General Mathematics Vol. 16, No. 4 (2008), 81–86

A characterization of the orthogonal polynomials

Ioan Gavrea

Abstract

We give a characterization of the orthogonal polynomials using certain inequalities linked to the scalar product between a fixed function ϕ and any convex function of order $n-1$.

2000 Mathematical Subject Classification: 26D05, 33C45.

1 Introduction

We denote by I the interval $[0,1]$, $\dot{I}=(0,1)$ and by e_i the monomial $e_i:[0,1]\to\mathbb{R}$, $e_i(x)=x^i$, $i=0,1,2,...$

Definition 1 *Let $J\subset\mathbb{R}$ be an interval. A function $f:J\to\mathbb{R}$ is called nonconcave of order $n-1$ on J if for every system $\{x_0,x_1,...,x_n\}$ of distinct points from J we have*

$$[x_0,x_1,...,x_n;f]\geq 0,$$

where $[x_0, x_1, ..., x_n; f]$ *is the divided difference of function* f *on a system of distinct points* $\{x_0, x_1, ..., x_n\}$, $x_k \in J$.

In the following we denote by $K_{n-1}(\dot{I})$ the set of nonconcave functions of order $n-1$ on $\dot{I}$ with the property that $\int_0^1 x^k f(x)dx$, $k \in \{0, 1, ..., n\}$ exist. Let us denote by P_n the Legendre polynomial of degree n on the interval I.

N. Ciorănescu ([1], [2], [3]) proved the following result:

Let $f \in K_{n-1}(\dot{I}) \cap C^n(\dot{I})$. Then there exists a point $\theta = \theta(f) \in \dot{I}$ such that

$$\int_0^1 P_n(x) f(x)dx = K\frac{f^{(n)}(\theta)}{n!}, \tag{1}$$

where $K = \int_0^1 x^n P_n(x)dx$.

A. Lupaş, [5], showed that for any $f \in K_{n-1}(\dot{I})$ there exist distinct points $c_i(f) \in \dot{I}$, $i = 0, 1, ..., n$ such that the following equation holds:

$$\int_0^1 P_n(x) f(x)dx = K[c_0, c_1, ..., c_n; f], \tag{2}$$

where $K = \int_0^1 x^n P_n(x)dx$. In fact Lupaş's result shows that the linear functional A, $A : C(I) \to \mathbb{R}$ is a P_n simple functional in the sense of T. Popoviciu ([7]).

In [5] we have extended the result obtained by A. Lupaş. We proved the following: let $A : C[0, 1] \to \mathbb{R}$ be a linear positive definite functional and P_n the orthogonal polynomial of degree n relative to the functional A. Then for every $f \in C[0, 1]$ there exist n distinct points $c_i := c_i(f)$, $c_i \in [0, 1]$, $i = 0, 1, ..., n$ such that:

$$A(fP_n) = K[c_0, c_1, ..., c_n; f], \; K = A(e_n). \tag{3}$$

In [6], A. Lupaş and A. Vernescu gave a characterization of the Legendre polynomials. They proved the following.

Theorem 1 *Let $p \in \Pi_n$ a monic polynomial. A necessary and sufficient condition such that the inequality*

$$\int_0^1 p(x)f(x)dx \geq 0 \tag{4}$$

holds for all $f \in K_{n-1}(\dot{I})$ is that $p = P_n^*$, where P_n^* is the Legendre monic polynomial of degree n.

The aim of this paper is to extend the result of Theorem 1.

2 Main Results

Let $\mathcal{L}$ be a $n+1$-dimensional space of $C^n(I)$ and $U_0, U_1, ..., U_n$ a basis of $\mathcal{L}$. The following definition is well known

Definition 2 *Let $(U_0, U_1, ..., U_n)$ be a basis of $\mathcal{L}$. The space $\mathcal{L}$ is said to be an* **extended Chebysev space on** I *if any nonzero element of $\mathcal{L}$ vanishes at most n times on $\dot{I}$ (with multiplicities).*

In the following we denote by $\Phi \subset C^{n-1}(\dot{I})$ the set of all functions ϕ for which the following conditions are satisfied:

1. For every $f \in K_{n-1}(\dot{I})$, $\int_0^1 f(x)\phi(x)dx$ is finite.
2. The space $\mathcal{L}$ spanned by the functions $\{e_0, e_1, ..., e_{n-1}, \phi\}$ is an $n+1$-dimensional extended Chebysev space on $\dot{I}$.

Theorem 2 *Let $\phi \in \Phi$ be a fixed function such that*

$$\int_0^1 \phi(x)f(x)dx \geq 0. \tag{5}$$

Then, there exists a weight function w such that function ϕ can be written as

$$\phi = P_n w,$$

where P_n is the monic orthogonal polynomial of degree n on $[0,1]$ relative to the scalar product

$$< f, g >= \int_0^1 f(x)g(x)w(x)dx.$$

Proof. The functions $\pm e_i \in K_{n-1},\ i = 0, 1, ..., n-1$. From (5) we get

$$\int_0^1 x^i f(x)dx = 0, \quad i = 0, 1, ..., n-1. \tag{6}$$

From (6) it follows that there exist n distinct points $x_i \in \dot{I}$, $i = 0, 1, ..., n$ where the function ϕ changes its sign. The function ϕ doesn't change its sign in any other points, because $\phi \in \mathcal{L}$. Therefore, the function ϕ can be written in the following form:

$$\phi = P_n w,$$

where

$$P_n(x) = (x - x_1)...(x - x_n), x \in I$$

and

$$w(x) = \begin{cases} \dfrac{\phi(x)}{P_n(x)}, & x \in \dot{I} \setminus \{x_1, ..., x_n\} \\ \\ \dfrac{\phi'(x)}{P_n'(x)}, & x \in \{x_1, ..., x_n\} \end{cases}$$

The function w is continuous and has the constant sign on $\dot{I}$. The function $P_n \in K_{n-1}(I)$ and therefore

$$\int_0^1 P_n(x)\phi(x)dx > 0$$

or

$$\int_0^1 P_n^2(x)w(x)dx > 0. \tag{7}$$

From (7), it follows that

$$w(x) > 0,$$

for every $x \in \dot{I}$ and therefore the proof is complete.

Corollary 1 *Let w be a positive function defined on $(0,1)$, such that for every $i \in \{0,1,...,n\}$, $\int_0^1 x^i w(x)dx < \infty$ and let p be a monic polynomial of degree n such that*

$$\int_0^1 f(x)p(x)w(x)dx \geq 0,$$

for every $f \in K_{n-1}(\dot{I})$. Then p coincides with the monic orthogonal polynomial of degree n relative to the scalar product

$$< f,g >= \int_0^1 f(x)g(x)w(x)dx.$$

Proof. The proof follows from the fact that the set span$\{e_0, e_1, ..., e_{n-1}, pw\}$, with p a monic polynomial of degree n is an extended Chebysev space on $\dot{I}$.

Remark 1 *For $w = 1$, we obtain the result from Theorem 1.*

References

[1] N. Ciorănescu, *Sur une nouvelle généralisation des polynômes de Legendre*, Acta Math, 61, Upsala 1932, 135–148.

[2] N. Ciorănescu, *La généralisation de la première formule de la moyenne*, L'enseignement Math. Genève, 37, 1938, 298–302.

[3] N. Ciorănescu, *The Mathematical Work (in Romanian)* , Publishing House of the Romanian Academy, Bucharest, 1975.

[4] I. Gavrea, *Operators which occur in approximation theory (doctoral thesis)*, Cluj-Napoca, 1982.

[5] A. Lupaş, *Mean theorems for the Fourier-Jacobi's coefficients*, Revue d'Analyse Numérique et Théorie de l'Approximation, 3(1974), No. 1, 79–84 (in Romanian).

[6] A. Lupaş, A. Vernescu *A characterization of the polynomial of Legendre*, Result. Math. (to appear).

[7] T. Popoviciu *Les Fonctions Convexes*, Actualités Scientifiques et Industrielles, Hermann & Cie, Paris, 1945.

Ioan Gavrea
Technical University of Cluj-Napoca
Department of Mathematics, Faculty of Automation and Computer Science
Str. G. Baritiu nr. 26-28, 400027 Cluj-Napoca, Romania
e-mail: Ioan.Gavrea@math.utcluj.ro