General Mathematics Vol. 15, No. 1 (2007), 67–74

The Hlawka–Djoković Inequality and Points in Unitary Spaces [1]

Walther Janous

Dedicated to Professor Ph.D. Alexandru Lupaş on the occasion of his 65^{th} birthday

Abstract

In this note several inequalities are proved. They are all in connection with the Hlawka–Djoković inequality. At the end a problem for further research is posed.

2000 Mathematics Subject Classification: 26D10.

1 Introduction

Starting points of this note was the following problem for planar triangles recently posed by O. Furdui in [5].

Let G be the centroid of ΔABC, and let A_1, B_1, C_1 be the mid-points of BC, CA, AB, respectively. If P is an arbitrary point in the plane of ΔABC, show that

$$PA + PB + PC + 3PG \geq 2(PA_1 + PB_1 + PC_1)\ .$$

[1] *Received 15 January, 2007*

Accepted for publication (in revised form) 28 February, 2007

(It should be noted that this result dates back at least to the paper [3] by M. Chiriţă and R. Constantinescu. See also the referential source [6], p. 410.)

The proof of this inequality is as follows.

Let a, b and c denote the three vectors $\overrightarrow{PA}$, $\overrightarrow{PB}$ and $\overrightarrow{PC}$, respectively.

Then the original Hlawka inequality says

$$|a| + |b| + |c| + |a + b + c| \geq |a + b| + |b + c| + |c + a|$$

(See for instance [7], p. 521.)

Because of $|a + b + c| = 3PG$ and $|a + b| = 2PA_1$ etc, the result is immediate.

In this note we extend this result to a finite number of points in arbitrary unitary spaces $\mathbf{X}$.

We then go on to exemplify the general inequality for some special spaces.

At the end we pose a problem concerning the asymptotic behavior of a difference playing a role in an inequality obtained in this note.

2 General Inequality

We are now in the position to state and prove the following

Theorem. *Let $A_1, A_2, \ldots, A_n$ be n fixed points ($n \geq 3$) in an arbitrary unitary space $\mathbf{X}$. Fix the entire number k such that $2 \leq k \leq n - 1$.*

Let furthermore $G_{i_1 \ldots i_k}$ be the "centroid" of the k points $A_{i_1}, \ldots A_{i_k}$ where $1 \leq i_1 < \ldots < i_k \leq n$, that is $G_{i_1 \ldots i_k} = \frac{1}{k}\left(A_{i_1} + \ldots + A_{i_k}\right)$.

Then for any point $P \in \mathbf{X}$ the following inequality is valid:

$$k \sum_{1 \leq i_1 < \ldots < i_k \leq n} PG_{i_1 \ldots i_k} \leq \left(\begin{array}{c} n-2 \\ k-2 \end{array} \right) \left(\frac{n-k}{k-1} \sum_{j=1}^{n} PA_j + nPG \right) .$$

(Here G denotes the centroid of all n points $A_1, A_2, \ldots, A_n$. Furthermore, AB is the distance between A and B, that is $AB = \|A - B\|$.)

In the spirit of the before-given proof this inequality is an immediate consequence of the Djoković–generalization of the original Hlawka–inequality, that is

$$\sum_{1\le i_1<\ldots<i_k\le n}\|x_{i_1}+\ldots+x_{i_k}\| \le \binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{j=1}^{n}\|x_j\| + \left\|\sum_{j=1}^{n}x_j\right\|\right)$$

where $x_1,\ldots,x_n \in \mathbf{X}$. (See [4] and also the referential source [7], p. 522–523.)

Remark. *The special case $k = 2$ of this inequality was originally proved by D. Adamović in [1]. It was reconsidered by M. Bencze in [2] where he also gave several applications of it.*

3 Some Applications

(1). For $\mathbf{X} = \mathbb{C}$ and $P = re^{i\varphi}$, $r \ge 0$, and $A_j = r_j e^{i\varphi_j}$, $r_j \ge 0$, $j = 1,\ldots,n$, there holds

$$\sum_{1\le i_1<\ldots<i_k\le n}\sqrt{k^2r^2+\sum_{j=1}^{n} r_{i_j}r_{i_m}\cos(\varphi_{i_j}-\varphi_{i_m})-2kr\sum_{j=1}^{n} r_{i_j}\cos(\varphi-\varphi_{i_j})} \le$$

$$\le \binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{j=1}^{n}\sqrt{r^2+r_j^2-2rr_j\cos(\varphi-\varphi_j)}+\right.$$

$$\left.+\sqrt{n^2r^2+\sum_{j=1}^{n}r_j^2+2+2\sum_{i\le j<m\le n} r_jr_m\cos(\varphi_j-\varphi_m)-2nr\sum_{j=1}^{n}r_j\cos(\varphi-\varphi_j)}\right)$$

- If all involved points are on the unit–circle this inequality reduces to

$$\sum_{1\le i_1<\ldots<i_k\le n}\sqrt{k(k+1)+2\sum_{1\le j<m\le n}\cos(\varphi_{i_j}-\varphi_{i_m})-2k\sum_{j=1}^{k}\cos(\varphi-\varphi_{i_j})} \le$$

$$\leq \binom{n-2}{k-2}\left(2\frac{n-k}{k-1}\sum_{j=1}^{k}\left|\sin\left(\frac{\varphi-\varphi_j}{2}\right)\right| + \right.$$

$$\left. + \sqrt{n(n+1)+2\sum_{i\leq j<m\leq n}\cos(\varphi_j-\varphi_m)-2n\sum_{j=1}^{n}\cos(\varphi-\varphi_j)}\right) \ .$$

- For $P=0$ and all the $A_j{}'$s on the unit–circle we get the inequality

$$\sum_{1\leq i_1<\ldots<i_k\leq n}\sqrt{k+2\sum_{i\leq j<m\leq n}\cos(\varphi_{i_j}-\varphi_{i_m})} \leq$$

$$\leq \binom{n-2}{k-2}\left(\frac{n-k}{k-1}n+\sqrt{n+2\sum_{1\leq j<m\leq n}\cos(\varphi_j-\varphi_m)}\right)$$

(2). Let $I=[a,b]$ be an interval on the real axis and consider the vector space $\mathbf{X}=\mathbb{C}[a,b]$ of all continuous functions on I with $\|f\|=\sqrt{\int_a^b (f(x))^2dx}$.

For $P=f\in\mathbb{C}[a,b]$ and $A_j=f_j\in\mathbb{C}[a,b]$, $j=1,\ldots,n$, there holds the "master–inequality"

$$\sum_{1\leq i_1<\ldots<i_k\leq n}\sqrt{\int_a^b\left(\sum_{j=1}^{k}f_{i_j}(x)-kf(x)\right)^2dx} \leq$$

$$\leq \binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{j=1}^{n}\sqrt{\int_a^b (f_j(x)-f(x))^2dx} + \right.$$

$$\left. + \sqrt{\int_a^b\left(\sum_{j=1}^{n}f_j(x)-nf(x)\right)^2dx}\right) \ .$$

This general integral inequality allows specifications in very many directions.

Of specific interest are, of course, orthogonal polynomials.

We investigate now the cases of Legendre, Laguerre and Hermite polynomials.

- $I = [-1,1]$ and Legendre polynomials P_n.

  We let $f_j = P_{j-1}$, $j = 1, \ldots, n$. Due to $\displaystyle\int_{-1}^{1} P_r(x)P_s(x)dx = \frac{2}{2s+1}\delta_{rs}$, $r, s \geq 0$, there holds

$$\sum_{1\le i_1<\ldots<i_k\le n}\sqrt{\sum_{j=1}^{k}\frac{2}{2i_j-1}+k^2\int_a^b(f(x))^2dx-2k\int_a^b\left(\sum_{j=1}^{k}P_{i_j}(x)\right)f(x)dx} \leq$$

$$\leq\binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{j=1}^{n}\sqrt{\frac{2}{2j-1}+\int_a^b(f(x))^2dx-2\int_a^b P_j(x)f(x)dx}+\right.$$

$$\left.+\sqrt{\sum_{j=1}^{n}\frac{2}{2j-1}+n^2\int_a^b(f(x))^2\,dx-2n\int_a^b\left(\sum_{j=1}^{n}P_j(x)\right)f(x)dx}\right).$$

  If we put in this inequality $f = P_N$, where $N \geq n$, then

$$\sum_{1\le i_1<\ldots<i_k\le n}\sqrt{\frac{k^2}{2N+1}+\sum_{j=1}^{k}\frac{1}{2i_j-1}} \leq$$

$$\leq\binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{j=1}^{n}\sqrt{\frac{1}{2j-1}+\frac{1}{2N+1}}+\sqrt{\sum_{j=1}^{n}\frac{1}{2j-1}+\frac{n^2}{2N+1}}\right).$$

  Letting $N \to \infty$ yields

$$\sum_{1\le i_1<\ldots<i_k\le n}\sqrt{\sum_{j=1}^{k}\frac{1}{2i_j-1}} \leq$$

$$\leq\binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{j=1}^{n}\sqrt{\frac{1}{2j-1}}+\sqrt{\sum_{j=1}^{n}\frac{1}{2j-1}}\right).$$

  If we set here $k = n-1$ then it follows

$$(*)\qquad \sum_{j=1}^{n}\sqrt{h_n-\frac{1}{2j-1}} \leq \sum_{j=1}^{n}\sqrt{\frac{1}{2j-1}}+\sqrt{\sum_{j=1}^{n}\frac{1}{2j-1}}+(n-2)\sqrt{h_n}$$

where $h_n = \sum_{k=1}^{n} \frac{1}{2j-1}$.

- $I = [0, \infty)$ and Laguerre functions $\Lambda_n(x) = e^{\frac{x}{2}} L_n(x)$, where L_n are the Laguerre polynomials.

  Letting $f_j = \Lambda_{j-1}$, $j = 1, \dots, n$, and noting $\sum_{0}^{\infty} \Lambda_r(x)dx = (s!)^2\delta_{rs}$, $r, s \geq 0$, we get for $f = \Lambda_N$, where $N \geq n$:

$$\sum_{1 \leq i_1 < \ldots < i_k \leq n} \sqrt{k^2(N!)^2 + \sum_{j=1}^{k} (i_j!)^2} \leq$$

$$\leq \binom{n-2}{k-2} \left(\frac{n-k}{k-1} \sum_{j=0}^{n-1} \sqrt{(N!)^2 + (j!)^2} + \sqrt{n^2(N!)^2 + \sum_{j=0}^{n-1} (j!)^2} \right) .$$

  This inequality becomes for $k = n-1$

$$\sum_{j=0}^{n-1} \sqrt{(n-1)^2 + s_n^{(N)} - \left(\frac{j!}{N!}\right)^2} \leq \sum_{j=0}^{n-1} \sqrt{(1 + \left(\frac{j!}{N!}\right)^2} + (n-2)\sqrt{n^2 + s_n^{(N)}}$$

  where $s_n^{(N)} = \sum_{j=0}^{n-1} \left(\frac{j!}{N!}\right)^2$.

  Dividing both of these inequalities by $N!$ and letting $N \longrightarrow \infty$ yields the identities $\binom{n}{k} = \binom{n}{k}$ and $(n-1)n = (n-1)n$, respectively. This means that these inequalities are sharp (at least "at infinity").

- $I = (-\infty, \infty)$ and Hermite functions $\psi_n(x) = e^{-\frac{x^2}{2}} H_n(x)$, where H_n are the Hermite polynomials.

  Proceeding as before we get due to $\int_{-\infty}^{\infty} \psi_r(x)\psi_s(x)dx = 2^s s!\delta_{rs}$, $r, s \geq 0$:

$$\sum_{1 \leq i_1 < \ldots < i_k \leq n} \sqrt{k^2 + \sum_{j=1}^{k} \frac{2^{i_j} i_j!}{2^N N!}} \leq$$

$$\leq \binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{j=0}^{n-1}\sqrt{\frac{2^{i_j} i_j!}{2^N N!}}+\sqrt{n^2+\sum_{j=0}^{n-1}\frac{2^j_j!}{2^N N!}}\right) .$$

From it similar conclusions as before can be drawn.

As a final example we let $I=[0,\pi]$ and $f_j(x)=\sin(jx)$, $j=1,\dots,n$. Then $\int_0^\pi \sin(rx)\sin(sx)dx=\frac{\pi}{2}\delta_{rs}$, $r,s\geq 1$.

This, the master–inequality and

$$\int_0^\pi \sin(rx)\cos(sx)dx=\begin{cases} 0, & r=s \\ \dfrac{r((-1)^{r+s}-1)}{s^2-r^2}, & r\neq s \end{cases}$$

would enable us to deduce various inequalities for Fourier coefficients of a functions $f(x)$ having $f(x)=\frac{a_0}{1}+\sum_{m=1}^{\infty}(a_m\cos(mx)+b_m\sin(mx))$ as its Fourier series. We leave this an exercise for the reader.

4 An Observation

In the light of the following recently published result many of the before proven results have natural extensions for the case $k=n-2$.

Indeed, the following inequality is valid ([8], p. 64).

Let $x_1,\dots,x_n$ be elements from a Hilbert space ($n\geq 3$) and $\mu_1\geq 1,\dots,\mu_n\geq 1$. Then

$$\left(\sum_{i=1}^n \mu_i-2\right)\left\|\sum_{i=1}^n \mu_i x_i\right\|+\sum_{i=1}^n \mu_i\|x_i\|\geq \sum_{i=1}^n \mu_i\left\|x_i-\sum_{j=1}^n \mu_j x_j\right\| .$$

5 An Open Problem

At the end of this note we raise the following question in connection with inequality (*). Unlike the inequalities obtained from the Hermite and Lagrange polynomials is the one stemming from Legendre polynomials, that is inequality (*), apparently far from being asymptotically sharp.

Therefore it is natural to find out the correct asymptotic behavior of the following difference we denote in honor of Alexandru Lupaş by $AL(n) = \sum_{j=1}^{n}\sqrt{1-a_j(n)} - \sum_{j=1}^{n}\sqrt{a_j(n)}$ where $a_j(n) = \dfrac{1}{h_n(2j-1)}$, $j = 1, \dots, n$.

This is, we are left to find $f(n)$, such that $AL(n) = n - f(n) + o(f(n))$, as $n \to \infty$.

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Ursulinengymnasium
Fürstenweg 86
A-6020 Innsbruck
Austria
E-mail address: *walter.janous@tirol.com*