Descent-Cycling in Schubert Calculus

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CONTENTS

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We prove two lemmata about Schubert calculus on generalized flag manifolds G/B, and in the case of the ordinary flag manifold GL_n/B we interpret them combinatorially in terms of descents, and geometrically in terms of missing subspaces. One of them gives a symmetry of Schubert calculus that we christen *descent-cycling*. Computer experiment shows these two lemmata are surprisingly powerful: they already suffice to determine all of GL_n Schubert calculus through $n = 5$, and 99.97%+ at $n = 6$. We use them to give a quick proof of Monk's rule. The lemmata also hold in equivariant ("double") Schubert calculus for Kac–Moody groups G.

1. BACKGROUND ON SCHUBERT PROBLEMS

Fix a pinning for a complex reductive Lie group G: a Borel subgroup B, an opposed Borel subgroup B_-, a Cartan subgroup $T = B \cap B_-$, the Weyl group $W = N(T)/T$, and R the Coxeter generators of W. There is a famous basis (as a free abelian group) for the cohomology of G/B given by the Poincaré duals of the closures of the B_- orbits on G/B; these are the *Schubert classes* $S_w := [\overline{B_-wB/B}]$, for $w \in W$, and are indexed by the Weyl group.

(In this introduction we will only consider ordinary cohomology and the case of finite-dimensional G. However, since the Schubert cycles $\overline{B_-wB/B}$ are T-invariant, they define elements not only of ordinary but of T-equivariant cohomology of G/B, and our results hold in that case also. In addition, our main arguments apply to the case of Kac–Moody G. Our references for equivariant cohomology of (possibly infinite-dimensional) G/B are [Graham 1999; Kostant and Kumar 1986].)

The degree of the cohomology class S_w is twice $l(w)$, the *length* of w (as a minimal product of Coxeter generators from R). Define a *Schubert problem* as a triple $(u, v, w) \in W^3$ such that

$$l(u) + l(v) + l(w) = \dim_{\mathbb{C}} G/B.$$

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In this case we can consider the *symmetric Schubert numbers*

$$c_{uvw} := \int_{G/B} S_u S_v S_w$$

which count the number of points in the intersection of three generic translates of Schubert cycles. Since this intersection is transverse (by a standard appeal to Kleiman's transversality theorem), and is of three complex subvarieties, the points are all counted with sign $+1$ and therefore the number is nonnegative. While formulae now exist for these numbers, it is a famous open problem to compute these numbers in a *manifestly positive* way. The analogous problem for G/P where $G = \mathrm{GL}_n$ and P is a maximal parabolic was solved first by the Littlewood–Richardson rule (or see [Knutson and Tao $\geq$ 2001]).

Recall the Bruhat order on W (due to Chevalley): $v > w$ if $v \in \overline{B_- wB/B}$. With this we can state our two lemmata:

Lemma 1.1. *Let (u, v, w) be a Schubert problem, and s a Coxeter generator. If $us > u$, $vs > v$, and $ws > w$, then $c_{uvw} = 0$.*

Lemma 1.2. *Let $(u, v, w) \in W^3$ be a triple with*

$$l(u) + l(v) + l(w) = \dim_{\mathbb{C}} G/B - 1,$$

and let s be a Coxeter generator. If $us > u$, $vs > v$, and $ws > w$, then

$$c_{us,v,w} = c_{u,vs,w} = c_{u,v,ws}.$$

In the case where $G = \mathrm{GL}_n(\mathbb{C})$, $W = S_n$, and s is the transposition $i \leftrightarrow i+1$, the statement $us > u$ says that $u(i) < u(i+1)$; one says that u *ascends* in the i-th place. Otherwise if $u(i) > u(i+1)$ one says that u *descends* in the i-th place, or that it has a descent there. For this reason we christen the symmetry of Lemma 1.2 *descent-cycling*, and call these three problems *dc-equivalent*. Extending this relation by transitivity, we get a very powerful notion of equivalence for solving Schubert problems; in particular many Schubert problems are dc-equivalent to ones that fall to Lemma 1.1, ones which we call *dc-trivial*.

We define a *Grassmannian Schubert problem* to be a Schubert problem (u, v, w) in which u, v each have only one descent, and in the same place. The name comes from the fact that the relevant integral can be performed on a Grassmannian; these Schubert problems are well-understood thanks to Littlewood–Richardson and other positive rules for their computation. Note that descent-cycling *cannot* be formulated in the context of Grassmannian problems alone: descent-cycling a nontrivial Grassmannian Schubert problem always produces a non-Grassmannian Schubert problem, and Grassmannian problems from different Grassmannians (the single descent in different places) can be dc-equivalent.

In Section 2 we define a graph whose vertices are Schubert problems and edges come from descent-cycling; by computer we were able to determine much about the structure of this graph in small examples. This we believe to be the main point of interest in the paper — that two such simple lemmata suffice to determine so many Schubert numbers.

It is our hope that this symmetry might help guide the search for a combinatorial formula for Schubert calculus; a rule generalizing Littlewood–Richardson (the case that π, ρ each have only one descent, and in the same place) and manifestly invariant under descent-cycling would have very strong evidence for it.[1]

In Section 3 we give the nearly-trivial proofs of the two lemmata, using standard properties of the (equivariant) BGG operators. We do this in terms of "Schubert structure constants" rather than symmetric Schubert numbers. This seems to be more appropriate for equivariant cohomology (in light of [Graham 1999]), and also gives results in the case of G a Kac–Moody group.

In the $\mathrm{GL}_n(\mathbb{C})$ case, there is an intuitive geometrical interpretation in terms of "reconstructing forgotten subspaces"; with this we can also say something about finding the actual flags in the intersection in synthetic-geometry terms, which we do in Section 4.

In Section 5 we prove Monk's rule via descent-cycling, to give an example of an interesting Schubert problem that falls to these techniques. It would be interesting to see if other known cases of $c_{uvw} = 0, 1$ (such as the Pieri rule [Lascoux and Schützenberger 1982; Robinson $\geq$ 2001; Sottile 1996]) are consequences of descent-cycling.

[1] I circulated a preprint a year ago entitled "A conjectural rule for $\mathrm{GL}_n(\mathbb{C})$ Schubert calculus", generalizing a Grassmannian theorem from [Knutson and Tao $\geq$ 2001]. Alas, the rule conjectured there is not invariant under descent-cycling.

2. THE SCHUBERT PROBLEMS GRAPH AND ITS STRUCTURE FOR SMALL $\mathrm{GL}_n(\mathbb{C})$

Let Γ_n be the graph whose vertices are Schubert problems for $\mathrm{GL}_n(\mathbb{C})$, with edges between two Schubert problems that are related by cycling a single descent. Then the descent-cycling Lemma 1.2 says that the symmetric Schubert number is constant on connected components of this graph. (This is perhaps more naturally a "3-regular hypergraph" than a graph, since the natural concept of "edge" here connects three, not two, vertices.) Recall that we define two Schubert problems to be dc-equivalent if they are in the same connected component, i.e., if one can be transformed into the other by a sequence of descent-cyclings. Also, we call a Schubert problem dc-trivial if it falls to Lemma 1.1; that is, if for some $(i,\ i{+}1)$ it has three ascents.

Example 2.1. We write a vertical bar to point out the descents, and a horizontal bar indicating to where we intend to cycle a descent. In the following line we descent-cycle our way to a dc-trivial problem; this shows $c_{1324,2143,2341} = 0$.

$$\begin{array}{ccccc}
1\ 3|2\ 4 & & 1\ 2\ 3\ 4 & & 1\ 2\ 3\ 4 \\
2|1\ 4|3 & \longrightarrow & 2|1\ 4|3 & \longrightarrow & 1\ 2\ 4|3 \\
2\ 3\text{-}4|1 & & 2\text{-}4|3|1 & & 4|2\ 3|1
\end{array}$$

Next we show that (1324,3142,1423) is dc-equivalent to (1234,1234,4321), so $c_{1324,3142,1423} = c_{1234,1234,4321}$, which is in turn easily seen to be 1.

$$\begin{array}{ccccccccc}
1\ 3|2\ 4 & & 1\ 3|2\ 4 & & 1\ 2\ 3\ 4 & & 1\ 2\ 3\ 4 & & 1\ 2\ 3\ 4 \\
3|1\ 4|2 & \rightarrow & 1\ 3\ 4|2 & \rightarrow & 1\ 3\ 4|2 & \rightarrow & 1\ 3|2\ 4 & \rightarrow & 1\ 2\ 3\ 4 \\
1\text{-}4|2\ 3 & & 4|1\text{-}2\ 3 & & 4|2|1\text{-}3 & & 4|2\text{-}3|1 & & 4|3|2|1
\end{array}$$

The reader may enjoy studying hands-on the properties of descent-cycling, using the descent-cycling Java applet found at http://www.math.berkeley.edu/~allenk/java/DCApplet.html.

We established the following facts by brute-force computation.

Fact 2.2. There are 35 Schubert problems for $\mathrm{GL}_3(\mathbb{C})$, of which 21 are not dc-trivial. All 21 live in the same connected component of Γ_3, which is pictured in Figure 1.

Fact 2.3. Let $n \leq 5$, and (π, ρ, σ) be a Schubert problem in dimension n. Then the symmetric Schubert number $c_{\pi\rho\sigma}$ equals zero if and only if (π, ρ, σ) is dc-equivalent to a dc-trivial problem. Otherwise, (π, ρ, σ) is dc-equivalent to $(\mathrm{id}, \mathrm{id}, w_0)$, where id denotes the identity permutation and w_0 the long word; therefore the symmetric Schubert number $c_{\pi\rho\sigma}$ is equal to one. Put another way, there is exactly one non-dc-trivial component of the Schubert problem graph for each $n \leq 5$.

In particular, the two lemmata (and the trivial calculation $c_{\mathrm{id},\mathrm{id},w_0} = 1$) suffice to completely determine Schubert calculus for $\mathrm{GL}_n(\mathbb{C})$ through $n = 5$. We know a priori that this connectedness cannot continue at $n = 6$, because the nonzero symmetric Schubert numbers are sometimes 2. (All symmetric Schubert numbers in this paper were computed with the Maple package ACE [Veigneau 1998].)

Fact 2.4. The graph Γ_6 has 8,881,334 vertices, of which all but 2,351,475 are dc-trivial. Throwing out the components with dc-trivial vertices we are left with 145 components comprising 411,582 vertices. The lion's share of those vertices, 409,023, are dc-equivalent to the easy case $(\mathrm{id}, \mathrm{id}, w_0)$, leaving 2559 cases (less than 0.03%) not succumbing to dc-equivalence/dc-triviality arguments.

Of the remaining 144 components, only one contains a Grassmannian Schubert problem, so in some sense the Littlewood–Richardson rule doesn't help much. (In fact this is the only one with any Grassmannian permutations, so a Schubert-times-Schur rule wouldn't help much either.)

Exactly one of these components has intersection number zero (despite containing no dc-trivial Schubert problems); one element of it is (231645, 231645, 326154). There are also 48 components of size one, that is, Schubert problems that admit no descent-cycling whatever; one example is (214365, 154326, 321654).

These computations were done in C and took 2.5 minutes on a Pentium 300. The limiting factor was that they just barely fit in 64 megabytes of RAM, putting the $n = 7$ case (which is roughly $7^3 = 343$ times bigger) out of reach without new ideas.

It seems likely that as n increases, the fraction of $\mathrm{GL}_n(\mathbb{C})$ Schubert problems having no place with three ascents (and so falling to Lemma 1.1 alone) goes to 0. We did not pursue this.

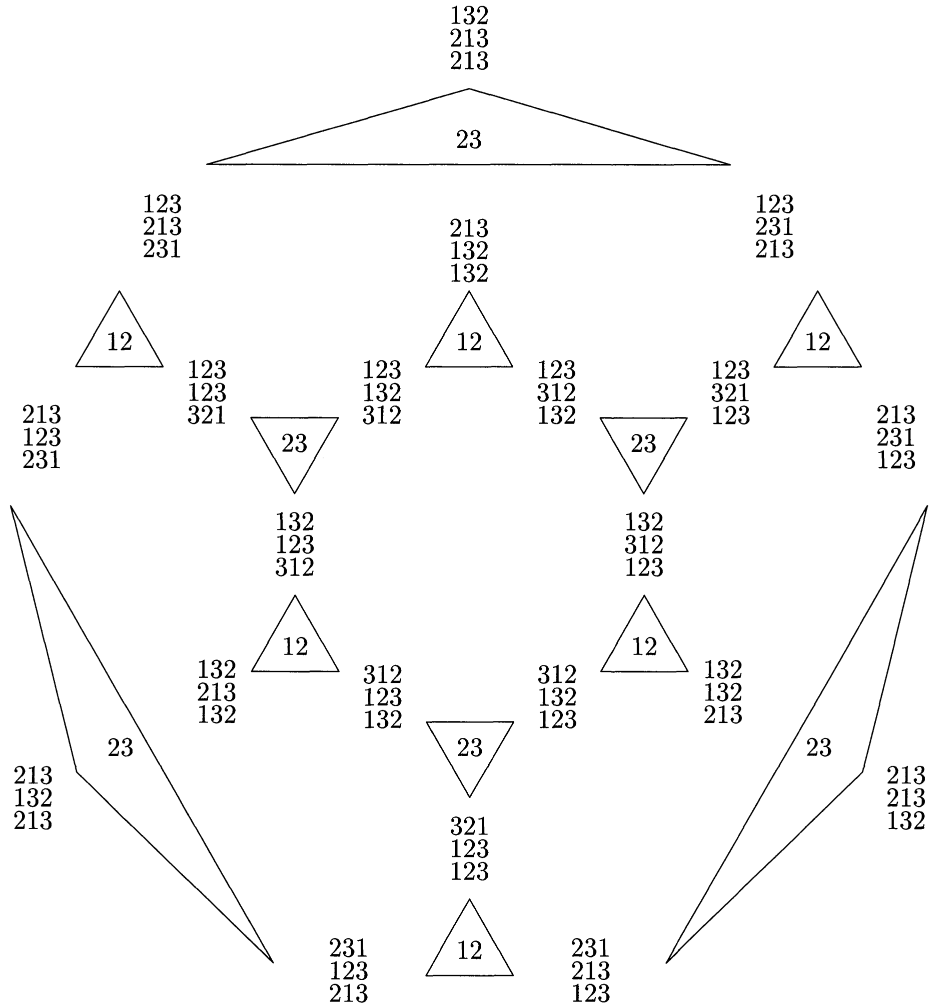

FIGURE 1. The non-dc-trivial component of Γ_3, drawn to make its S_3 symmetry manifest. The edges, which always come in sets of three, are drawn as triangles and labeled with the column where descents are being cycled. Note that not all vertices have degree 4; one cannot descent-cycle in a column which has *two* descents.

It was very tempting to believe that a vanishing Schubert number could always be "blamed" on dc-equivalence to a dc-trivial problem, and it was very sad to find the lone component in Γ_6 that belies this. Mark Haiman pointed out a "stabilization" map $\Gamma_{n-1} \hookrightarrow \Gamma_n$ taking (u,v,w) to $\big(un, vn, (w{+}1)1\big)$, where $w+1$ means w with 1 added to each element. For example,

$$(2143, 1243, 3214) \mapsto (21435, 12435, 43251).$$

Unfortunately, it turns out that stabilizing the bad problem $(231645, 231645, 326154)$ one, two, or three times (to $(231645789, 231645789, 659487321)$ — this was again limited by computer memory) does not make it dc-equivalent to a dc-trivial problem. Question: is dc-equivalence already a stable relation?

3. PROOFS OF THE LEMMATA

The statements in this section are slightly different from those in the introduction, in that they are phrased in terms of *structure constants* c_{uv}^w, rather than symmetric Schubert numbers c_{uvw}. We first remind the reader of the partial relation between these and explain why we switch to the less-symmetric formulation.

In ordinary cohomology of G/B (if $\dim G < \infty$), we have the Poincaré-pairing duality relation

$$\int_{G/B} S_u S_v = \delta_{u,w_0 v}$$

where w_0 is the long element of the Weyl group, and δ is the Kronecker delta. (Note that the duality discussed here is in the sense of dual bases, not Poincaré duality!)

One way to see this is to realize the class S_v not by the Schubert cycle $\overline{B_{-}vB}$ but the opposite Schubert cycle $\overline{Bw_0vB} = w_0\overline{B_{-}vB}$. Since w_0 is connected to the identity in G, these two cycles define the same element of cohomology.

From this we derive that

$$S_uS_v = \sum_w c_{uv}^w\, S_w \quad \text{implies} \quad c_{uv}^w = c_{u,v,w_0w}.$$

So in ordinary cohomology, one can work instead with these *Schubert structure constants*, though our results from the first section are prettier to state symmetrically. However, in T-equivariant cohomology the dual basis to the Schubert basis is *not* once again the Schubert basis (essentially because w_0 is not connected to the identity *through T-invariant maps* of G/B) and these two concepts part ways.

In [Graham 1999] a certain positivity result was proven for the equivariant $c_{uv}^w \in H_T^*$ (which must be carefully stated, insofar as these are polynomials not numbers). This implies a much weaker positivity result for the c_{uvw}, and so it seems more interesting to prove results about the structure constants.

Also, in the case of G an infinite-dimensional Kac–Moody group, one cannot so blithely do integrals on G/B, and the c_{uv}^w are the only concept that makes sense. This concludes the advertisement for Schubert structure constants over symmetric Schubert numbers. (In fact almost all work on Littlewood–Richardson is in terms of the structure constants; see [Knutson and Tao $\geq$ 2001] for a discussion of this.)

Note that the condition we gave in Section 1 for a "Schubert problem" corresponds to $l(w) = l(u) + l(v)$, which seems a reasonable thing to ask since cohomology is a graded ring. But we will not insist on this in what follows because, in equivariant cohomology, the structure constants can be nonzero even if one only has $l(w) \leq l(u) + l(v)$. (We only imposed this condition before to keep the graph of Schubert problems a reasonable size.)

Let $s \in R$ be a simple reflection, and $P_s = \overline{BsB} \leq G$ be the corresponding minimal parabolic. Let $p_s : G/B \to G/P_s$ be the corresponding projection; it is G-equivariant and therefore T-equivariant. Composing pushforward with pullback defines a degree -2 endomorphism ∂_s on $H_T^*(G/B)$. (This map was first introduced in [Bernstein et al. 1973; Demazure 1974], in a nonequivariant formulation, though equivariant K-theory is implicit in the second of these works.)

We refer to [Kostant and Kumar 1986] for the four properties we need of these BGG operators ∂_s:

1. If $ws > w$, then $\partial_s S_w = 0$.
2. If $ws < w$, then $\partial_s S_w = S_{ws}$.
3. With respect to a certain natural action of W on $H_T^*(G/B)$, the map ∂_s is a twisted derivation:
$$\partial_s(\alpha\beta) = \alpha\, \partial_s(\beta) + \partial_s(\alpha)\,(s \cdot \beta).$$
(We won't need to understand this action of W.)
4. If $s_1s_2 \ldots s_l = r_1r_2 \ldots r_l$ are two reduced expressions for a Weyl group element w, then
$$\partial_{s_1} \ldots \partial_{s_l} = \partial_{r_1} \ldots \partial_{r_l},$$
and so we have a well-defined operator ∂_w.

Since the proofs of both lemmata have much in common, we gather them into a single proposition. Trivial as it is, this is the heart of the paper.

Proposition 3.1. *Let $(u, v, w) \in W^3$, and s a simple reflection, such that $us > u$ but $ws < w$.*

- *If $vs > v$, then $c_{uv}^w = 0$.*
- *If $vs < v$, then $c_{uv}^w = c_{u,vs}^{ws}$.*

Proof. The main formula we need is

$$c_{xy}^z = \text{coefficient of } S_1 \text{ in } \partial_z(S_xS_y)$$

which follows easily from the properties stated of the BGG operators.

Since $ws < w$, we have

$$\begin{aligned}\partial_w(S_uS_v) &= \partial_{ws}\partial_s(S_uS_v)\\ &= \partial_{ws}(S_u\partial_s(S_v) + \partial_s(S_u)(s \cdot S_v))\\ &= \partial_{ws}(S_u\partial_s(S_v)),\end{aligned}$$

this last because by the $us > u$ assumption, ∂_s annihilates S_u.

If $vs > v$, then ∂_s annihilates S_v too, and the RHS is zero. Combining that with the formula for c_{xy}^z gives the first result.

If $vs < v$, then the RHS is $\partial_{ws}(S_uS_{vs})$, and two applications of the formula for c_{xy}^z give the second result. □

The conditions on ws versus w in this proposition are backwards from how they were in Lemmata 1.1 and 1.2; that's because of the multiplication by w_0 in comparing Schubert structure constants with symmetric Schubert numbers. With this in mind the two lemmata follow.

4. A GEOMETRICAL INTERPRETATION

For $w \in W$, let $D_w := \overline{G \cdot (wB, B)} \subseteq (G/B)^2$. Given a simple reflection $s \in R$, let P_s again be the corresponding minimal parabolic $\overline{BsB}$, and consider the composite map $D_w \hookrightarrow (G/B)^2 \to G/P_s \times G/B$.

Lemma 4.1. *Let $w \in W, s \in R, P = \overline{BsB}$. The fibers of $D_w \hookrightarrow (G/B)^2 \to G/P_s \times G/B$ are*

- *$\mathbb{CP}^1$'s if $ws < w$*
- *single points (generically), if $ws > w$.*

Proof. We reduce to the well-studied case (see [Demazure 1974]) of a single flag manifold. Let $X := G/B \times \{B/B\}$; since $G \cdot X = G/B \times G/B$ it suffices to consider the map $D_w \cap X \to G/P_s \times \{B/B\}$. And $D_w \cap X = \overline{BwB/B} \times \{B/B\}$, so (omitting the $\{B/B\}$) we're studying the fibers of the composite $\overline{BwB/B} \to G/B \to G/P_s$, as already done in [Demazure 1974]. □

In the case of $G = \mathrm{GL}_n(\mathbb{C})$, D_w is the variety of pairs of flags (F, G) in $\mathbb{C}^n$ such that "F is w-close or closer to G". In this case, the generators R correspond one-to-one to the subspaces in a flag (other than the zero subspace and the whole space), and the map $G/B \to G/P_s$ corresponds to "forgetting" the subspace. Then we can interpret the lemma in very familiar terms:

Corollary 4.2. *Let $w \in S_n$ and $i \in \{2, \ldots, n-1\}$. Let F, G be two flags in $\mathbb{C}^n$ such that F is w-close or closer to G. Let F' be the partial flag obtained by forgetting F's i-dimensional subspace. Can we reconstruct F knowing only F', G, and w?*

- *If w ascends at $(i, i+1)$, there is no hope — any i-dimensional space between F_{i-1} and F_{i+1} will do.*
- *If w descends at $(i, i+1)$, then (for a Zariski-open set of such G) the subspace F_i is uniquely determined.*

Another way to interpret this is that if w does not descend at $(i, i+1)$, then G "does not care" what F_i is used (to get F w-close to G). Conversely, if w does descend there, then G "usually insists" on a particular F_i, when presented with the rest of F.

Proof of Lemma 1.1. Let $\{F\}$ be the set of flags in relative position u to A, v to B, and w to C where A, B, C are three flags in generic relative position. Then by codimension count (and the usual appeal to Kleiman's transversality theorem) the set $\{F\}$ is finite. However, since none of A, B, C care what F_i is (since by assumption none of them have a descent at $(i, i+1)$), the set $\{F\}$ is a union of $\mathbb{CP}^1$'s. These two facts are only compatible if $\{F\}$ is empty. □

Proof of Lemma 1.2. Let $\{F\}$ be the set of flags in relative position u to A, v to B, and w to C where A, B, C are three flags in generic relative position. Then as in the previous proof, the set $\{F\}$ is a union of $\mathbb{CP}^1$'s, reflecting the ambiguity in F_i. If we change one of u, v, w to have a descent at $(i, i+1)$, each of these $\mathbb{CP}^1$'s is cut down to a single point. But it doesn't matter which of u, v, w gets this new descent. □

This geometric description of descent-cycling suggests that additional symmetries may come from forgetting multiple subspaces at a time. It appears, though, that all of these are implied by the single-subspace case.

One application of this geometric description is to actually locate the flag satisfying the desired intersection conditions, in the case that (π, ρ, σ) is dc-equivalent to the easy case $(\mathrm{id}, \mathrm{id}, w_0)$. We illustrate this in the case of the Schubert problem

$$(132, 213, 213),$$

which we can descent-cycle to $(123, 213, 231)$, and from there to $(123, 123, 321)$. Working from the end, the unique F satisfying $(123, 123, 321)$ is given by $F_1 = C_1$, $F_2 = C_2$. When we cycle the descent in the $(1, 2)$ column, we have to replace $F_1 = C_1$ by $F_1 = B_2 \cap C_2$. Then when we cycle the descent in the $(2, 3)$ column we have to replace $F_2 = C_2$ by $F_2 = A_1 \oplus (B_2 \cap C_2)$.

There is an alternate way to prove the vanishing condition in Proposition 3.1 cohomologically, involving the projection $G/B \to G/P_s$. A Schubert class S_u is in the image of the pullback of $H^*_T(G/P_s)$ if and only if $us > u$. Since this pullback is a ring homomorphism, the product of two pulled-back classes is also in this image, and cannot involve any S_w with $ws < w$.

5. MONK'S RULE

Monk's rule [Monk 1959] is concerned with the case of $\mathrm{GL}_n(\mathbb{C})$ Schubert problems in which ρ is a simple reflection $s_i = (i \leftrightarrow i+1)$.

Theorem 5.1 (Monk's rule). *Let σw_0 cover π in the Bruhat order; that is, σ is π with each number j replaced by $(n+1)-j$, and two numbers inverted in πw_0 have been put back in correct order, decreasing the number of inversions by exactly one. Then $c_{\pi,s_i,\sigma} = 1$ if the numbers switched straddled the position between i and $i+1$, whereas $c_{\pi,s_i,\sigma} = 0$ if the numbers switched were both physically on one side of $(i,\, i{+}1)$.*

Remark. Some may object that Monk's rule says more — that $c_{\pi,s_i,\sigma} = 0$ unless σw_0 covers π — but we prefer to see this as a more general property of symmetric Schubert numbers, that if π, ρ are not less than σw_0 in the Bruhat order, then $c_{\pi\rho\sigma} = 0$.

For example, let $\pi = 34152$, $i = 2$. Then $\pi w_0 = 32514$, which covers 23514, 31524, 32154, 32415. But only 31524 involves switching a number in the first 2 places with a number in the last $5-2$ places. So $c_{34152,s_i,31524} = 1$, but $c_{34152,s_i,23514} = c_{34152,s_i,32154} = c_{34152,s_i,32415} = 0$.

Theorem 5.2. *Let $f : V(\Gamma_n) \to \mathbb{Z}$ be a functional on the set of Schubert problems. If f satisfies the properties*

1. *f is invariant under descent-cycling (i.e., is constant on components),*
2. *$f = 0$ on dc-trivial Schubert problems, and*
3. *$f(\mathrm{id}, \mathrm{id}, w_0) = 1$,*

then f obeys Monk's rule; that is, $f(\pi, s_i, \sigma)$ equals 0 or 1 according to the criterion of Monk's rule.

We first prove a lemma:

Lemma 5.3. *Let f satisfy the conditions of Theorem 5.2, and $\pi, \sigma \in S_n$ such that $l(\pi) + l(\sigma) = \binom{n}{2}$. Then if $\pi = \sigma w_0$, we have $f(\pi, \mathrm{id}, \sigma) = 1$, and otherwise $f(\pi, \mathrm{id}, \sigma) = 0$.*

Proof. Since the second argument has no descents, any place $(i,\, i{+}1)$ that π has a descent and σ does not gives us an opportunity to cycle a descent from the first argument to the third, replacing $\pi \mapsto s_i\pi$ and $\sigma \mapsto s_i\sigma$. This modification keeps the sum of the lengths $= \binom{n}{2}$ and neither causes nor breaks the condition $\pi = \sigma w_0$. So we can reduce to the case that any descent in π occurs at a descent of σ.

If $\pi = \sigma w_0$, each ascent in σ occurs at a descent of π. By our reduction above, this means that σ has no ascents. So we're looking at $f(\mathrm{id}, \mathrm{id}, w_0)$ which by assumption is 1.

Conversely, if $f(\pi, \mathrm{id}, \sigma) \neq 0$, then no column $(i,\, i{+}1)$ has three ascents (the Schubert problem $(\pi, \mathrm{id}, \sigma)$ is not dc-trivial). By our reduction, this means that σ has no ascents. So $\sigma = w_0$. By the assumption on the total length, $\pi = \mathrm{id}$, so $\pi = \sigma w_0$ as desired. □

Proof. Theorem 5.2] Let σ be πw_0 with the numbers in the j-th and k-th positions switched, decreasing the number of inversions by exactly one (and so that $j < k, \sigma(j) < \sigma(k)$). In particular every number in σ physically between the j-th and k-th positions is *not* numerically between $\sigma(j)$ and $\sigma(k)$. We want to show that $f(\pi, s_i, \sigma) = 0$ unless $j \leq i < k$, in which case $f(\pi, s_i, \sigma) = 1$.

First we treat the case $k = j+1$. If $j = i$ and $k = i+1$, neither π nor σ has a descent at $(i,\, i{+}1)$. So we can cycle the descent from the second argument of f into the third, making them $(\pi, \mathrm{id}, \pi w_0)$. Now Lemma 5.3 tells us that this 1. If, on the other hand, $j, k \leq i$ or $j, k \geq i+1$ (still in the case $k = j+1$), then none of π, id, or σ have a descent at (j, k), and therefore f vanishes as it's supposed to.

Now take the case $k > j{+}1$. Then since σ has only one fewer inversion than πw_0, σ must have the same descent-pattern as πw_0. Now we reduce (much as in Lemma 5.3) by cycling descents between the first and third arguments, in order to move the positions j and k closer together.

We can do this descent-cycling in the $(j,\, j{+}1)$ column as long as $j \neq i$, and the $(k{-}1,\, k)$ column as long as $k \neq i+1$. If j, k are both on the same side of the $(i,\, i{+}1)$ divide, they can be brought next to each other (by e.g. just moving one of them). If j, k are on opposite sides of the divide, we can at least get j up to i, and k down to $i+1$. Either way we reduce to the $k = j+1$ case and therefore get the same answer as Monk's rule. □

In particular, this gives us an explicit sequence of descent-cyclings to turn a Monk's rule problem into $(\mathrm{id}, \mathrm{id}, w_0)$. So in principle one can reverse the steps and construct the flag in the intersection of these three Schubert varieties, as an expression in the lattice of subspaces.

There are other special cases known for symmetric Schubert numbers where the answer is 0 or 1, mostly notably the Pieri rule [Bergeron and Sottile 1998; Lascoux and Schützenberger 1982; Robinson $\geq$ 2001; Sottile 1996]; it would be interesting to see if they too are consequences of descent-cycling. Probably the best version of this would be a "descent-cycling normal form" for Schubert problems, and an effective way to test whether a Schubert problem is dc-equivalent to $(\mathrm{id}, \mathrm{id}, w_0)$.

6. SYNTHETICITY VERSUS c = 1 QUESTIONS

Recall that given a Schubert problem $P = (\pi, \rho, \sigma)$, and three generically situated flags A, B, C, one can think of the symmetric Schubert number $c_{\pi\rho\sigma}$ as the number of flags F such that F is π-close to A, ρ-close to B, and σ-close to C.

The Schubert problem $(\mathrm{id}, \mathrm{id}, w_0)$ is then easily seen to have symmetric Schubert number one; to be id-close to A or B is no condition at all, and to be w_0-close to C requires $F_i = C_i$ for all $i = 1, \dots, n$.

Consider the following four statements one might make about a Schubert problem $P = (\pi, \rho, \sigma)$:

- *dc-easiness:* P is dc-equivalent to the Schubert problem $(\mathrm{id}, \mathrm{id}, w_0)$.
- *partial syntheticity:* There is a flag in the free modular lattice on three flags A, B, C satisfying P.
- *full syntheticity:* Every flag satisfying P is in the free modular lattice on three flags A, B, C.
- $c = 1$: The symmetric Schubert number $c_{\pi\rho\sigma}$ equals 1.

So P dc-easy implies P fully synthetic and $c_P = 1$. The other possible implications seem to be unknown.

Question. Does $c = 1$ imply partial (and thus full) syntheticity? This would seem to be a Galois theory argument, with "synthetic" the analogue of "rational."

Question. Does P partially synthetic imply P stably dc-easy? If the flag F is a synthetic solution to the Schubert problem P, i.e., F_i is a lattice word in A, B, C for each i, perhaps there is an algorithm to "simplify" the "worst" subspace in P using descent-cycling, with the only unsimplifiable subspaces being those in A, B, C. In particular, this would say that P partially synthetic implies P fully synthetic.

ADDENDUM

Since the submission of this article, Kevin Purbhoo has shown me a very simple argument that partial syntheticity already implies $c = 1$ (and therefore full syntheticity). Contrapositively, if $c > 1$, none of the solutions can be synthetic. Details will appear elsewhere.

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REFERENCES

[Bergeron and Sottile 1998] N. Bergeron and F. Sottile, "Schubert polynomials, the Bruhat order, and the geometry of flag manifolds", *Duke Math. J.* **95**:2 (1998), 373–423.

[Bernstein et al. 1973] I. N. Bernšteĭn, I. M. Gel'fand, and S. I. Gel'fand, "Schubert cells, and the cohomology of the spaces G/P", *Uspehi Mat. Nauk* **28**:3 (1973), 3–26. In Russian.

[Demazure 1974] M. Demazure, "Désingularisation des variétés de Schubert généralisées", *Ann. Sci. École Norm. Sup.* (4) **7** (1974), 53–88. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.

[Graham 1999] W. Graham, "Positivity in equivariant Schubert calculus", preprint, 1999. See www.arxiv.org/abs/math.AG/9908172.

[Knutson and Tao $\geq$ 2001] A. Knutson and T. Tao, "Puzzles, (equivariant) cohomology of Grassmannians, and the Molev–Sagan rule". In preparation.

[Kostant and Kumar 1986] B. Kostant and S. Kumar, "The nil Hecke ring and cohomology of G/P for a Kac–Moody group G", *Adv. in Math.* **62**:3 (1986), 187–237.

[Lascoux and Schützenberger 1982] A. Lascoux and M.-P. Schützenberger, "Polynômes de Schubert", *C. R. Acad. Sci. Paris Sér. I Math.* **294**:13 (1982), 447–450.

[Monk 1959] D. Monk, "The geometry of flag manifolds", *Proc. London Math. Soc.* (3) **9** (1959), 253–286.

[Robinson $\geq$ 2001] S. Robinson, "The equivariant Pieri rule". In preparation.

[Sottile 1996] F. Sottile, "Pieri's formula for flag manifolds and Schubert polynomials", *Ann. Inst. Fourier (Grenoble)* **46**:1 (1996), 89–110.

[Veigneau 1998] S. Veigneau, "ACE, an algebraic combinatorics environment for the computer algebra system Maple: user's reference manual, version 3.0", Technical Report IGM 98–11, Université de Marne-la-Vallée, France, 1998.

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