (1.4). Then $f(z)\in\sum_p^*(\alpha,\beta)$ if and only if*

$$\sum_{n=p}^{\infty}\binom{n}{p}\;a_n\leq\frac{2\beta(1-\frac{\alpha}{p!})}{1+\beta}\;. \tag{2.10}$$

3 Distortion Theorem

Theorem 2 *If the function $f(z)=\dfrac{1}{z}+\sum_{n=p}^{\infty}a_n\;z^n(a_n\geq0)$ is in the class $\sum_p^*(A,B,\beta)$, then*

$$\left|f^{(j)}(z)\right|\geq\frac{j!}{|z|^{j+1}}-\frac{p!(B-A)\beta}{(p-j)!(1+\beta B)}\;|z|^{p-j} \tag{3.1}$$

and

$$\left|f^{(j)}(z)\right| \leq \frac{j!}{|z|^{j+1}} + \frac{p!(B-A)\beta}{(p-j)!(1+\beta B)}\ |z|^{p-j} \tag{3.2}$$

for $z \in U^*$, where $0 \leq j \leq p$ and $0 < \beta \leq \dfrac{j!(p-j)!}{p!(B-A) - j!(p-j)!B}$.
Equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(B-A)\beta}{1+\beta B}\ z^p \quad (p \in N^*)\ . \tag{3.3}$$

Proof. It follows from Theorem 1, that

(3.4)
$$\frac{(p-j)!(1+\beta B)}{p!} \sum_{n=p}^{\infty} \frac{n!}{(n-j)!}\, a_n \leq \sum_{n=p}^{\infty} \begin{pmatrix} n \\ p \end{pmatrix} (1+\beta B) a_n \leq (B-A)\beta\ .$$

Therefore, we have

$$\sum_{n=p}^{\infty} \frac{n!}{(n-j)!}\ a_n \leq \frac{p!(B-A)\beta}{(p-j)!(1+\beta B)}\ . \tag{3.5}$$

Thus we have

$$\begin{aligned} \left|f^{(j)}(z)\right| &\geq \frac{j!}{|z|^{j+1}} - \sum_{n=p}^{\infty} \frac{n!}{(n-j)!} a_n |z|^{n-j} \\ &\geq \frac{j!}{|z|^{j+1}} - \frac{p!(B-A)\beta}{(p-j)!(1+\beta B)}\ |z|^{p-j}, \end{aligned} \tag{3.6}$$

and

$$\left|f^{(j)}(z)\right| \leq \frac{j!}{|z|^{j+1}} + \sum_{n=p}^{\infty} \frac{n!}{(n-j)!}\ a_n\ |z|^{n-j} \tag{3.7}$$

$$\leq \frac{j!}{|z|^{j+1}} + \frac{p!(B-A)\beta}{(p-j)!(1+\beta B)} \ |z|^{p-j}.$$

Hence we have Theorem 2.

Putting $j = 0$ in Theorem 2, we have

Corollary 4 *If* $f(z) \in \sum_p^*(A, B, \beta)$, *then*

$$\frac{1}{|z|} - \frac{(B-A)\beta}{1+\beta B}|z|^p \leq |f(z)| \leq \frac{1}{|z|} + \frac{(B-A)\beta}{1+\beta B} \ |z|^p \tag{3.8}$$

for $z \in U^*$. Equalities in (3.8) are attained for the function $f(z)$ given by (3.3).

Putting $j = 1$ in Theorem 2, we have

Corollary 5 *If* $f(z) \in \sum_p^*(A, B, \beta)$, *then*

$$\frac{1}{|z|^2} - \frac{p(B-A)\beta}{1+\beta B}|z|^{p-1} \leq |f^{'}(z)| \leq \frac{1}{|z|^2} + \frac{p(B-A)\beta}{1+\beta B} \ |z|^{p-1} \tag{3.9}$$

for $z \in U^*$, where $0 < \beta \leq \dfrac{1}{p(B-A)-B}$. Equalities in (3.9) are attained for the function $f(z)$ given by (3.3).

Putting (i) $p = 1$ and $j = 0$ (ii) $p = j = 1$ in Theorem 2, we obtain

Corollary 6 *If* $f(z) \in \sum_1^*(A, B, \beta)$, *then*

$$\frac{1}{|z|} - \frac{(B-A)\beta}{1+\beta B} \ |z| \leq |f(z)| \leq \frac{1}{|z|} + \frac{(B-A)\beta}{1+\beta B} \ |z| \ , \tag{3.10}$$

and

$$\frac{1}{|z|^2} - \frac{(B-A)\beta}{1+\beta B} \leq |f'(z)| \leq \frac{1}{|z|^2} + \frac{(B-A)\beta}{1+\beta B} \tag{3.11}$$

for $z \in U^*$. Equalities in (3.10)and (3.11) are attained for the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(B-A)\beta}{1+\beta B}\ z\ . \tag{3.12}$$

4 Radii of starlikeness and convexity

Theorem 3 *Let the function* $f(z)$ *defined by* (1.4) *be in the class* $\sum_p^*(A, B, \beta)$, *then* $f(z)$ *is meromorphically starlike of order* $\delta (0 \leq \delta < 1)$ *in* $|z| < r_1$, *where*

$$r_1 = \inf_{n \geq p} \left\{ \frac{\binom{n}{p} (1+\beta B)(1-\delta)}{(B-A)\beta(n+2-\delta)} \right\}^{\frac{1}{n+1}} . \tag{4.1}$$

The result is sharp for the function $f(z)$ given by (2.8).
Proof. Let $f(z) \in \sum_p^*(A, B, \beta)$. Then, by Theorem 1

$$\sum_{n=p}^{\infty} \binom{n}{p} \frac{(1+\beta B)}{(B-A)\beta}\ a_n \leq 1\ . \tag{4.2}$$

It is sufficient to show that

$$\left| \frac{z\, f'(z)}{f(z)} + 1 \right| \leq 1 - \delta \tag{4.3}$$

for $|z| < r_1$. We note that

$$\left|\frac{z\,f'(z)}{f(z)}+1\right| = \left|\frac{\sum\limits_{n=p}^{\infty}(n+1)a_n z^n}{\frac{1}{z}+\sum\limits_{n=p}^{\infty} a_n z^n}\right| \le \frac{\sum\limits_{n=p}^{\infty}(n+1)a_n\,|z|^{n+1}}{1-\sum\limits_{n=p}^{\infty} a_n\,|z|^{n+1}} \tag{4.4}$$

This will be bounded by $1-\delta$ if

$$\sum_{n=p}^{\infty}\frac{(n+2-\delta)}{(1-\delta)}\,a_n\,|z|^{n+1} \le 1\ . \tag{4.5}$$

In view of (4.3), it follows that (4.5) is true if

$$\frac{(n+2-\delta)}{(1-\delta)}\,|z|^{n+1} \le \binom{n}{p}\frac{(1+\beta B)}{(B-A)\beta}\ \ (n \ge p), \tag{4.6}$$

or

$$|z| \le \left\{\frac{\binom{n}{p}(1+\beta B)(1-\delta)}{(B-A)\beta(n+2-\delta)}\right\}^{\frac{1}{n+1}} \qquad (n \ge p; p \in N^*). \tag{4.7}$$

Setting $|z| = r_1$ in (4.7), the result follows.

Theorem 4 *Let the function $f(z)$ defined by (1.4) be in the class $\sum_p^*(A,B,\beta)$, then $f(z)$ is meromorphically convex of order δ in $|z| < r_2$, where*

$$r_2 = \inf_{n\ge p}\left\{\frac{\binom{n}{p}(1+\beta B)(1-\delta)}{(B-A)\beta\ n(n+2-\delta)}\right\}^{\frac{1}{n+1}}. \tag{4.9}$$

The result is sharp for the function $f(z)$ defined by (2.8).

5 Convolution properties

For functions

$$(5.1) \qquad f_j(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geq 0,\ j = 1, 2; p \in N^*)$$

belonging to the class $\sum_p^*$, we denote by $(f_1 * f_2)(z)$ the convolution (or Hadamard product) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(5.2) \qquad (f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_{n,1}\ a_{n,2}\, z^n \ .$$

Theorem 5 *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (5.1)*be in the class* $\sum_p^*(\alpha, \beta)$, *then* $(f_1 * f_2)(z) \in \sum_p^*(\gamma, \beta)$, *where*

$$(5.3) \qquad \gamma = p! - \frac{2\beta(p! - \alpha)^2}{p!(1 + \beta)} \ .$$

The result is sharp for the functions

$$(5.4) \qquad f_j(z) = \frac{1}{z} + \frac{2\beta(p! - \alpha)}{p!(1 + \beta)}\ z^p \ \ (j = 1, 2, p \in N^*).$$

Proof. Employing the technique used earlier Schild and Silverman [9] , we need to find the largest γ such that

$$(5.5) \qquad \sum_{n=p}^{\infty} \binom{n}{p} \frac{(1 + \beta)}{2\beta(1 - \frac{\gamma}{p!})} a_{n,1}\ a_{n,2}\ \leq 1 \ .$$

for $f_j(z) \in \sum_p^*(\alpha, \beta)(j = 1, 2)$. Since $f_j(z) \in \sum_p^*(\alpha, \beta)(j = 1, 2)$, we readily see that

$$\sum_{n=p}^{\infty}\left(\begin{array}{c} n \\ p \end{array}\right)\frac{(1+\beta)}{2\beta(1-\frac{\alpha}{p!})}\,a_{n,j}\ \leq 1\quad (j=1,2). \tag{5.6}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=p}^{\infty}\left(\begin{array}{c} n \\ p \end{array}\right)\frac{(1+\beta)}{2\beta(1-\frac{\alpha}{p!})}\sqrt{a_{n,1}\ a_{n,2}}\ \leq 1\ , \tag{5.7}$$

this implies that, we need only show that

$$\frac{a_{n,1}\ a_{n,2}}{1-\frac{\gamma}{p!}}\leq\frac{\sqrt{a_{n,1}\ a_{n,2}}}{1-\frac{\alpha}{p!}}\quad (n\geq p) \tag{5.8}$$

or , equivalently, that

$$\sqrt{a_{n,1}\ a_{n,2}}\quad\leq\frac{1-\frac{\gamma}{p!}}{1-\frac{\alpha}{p!}}\quad (n\geq p). \tag{5.9}$$

Hence, by the inequality (5.7), it is sufficient to prove that

$$\frac{2\beta(1-\frac{\alpha}{p!})}{\left(\begin{array}{c} n \\ p \end{array}\right)(1+\beta)}\ \leq\frac{1-\frac{\gamma}{p!}}{1-\frac{\alpha}{p!}}\quad (n\geq p). \tag{5.10}$$

It follows from (5.10) that

$$\gamma\leq p!-\frac{2\beta(p!-\alpha)^2}{p!\left(\begin{array}{c} n \\ p \end{array}\right)(1+\beta)}\quad (n\geq p). \tag{5.11}$$

Now, defining the function $\phi(n)$ by

$$\phi(n) = p! - \frac{2\beta(p! - \alpha)^2}{p! \binom{n}{p} (1+\beta)} \quad (n \geq p). \tag{5.12}$$

We see that $\varphi(n)$ is an increasing function of n. Therefore, we conclude that

$$\gamma \leq \phi(p) = p! - \frac{2\beta(p! - \alpha)^2}{p!(1+\beta)} \ , \tag{5.13}$$

which evidently completes the proof of Theorem 5.

Using similar arguments as in the proof of Theorem 5, we can obtain the following result.

Theorem 6 *If* $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_{n,1} \ z^n \in \sum_p^{*}(\alpha, \beta)$ *and* $f_2(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_{n,2} \ z^n \in \sum_p^{*}(\gamma, \beta)$, *then* $(f_1 * f_2) \in \sum_p^{*}(\Im, \beta)$, *where*

$$\Im = p! - \frac{2\beta(p! - \alpha)(p! - \gamma)}{p!(1+\beta)} \ . \tag{5.14}$$

The result is possible for the functions $f_j(z)(j = 1, 2)$ given by

$$f_1(z) = \frac{1}{z} + \frac{2\beta(p! - \alpha)}{p!(1+\beta)} \ z^p \ \ (p \in N^*) \tag{5.15}$$

and

$$f_2(z) = \frac{1}{z} + \frac{2\beta(p! - \gamma)}{p!(1+\beta)} \ z^p \ (p \in N^*). \tag{5.16}$$

Theorem 7 *Let the functions $f_j(z)(j=1,2)$ defined by* (5.1) *be in the class $\sum_p^*(\alpha,\beta)$. Then the function $h(z)$ defined by*

$$h(z)=\frac{1}{z}+\sum_{n=p}^{\infty}(a_{n,1}^2+a_{n,2}^2)\,z^n \tag{5.17}$$

also belongs to the class $\sum_p^*(\delta,\beta)$, where

$$\delta=p!-\frac{4\beta(p!-\alpha)^2}{p!(1+\beta)}\ . \tag{5.18}$$

The result is sharp for the functions $f_j(z)(j=1,2)$ defined by (5.4).
Proof. Noting that

$$\sum_{n=p}^{\infty}\left[\binom{n}{p}\frac{1+\beta}{2\beta(1-\frac{\alpha}{p!})}\right]^2 a_{n,j}^2 \tag{5.19}$$

$$\leq\left[\sum_{n=p}^{\infty}\binom{n}{p}\frac{1+\beta}{2\beta(1-\frac{\alpha}{p!})}\,a_{n,j}\right]^2\leq 1\quad (j=1,2)$$

for $f_j(z)\in\sum_p^*(\alpha,\beta)$ $(j=1,2)$, we have

$$\sum_{n=p}^{\infty}\frac{1}{2}\left[\binom{n}{p}\frac{1+\beta}{2\beta(1-\frac{\alpha}{p!})}\right]^2(a_{n,1}^2+a_{n,2}^2)\leq 1\ . \tag{5.20}$$

Therefore, we have to find the largest δ such that

$$\frac{\binom{n}{p}(1+\beta)}{2\beta(1-\frac{\delta}{p!})}\leq\frac{1}{2}\binom{n}{p}^2\frac{(1+\beta)^2}{4\beta^2(1-\frac{\alpha}{p!})^2}\quad (n\geq p), \tag{5.21}$$

that is, that

$$\delta \leq p! - \frac{4\beta(p! - \alpha)^2}{\binom{n}{p} p!(1+\beta)} \quad (n \geq p). \tag{5.22}$$

Now, defining a function $\Psi(n)$ by

$$\Psi(n) = p! - \frac{4\beta(p! - \alpha)^2}{\binom{n}{p} p!(1+\beta)} \quad (n \geq p). \tag{5.23}$$

We observe that $\Psi(n)$ is an increasing function of n. Thus, we conclude that

$$\delta \leq \Psi(p) = p! - \frac{4\beta(p! - \alpha)^2}{p!(1+\beta)}\ , \tag{5.24}$$

which completes the proof of Theorem 7.

In the remaining theorems, we consider the functions in the class $\sum_p^*(A, B, \beta)$.

Theorem 8 *If* $f(z) = \frac{1}{z} + \sum\limits_{n=p}^{\infty} a_{n,1}\ z^n \in \sum_p^*(A, B, \beta)$ *and* $g(z) = \frac{1}{z} + \sum\limits_{n=p}^{\infty} a_{n,2}\ z^n$ *with* $|a_{n,2}| \leq 1(n \geq p)$*, then* $(f_1 * f_2)(z) \in \sum_p^*(A, B, \beta)$.

Proof. Since

$$\begin{aligned}\sum_{n=p}^{\infty} \binom{n}{p} \frac{1+\beta B}{(B-A)\beta} |a_{n,1}\ a_{n,2}| &= \sum_{n=p}^{\infty} \binom{n}{p} \frac{1+\beta B}{(B-A)\beta} a_{n,1} |a_{n,2}| \\ &\leq \sum_{n=p}^{\infty} \binom{n}{p} \frac{1+\beta B}{(B-A)\beta}\ a_{n,1} \leq 1,\end{aligned}$$

by Theorem 1, it follows that $(f_1 * f_2)(z) \in \sum_p^*(A, B, \beta)$.

Corollary 7 *If* $f_1(z) = \dfrac{1}{z} + \sum\limits_{n=p}^{\infty} a_{n,1}\ z^n \in \sum\limits_p^*(A,B,\beta)$ *and* $f_2(z) = \frac{1}{z} + \sum\limits_{n=p}^{\infty} a_{n,2}\ z^n$ *with* $0 \leq a_{n,2} \leq 1, n \geq p$, *then* $(f_1 * f_2)(z) \in \sum_p^*(A,B,\beta)$.

Theorem 9 *Let the functions* $f_j(z)(j = 1,2)$ *defined by* (5.1) *be in the class* $\sum_p^*(A,B,\beta)$ *and* $1-\beta B+2\beta A \geq 0$, *then the function* $h(z)$ *defined by* (5.17) *also belongs to* $\sum_p^*(A,B,\beta)$.

Proof. Since $f_1(z) \in \sum_p^*(A,B,\beta)$, we have

$$\sum_{n=p}^{\infty} \binom{n}{p} \frac{1+\beta B}{(B-A)\beta}\ a_{n,1} \leq 1$$

and so

$$\sum_{n=p}^{\infty} \left[\binom{n}{p} \frac{(1+\beta B)}{(B-A)\beta}\right]^2\ a_{n,1}^2 \leq 1\ .$$

Similarly, since $f_2(z) \in \sum_p^*(A,B,\beta)$, we have

$$\sum_{n=p}^{\infty} \left[\binom{n}{p} \frac{(1+\beta B)}{(B-A)\beta}\right]^2\ a_{n,2}^2 \leq 1\ .$$

Hence

$$\sum_{n=p}^{\infty} \frac{1}{2} \left[\binom{n}{p}\ \frac{1+\beta B}{(B-A)\beta}\right]^2\ (a_{n,1}^2 + a_{n,2}^2) \leq 1\,.$$

In view of Theorem 1, it is sufficient to show that

$$\sum_{n=p}^{\infty} \binom{n}{p} \frac{(1+\beta B)}{(B-A)\beta}\,(\,a_{n,1}^2 + a_{n,2}^2) \leq 1\ . \tag{5.25}$$

Thus the inequality (5.25) will be satisfied if for $n \geq p$,

$$(5.26) \qquad \binom{n}{p} \frac{(1+\beta B)}{(B-A)\beta} \leq \frac{1}{2} \binom{n}{p}^2 \frac{(1+\beta B)^2}{(B-A)^2\beta^2} \,,$$

or if

$$(5.27) \qquad \binom{n}{p} (1+\beta B) + 2\beta(A-B) \geq 0$$

for $n \geq p \ \ (p \in N^*)$. The left hand side of (5.27) is an increasing function of n , hence (5.27) is satisfied for all n if

$$1 - \beta B + 2\beta A \geq 0 \,,$$

which is true by our assumption . Hence the theorem.

References

[1] M. K. Aouf, *A certain subclass of meromorphically starlike functions with positive coefficients*, Rend. Mat. 9(1989), 225-235.

[2] M. K. Aouf, *On a certain class of meromorphic univalent functions with positive coefficients*, Rend. Mat. 11(1991), 209-219.

[3] N. E. Cho, *On certain class of meromorphic functions with positive coefficients*, J. Inst. Math. Comput. Sci. 3(1990), no.2, 119-125.

[4] N. E. Cho, S. H. Lee, S. Owa, *A class of meromorphic univalent functions with positive coefficients*, Kobe J. Math. 4(1987), 43-50.

[5] Y. G. Kim, S. H. Lee, S. Owa, *On certain meromorphic functions with positive coefficients*, Internat. J. Math. Math. Sci. 16(1993), no.2, 409-412.

[6] J. E. Miller, *Convex meromorphic mapping and related functions*, Proc. Amer. Math. Soc. 25(1970), 220-228.

[7] M. L. Morga, T. Reddy, O. P. Juneja, *Meromorphic univalent functions with positive coefficients*, Bull. Austral. Math. Soc. 32(1985), 161-176.

[8] Ch. Pommerenke, *On meromorphic starlike functions*, Pacific J. Math. 13(1963), 221-235.

[9] A. Schild,H.Silverman , *Convolution of univalent functions with negative coefficients*, Ann. Univ Mariae Curre-Sklodowska sect. A29 (1975), 99-107.

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