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On a sequence of linear and positive operators

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Abstract

In order to approximate function $f : [0,\infty) \to \mathbb{R}$, with $|f(x)| \leq Mx^\alpha$ for $x > 0$ and $M = M(f) > 0$, we introduce the approximation operators $\mathcal{F}_n : f \to \mathcal{F}_n f$, with

$$(\mathcal{F}_n f)(x) = \frac{(nx)_{n+1}}{n!} \int_0^1 t^{nx-1}(1-t)^n f\left(\frac{t}{1-t}\right) dt, \quad x > 0, \quad \alpha > 0,$$

where $n \geq n_0$ with $n_0 = [\alpha] + b + 1$ and $n \in \mathbb{N}^*$ is fixed.

Our aim is to find some properties for above operator.

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Let Y_α be the linear space of all functions $f : [0,\infty) \to \mathbb{R}$, with the property that there exist M, $M = M(f) > 0$ and $\alpha > 0$ such that $|f(x)| \leq Mx^\alpha$, for all $x > 0$. We define the operators $\mathcal{F}_n : f \to \mathcal{F}_n f$,

$$(1) \quad (\mathcal{F}_n f)(x) = \frac{(nx)_{n+1}}{n!} \int_0^1 t^{nx-1}(1-t)^n f\left(\frac{t}{1-t}\right) dt, \quad x > 0, \quad \alpha > 0$$

where $n \geq n_0$, $n_0 = [\alpha] + b + 1$ and $n \in \mathbb{N}^*$ is fixed.

Now, we demonstrate that if $f \in Y_\alpha$, then $\mathcal{F}_n f \in Y_\alpha$.

Theorem 1 *If* $(\mathcal{F}_n)_{n\geq n_0}$ *are the operators defined in relation (1) and* $f \in Y_\alpha$, $|f(x)| \leq M(f)x^\alpha$, $\alpha > 0$, $x > 0$ *then, there exist* $M(\mathcal{F}_n f) > 0$ *such that for all* $x > c > 0$ *the following relation hold*

$$|(\mathcal{F}_n f)(x)| \leq M(\mathcal{F}_n f)x^\alpha,$$

where $M(\mathcal{F}_n f) = M(f)e^{\frac{2\alpha^2}{b\sqrt{c}}}$.

Proof. We have successive

$$\begin{aligned} |(\mathcal{F}_n f)(x)| &\leq \frac{(nx)_{n+1}}{n!}\int_0^1 t^{nx-1}(1-t)^n \left| f\left(\frac{t}{1-t}\right)\right| dt \\ &\leq M\frac{(nx)_{n+1}}{n!}\int_0^1 t^{nx-1}(1-t)^n \left(\frac{t}{1-t}\right)^\alpha dt \\ &= M\frac{(nx)_{n+1}}{n!}\int_0^1 t^{nx+\alpha-1}(1-t)^{n-\alpha+1-1}dt \\ &= M\frac{\Gamma(nx+n+1)}{n!\Gamma(nx)}\frac{\Gamma(nx+\alpha)\Gamma(n-\alpha+1)}{\Gamma(n+nx+1)}. \end{aligned}$$

Therefore, we obtain

$$|(\mathcal{F}_n f)(x)| \leq M\frac{\Gamma(n-\alpha+1)\Gamma(nx+\alpha)}{\Gamma(n+1)\Gamma(nx)}. \tag{2}$$

To obtain our results we need the following theorem

Theorem 2 *(Bohr & Mollerup) There is only one function* $g : (0,\infty) \to (0,\infty)$ *which verifies:*

1. $g(1) = 1$
2. $g(x+1) = xg(x)$
3. $\ln g$ *is a convex function on* $(0,\infty)$,

then $g(x) = \Gamma(x)$*, for all* $x > 0$*.*

From Theorem 2 we have

$$[x_1, x_2, x_3; \ln \Gamma] \geq 0, \text{ for all } 0 < x_1 < x_2 < x_3 < \infty \tag{3}$$

namely,

$$(\Gamma(x_2))^{x_3 - x_1} \geq (\Gamma(x_1))^{x_3 - x_2} (\Gamma(x_3))^{x_2 - x_1}. \tag{4}$$

We choose $0 < x_1 = z + 1 - \alpha < x_2 = z + 1 < x_3 = z + 2 < \infty$. From relation (4) we obtain

$$(\Gamma(z+1))^{1+\alpha} \geq (\Gamma(z + 1 - \alpha))((z+1)\Gamma(z+1))^{\alpha},$$

therefore

$$\frac{\Gamma(z+1)}{\Gamma(z+1-\alpha)} \leq (z+1)^{\alpha}, \quad \text{for all } z + 1 > \alpha > 0. \tag{5}$$

If we choose $0 < x_1 = z - \alpha < x_2 = z + 1 - \alpha < x_3 = z + 1 - \alpha < z + 1 < \infty$, then the following relation holds

$$(\Gamma(z + 1 - \alpha))^{1+\alpha} \geq \left(\frac{\Gamma(z + 1 - \alpha)}{z - \alpha} \right)^{\alpha} \Gamma(z+1),$$

namely

$$\frac{\Gamma(z + 1 - \alpha)}{\Gamma(z+1)} \leq \frac{1}{(z - \alpha)^{\alpha}}, \quad \text{for all } z > \alpha > 0. \tag{6}$$

From (5) and (6) we obtain

$$\frac{1}{(z+1)^{\alpha}} \leq \frac{\Gamma(z + 1 - \alpha)}{\Gamma(z+1)} \leq \frac{1}{(z - \alpha)^{\alpha}}, \quad \text{for all } z > \alpha > 0. \tag{7}$$

In relation (4) we choose $0 < x_1 = nx < x_2 = nx+\alpha < x_3 = nx+\alpha+1 < \infty$ and we obtain

$$\left(\frac{\Gamma(nx+\alpha+1)}{(nx+\alpha)}\right)^{\alpha+1} \leq (\Gamma(nx))(\Gamma(nx+\alpha+1))^{\alpha},$$

namely

$$\frac{\Gamma(nx+\alpha+1)}{\Gamma(nx)} \leq (nx+\alpha)^{\alpha+1}, \quad \text{for all } x > 0,\ \alpha > 0,$$

$$\frac{(nx+\alpha)\Gamma(nx+\alpha)}{\Gamma(nx)} \leq (nx+\alpha)^{\alpha+1}.$$

From above relation we have

$$\text{(8)} \qquad \frac{\Gamma(nx+\alpha)}{\Gamma(nx)} \leq (nx+\alpha)^{\alpha}, \quad \text{for all } x > 0,\ \alpha > 0.$$

In relation (4) we choose $0 < x_1 = nx-1 < x_2 = nx < x_3 = nx+\alpha < \infty$and we obtain

$$\Gamma(nx)^{\alpha+1} \leq \left(\frac{\Gamma(nx)}{nx-1}\right)^{\alpha} \Gamma(nx+\alpha),$$

namely

$$\text{(9)} \qquad 0 < (nx-1)^{\alpha} \leq \frac{\Gamma(nx+\alpha)}{\Gamma(nx)},\ nx-1 > 0,\ \alpha > 0.$$

From (8) and (9) we have

$$\text{(10)} \quad (nx-1)^{\alpha} \leq \frac{\Gamma(nx+\alpha)}{\Gamma(nx)} \leq (nx+\alpha)^{\alpha}, \quad \text{for all } x > \frac{1}{x},\ x > 0,\ \alpha > 0.$$

If in relation (2) we use the inequalities (7)and (10), we obtain

$$\begin{aligned} |(\mathcal{F}_n f)(x)| &\leq M(f)\frac{(nx+\alpha)^{\alpha}}{(n-\alpha)^{\alpha}} = M(f)x^{\alpha}\frac{(nx+\frac{\alpha}{x})^{\alpha}}{(n-\alpha)^{\alpha}} \\ &= M(f)x^{\alpha}\left(1+\frac{\alpha+\frac{\alpha}{x}}{n-\alpha}\right)^{\alpha} = M(f)x^{\alpha}\left(1+\frac{\alpha+\frac{\alpha}{x}}{n-\alpha}\right)^{\frac{n-\alpha}{\alpha+\frac{\alpha}{x}}\frac{\alpha+\frac{\alpha}{x}}{n-\alpha}\alpha}, \end{aligned}$$

namely

$$|(\mathcal{F}_n f)(x)| < M(f)x^\alpha e^{\frac{\alpha^2\left(1+\frac{1}{x}\right)}{n-\alpha}} < M(f)x^\alpha e^{\frac{2\alpha^2}{\sqrt{x}(n-\alpha)}}. \tag{11}$$

Let $b \in \mathbb{N}^*$ be a fixed number and we denote $n_0 = [\alpha+b+1] = [\alpha]+b+1 > \alpha + b$. We consider $n \geq n_0$ and we have $n - \alpha > b$, namely $\dfrac{1}{n-\alpha} < \dfrac{1}{b}$.
From (11) we obtain

$$|(\mathcal{F}_n f)(x)| \leq M(f)x^\alpha e^{\frac{2\alpha^2}{\sqrt{x}b}} \leq M(f)x^\alpha e^{\frac{2\alpha^2}{b\sqrt{c}}} =: M(\mathcal{F}_n f)x^\alpha,$$

where $M(\mathcal{F}_n f) = M(f)x^\alpha e^{\frac{2\alpha^2}{b\sqrt{c}}}$.

Next to calculate $(\mathcal{F}_n e_j)(x)$, where $e_j(x) = x^j$. We have

$$(\mathcal{F}_n e_j)(x) = \frac{(nx)_{n+1}}{n!}\int_0^1 t^{nx+j-1}(1-t)^{n-j}dt = \frac{\Gamma(nx+n+1)}{\Gamma(nx)n!}B(nx+j, n-j+1)$$

$$= \frac{\Gamma(nx+n+1)}{\Gamma(nx)n!}\frac{\Gamma(nx+j)\Gamma(n-j+1)}{\Gamma(nx+n+1)} = \frac{(nx)_j\Gamma(n-j+1)}{\Gamma(n+1)} = \frac{(nx)_j}{(n-j+1)_j}.$$

Therefore, we have $(\mathcal{F}_n e_0)(x) = 1$, $(\mathcal{F}_n e_1)(x) = x$, respectively

$$(\mathcal{F}_n e_2)(x) = \frac{(nx)(nx+1)}{(n-2+1)_2} = \frac{nx(nx+1)}{n(n-1)} = x^2 + \frac{x(1+x)}{n-1} \xrightarrow{n\to\infty} x^2.$$

We need the following theorem:

Theorem 3 *(A. Lupaş [4]) If* $\lim\limits_{n\to\infty}(\mathcal{L}e_j)(x) = [\varphi(x)]^j$, $j = 0, 1, 2$, *then*

$$\lim_{n\to\infty}(\mathcal{L}f)(x) = f(\varphi(x)),$$

for f *a continuous function on the interval* $[0, M]$, $M > 0$.

Using the above theorem, we obtain the following result:

Theorem 4 *Let $f : [0, \infty) \to \mathbb{R}$ be a function which verifies $|f(x)| \leq Mx^{\alpha}$, $\alpha > 0$, $M > 0$, for $x \to \infty$. If $\mathcal{F}_n$ are the linear and positive operators defined in relation (1), then*

$$\lim_{n\to\infty} (\mathcal{F}f)(x) = f(x),$$

for f a continuous function on the interval $[0, M]$, $M > 0$.

In [6], A. Lupaş has demonstrated the following result:

Theorem 5 *If $L : C(K) \to C(K_1)$, $K_1 = [a_1, b_1] \subseteq K$ is a linear operator, then for all function $f \in C(K)$ and $\delta > 0$, the following relation is verified*

$$||f - Lf||_{K_1} \leq ||f|| \cdot ||e_0 - Le_0||_{K_1} + \inf_{m=1,2,\ldots} \{||Le_0||_{K_1} + \delta^{-m} ||L\Omega_m||_{K_1}\} \omega(f; \delta),$$

where $|| \cdot || = \max_K | \cdot |$ and $\Omega_m(t) = (t - x)^m$.

Using the above theorem, we obtain the following result.

Theorem 6 *Let $\mathcal{F}_n f$ be the operators defined in (1). Then for all $f \in Y_\alpha \cap C[0, \infty)$, $\alpha \geq 2$ we have*

$$||f - \mathcal{F}_n f|| \leq \frac{5}{4} \omega \left(f; \frac{1}{\sqrt{n-1}} \right).$$

Proof. We consider the case $m = 2$, $\Omega_2(t) = (t - x)^2$.

From (1) we have (see [7]):

$$(\mathcal{F}_n \Omega_2)(x) = x^2 + \frac{x(1-x)}{n-1} - 2x^2 + x^2 = \frac{x(1-x)}{n-1}.$$

If we choose $\delta = \dfrac{1}{\sqrt{n-1}}$ and use the inequality $x(1-x) \leq \dfrac{1}{4}$, we obtain

$$||f - \mathcal{F}_n f|| \leq \frac{5}{4} \omega \left(f; \frac{1}{\sqrt{n-1}} \right).$$

Let $Y_B = \{f : [0,\infty) \to \mathbb{R};\ |f(x)| \le A(f)e^{Bx},\ x \ge 0\}$ be a linear space, where $B > 0$. We consider Favard - Sasz linear and positive operators, defined so $S_n : f \to S_n f$,

$$(12)\quad (S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \quad (n = 1, 2, \dots,),\ \text{where}\ S_n f \in Y_B.$$

It is know that

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt,$$

and using the change of variable $t = ay$, we have

$$\frac{1}{a^{\alpha}}\Gamma(\alpha) = \int_0^{\infty} e^{-ay} y^{\alpha-1} dy.$$

For the Favard - Sasz operator we have

$$\begin{aligned}
\int_0^{\infty} e^{-ay} y^{\alpha-1} (S_n f)(y) dy &= \int_0^{\infty} e^{-(a+n)y} \sum_{k=0}^{\infty} \frac{n^k}{k!} y^{k+\alpha-1} f\left(\frac{k}{n}\right) dy \\
&= \sum_{k=0}^{\infty} \frac{n^k}{k!} f\left(\frac{k}{n}\right) \int_0^{\infty} e^{-(a+n)y} y^{k+\alpha-1} dy \\
&= \sum_{k=0}^{\infty} \frac{n^k}{k!} \frac{1}{(a+n)^{k+\alpha}} \Gamma(k+\alpha) f\left(\frac{k}{n}\right) \\
&= \frac{1}{(a+n)^{\alpha}} \sum_{k=0}^{\infty} \frac{(\alpha)_k \Gamma(\alpha)}{k!} \left(\frac{n}{(a+n)}\right)^k f\left(\frac{k}{n}\right).
\end{aligned}$$

If we consider the case $\alpha = nx$; $\dfrac{n}{(a+n)} = \dfrac{1}{2}$ and use the notation $(z)_k = \dfrac{\Gamma(z+k)}{\Gamma(z)}$ we obtain the Lupaş linear and positive operators (see [5])

$$(13)\qquad (\mathcal{L}_n f)(x) = (2)^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \quad x \ge 0$$

where $f : [0,\infty) \to \mathbb{R}$.

We consider the positive operator

$$(G_n f)(x) = \frac{n^{nx}}{\Gamma(nx)} \int_0^\infty e^{-nt} t^{nx-1} f(t) dt, \quad x > 0. \tag{14}$$

If we use the change of variable $t = \dfrac{T}{n}$, $dt = \dfrac{1}{n} dT$, we obtain

$$(G_n f)(x) = \frac{n^{nx}}{\Gamma(nx)} \frac{1}{n} \int_0^\infty e^{-T} \frac{T^{nx-1}}{n^{nx-1}} f\left(\frac{T}{n}\right) dT,$$

namely, we have the Post-Widder operator

$$(W_n f)(x) = \frac{1}{\Gamma(nx)} \int_0^\infty e^{-t} t^{nx-1} f\left(\frac{t}{n}\right) dt. \tag{15}$$

Theorem 7 *The operators $\mathcal{F}_n f$ defined in (1)verify the following relation*

$$\mathcal{F}_n f = W_n G_n f,$$

where W_n are Post Widder operators, respectively $G_n f$ are Gamma operators.

Proof. We use the following representation of Gamma operators

$$(G_n f)(x) = \frac{1}{n!} \int_0^\infty e^{-t} t^n f\left(\frac{nx}{t}\right) dt,$$

and for $x = \dfrac{t}{n}$ we have

$$(G_n f)\left(\frac{t}{n}\right) = \frac{1}{n!} \int_0^\infty e^{-s} s^n f\left(\frac{t}{s}\right) ds.$$

We obtain

$$\begin{aligned}
(W_n G_n f)(x) &= \frac{1}{n!\Gamma(nx)} \int_0^\infty e^{-t} t^{nx-1} \left(\int_0^\infty e^{-s} s^n f\left(\frac{t}{s}\right) ds \right) dt \\
&= \frac{1}{n!\Gamma(nx)} \int_0^\infty e^{-s} s^n \left(\int_0^\infty e^{-t} t^{nx-1} f\left(\frac{t}{s}\right) dt \right) ds.
\end{aligned}$$

If we use the change of variable $\dfrac{t}{s} = y$, namely $t = ys$ we have

$$\begin{aligned}(W_nG_nf)(x) &= \frac{1}{n!\Gamma(nx)}\int_0^\infty e^{-s}s^{n+nx}\left(\int_0^\infty e^{-ys}y^{nx-1}f(y)dy\right)ds\\ &= \frac{1}{n!\Gamma(nx)}\int_0^\infty \left(\int_0^\infty e^{-s(1+y)}s^{n+nx}ds\right)y^{nx-1}f(y)dy.\end{aligned}$$

Denote $s(1+y) = T$, $ds = \dfrac{1}{1+y}dT$ and we obtain

$$(W_nG_nf)(x) = \frac{1}{n!\Gamma(nx)}\int_0^\infty \frac{1}{(1+y)^{n+nx+1}}\left(\int_0^\infty e^{-T}T^{n+nx}dT\right)y^{nx-1}f(y)dy,$$

Since

$$\Gamma(n+nx+1) = \int_0^\infty e^{-T}T^{n+nx}dT,$$

we have

$$\begin{aligned}(W_nG_nf)(x) &= \frac{\Gamma(n+nx+1)}{n!\Gamma(nx)}\int_0^\infty \frac{y^{nx-1}}{(1+y)^{n+nx+1}}f(y)dy\\ &= \frac{(nx)_{n+1}}{n!}\int_0^\infty \frac{y^{nx-1}}{(1+y)^{n+nx+1}}f(y)dy.\end{aligned}$$

If we use the change of variable $\dfrac{y}{1+y} = t$, $dy = \dfrac{1}{(1-t)^2}dt$, we obtain

$$\begin{aligned}(W_nG_nf)(x) &= \frac{(nx)_{n+1}}{n!}\int_0^1 \left(\frac{t}{1-t}\right)^{nx-1}\frac{(1-t)^{n+nx+1}}{(1-t)^2}f\left(\frac{t}{1-t}\right)dt\\ &= \frac{(nx)_{n+1}}{n!}\int_0^1 t^{nx-1}(1-t)^n f\left(\frac{t}{1-t}\right)dt = (\mathcal{F}_nf)(x).\end{aligned}$$

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A lower bound for the second moment of Schoenberg operator

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Abstract

In this paper we represent a new lower bound for the second moment for Schoenberg variation-diminishing spline operator. We apply this estimate for $f \in C^2[0,1]$ and generalize the results obtained earlier by Gonska, Pitul and Rasa.

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1 Main result

We start with the definition of variation-diminishing operator, introduced by I.Schoenberg. For the case of equidistant knots we denote it by $S_{n,k}$. Consider the knot sequence $\Delta_n = \{x_i\}_{-k}^{n+k}$, $n \geq 1$, $k \geq 1$ with equidistant "interior knots", namely

$$\Delta_n : x_{-k} = \cdots = x_0 = 0 < x_1 < x_2 < \cdots < x_n = \cdots = x_{n+k} = 1$$