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On analytic functions with positive real part [1]

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Abstract

We find conditions on the complex-valued functions $A, B, C : U \to \mathbb{C}$ defined in the unit disc U such that the differential inequality

$$\text{Re } [A(z)p^{4k}(z) + B(z)p^{4k-1}(z) + C(z)p^{4k-2}(z) + \alpha(zp'(z) - 1)^3 -$$

$$-3\beta(zp'(z))^2 + 3\gamma zp'(z) + \delta] > 0$$

implies Re $p(z) > 0$, where $p \in \mathcal{H}[1, n]$, $\alpha, \beta, \gamma \in \mathbb{C}$ and n, k are two positive integers.

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1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots, \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i [1, p. 35].

Lemma A. [1, p. 35] *Let $\psi : \mathbb{C}\times U \rightarrow \mathbb{C}$ be a function which satisfies*

$$Re \ \psi(\rho i, \sigma; z) \leq 0,$$

where $\rho, \sigma \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1+\rho^2)$, $z \in U$ and $n \geq 1$.

If $p \in \mathcal{H}[1, n]$ and

$$Re \ \psi(p(z), zp'(z); z) > 0,$$

then

$$Re \ p(z) > 0.$$

2 Main results

Theorem. *Let* $\alpha \in \mathbb{C}(\mathrm{Re}\alpha \geq 0)$, $\beta, \gamma \in \mathbb{C}$,$(\alpha+\beta),(\alpha+\gamma) \in \mathbb{R}^+$, $\delta \leq \frac{n^3}{4}\mathrm{Re}\alpha + \frac{3}{4}n^2(\alpha+\beta) + \frac{3}{2}n(\alpha+\gamma)$ *and* n, k *be two positive integer. Suppose that the functions* $A, B, C : U \to \mathbb{C}$ *satisfy:*

$$i)\ -\left(\frac{n^3}{8}+1\right)\mathrm{Re}\alpha - \frac{3}{4}n^2(\alpha+\beta) - \frac{3}{2}n(\alpha+\gamma) < \mathrm{Re}A(z) \leq 0$$

(2.1) $$ii)\ \mathrm{Re}C(z) \geq 0$$

$$iii)\ \mathrm{Im}\ ^2B(z) \leq -4\mathrm{Re}A(z)\mathrm{Re}C(z).$$

If $p \in \mathcal{H}[1, n]$ *and*

$$\mathrm{Re}\ [A(z)p^{4k}(z) + B(z)p^{4k-1}(z) + C(z)p^{4k-2}(z) + \alpha(zp'(z)-1)^3-$$

(2.2) $$-3\beta(zp'(z))^2 + 3\gamma zp'(z) + \delta] > 0$$

then

$$\mathrm{Re}\ p(z) > 0.$$

Proof. We let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

$$\psi(p(z), zp'(z); z) = A(z)p^{4k}(z)+B(z)p^{4k-1}(z)+C(z)p^{4k-2}(z)+\alpha(zp'(z)-1)^3-$$

$$-3\beta(zp'(z))^2 + 3\gamma zp'(z) + \delta.$$

From (2.2) we have

$$\text{Re } \psi(p(z), zp'(z); z) > 0 \text{ for } z \in U.$$

For $\sigma, \rho \in \mathbb{R}$ satisfying $\sigma \leq -\frac{n}{2}(1+\rho^2)$, hence $-\sigma^2 \leq -\frac{n^2}{4}(1+\rho^2)^2$, $\sigma^3 \leq -\frac{n^3}{8}(1+\rho^2)^3$ and $z \in U$, by using (2.1) we obtain:

$$\text{Re } \psi(\rho i, \sigma; z) =$$

$$= \text{Re } [A(z)(\rho i)^{4k} + B(z)(\rho i)^{4k-1} + C(z)(\rho i)^{4k-2} + \alpha(\sigma-1)^3 - 3\beta\sigma^2 + 3\gamma\sigma + \delta] =$$

$$= \rho^{4k}\text{Re}A(z) + \rho^{4k-1}\text{Im}B(z) - \rho^{4k-2}\text{Re}C(z) + (\sigma^3 - 1)\text{Re}\alpha -$$

$$-3(\alpha+\beta)\sigma^2 + 3(\alpha+\gamma)\sigma + \delta \leq$$

$$\leq \rho^{4k}\text{Re}A(z) + \rho^{4k-1}\text{Im}B(z) - \rho^{4k-2}\text{Re}C(z) - \frac{n^3}{8}(1+\rho^2)^3\text{Re}\alpha - \text{Re}\alpha -$$

$$-\frac{3}{4}n^2(\alpha+\beta)((1+\rho^2)^2) - \frac{3}{2}n(\alpha+\gamma)(1+\rho^2) + \delta =$$

$$= \rho^{4k-2}[\rho^2\text{Re}A(z) + \rho\text{Im}B(z) - \text{Re}C(z)] - \frac{n^3}{8}\rho^6\text{Re}\alpha -$$

$$-\left(\frac{3n^3}{8}\text{Re}\alpha + \frac{3}{4}n^2(\alpha+\beta)\right)\rho^4 -$$

$$-\left(\frac{3n^3}{8}\text{Re}\alpha + \frac{3}{2}n^2(\alpha+\beta) + \frac{3}{2}n(\alpha+\gamma) + \text{Re}A(z)\right)\rho^2 -$$

$$-\left(\frac{n^3}{8}+1\right)\text{Re}\alpha - \frac{3}{4}n^2(\alpha+\beta) - \frac{3}{2}n(\alpha+\gamma) + \delta \leq 0.$$

By using Lemma A we have that Re $p(z) > 0$.

If $\delta = \left(\frac{n^3}{8}+1\right)\text{Re}\alpha + \frac{3}{4}n^2(\alpha+\beta) + \frac{3}{2}n(\alpha+\gamma)$, then the Theorem can be rewritten as follows:

Corollary 1. *Let* $\alpha \in \mathbb{C}(\mathrm{Re}\alpha \geq 0)$, $\beta, \gamma \in \mathbb{C}$,$(\alpha+\beta), (\alpha+\gamma) \in \mathbb{R}^{+}$, *and* n, k *be two positive integer. Suppose that the functions* $A, B, C : U \to \mathbb{C}$ *satisfy:*

$$i) - \frac{3n^3}{8}\mathrm{Re}\alpha - \frac{3}{4}n^2(\alpha+\beta) - \frac{3}{2}n(\alpha+\gamma) < \mathrm{Re}A(z) \leq 0$$

$$ii)\ \mathrm{Re}C(z) \geq 0 \tag{2.3}$$

$$iii)\ \mathrm{Im}\ ^2B(z) \leq -4\mathrm{Re}A(z)\mathrm{Re}C(z).$$

If $p \in \mathcal{H}[1, n]$ *and*

$$\mathrm{Re}\ [A(z)p^{4k}(z) + B(z)p^{4k-1}(z) + C(z)p^{4k-2}(z) + \alpha(zp'(z) - 1)^3 -$$

$$-3\beta(zp'(z))^2 + 3\gamma zp'(z) +$$

$$+ \left(\frac{n^3}{8} + 1\right) Re\alpha + \frac{3}{4}n^2(\alpha+\beta) + \frac{3}{2}n(\alpha+\gamma)] > 0 \tag{2.4}$$

then

$$\mathrm{Re}\ p(z) > 0.$$

Taking $\beta = \gamma = \overline{\alpha}$ in the above Theorem, we have

Corollary 2. *Let* $\alpha \in \mathbb{C}(\mathrm{Re}\alpha \geq 0)$, $\delta {\leq} (\frac{n^3}{8} + \frac{3}{2}n^2 + 3n + 1)\mathrm{Re}\alpha$ *and* n, k *be two positive integer. Suppose that the functions* $A, B, C : U \to \mathbb{C}$ *satisfy:*

$$i)\ (-\frac{3n^3}{8} - \frac{3}{2}n^2 - 3n)\mathrm{Re}\alpha < \mathrm{Re}A(z) \leq 0$$

$$ii)\ \mathrm{Re}C(z) \geq 0 \tag{2.5}$$

$$iii)\ \mathrm{Im}\ ^2B(z) \leq -4\mathrm{Re}A(z)\mathrm{Re}C(z).$$

If $p \in \mathcal{H}[1,n]$ *and*

$$(2.6)\qquad Re\ [A(z)p^{4k}(z) + B(z)p^{4k-1}(z) + C(z)p^{4k-2}(z) + \alpha(zp'(z)-1)^3-$$

$$-3\overline{\alpha}(zp'(z))^2 + 3\overline{\alpha}zp'(z) + \delta] > 0$$

then

$$\mathrm{Re}\ p(z) > 0.$$

Taking $\alpha + \beta = \alpha + \gamma = 1$ in the above Theorem, we have

Corollary 3. *Let* $\alpha \in \mathbb{C}(\mathrm{Re}\alpha \geq 0)$, $\delta \leq \left(\frac{n^3}{8}+1\right)\mathrm{Re}\alpha + \frac{3}{4}n^2 + \frac{3}{2}n$ *and* n, k *be two positive integer. Suppose that the functions* $A, B, C : U \rightarrow \mathbb{C}$ *satisfy:*

$$i) -\frac{3n^3}{8}\mathrm{Re}\alpha - \frac{3}{4}n^2 - \frac{3}{2}n < \mathrm{Re}A(z) \leq 0$$

$$(2.7)\qquad ii)\ \mathrm{Re}C(z) \geq 0$$

$$iii)\ \mathrm{Im}\ ^2B(z) \leq -4\mathrm{Re}A(z)\mathrm{Re}C(z).$$

If $p \in \mathcal{H}\ [1,n]$ *and*

$$\mathrm{Re}\ [A(z)p^{4k}(z) + B(z)p^{4k-1}(z) + C(z)p^{4k-2}(z) + \alpha(zp'(z)-1)^3-$$

$$(2.8)\qquad -3(1-\alpha)(zp'(z))^2 + 3(1-\alpha)zp'(z) + \delta] > 0$$

then

$$\text{Re } p(z) > 0.$$

Taking $\alpha = 0$ in the above Theorem, we have

Corollary 4. *Let $\beta, \gamma > 0$, $\delta \leq \frac{3}{4}n^2\beta + \frac{3}{2}n\gamma$ and n, k be two positive integer. Suppose that the functions $A, B, C : U \rightarrow \mathbb{C}$ satisfy:*

$$i) \ -\frac{3}{4}n^2\beta - \frac{3}{2}n\gamma < \text{Re}A(z) \leq 0$$

$$(2.9) \qquad ii) \ \text{Re}C(z) \geq 0$$

$$iii) \ \text{Im }^2B(z) \leq -4\text{Re}A(z)\text{Re}C(z).$$

If $p \in \mathcal{H}[1, n]$ and

(2.10)

$$\text{Re }[A(z)p^{4k}(z)+B(z)p^{4k-1}(z)+C(z)p^{4k-2}(z)-3\beta(zp'(z))^2+3\gamma zp'(z)+\delta] > 0,$$

then

$$\text{Re } p(z) > 0.$$

Taking $\beta = \gamma = 0$ in the above Theorem, we have

Corollary 5. *Let $\alpha > 0$, $\delta \leq \left(\frac{n^3}{8} + 1\right)\alpha + \frac{3}{4}n^2\alpha + \frac{3}{2}n$ and n, k be two positive integer. Suppose that the functions $A, B, C : U \rightarrow \mathbb{C}$ satisfy:*

$$i) - \left(\frac{n^3}{8} + 1\right)\text{Re}\alpha - \frac{3}{4}n^2\alpha - \frac{3}{2}n\alpha < \text{Re}A(z) \leq 0$$

$$(2.11) \qquad ii) \ \text{Re}C(z) \geq 0$$

$$iii)\ \text{Im}\ {}^2B(z) \leq -4\text{Re}A(z)\text{Re}C(z).$$

If $p \in \mathcal{H}[1,n]$ *and*

$$\text{Re}\ [A(z)p^{4k}(z) + B(z)p^{4k-1}(z) + C(z)p^{4k-2}(z) + \alpha(zp'(z)-1)^3 + \delta] > 0$$

then

$$\text{Re}\ p(z) > 0.$$

References

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