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Voronovskaja's theorem and the exact degree of approximation for the derivatives of complex Riesz-Zygmund means

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Abstract

In this paper we obtain a quantitative Voronovskaja result and the exact orders in approximation by the derivatives of complex Riesz-Zygmund means in compact disks.

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1 Introduction

Of great importance in approximation theory, Fejér's theorem states that if $f : \mathbb{R} \to \mathbb{C}$ is a continuous function with period 2π, then the sequence $(F_n^*(g)(x))_n$ of arithmetic mean of the sequence of partial sums of the Fourier series of g given by

$$F_n^*(g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)K_n(x-t)dt, x \in \mathbb{R},$$

where the kernel $K_n(u)$ is given by $K_n(u) = \frac{1}{n}\left(\frac{\sin(nu/2)}{\sin(u/2)}\right)^2$, converges uniformly to g on $[-\pi, \pi]$ (see Fejér [2]).

Their complex form attached to an analytic function $f(z) = \sum_{k=0}^{\infty} c_k z^k$ in a disk $\mathbb{D}_R = \{z \in \mathbb{C}, |z| < R\}$ and given by

$$F_n(f)(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(ze^{iu})K_n(u)du = \frac{1}{2\pi n}\int_{-\pi}^{\pi} f(ze^{iu})\left(\frac{\sin(nu/2)}{\sin(u/2)}\right)^2 du =$$

$$\sum_{k=0}^{n-1} c_k \frac{n-k}{n} z^k,$$

(for the last formula see e.g. Gal [4, Theorem 3.1]) also have nice approximation properties, satisfying the estimate (see Gaier [3, Theorem 1])

$$\|F_n(f) - f\|_r \le M\omega_1\left(f; \frac{1}{n}\right)_{\overline{\mathbb{D}}_r} \le \frac{M\|f\|_r}{n} := \frac{C_r(f)}{n}, \text{for all } n \in \mathbb{N},$$

where $M > 0$, $0 < r < R$, $\|f\|_r = \sup\{|f(z)|; |z| \le r\}$ and $\omega_1\left(f; \frac{1}{n}\right)_{\overline{\mathbb{D}}_r}$ is the classical modulus of continuity of f in $\overline{\mathbb{D}}_r$.

In fact, from the saturation result in Bruj-Schmieder [1, p. 161-162], it follows that if f is not a constant then the approximation order by $F_n(f)$ is exactly $\frac{1}{n}$.

The Fejér means, both trigonometric and complex cases were generalized by Riesz and Zygmund, their complex form being defined for any $s \in \mathbb{N}$ by

$$R_{n,s}(f)(z) = \sum_{k=0}^{n-1} c_k \left[1 - \left(\frac{k}{n}\right)^s\right] z^k, n \in \mathbb{N}.$$

For $s = 1$ one recapture the Fejér means and for $s = 2$ one get the means introduced by Riesz.

From the saturation result in Bruj-Schmieder [1, p. 161-162] it follows that if f is not a constant then the approximation order by $R_{n,s}(f)$ is exactly $\frac{1}{n^s}$.

In this note we prove that the approximation order by the derivatives of $R_{n,s}(f)$ also is exactly $\frac{1}{n^s}$. Useful in the proof will be the Voronovskaja's formula for $R_{n,s}(f)(z)$ with a quantitative estimate. It is worth noting that our method is different from that in Bruj-Schmieder [1]. Also, in addition we obtain a Voronovskaja result with a quantitative upper estimate and the exact orders in simultaneous approximation by derivatives.

2 Main Results

First we prove an upper estimate in approximation by the derivatives of $R_{n,s}(f)$.

Theorem 1 *Let $R > 1$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$. Also, denote $C_{r_1,s}(f) = \sum_{k=0}^{\infty} |c_k| k^s r_1^k < \infty$.*

If $1 \leq r < r_1 < R$ and $s, p \in \mathbb{N}$ then we have

$$\|R_{n,s}^{(p)}(f) - f^{(p)}\|_r \leq \frac{r_1 p! C_{r_1,s}(f)}{(r_1 - r)^{p+1} n^s}, n \in \mathbb{N}.$$

Proof. An estimate of the form

$$\|R_{n,s}(f) - f\|_r \leq \frac{C_{r,s}(f)}{n^s}, n \in \mathbb{N},$$

essentially follows from Bruj-Schmieder [1]. Below we reprove this estimate in a different and simple way, with an explicit constant $C_{r,s}(f)$. Thus, denoting $e_k(z) = z^k$ and writing $f(z) = \sum_{k=0}^{\infty} c_k e_k(z)$ valid for all $|z| < R$, for all $|z| \le r$ we obtain

$$|R_{n,s}(f)(z) - f(z)| \le$$

$$\sum_{k=0}^{\infty} |c_k| \cdot |R_{n,s}(e_k)(z) - e_k(z)| = \sum_{k=0}^{n-1} |c_k| \cdot |R_{n,s}(e_k)(z) - e_k(z)| +$$

$$\sum_{k=n}^{\infty} |c_k| \cdot |R_{n,s}(e_k)(z) - e_k(z)| \le \frac{1}{n^s} \sum_{k=0}^{n-1} |c_k| k^s r^k + \sum_{k=n}^{\infty} |c_k| r^k.$$

Here we used that $e_k(z) = \sum_{j=0}^{\infty} c_j z^j$, with $c_k = 1$ and $c_j = 0$ for all $j \neq k$.

Taking into account that for $k \ge n$ we have $1 \le \frac{k^s}{n^s}$, from the previous inequality we get

$$|R_{n,s}(f)(z) - f(z)| \le \frac{1}{n^s} \sum_{k=0}^{n-1} |c_k| k^s r^k + \frac{1}{n^s} \sum_{k=n}^{\infty} |c_k| k^s r^k = \frac{1}{n^s} \sum_{k=0}^{\infty} |c_k| k^s r^k,$$

for all $|z| \le r$, therefore we can take $C_{r,s}(f) = \sum_{k=0}^{\infty} |c_k| k^s r^k < \infty$.

Now, denoting by γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \le r$ and $v \in \gamma$, we have $|v - z| \ge r_1 - r$, by the Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$|R_{n,s}^{(p)}(f)(z) - f^{(p)}(z)| = \frac{p!}{2\pi} \left| \int_{\gamma} \frac{R_{n,s}(f)(v) - f(v)}{(v - z)^{p+1}} dv \right| \le$$

$$\frac{C_{r_1,s}(f)}{n^s} \cdot \frac{p!}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{p+1}},$$

which proves the theorem.

Useful in the proof of the exact degree of approximation will be the Voronovskaja's formula for $R_{n,s}(f)(z)$. For this purpose, first we need the following simple lemma.

Lemma 1 *Let $k, s \in \mathbb{N}$. If we denote $k^s = \sum_{j=1}^{s} \alpha_{j,s} k(k-1)...(k-(j-1))$, then the coefficients $\alpha_{j,s}$ can be chosen independent of k and to satisfy $\alpha_{1,s} = \alpha_{s,s} = 1$ for all $s \geq 1$ and $\alpha_{j,s+1} = \alpha_{j-1,s} + j\alpha_{j,s}$, $j = 2, ..., s$, $s \geq 2$.*

Proof. We have

$$k^{s+1} = \sum_{j=1}^{s+1} \alpha_{j,s+1} k(k-1)...(k-(j-1)) = k \sum_{j=1}^{s} \alpha_{j,s} k(k-1)...(k-(j-1)) =$$

$$\sum_{j=1}^{s} \alpha_{j,s}(k-j+j)k(k-1)...(k-(j-1)) = \sum_{j=1}^{s} \alpha_{j,s} k(k-1)...(k-j) +$$

$$\sum_{j=1}^{s} j\alpha_{j,s} k(k-1)...(k-(j-1)) = \sum_{j=2}^{s+1} \alpha_{j-1,s} k(k-1)...(k-(j-1)) +$$

$$\sum_{j=1}^{s} j\alpha_{j,s} k(k-1)...(k-(j-1)) = \alpha_{s,s} + \sum_{j=2}^{s} \alpha_{j-1,s} k(k-1)...(k-(j-1)) +$$

$$\sum_{j=2}^{s} j\alpha_{j,s} k(k-1)...(k-(j-1)) + \alpha_{1,s},$$

which implies

$$\sum_{j=1}^{s+1} \alpha_{j,s+1} k(k-1)...(k-(j-1)) = \alpha_{1,s+1} + \sum_{j=2}^{s} \alpha_{j,s+1} k(k-1)...(k-(j-1)) +$$

$$\alpha_{s+1,s+1} = \alpha_{s,s} + \sum_{j=2}^{s} \alpha_{j-1,s} k(k-1)...(k-(j-1)) +$$

$$\sum_{j=2}^{s} j\alpha_{j,s} k(k-1)...(k-(j-1)) + \alpha_{1,s},$$

and proves the lemma.

Now, the Voronovskaja's theorem for $R_{n,s}(f)(z)$ one states as follows.

Theorem 2 *Let $R > 1$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. For any $r \in [1, R)$ we have*

$$\left\| R_{n,s}(f) - f + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} e_j f^{(j)} \right\|_r \leq \frac{A_{r,s}(f)}{n^{s+1}}, n \in \mathbb{N}$$

where $A_{r,s}(f) = \sum_{k=1}^{\infty} |c_k| k^{s+1} r^k < \infty$, $e_j(z) = z^j$ and $\alpha_{j,s}$ are defined by Lemma 1.

Proof. Denoting

$$E_{k,n,s}(z) = R_{n,s}(e_k)(z) - e_k(z) + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} k(k-1)...(k-(j-1)) z^k,$$

we obtain

$$\left| R_{n,s}(f)(z) - f(z) + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} z^j f^j(z) \right| \leq \sum_{k=0}^{\infty} |c_k| \cdot |E_{k,n,s}(z)| =$$

$$\sum_{k=1}^{n-1} |c_k| \cdot |E_{k,n,s}(z)| + \sum_{k=n}^{\infty} |c_k| \cdot |E_{k,n,s}(z)| =$$

$$\sum_{k=1}^{n-1} |c_k| \cdot \left| \left(1 - \left(\frac{k}{n} \right)^s \right) - 1 + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} k(k-1)...(k-(j-1)) \right| \cdot |z|^k +$$

$$\sum_{k=n}^{\infty} |c_k| \cdot \left| -1 + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} k(k-1)...(k-(j-1)) \right| \cdot |z|^k =$$

$$0 + \sum_{k=n}^{\infty} |c_k| \cdot \left| -1 + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} k(k-1)...(k-(j-1)) \right| \cdot |z|^k,$$

where for the first sum we used Lemma 1.

Taking into account that by Lemma 1 we have $\sum_{j=1}^{s} \alpha_{j,s} n(n-1)...(n-(j-1)) = n^s$, for all $|z| \leq r$ it follows

$$\left| R_{n,s}(f)(z) - f(z) + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} z^j f^j(z) \right| \leq$$

$$\left(\sum_{k=n}^{\infty} |c_k| r^k \right) \cdot \left(\sum_{j=1}^{s} \alpha_{j,s} \frac{k(k-1)...(k-(j-1))}{n^s} \right) =$$

$$\frac{1}{n^s} \sum_{k=n}^{\infty} |c_k| k^s r^k \leq \frac{1}{n^{s+1}} \sum_{k=n}^{\infty} |c_k| k^{s+1} r^k \leq \frac{1}{n^{s+1}} \sum_{k=1}^{\infty} |c_k| k^{s+1} r^k,$$

which proves the theorem.

By using Theorems 1 and 2 we are in position to obtain the exact degree of approximation by the derivatives of $R_{n,s}(f)(z)$.

Theorem 3 *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Also, let $1 \leq r < r_1 < R$ and $p, s \in \mathbb{N}$ be fixed. If f is not a polynomial of degree $\leq p-1$ then we have*

$$\|R_{n,s}^{(p)}(f) - f^{(p)}\|_r \sim \frac{1}{n^s},$$

where the constants in the equivalence depend on f, r, r_1, s and p.

Proof. Denoting by Γ the circle of radius $r_1 >$ and center 0 (where $r_1 > r \geq 1$), by the Cauchy's formulas it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$ we have

$$R_{n,s}^{(p)}(f)(z) - f^{(p)}(z) = \frac{p!}{2\pi i} \int_{\Gamma} \frac{R_{n,s}(f)(v) - f(v)}{(v-z)^{p+1}} dv,$$

where the inequality $|v-z| \geq r_1 - r$ is valid for all $|z| \leq r$ and $v \in \Gamma$.

Taking into account Theorem 1, it remains to prove the lower estimate for $\|R^{(p)}_{n,s}(f) - f^{(p)}\|_r$. For this purpose, for all $v \in \Gamma$ and $n \in \mathbb{N}$ we have

$$R_{n,s}(f)(v) - f(v) = \frac{1}{n^s}\left\{-\sum_{j=1}^{s}\alpha_{j,s}v^j f^{(j)}(v) + \right.$$

$$\left.\frac{1}{n}\left[n^{s+1}\left(R_{n,s}(f)(v) - f(v) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}v^j f^{(j)}(v)\right)\right]\right\},$$

which replaced in the above Cauchy's formula implies

$$R^{(p)}_{n,s}(f)(z) - f^{(p)}(z) = \frac{1}{n^s}\left\{\frac{p!}{2\pi i}\int_{\Gamma}\frac{-\sum_{j=1}^{s}\alpha_{j,s}v^j f^{(j)}(v)}{(v-z)^{p+1}}dv + \right.$$

$$\left.\frac{1}{n}\cdot\frac{p!}{2\pi i}\int_{\Gamma}\frac{n^{s+1}\left(R_{n,s}(f)(v) - f(v) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}v^j f^{(j)}(v)\right)}{(v-z)^{p+1}}dv\right\} =$$

$$\frac{1}{n^s}\left\{\left[-\sum_{j=1}^{s}\alpha_{j,s}v^j f^{(j)}(v)\right]^{(p)} + \right.$$

$$\left.\frac{1}{n}\cdot\frac{p!}{2\pi i}\int_{\Gamma}\frac{n^{s+1}\left(R_{n,s}(f)(v) - f(v) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}v^j f^{(j)}(v)\right)}{(v-z)^{p+1}}dv\right\}$$

Passing now to $\|\cdot\|_r$ it follows

$$\|R^{(p)}_{n,s}(f) - f^{(p)}\|_r \geq \frac{1}{n^s}\left\{\left\|\left[\sum_{j=1}^{s}\alpha_{j,s}e_j f^{(j)}\right]^{(p)}\right\|_r - \right.$$

$$\frac{1}{n}\left\|\frac{p!}{2\pi}\int_{\Gamma}\frac{n^{s+1}\left(R_{n,s}(f)(v)-f(v)+\frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}v^j f^{(j)}(v)\right)}{(v-z)^{p+1}}dv\right\|_r\Bigg\},$$

where by using Theorem 2 we obtain

$$\left\|\frac{p!}{2\pi}\int_{\Gamma}\frac{n^{s+1}\left(R_{n,s}(f)(v)-f(v)+\frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}v^j f^{(j)}(v)\right)}{(v-z)^{p+1}}dv\right\|_r \leq$$

$$\frac{p!}{2\pi}\cdot\frac{2\pi r_1 n^{s+1}}{(r_1-r)^{p+1}}\left\|R_{n,s}(f)-f+\frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}e_j f^{(j)}\right\|_{r_1} \leq \frac{A_{r_1,s}(f)p!r_1}{(r_1-r)^{p+1}}.$$

But by hypothesis on f we have $\left\|\left[\sum_{j=1}^{s}\alpha_{j,s}e_j f^{(j)}\right]^{(p)}\right\|_r > 0.$

Indeed, supposing the contrary it follows that

$$\sum_{j=1}^{s}\alpha_{j,s}z^j f^{(j)}(z) = Q_{p-1}(z),$$

for all $z \in \overline{\mathbb{D}}_r$, where $Q_{p-1}(z) = \sum_{j=1}^{p-1} A_j z^j$ necessarily is a polynomial of degree $\leq p-1$.

Seeking now the analytic solution in the form $f(z) = \sum_{k=0}^{\infty}\beta_k z^k$, replacing in the differential equation and taking into account again Lemma 1, by identification of the coefficients β_k we easily obtain $\beta_k = 0$, for all $k \geq p$, that is $f(z)$ necessarily is a polynomial of degree $\leq p-1$, in contradiction with the hypothesis.

Therefore, there exists an index n_0 depending only on f, s, p, r and r_1, such that for all $n \geq n_0$ we have

$$\left\| \left[\sum_{j=1}^{s} \alpha_{j,s} e_j f^{(j)} \right]^{(p)} \right\|_r -$$

$$\frac{1}{n} \left\| \frac{p!}{2\pi} \int_\Gamma \frac{n^{s+1} \left(R_{n,s}(f)(v) - f(v) + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} v^j f^{(j)}(v) \right)}{(v-z)^{p+1}} dv \right\|_r \geq$$

$$\frac{1}{2} \left\| \left[\sum_{j=1}^{s} \alpha_{j,s} e_j f^{(j)} \right]^{(p)} \right\|_r ,$$

which immediately implies

$$\|R_{n,s}^{(p)}(f) - f^{(p)}\|_r \geq \frac{1}{n^s} \cdot \frac{1}{2} \left\| \left[\sum_{j=1}^{s} \alpha_{j,s} e_j f^{(j)} \right]^{(p)} \right\|_r ,$$

for all $n \geq n_0$.

For $n \in \{1, ..., n_0 - 1\}$ we obviously have $\|R_{n,s}^{(p)}(f) - f^{(p)}\|_r \geq \frac{M_{r,s,p,n}(f)}{n}$ with $M_{r,s,,p,n}(f) = n \cdot \|R_{n,s}^{(p)}(f) - f^{(p)}\|_r > 0$, which finally implies $\|R_{n,s}^{(p)}(f) - f^{(p)}\|_r \geq \frac{C_{r,s,p}(f)}{n}$ for all n, where

$$C_{r,s,p}(f) = \min \left\{ M_{r,s,p,1}(f), ..., M_{r,s,p,n_0-1}(f), \frac{1}{2} \left\| \left[\sum_{j=1}^{s} \alpha_{j,s} e_j f^{(j)} \right]^{(p)} \right\|_r \right\}.$$

This completes the proof.

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