Error estimates for approximating fixed points of quasi contractions

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

The main purpose of the paper is to give error estimates for two important fixed point theorems involving quasi contractive type operators.

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1 Introduction

The literature of the last four decades abounds with papers which establish fixed point theorems for self and nonself operators of a certain ambient space and satisfying a variety of conditions [1-2], [4-30]. For most of them, their reference result is the well known Banach′s fixed point theorem - one of the most useful results in fixed point theory.

In a metric space setting its complete statement is the following, see for example Berinde [2].

Theorem B. *Let (X, d) be a complete metric space and $T : X \to X$ an a-contraction, that is, an operator satisfying*

$$(1) \qquad d(Tx, Ty) \le a\, d(x, y)\,, \quad \textit{for all } x, y \in X$$

with $a \in [0, 1)$ fixed. Then

(*i*) *T has a unique fixed point, i.e. $F_T = \{x^*\}$;*

(*ii*) *The Picard iteration associated to T, that is, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by*

$$(2) \qquad x_n = Tx_{n-1} = T^n x_0\,, \quad n = 1, 2, \ldots$$

converges to x^*, for any initial guess $x_0 \in X$;

(*iii*) *The a priori and a posteriori error estimates*

$$(3) \qquad d(x_n, x^*) \le \frac{a^n}{1-a}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \ldots$$

$$(4) \qquad d(x_n, x^*) \le \frac{a}{1-a}\, d(x_{n-1}, x_n)\,, \quad n = 0, 1, 2, \ldots$$

hold.

(*iv*) *The rate of convergence of the Picard iteration is given by*

$$(5) \qquad d(x_n, x^*) \le a \cdot d(x_{n-1}, x^*)\,, \quad n = 1, 2, \ldots$$

Note. A map satisfying (*i*) and (*ii*) in the previous theorem, is said to be a *Picard operator*, see Rus [23]-[27].

Theorem B, together with its direct generalizations, has many applications in solving nonlinear functional equations, but suffers from one drawback - the strong contractive condition (1) forces that T be continuous throughout X.

It is then natural to ask if there exist contractive conditions which do not imply the continuity of T. This was answered in the affirmative by R. Kannan [15] in 1968, who proved a fixed point theorem which extends Theorem B to mappings that need not be continuous, by considering instead of (1) the next condition: there exists $b \in \left[0, \dfrac{1}{2}\right)$ such that

$$(6) \qquad d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X.$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T, see for example, Rus [23], [26], Taskovic [29], and references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [7], is based on a condition similar to (6): there exists $c \in \left[0, \dfrac{1}{2}\right)$ such that

$$(7) \qquad d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X.$$

It is well known, see Rhoades [17], that the contractive conditions (1) and (6), (6) and (7), as well as (1) and (7), respectively, are independent.

In 1972, Zamfirescu [30] obtained a very interesting fixed point theorem, by combining (1), (6) and (7).

Theorem Z. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a map for which there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b, c \leq 1/2$ such that for each pair x, y in X, at least one of the following is true:*

(z_1) $d(Tx, Ty) \leq a\, d(x, y)$;

(z_2) $d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big]$;

(z_3) $d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big]$.

Then T is a Picard operator.

One of the most general contraction condition for which the map satisfying it is still a Picard operator, has been obtained by Ciric [9] in 1974: there exists $0 < h < 1$ such that

$$(8) \quad d(Tx, Ty) \leq h \cdot \max\big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\},$$

$$\text{for all } x, y \in X.$$

Remark 1. *1) As shown by Rhoades ([20], Theorem 2), a quasi contraction, i.e., a mapping satisfying (8), is still continuous at the fixed point.*

2) The fixed point theorems for contractive definitions of the form (1); (6); (7); (z_1) - (z_3); (8) were unified by various authors, see for example Berinde [1], [2], Rus [23], [26].

As in the case of Theorem B, for all these general theorems, the fixed points are constructed by means of the Picard iteration (or method of successive approximations). Since the successive approximation method is an iterative method that produces an **approximate** fixed point, it is very important that a such fixed point theorem also provide an error estimate. There are some general theorems in this area (see, for example, Rus [23], [26], Berinde [2], [3]) that give an error estimate for the Picard iteration but, to our best knowledge, there are not direct results of this type for Kannan′s and Zamfirescu′s theorems.

Consequently, the main aim of this paper is to obtain complete statements for the fixed point theorems of Kannan and Zamfirescu, including both a priori and a posteriori error estimates, when the Picard iteration is used to approximate the fixed points.

Recall that for Ciric's fixed point theorem, there exists an a priori error estimate of the form (3), with $a = h$, the constant appearing in (8), see Ciric [11].

2   Error estimates for Kannan's and Zamfirescu's fixed point theorems

The main results of this section are Theorems 1 and 2.

Theorem 1. *Let (X, d) be a complete metric space and T be a self map of X satisfying condition (6).*

For any $x_0 \in X$, consider the Picard iteration $\{x_n\}_{n=0}^{\infty}$ associated to T, defined by (2). Then

1) *$F_T = \{x^*\}$;*

2) *The Picard iteration converges to x^*, for any $x_0 \in X$;*

3) *The following error estimates*

$$(9) \qquad d(x_n, x^*) \leq \frac{\alpha^n}{1 - \alpha} d(x_0, x_1), \quad n = 0, 1, 2, \ldots$$

$$(10) \qquad d(x_n, x^*) \leq \frac{\alpha}{1 - \alpha} d(x_{n-1}, x_n), \quad n = 1, 2, \ldots$$

hold, where $\alpha = a/(1 - a)$.

4) *The rate of convergence of the Picard iteration is given by*

$$(11) \qquad d(x_n, x^*) \leq \alpha \cdot d(x_{n-1}, x^*), \quad n = 1, 2, \ldots$$

Proof.   First note that if T satisfies (6), then card $F_T \leq 1$, i.e. T has at most one fixed point.

Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration, starting from $x_0 \in X$ arbitrary. Then by (6) we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq a\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big]$$

which implies

(12) $\qquad d(x_n, x_{n+1}) \leq \dfrac{a}{1-a} d(x_{n-1}, x_n), \quad$ for all $n = 1, 2, \ldots$

Since $0 \leq a < \dfrac{1}{2}$, it results $0 \leq \dfrac{a}{1-a} < 1.$

Denote $\alpha = \dfrac{a}{1-a}.$
Using (4) we obtain by induction

(13) $\qquad\qquad\qquad d(x_{n+k}, x_{n+k-1}) \leq \alpha^k d(x_n, x_{n-1}), k \in \mathbb{N}^*$

and hence

$$d(x_{n+p}, x_n) \leq \left(\alpha + \alpha^2 + \cdots + \alpha^p\right) d(x_n, x_{n-1})$$

which yields

(14) $\qquad d(x_{n+p}, x_n) \leq \dfrac{\alpha(1-\alpha^p)}{1-\alpha} d(x_n, x_{n-1}), n, p \in \mathbb{N}^*.$

Since by (5),

$$d(x_n, x_{n-1}) \leq \alpha^{n-1} d(x_0, x_1), \quad n \geq 1$$

from (6) we obtain

(15) $\qquad d(x_{n+p}, x_n) \leq \dfrac{\alpha^n(1-\alpha^p)}{1-\alpha} d(x_0, x_1), \quad n, p \in \mathbb{N}^*$

Now by letting $p \to \infty$ in (15) and (14) we obtain the estimates (9) and (10), respectively.

Again, by (6) we have

$$d(Tx, Ty) \leq a\big[d(x, Tx) + d(y, Ty)\big] \leq$$

$$\leq a\big\{d(x, tx) + \big[d(y, x) + d(x, Tx) + d(Tx, Ty)\big]\big\}$$

which implies

(16) $$\qquad d(Tx, Ty) \leq \frac{a}{1-a} \cdot d(x, y) + \frac{2a}{1-a} d(x, Tx).$$

Take $x := x^*$, $y := x_{n-1}$ in (16) to obtain

$$d(x_n, x^*) \leq \frac{a}{1-a} d(x_{n-1}, x^*),$$

that is, the estimate (11).

Remark 2. *1) Note that the estimates (9) - (11) in Theorem 1 are the same as the corresponding ones in Theorem B, the only difference is that a is replaced by $\dfrac{a}{1-a}$.*

2) The estimates (9) and (10) show that the Picard iteration converges to x^*, the unique fixed point of T, at least as fast as a geometric progression;

3) The estimate (11) shows that the rate of convergence of the Picard iteration is linear.

Theorem 2. *Let (X, d) and T be as in Theorem Z.*

For any $x_0 \in X$, consider the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (2). Then all conclusions in Theorem 1 hold, with

$$\alpha := \delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\},$$

where a, b, c are the constants involved in (z_1), (z_2) and (z_3), respectively.

Proof. We first fix $x, y \in X$. At least one of (z_1), (z_2) or (z_3) is true. If (z_2) holds, then as we have seen in proving Theorem 1, the following inequality

$$(17) \qquad d(Tx, Ty) \leq \frac{b}{1-b}\, d(x, y) + \frac{2b}{1-b}\, d(x, Tx)\,,$$

holds. If (z_3) holds, then similarly we get

$$(18) \qquad d(Tx, Ty) \leq \frac{c}{1-c}\, d(x, y) + \frac{2c}{1-c} \cdot d(x, Tx)\,.$$

Therefore by denoting

$$\delta = \max\left\{ a, \frac{b}{1-b}, \frac{c}{1-c} \right\}$$

we have $0 \leq \delta < 1$ and, for all $x, y \in X$, the following inequality

$$(19) \qquad d(Tx, Ty) \leq \delta \cdot d(x, y) + 2\delta d(x, Tx)\,,$$

holds.

In a similar manner we obtain

$$(20) \qquad d(Tx, Ty) \leq \delta \cdot d(x, y) + 2\delta(x, Ty)\,,$$

valid for all $x, y \in X$.

But by (19) it follows that card $F_T \leq 1$ and remains to show that there exists a fixed point.

If we take $x := x_n$, $y := x_{n-1}$ in (19), then we get

$$d(x_{n+1}, x_n) \leq \delta \cdot d(x_n, x_{n-1})$$

and the rest of the proof is similar to that of Theorem 1.

Remark 3. *1) The a priori and a posteriori error estimates as well as the rate of convergence given here for both Kannan's and Zamfirescu's fixed point theorems are formally the same as in Banach's fixed point theorem.*

2) The a priori error estimates can be obtained from a more general result given in Berinde [1], see also Berinde [2] Theorem 1.5.4 for a corrected version.

3) Recently, the author Berinde ([6], [4], [5]), obtained similar error estimates for T satisfying a more general contractive condition that includes (1), (6) and (7) and that does not forces the uniqueness of the fixed point

References

[1] Berinde,V., *A fixed point theorem for mappings with contracting orbital diameters*, Bul. Stiint. Univ. Baia Mare 10 (1994) 29-38

[2] Berinde, V., *Generalized Contractions and Applications*, Editura Cub Press 22, Baia Mare, 1997 (in Romanian)

[3] Berinde, V., *Iterative Approximation of Fixed Points*, Editura Efemeride, Baia Mare, 2002

[4] Berinde, V., *Approximating fixed points of weak φ-contractions using the Picard iteration*, Fixed Point Theory 4 (2003), No. 2, 131-142

[5] Berinde, V., *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math., 19 (2003), 7-22

[6] Berinde, V., *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Analysis Forum 9 (2004), No. 1, 43-53

[7] Chatterjea, S.K., *Fixed-point theorems*, C.R. Acad. Bulgare Sci. 25 (1972) 727-730

[8] Ciric, Lj.B., *Generalized contractions and fixed-point theorems*, Publ. Inst. Math. (Beograd) 12 (1971) 19-26

[9] Ciric, Lj.B., *A generalization of Banach's contraction principle*, Proc. Am. Math. Soc. 45 (1974) 267-273

[10] Ciric, Lj.B., *Convergence theorems for a sequence of Ishikawa iteration for nonlinear quasi-contractive mappings*, Indian J. Pure Appl. Appl. Math. 30 (4) (1999) 425-433

[11] Ciric, Lj.B., *On contraction type mappings*, Math. Balkanica 1 (1971) 52-57

[12] Ciric, Lj.B., *Quasicontraction nonself mappings on Banach spaces*, Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur. Sci. Math 23 (1998) 25-31

[13] Hicks, T.L. and Rhoades, B.E. , *A Banach type fixed point theorem*, Math. Japonica 24 (3) (1979) 327-330

[14] Ivanov, A.A., *Fixed points of metric space mappings* (in Russian), Isledovaniia po topologii. II, Akademia Nauk, Moskva, 1976, pp. 5-102

[15] Kannan, R., *Some results on fixed points*, Bull. Calcutta Math. Soc. 10 (1968) 71-76

[16] Rhoades, B.E., *Fixed point iterations using infinite matrices* Trans. Amer. Math. Soc. 196 (1974), 161-176

[17] Rhoades, B.E., *A comparison of various definitions of contractive mappings* Trans. Amer. Math. Soc. 226 (1977) 257-290

[18] Rhoades, B.E., *Extensions of some fixed theorems of Ciric, Maiti and Pal*, Math. Seminar Notes, 6 (1978) 41-46

[19] Rhoades, B.E., *Contractive definitions revisited*, Contemporary Mathematics 21 (1983) 189-205

[20] Rhoades, B.E., *Contractive definitions and continuity*, Contemporary Mathematics 72 (1988) 233-245

[21] Rus, I.A., *The method of successive approximations* (in Russian), Revue Roum. de Math. Pures et Appl. 17 (1972), 1433-1437

[22] Rus, I.A., *Metrical Fixed Point Theorems*, Univ. of Cluj-Napoca, 1979

[23] Rus, I.A., *Principles and Applications of the Fixed Point Theory* (in Romanian), Editura Dacia, Cluj-Napoca, 1979

[24] Rus, I.A., *Generalized contractions*, Seminar on Fixed Point Theory 3 (1983) 1-130

[25] Rus, I.A., *Picard operators and applications*, Babes-Bolyai Univ., 1996

[26] Rus, I.A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001

[27] Rus, I.A., *Picard operators and applications*, Scientiae Math. Japon. 58 (2003), No. 1, 191-219

[28] Singh, S.L. , Mishra, S.N., *On a Liubomir Ciric fixed point theorem for nonexpansive type maps with applications*, Indian J. Pure Appl. Math. 33 (4) (2002) 531-542

[29] Taskovic, M., *Osnove teorije fiksne tacke (Fundamental Elements of Fixed Point Theory)*, Matematicka biblioteka 50, Beograd, 1986

[30] Zamfirescu, T., *Fix point theorems in metric spaces* Arch. Math. (Basel), 23 (1972), 292-298

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